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Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux, Patrick Joly

► **To cite this version:**

Sébastien Tordeux, Patrick Joly. Matching of asymptotic expansions for the wave propagation in media with thin slot. TiSCoPDE workshop (New Trends in Simulation and Control of PDEs), WIAS, 2005, Berlin, Germany. inria-00528072

HAL Id: inria-00528072

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Submitted on 21 Oct 2010

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Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux and Patrick Joly

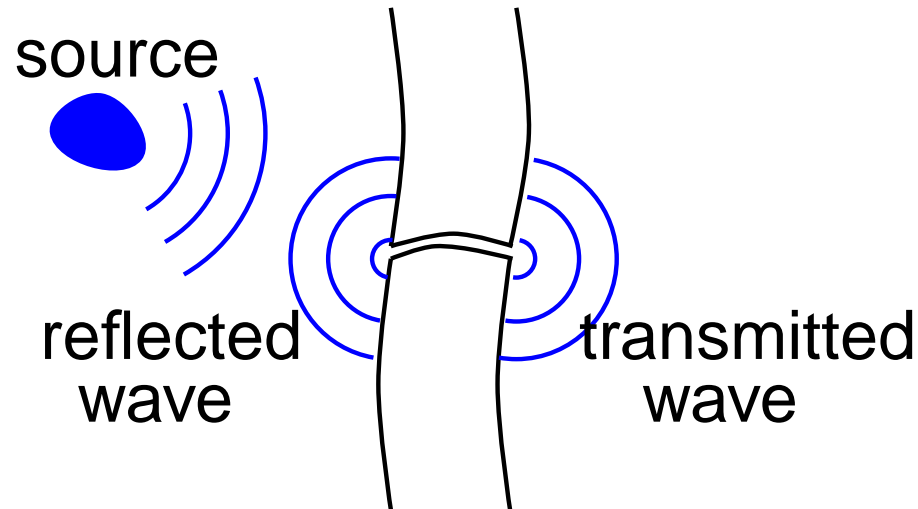
TiSCoPDE workshop, Berlin, September 2005

INRIA-Rocquencourt-Projet POEMS

ETH-SAM

A typical application

How can we study the scattering in media with **thin slot** ?

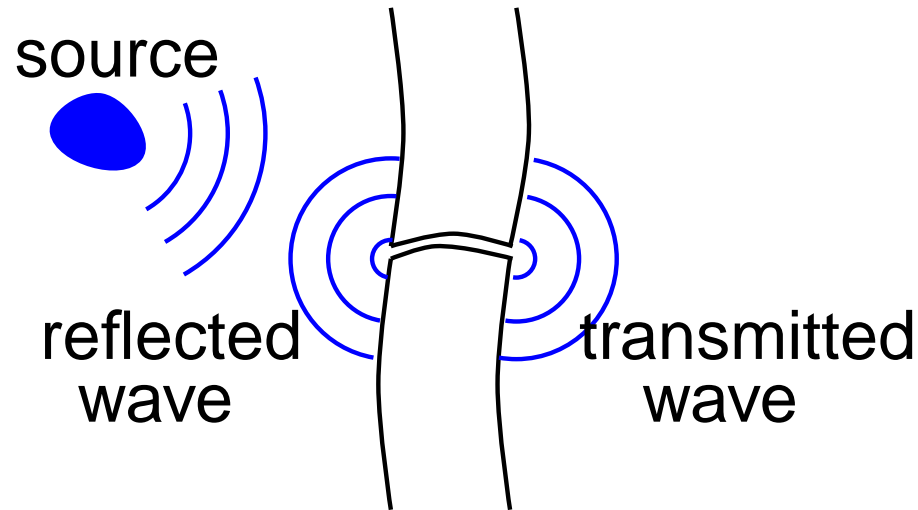


A physical problem with two **characteristical** lengthes

- The **wavelength** λ
- The **width** of the slot ε

A typical application

How can we study the scattering in media with **thin slot** ?

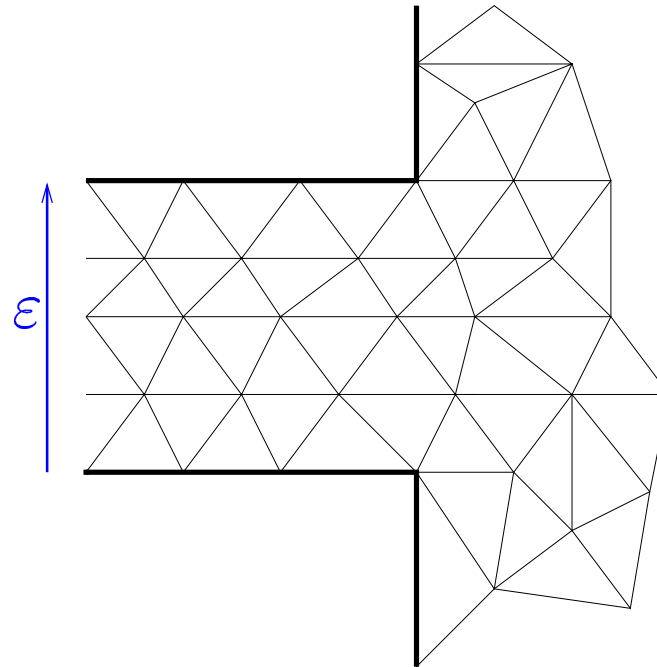


An **asymptotic** case:

$$\varepsilon \ll \lambda$$

The numerical difficulty

A **mesh step** smaller than ε



This leads to **costly** computations

Some references

- Thin slot:
[Harrington, Auckland](#) (1980), [Tatout](#) (1996).
- Finite differences:
[Taflove](#) (1995).
- Thin plates and junction theory,...
[Ciarlet, Le Dret, Dauge-Costabel](#).
- Matching of asymptotic expansions:
[McIver, Rawlins](#) (1993), [Il'in](#) (1992).
- multiscale analysis
[Maz'ya, Nazarov, Plamenevskii](#) (1991).
[Oleinik, Shamaev, Yosifian](#) (1992).

A simple problem

Scalar wave equation:

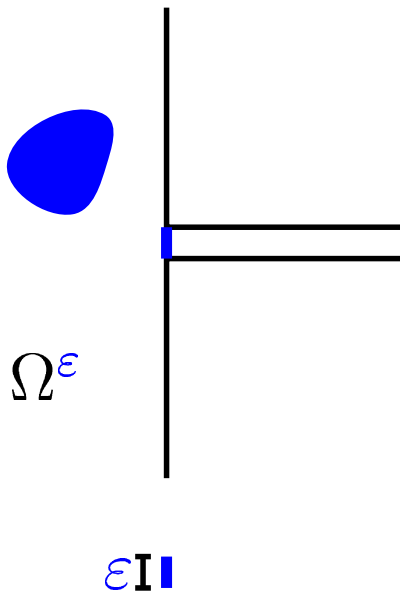
$$\frac{\partial^2 p^\varepsilon}{\partial t^2} - \Delta p^\varepsilon = f$$

Harmonic solution:

$$p^\varepsilon(x, y, t) = \exp(-i\omega t) u^\varepsilon(x, y)$$

Helmholtz Equation:

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = -f \quad \text{in } \Omega^\varepsilon$$



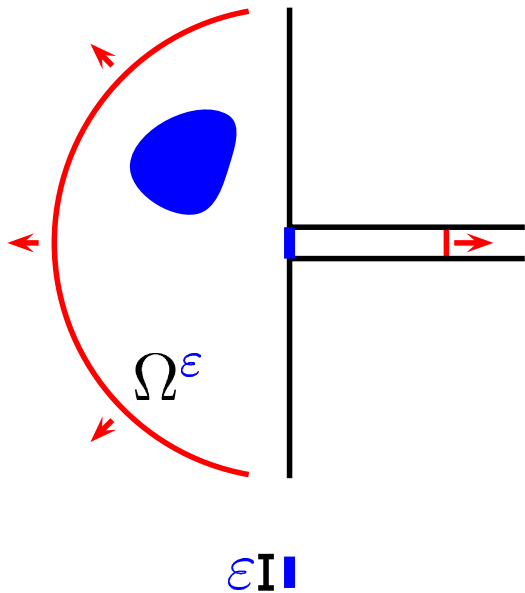
A simple problem

Outgoing solution at infinity:

$$\frac{\partial u^\varepsilon}{\partial n} - i\omega u^\varepsilon \leq \frac{C}{r^2}, \quad \text{for } r \text{ large,}$$

Neumann limit condition
(rigid wall)

$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$



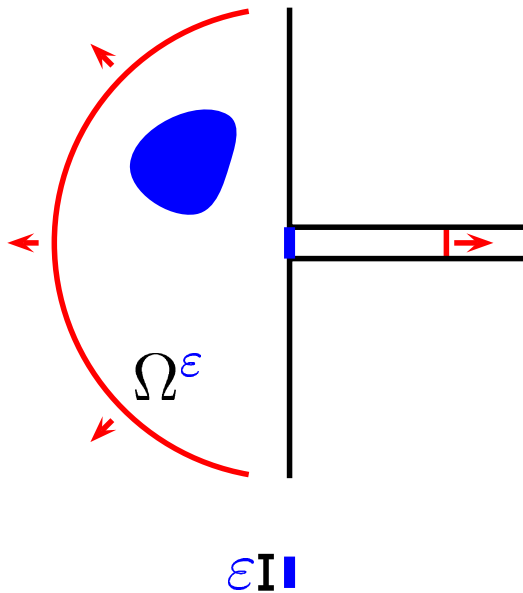
A simple problem

Outgoing solution at infinity:

$$\frac{\partial u^\varepsilon}{\partial n} - i\omega u^\varepsilon \leq \frac{C}{r^2}, \quad \text{for } r \text{ large,}$$

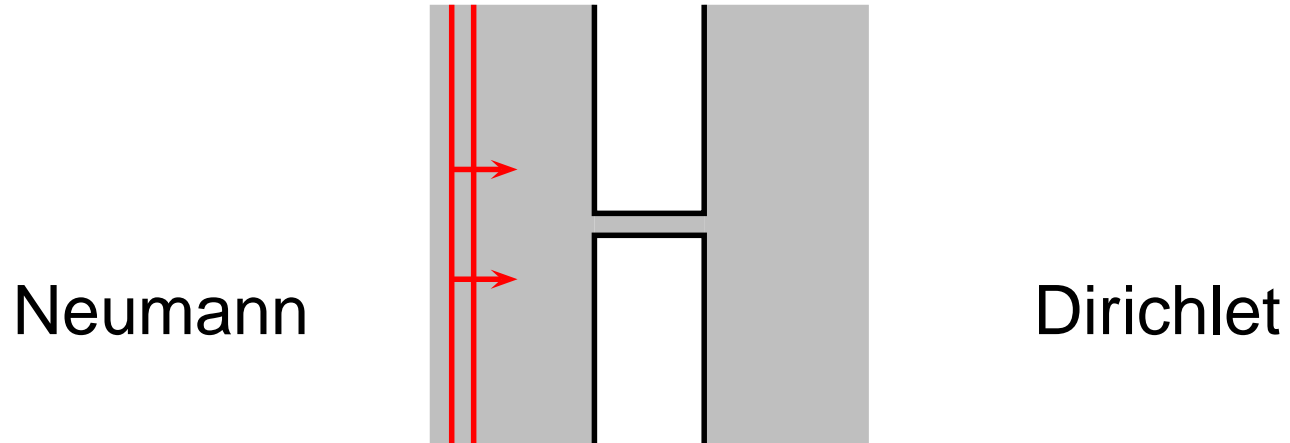
Neumann limit condition
(rigid wall)

$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$



With the **Dirichlet** limit condition, the transmission inside the slot is **negligible** ($o(\varepsilon^\infty)$).

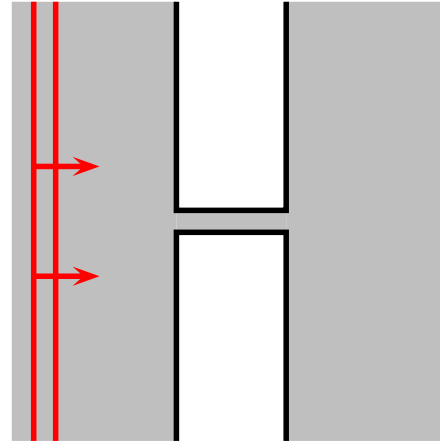
A numerical computation



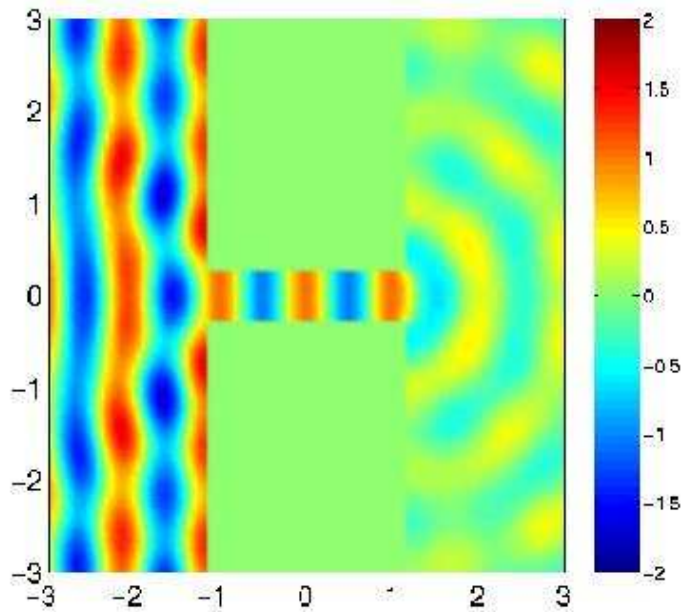
Numerical computation done with the **high order finite elements code** of (M. Duruflé, INRIA)

A numerical computation

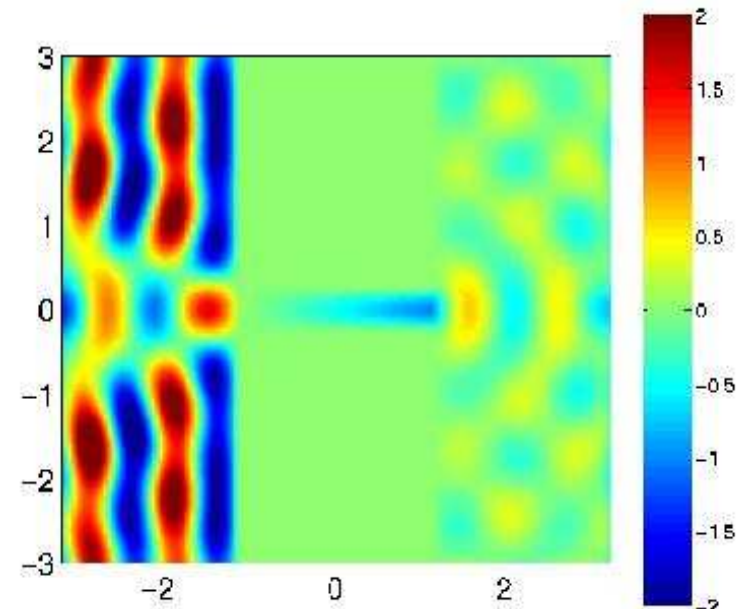
Neumann



Dirichlet

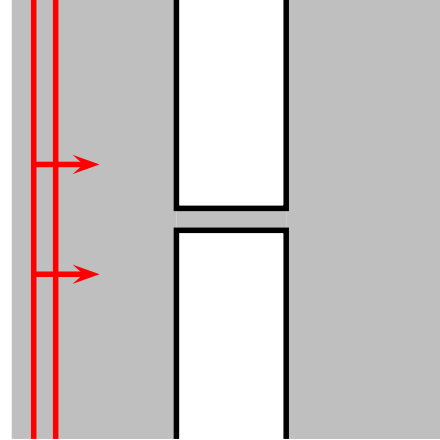


$$\frac{\varepsilon}{\lambda} = 0.5$$

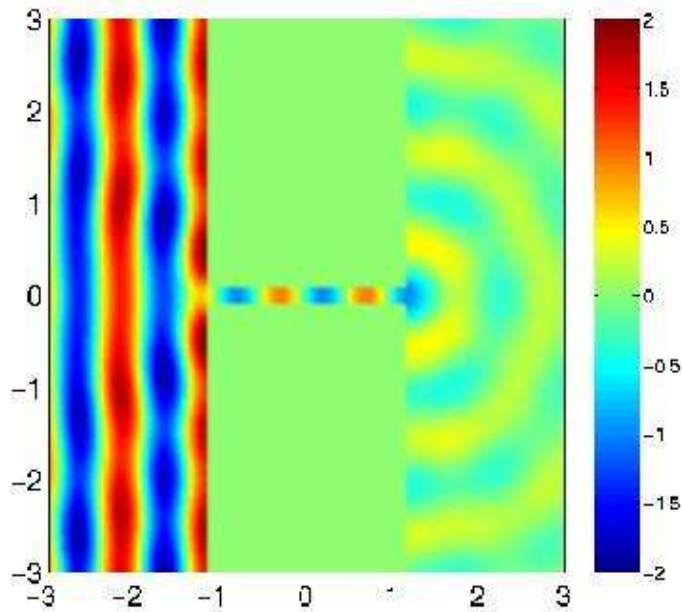


A numerical computation

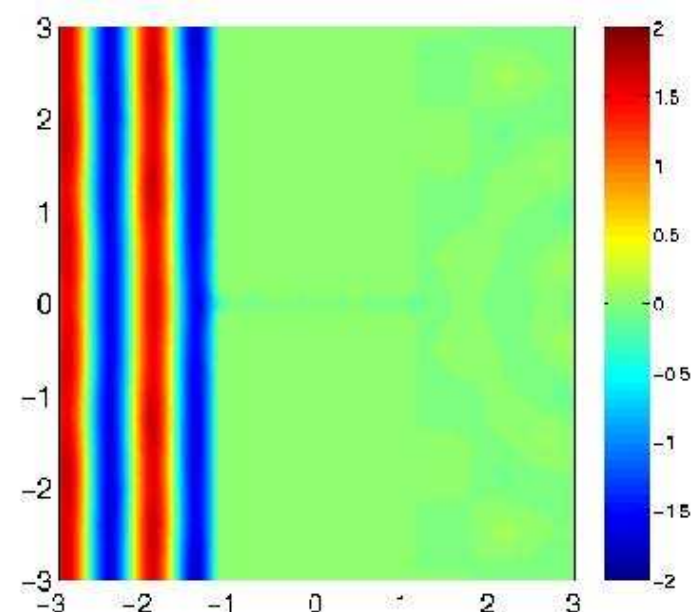
Neumann



Dirichlet

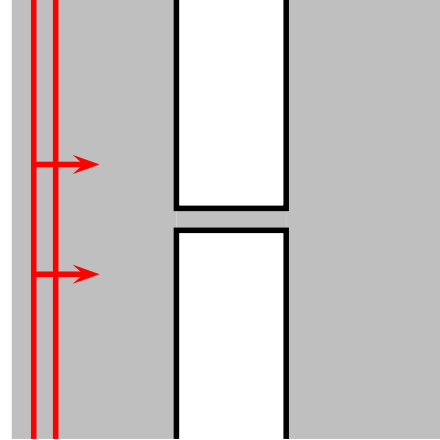


$$\frac{\varepsilon}{\lambda} = 0.2$$

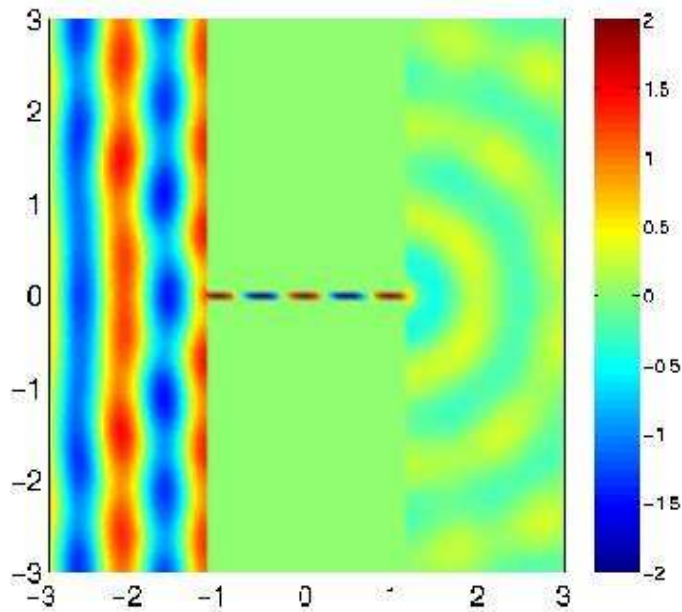


A numerical computation

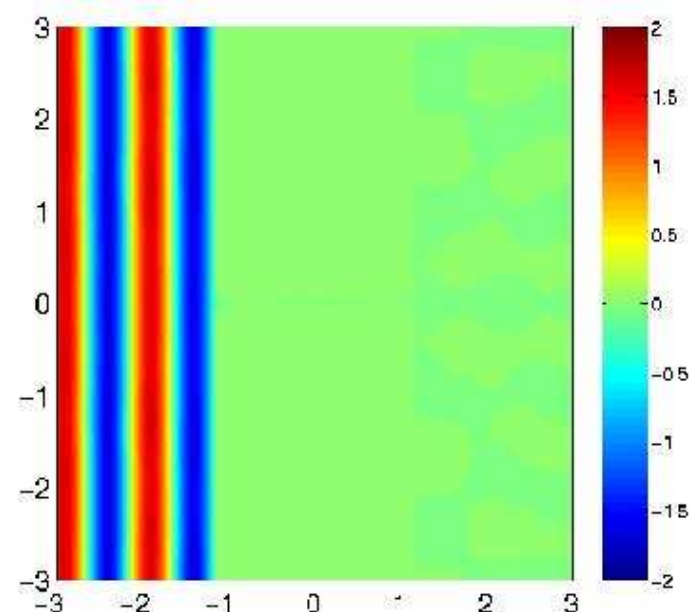
Neumann



Dirichlet



$$\frac{\varepsilon}{\lambda} = 0.1$$

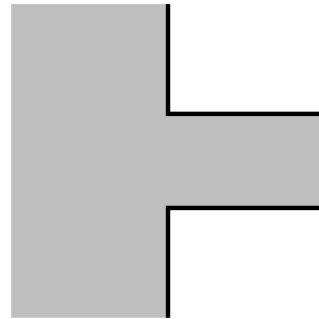


Objectives

- Introduce **accurate** numerical methods

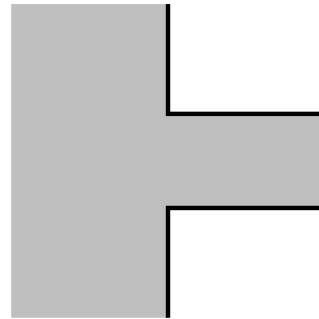
Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



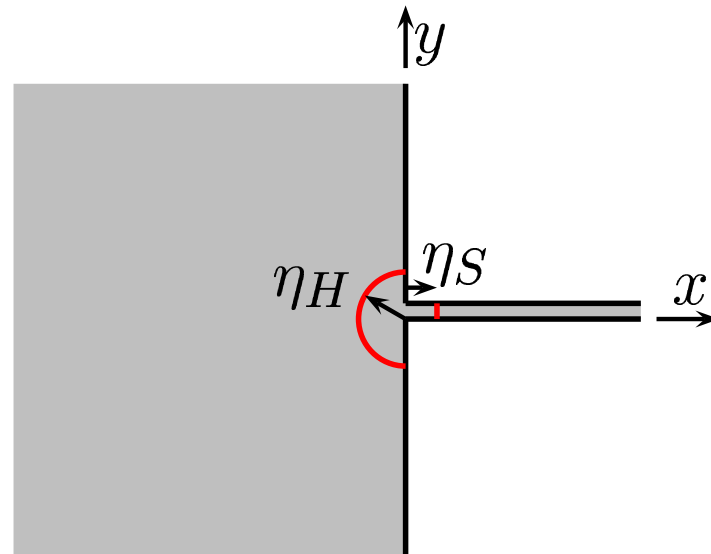
Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



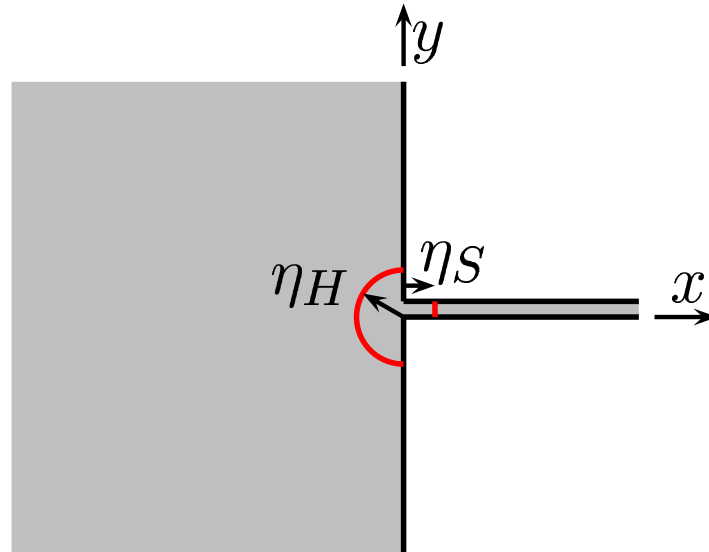
- A technique **the matching of asymptotic expansions**
 - Define **new approximate models** to compute the solution.
 - Use effectively “universal” technique of numerical computation (mesh refinement).

Three zones



- Far field (2D field)
- Near field (boundary layer)
- Slot field (1D field)

Three zones



The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda,$$

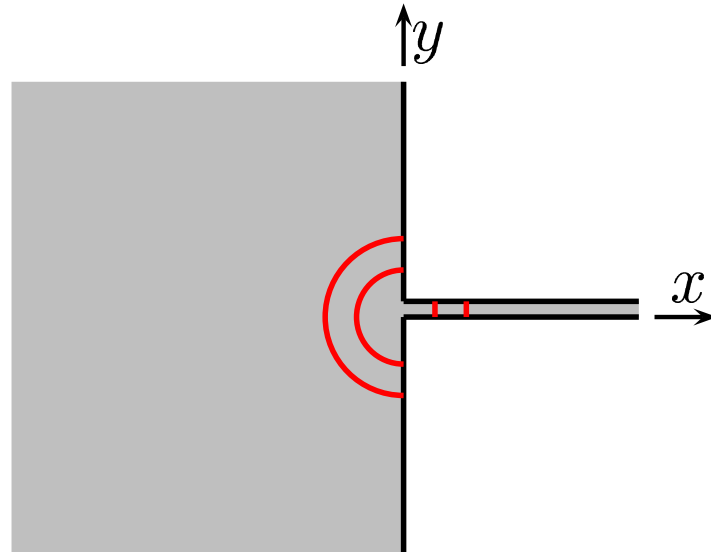
$$\varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0$$

$$\eta(\varepsilon) \rightarrow 0$$

$$\eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones

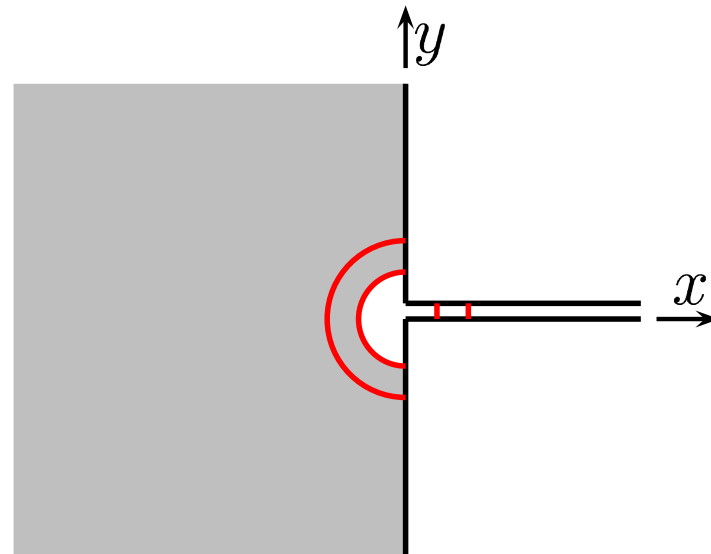


The asymptotic assumptions:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



Far field

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda,$$

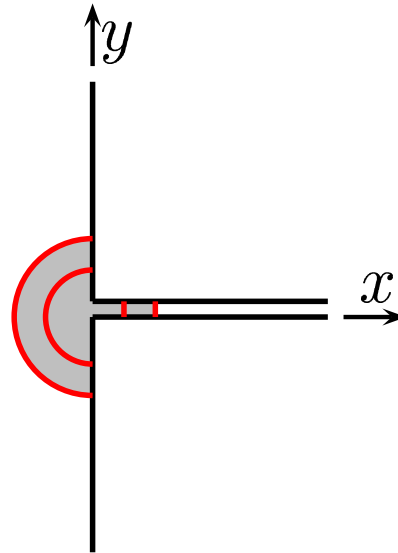
$$\varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0$$

$$\eta(\varepsilon) \rightarrow 0$$

$$\eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



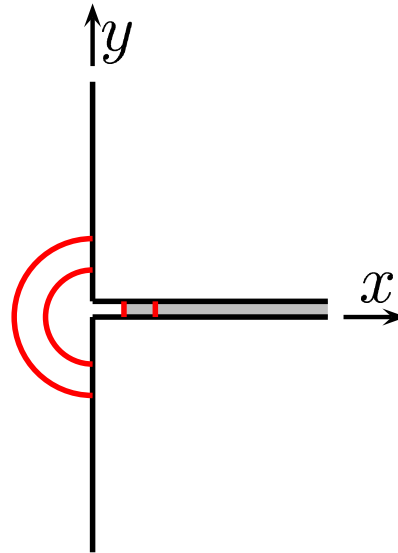
Near field

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



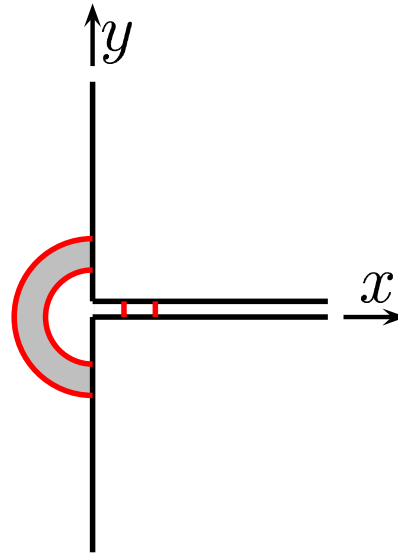
Slot field

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



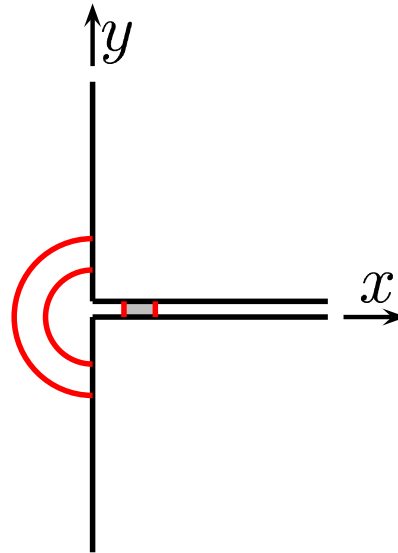
Far and near

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



Slot and near

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

The different steps of the method

- **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible

The different steps of the method

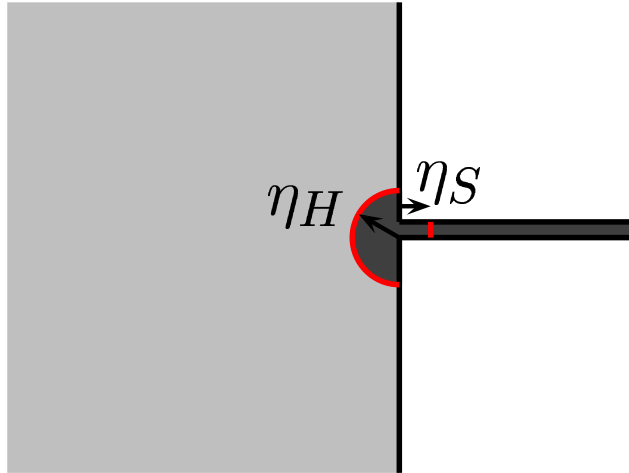
- **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible
- **Describe** the asymptotic expansions
 - **Rigorous** part
 - **Definition** of the terms of the asymptotic expansions

The different steps of the method

- **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible
- **Describe** the asymptotic expansions
 - **Rigorous** part
 - **Definition** of the terms of the asymptotic expansions
- **Mathematical validation** of the asymptotic expansions
 - **Rigorous** part
 - **Error estimates**

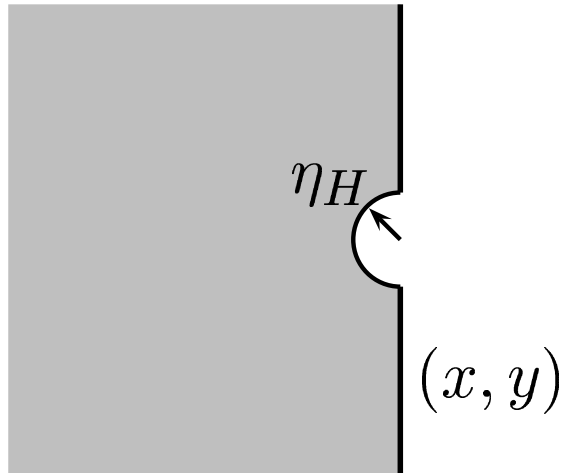
Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda.$



Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda.$



No **normalization**:

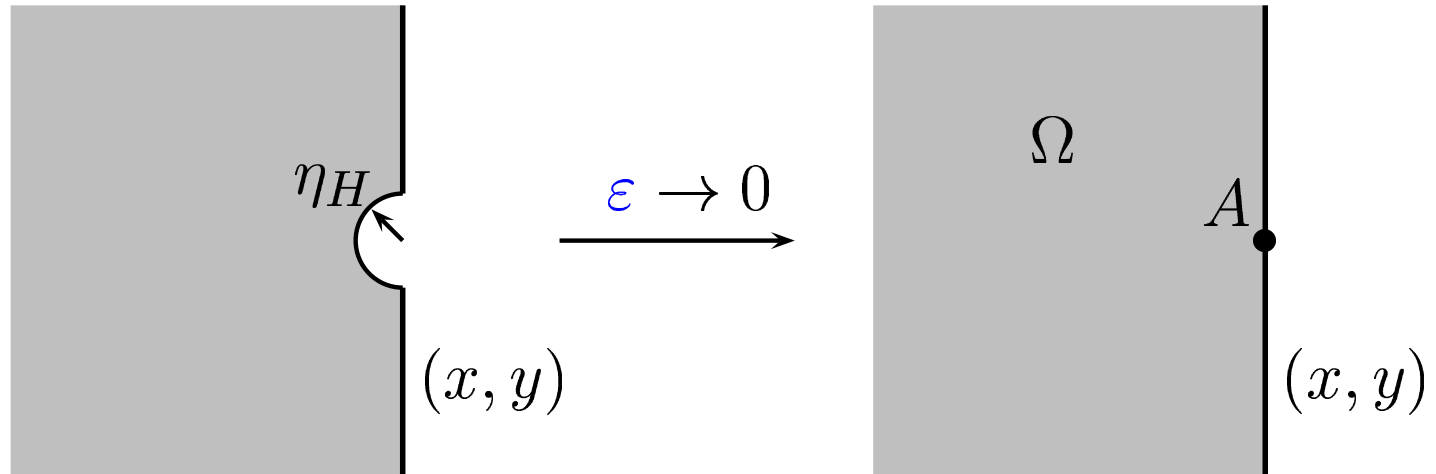
$$X = x,$$

$$Y = y.$$

Far field

Asymptotic context:

$$\varepsilon \ll \eta_H \ll \lambda.$$



No **normalization**:

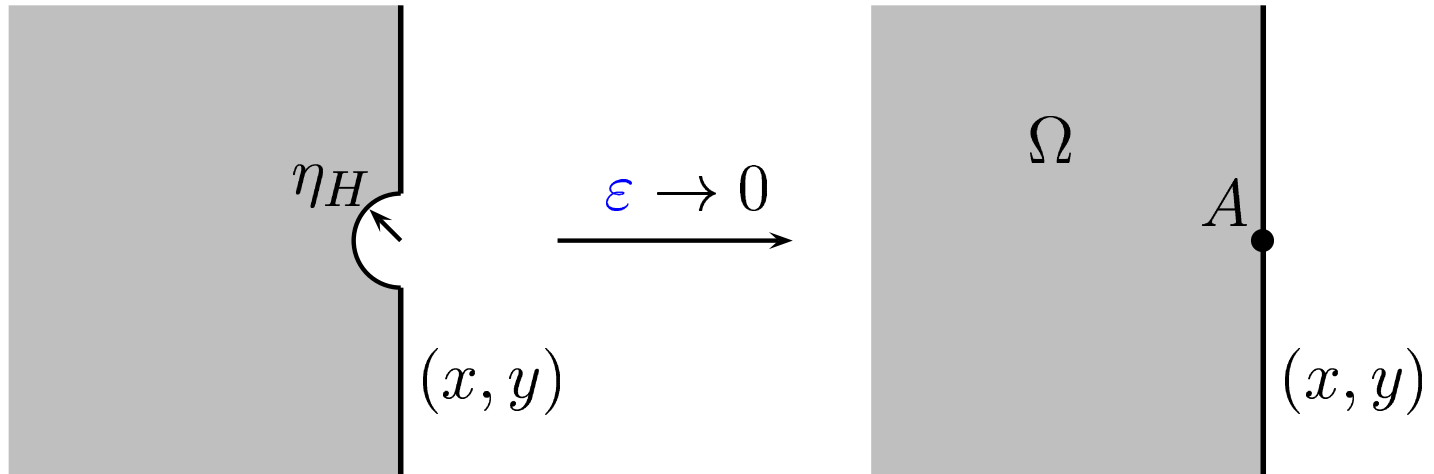
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Far field

Asymptotic context:

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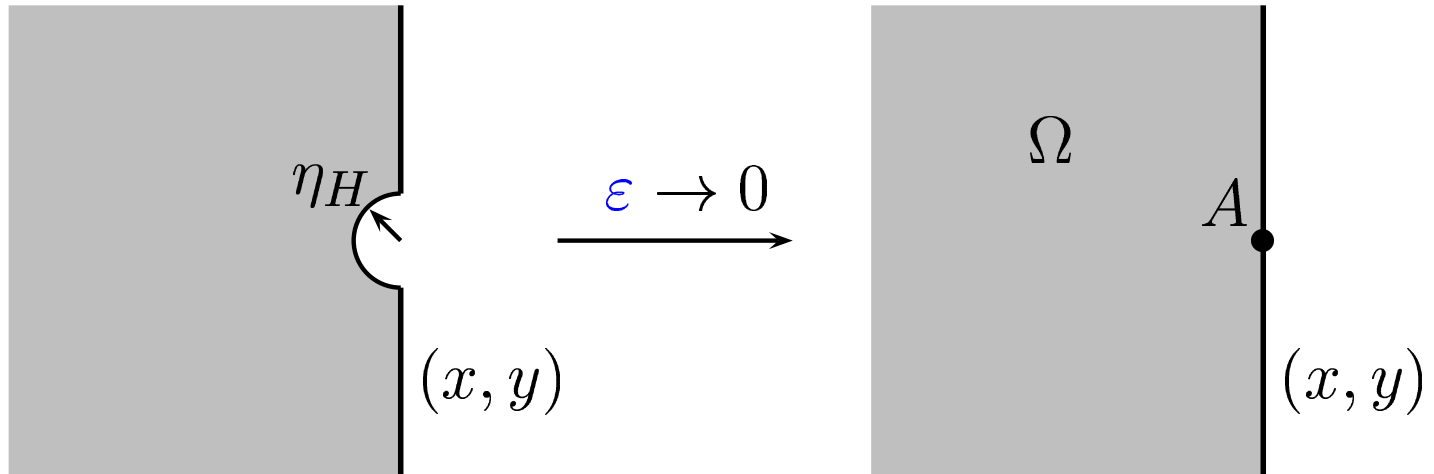


$$u^\varepsilon = u^0 + \sum_{i=1}^{+\infty} \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k + o(\varepsilon^\infty), \quad \text{in } \Omega.$$

Far field

Asymptotic context:

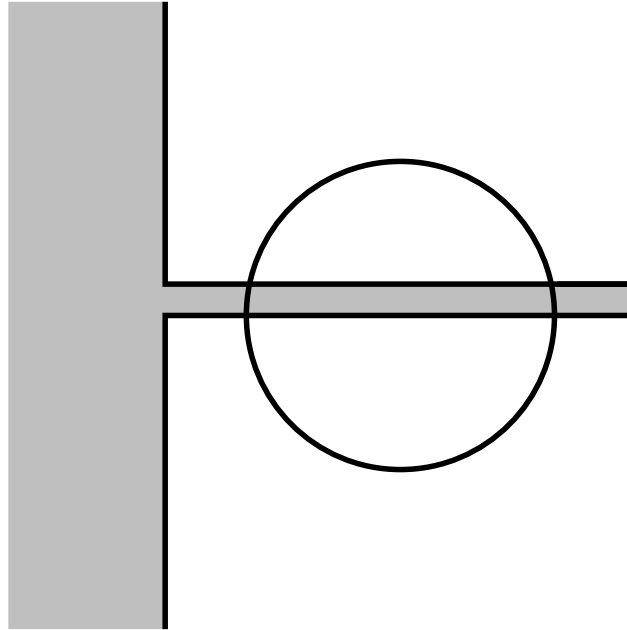
$$\varepsilon \ll \eta_H \ll \lambda.$$



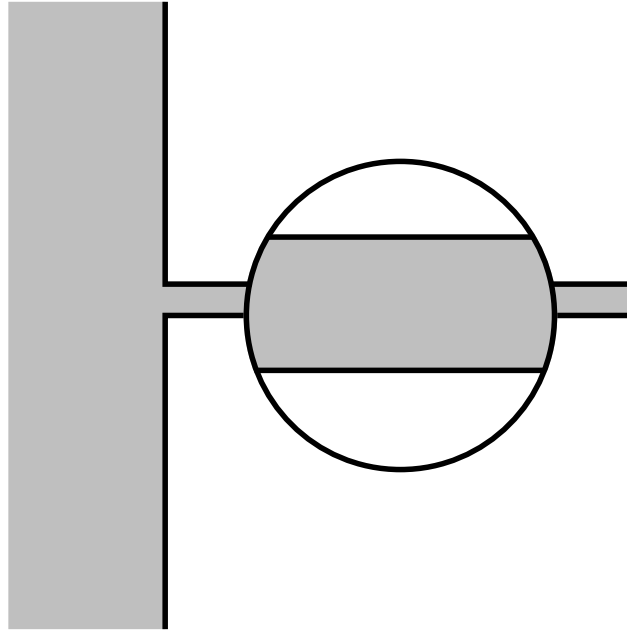
where the u_i^k satisfy the **homogeneous Helmholtz** equation

$$\Delta u_i^k + \omega^2 u_i^k = 0.$$

Slot field

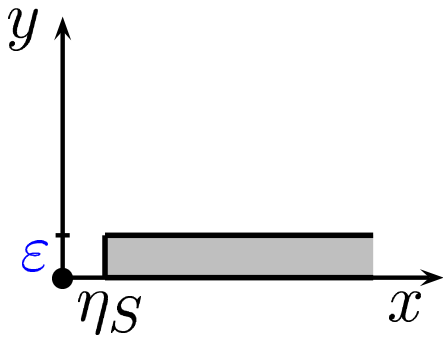


Slot field



$$u^\varepsilon(x, y) = U^\varepsilon\left(x, \frac{y}{\varepsilon}\right)$$

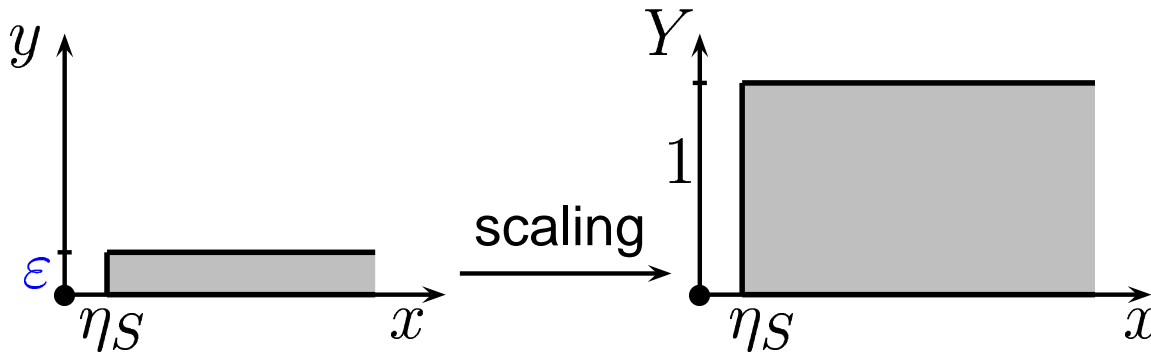
Slot field



The **asymptotic** context: $\varepsilon \ll \eta_S \ll \lambda$.

The **normalization**: $X = x, \quad Y = \frac{y}{\varepsilon}$

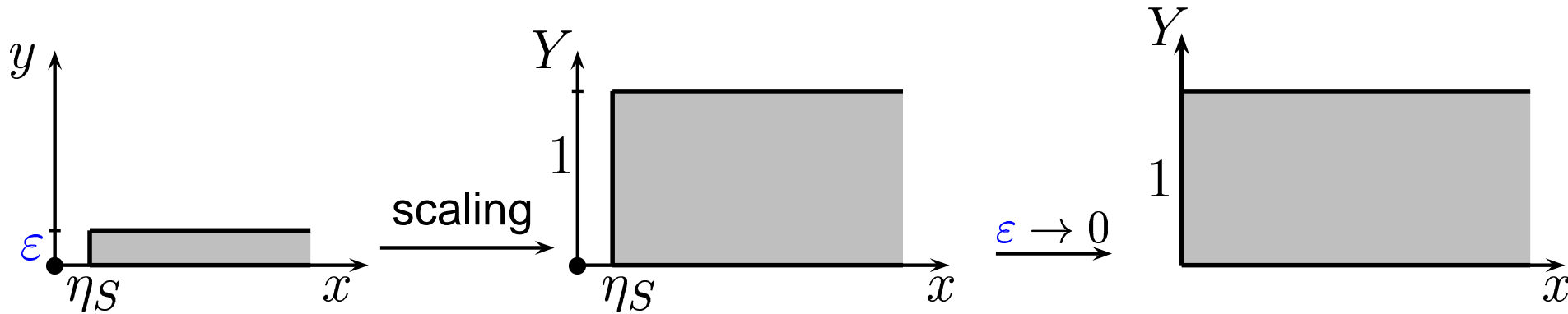
Slot field



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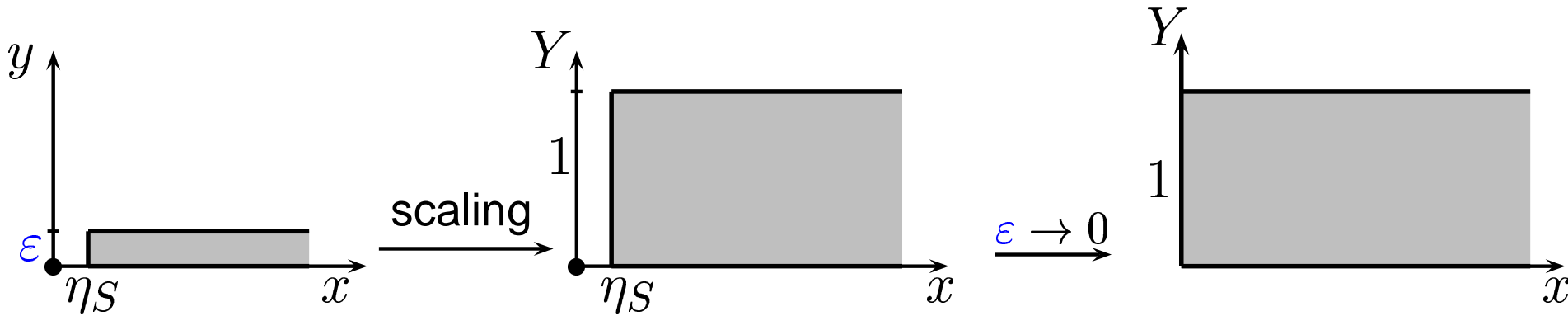
Slot field



The **asymptotic** context: $\varepsilon \ll \eta_S \ll \lambda$.

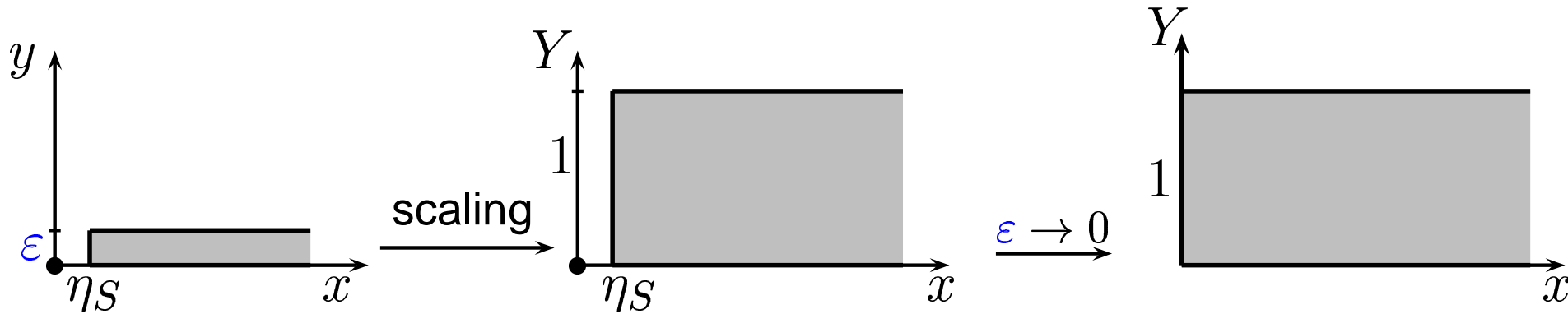
The **normalization**: $X = x, \quad Y = \frac{y}{\varepsilon}$

Slot field



$$u^\varepsilon(x, Y_\varepsilon) = U^\varepsilon(x, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k(x, Y) + o(\varepsilon^\infty),$$

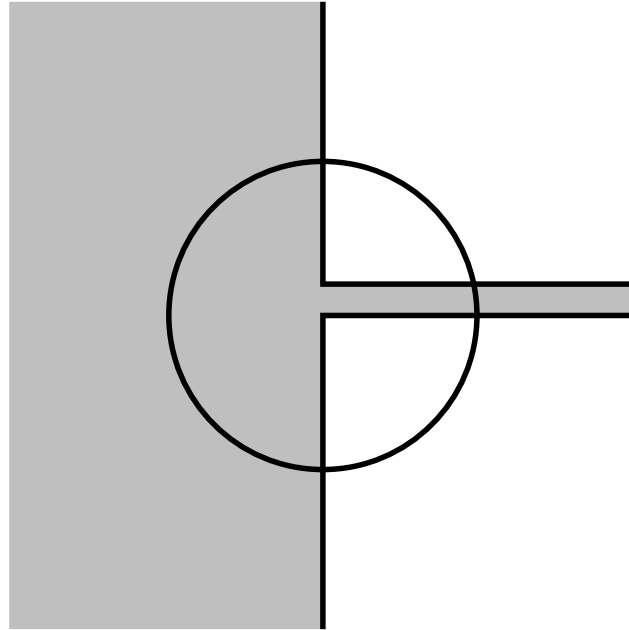
Slot field



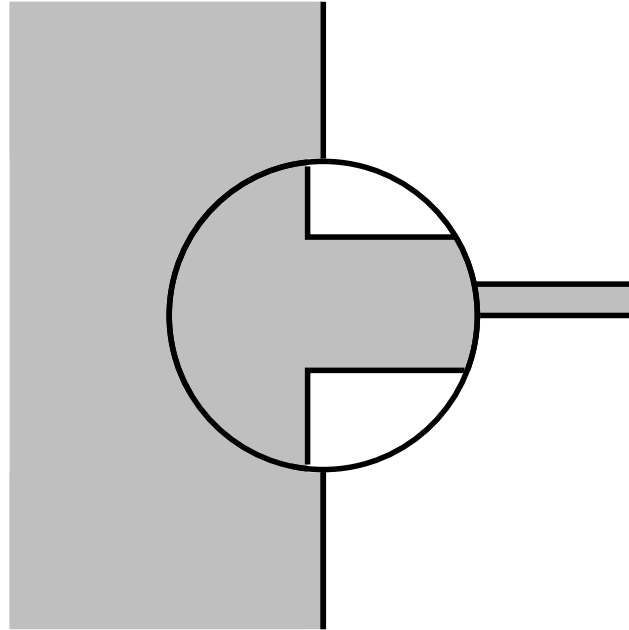
where the U_i^k satisfy the **1D Helmholtz** equation:

$$\frac{d^2 U_i^k}{dx^2} + \omega^2 U_i^k = 0$$

Near field

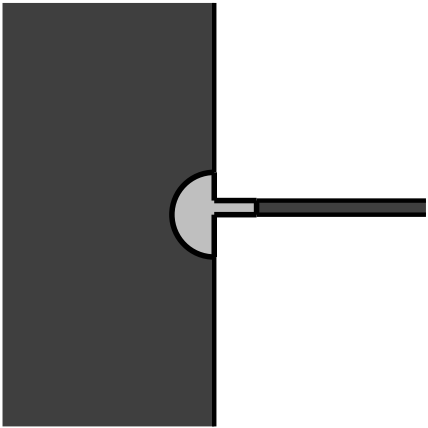


Near field



$$u^\varepsilon(x, y) = u_p^\varepsilon\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

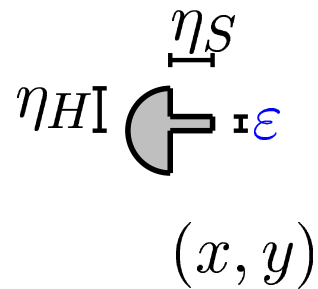
Near field



The **Asymptotic** context: $\varepsilon \ll \eta_H \ll \lambda$, $\varepsilon \ll \eta_S \ll \lambda$.

The **normalization**: $X = \frac{x}{\varepsilon}$, $Y = \frac{y}{\varepsilon}$

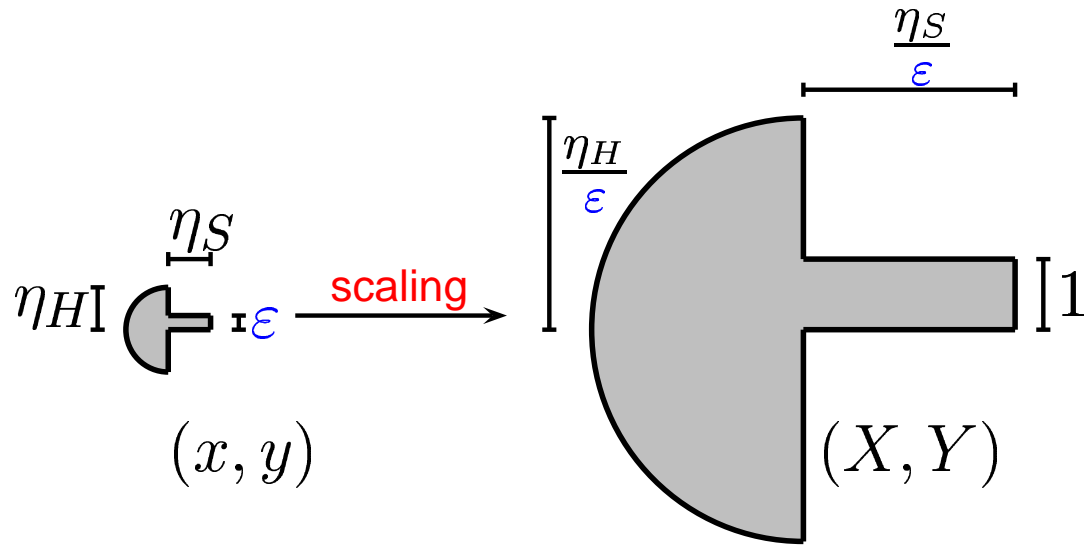
Near field



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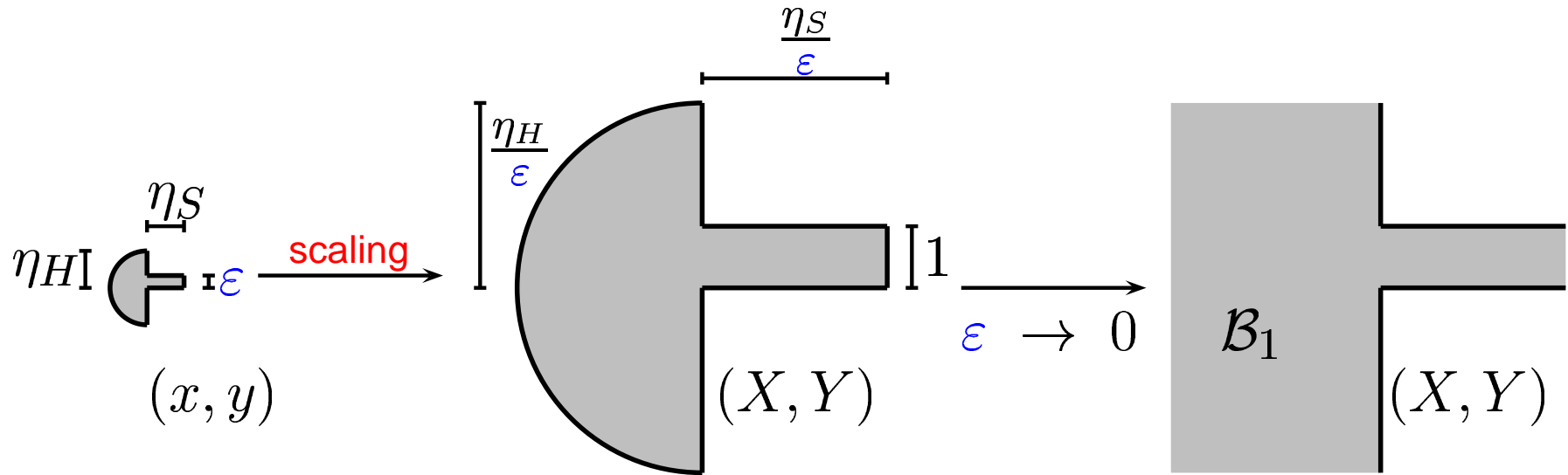
Near field



The **Asymptotic** context: $\epsilon \ll \eta_H \ll \lambda$, $\epsilon \ll \eta_S \ll \lambda$.

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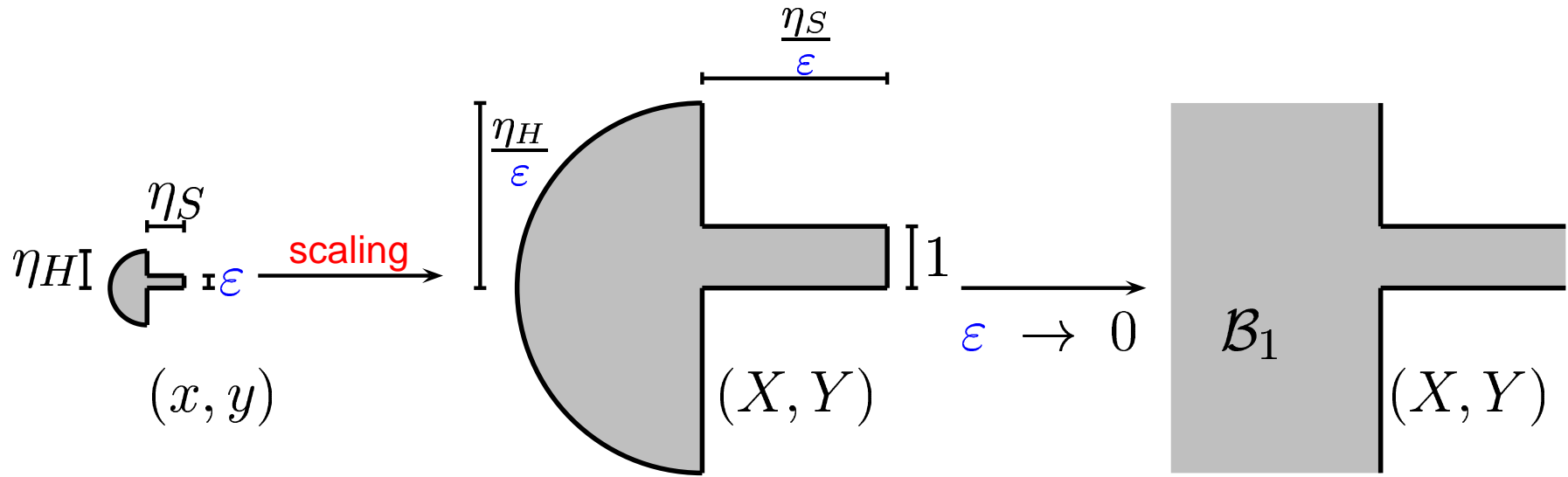
Near field



The **Asymptotic** context: $\epsilon \ll \eta_H \ll \lambda$, $\epsilon \ll \eta_S \ll \lambda$.

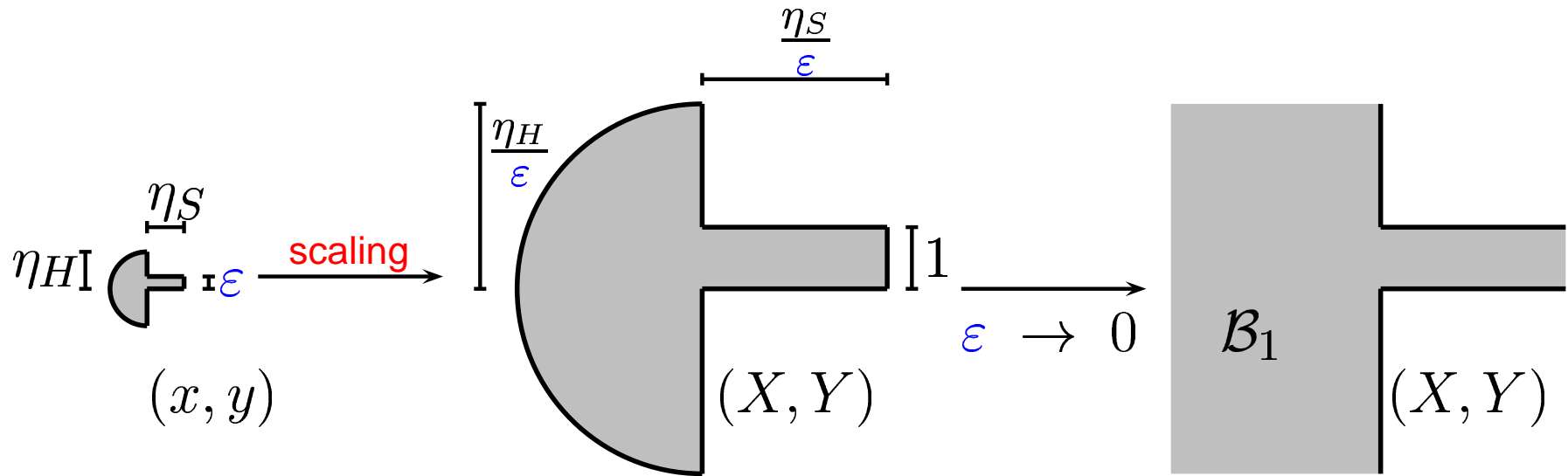
The **normalization**: $X = \frac{x}{\epsilon}$, $Y = \frac{y}{\epsilon}$

Near field



$$u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k(X, Y) + o(\varepsilon^\infty)$$

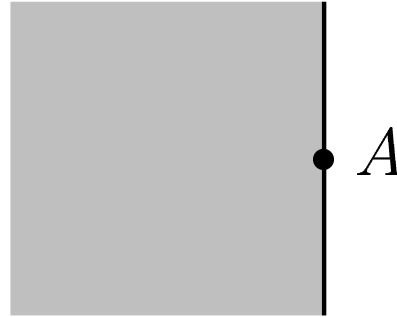
Near field



where the $(u_p)_i^k$ satisfy the (in)-homogeneous Laplace equation.

$$\begin{cases} \Delta(u_p)_i^k = 0, & \text{if } i = k \text{ or } k + 1, \\ \Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, & \text{else.} \end{cases}$$

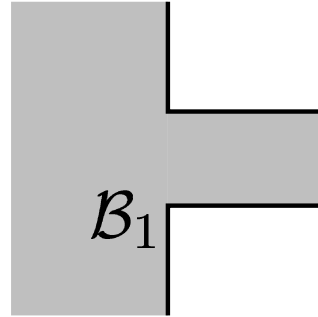
Order 0 : \underline{u}^0 , $(u_p)_0^0$, U_0^0



Far field:

$$\left\{ \begin{array}{l} \text{Find } u^0 \in H_{loc}^1(\Omega) \text{ such that :} \\ -\Delta u^0 - \omega^2 u^0 = f, \quad \text{in } \Omega, \\ \frac{\partial u^0}{\partial n} = 0, \quad \text{on } \partial\Omega, \\ u^0 \text{ is outgoing.} \end{array} \right.$$

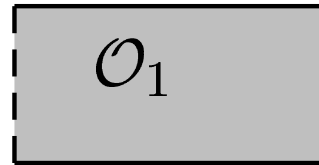
Order 0 : u^0 , $\underline{(u_p)_0^0}$, U_0^0



Near field:

$$(u_p)_0^0(X, Y) = u^0(A), \quad \text{in } B_1.$$

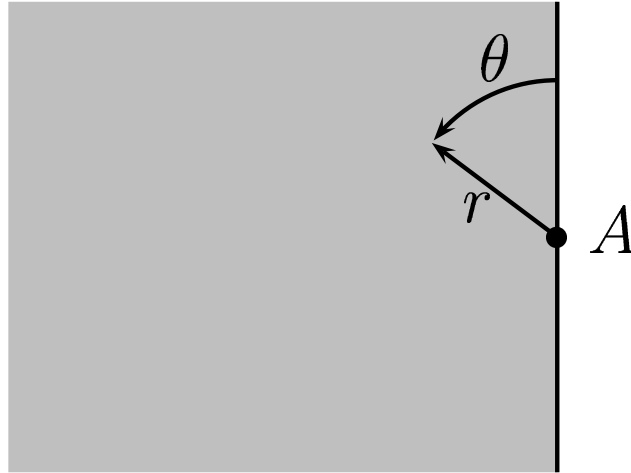
Order 0 : u^0 , $(u_p)_0^0$, U_0^0



Slot field:

$$U_0^0(x, Y) = u^0(A) \exp(i\omega x), \quad \text{in } \mathcal{O}_1.$$

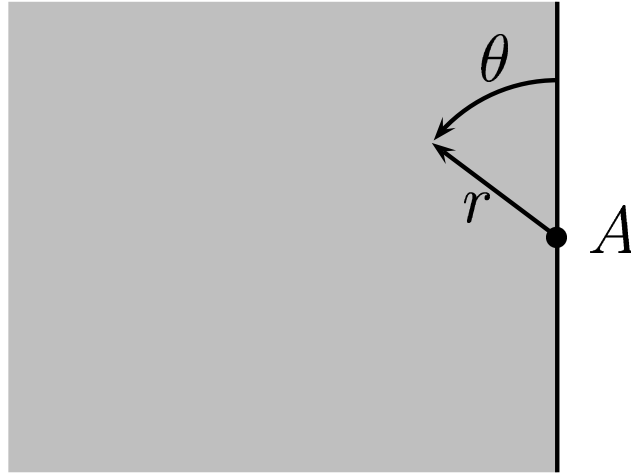
Order 1 : \underline{u}_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1



Approximation of the exact Solution:

$$u^\varepsilon \simeq u^0 + \varepsilon u_1^0$$

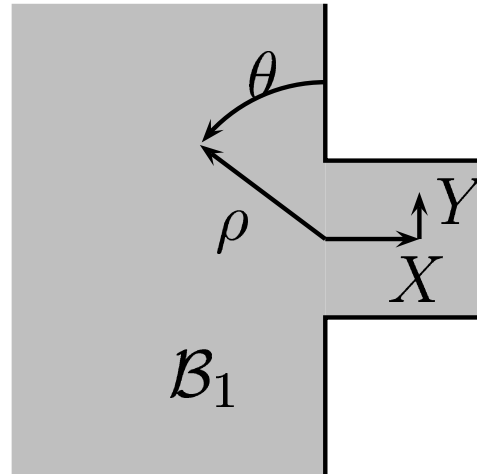
Order 1 : \underline{u}_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1



explicit form of u_1^0

$$u_1^0(r, \theta) = -\frac{\omega}{2} u^0(A) H_0^{(1)}(\omega r).$$

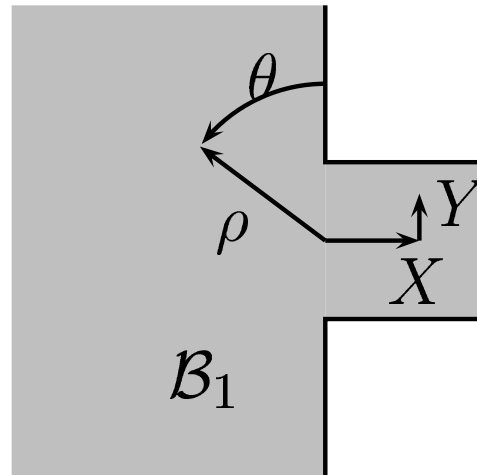
Order 1 : u_1^0 , $\underline{(u_p)_1^0}$, $\underline{(u_p)_1^1}$, U_1^0 , U_1^1



Approximation of the exact solution:

$$\begin{cases} u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y), \\ u_p^\varepsilon \simeq (u_p)_0^0 + \varepsilon (u_p)_1^0 + \varepsilon \log \varepsilon (u_p)_1^1. \end{cases}$$

Order 1 : u_1^0 , $\underline{(u_p)_1^0}$, $(u_p)_1^1$, U_1^0 , U_1^1

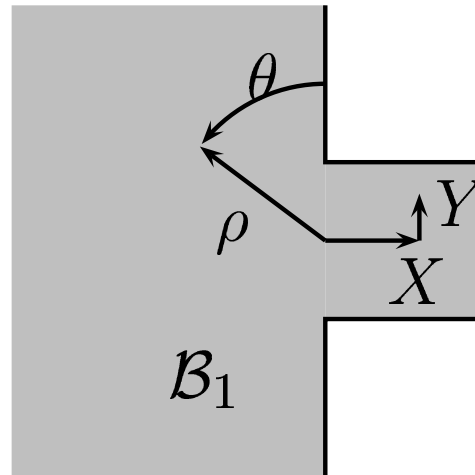


Near field:

Find $(u_p)_1^0 \in H_{loc}^1(\mathcal{B}_1)$ such that:

$$\begin{cases} \Delta(u_p)_1^0 = 0, & \text{in } \mathcal{B}_1 \\ \frac{\partial(u_p)_1^0}{\partial n} = 0, & \text{on } \partial\mathcal{B}_1. \end{cases}$$

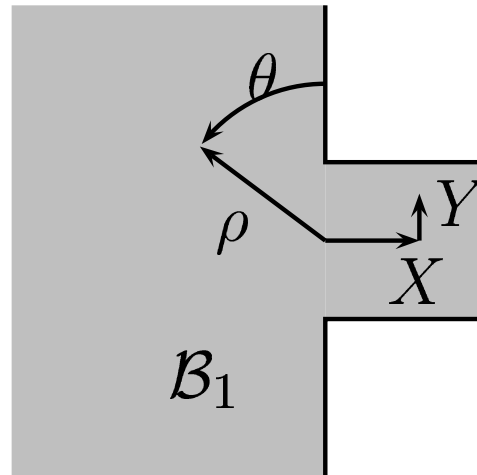
Order 1 : u_1^0 , $\underline{(u_p)_1^0}$, $(u_p)_1^1$, U_1^0 , U_1^1



Behavior at infinity in the half-space:

$$(u_p)_1^0(\rho, \theta) - \frac{\partial u^0}{\partial y}(A) \rho \cos \theta + \frac{\omega}{2} u^0(A) \left[1 + \frac{2i}{\pi} (\log \rho + \gamma) \right] = O\left(\frac{1}{\rho}\right).$$

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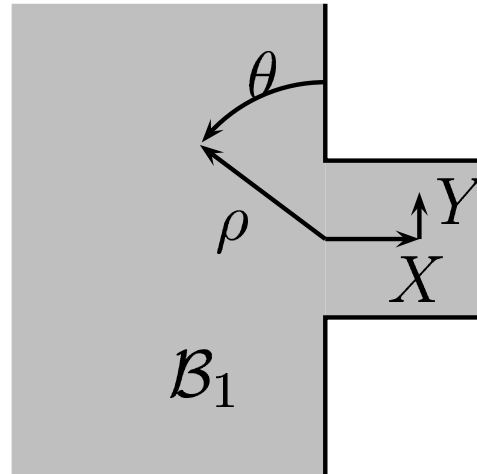
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Behavior at infinity in the slot:

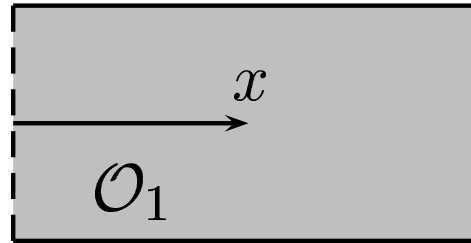
$$(u_p)_1^0(X, Y) - i \omega u^0(A) X = O(1).$$

Order 1 : u_1^0 , $(u_p)_1^0$, $\underline{(u_p)_1^1}$, U_1^0 , U_1^1



$$(u_p)_1^1 = -\frac{\mathbf{i}\omega}{\pi} u^0(A)$$

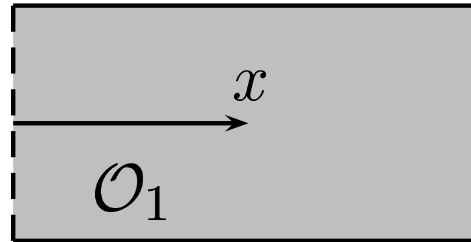
Order 1 : $u_1^0, (u_p)_1^0, (u_p)_1^1, \underline{U}_1^0, \underline{U}_1^1$



Approximation of the exact solution:

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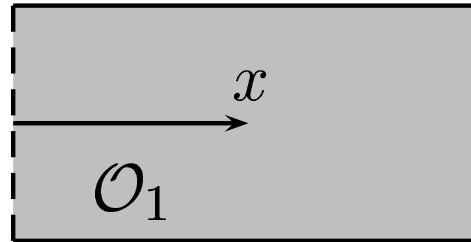
Order 1 : u_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1



The slot field:

$$U_1^0(x) = \int_0^1 (u_p)_1^0(0, Y) dY \exp(i\omega x),$$

Order 1 : u_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1

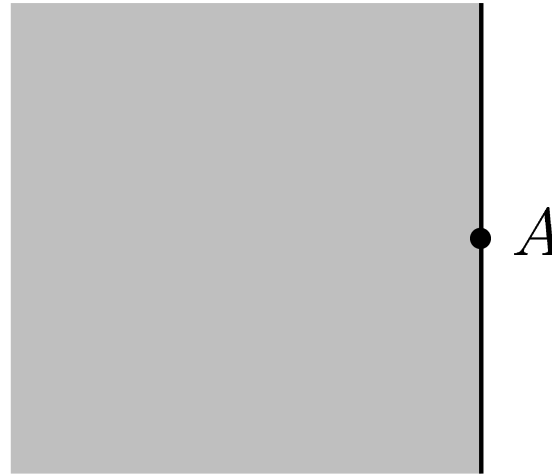


The slot field:

$$U_1^1(x) = -\frac{\mathbf{i}\omega}{\pi} u^0(A) \exp(\mathbf{i}\omega x).$$

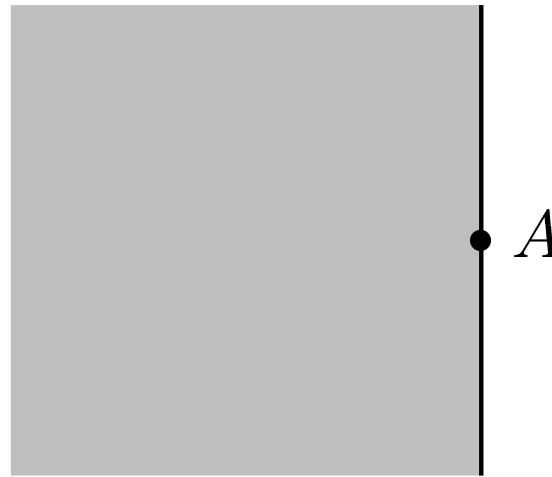
The far field of order $i > 1$

- The fields u_i^k are defined in the **half space**:



The far field of order $i > 1$

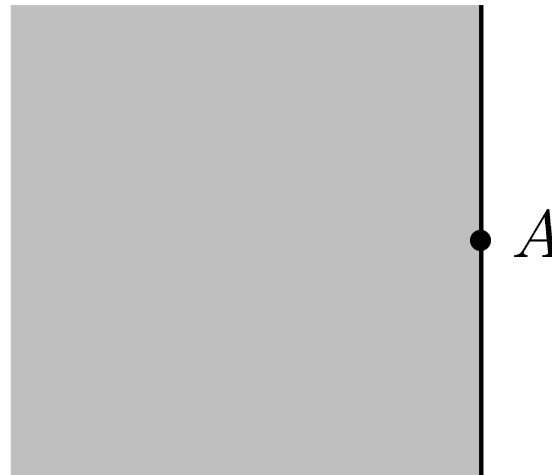
- The fields u_i^k are defined in the **half space**:



- The far fields u_i^k
 - satisfy the **homogeneous Helmholtz** equation
 - are **singular** at the neighborhood of the origin
 - are outgoing at infinity

The far field of order $i > 1$

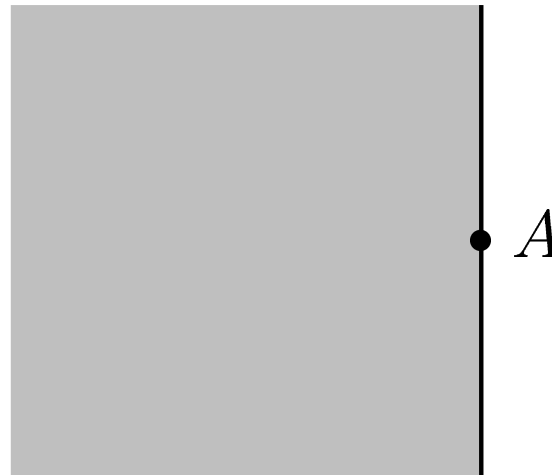
- The fields u_i^k are defined in the **half space**:



- $$u_i^k = \sum_{p=0}^{+\infty} a_p H_p^{(1)}(\omega r) \cos p\theta$$

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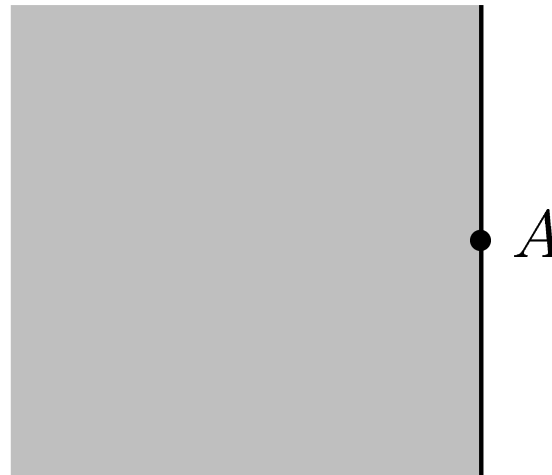
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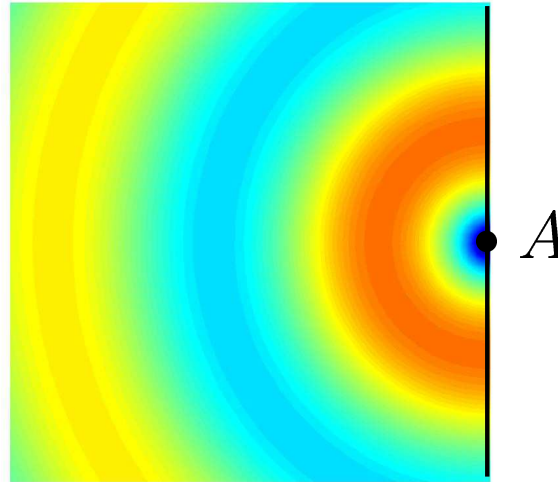
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The a_p are functions of **lower order** terms

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- The fields u_i^k are defined in the **half space**:

$$\text{Im}(H_0^{(1)}(\omega r))$$



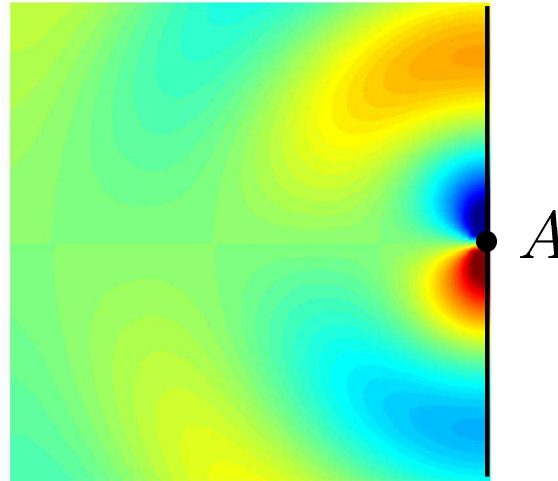
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$$\text{Im}(H_1^{(1)}(\omega r) \cos \theta)$$

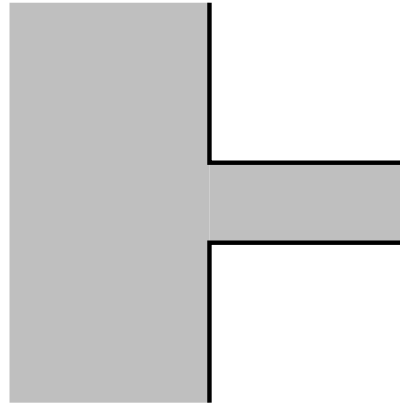


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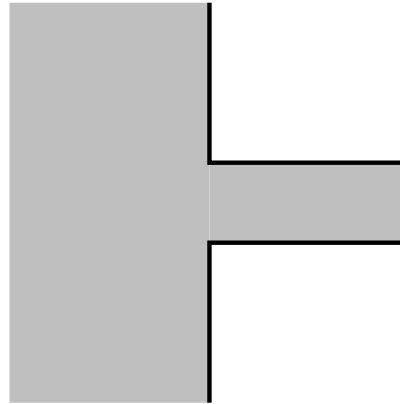
The near fields of order $i > 1$

- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



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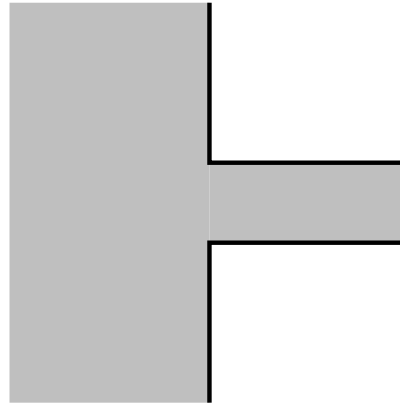
- by **Laplace** equation:

$$\Delta(u_p)_i^k = 0, \quad (i = k \text{ ou } k + 1),$$

$$\Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, \quad (i \geq k + 2),$$

The near fields of order $i > 1$

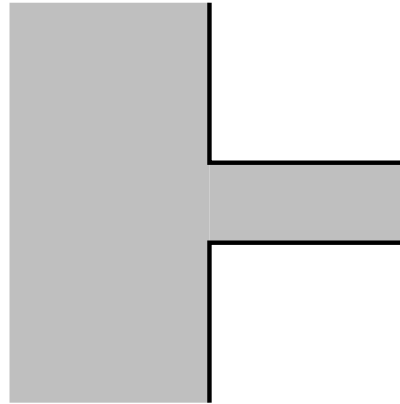
- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



- by **Laplace** equation:
- by polynomial **growings** at infinity:
 - The **growings** in the half space are functions of **far field of lower (or equal) order**
 - The **growings** in the slot are functions of the slot fields **of lower order**

The near fields of order $i > 1$

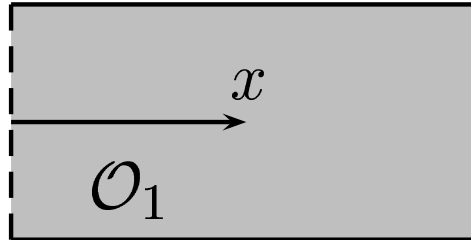
- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



- Proof of the **existence-unicity**:
 - with truncature functions, we subtract the growing behavior at infinity of the $(u_p)_i^k$
 - We use the “classical” **variational theory** (wheighted Sobolev spaces, Leroux, Hardy,...)

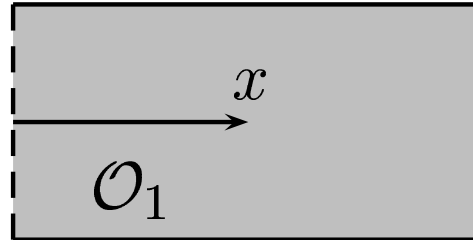
The slot field of order $i > 1$

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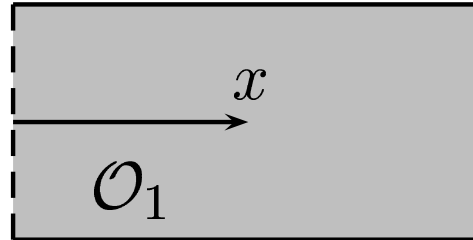
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- $$U_i^k(x) = \left(\int_0^1 (u_p)_i^k(0, Y) dY \right) \exp i\omega x$$

Some properties

We see that:

- More $i - k$ is **large** more u_i^k is **singular** at the origin:

$$r^{-p} \text{ terms, } p = 0, \dots, i - k - 1$$

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$$\begin{cases} \rho^p \text{ terms, } & p = 0, \dots, i - k, \\ X^p \text{ terms, } & p = 0, \dots, i - k, \end{cases}$$

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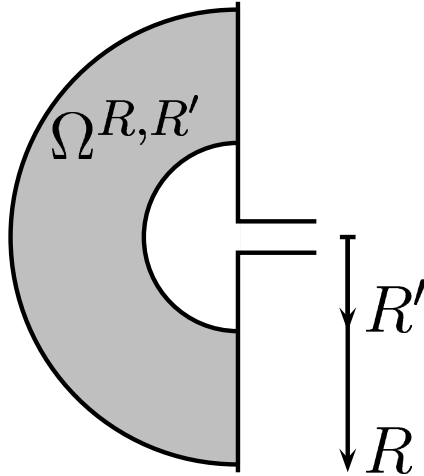
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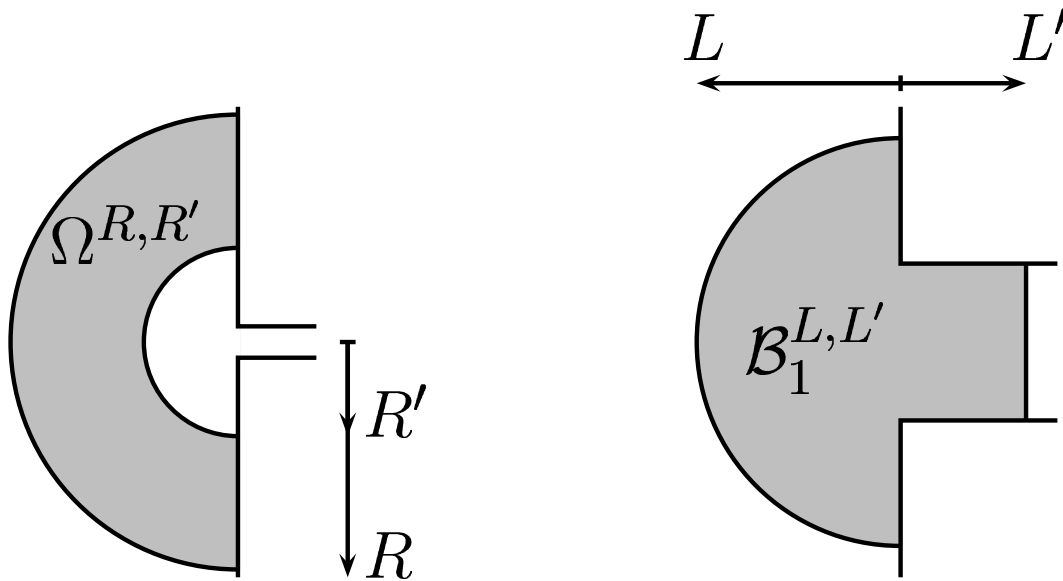
- When the **order** i grows, one has $O(\frac{i^2}{2})$ ($\times 3$) terms to compute...

Mathematical analysis



$$\left\| u^\epsilon - u^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \epsilon^i (\log \epsilon)^k u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \epsilon^{n+1} (\log \epsilon)^n \|f\|_{L^2(\Omega)}.$$

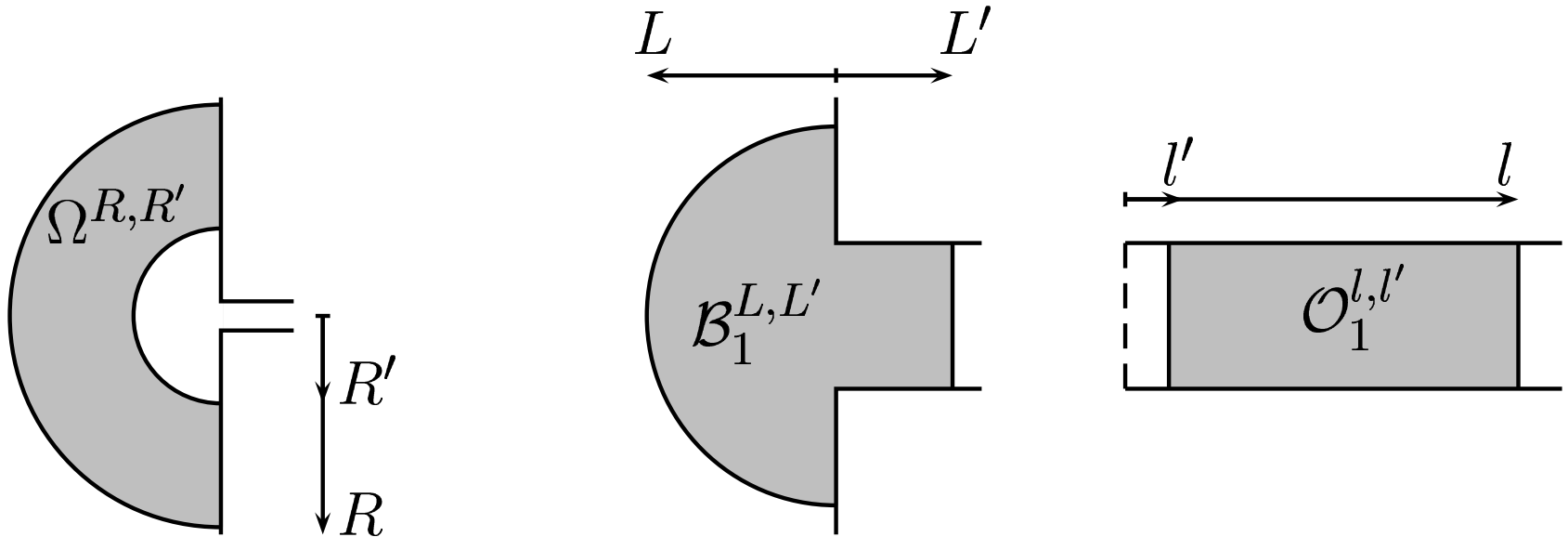
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$$\left\| u_p^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

Mathematical analysis



$$\left\| \mathbf{u}^\varepsilon - \mathbf{u}^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k \mathbf{u}_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^n \|f\|_{L^2(\Omega)}.$$

$$\left\| \mathbf{u}_p^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (\mathbf{u}_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

$$\left\| \mathbf{U}^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k \mathbf{U}_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$