A short introduction to rough paths: outline and selected bibliography

Antoine Lejay

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Antoine Lejay[1]

Lecture given at the Taras Schevchenko University during the *Workshop on Long Range Dependence: from Fractional Calculus to Financial Applications*[2]

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Outline of the lectures

Lecture 1: The Young case

- Presentation and a short history
- Young integral: how to define $\int y\,\mathrm{d}x$ for x and y respectively α-Hölder and β-Hölder continuous with $\alpha+\beta>1$? Main estimate

$$\left|\int_s^t y_r\,\mathrm{d}x_r - y_s(x_t-x_s)\right| \leqslant C(t-s)^{\alpha+\beta}H_\alpha(x)H_\beta(y).$$

- The sewing lemma: main tool for the construction of integrals.
- Definition of $\int_0^t f(x_s)\,\mathrm{d}x_s$ and main properties (mainly continuity) when $\alpha>1/2$, f γ-Hölder, $\alpha(1+\gamma)>1$.
- Definition of a controlled ordinary differential equation $y_t=y_0+\int_0^t f(y_s)\,\mathrm{d}x_s$ when $\alpha>1/2$ and main properties : continuity, uniqueness, flow property, convergence of the Euler scheme, ...

☞ The material for this first lecture is taken from [1, 3, 5, 12] regarding the Young integral, and from [24] for controlled differential equation. We also gave a proof of the sewing Lemma in a simplified form that comes from [21]. In [9], M. Gubinelli gives an nice algebraic presentation of the sewing lemma.

Lecture 2: The area and continuity issue

- Definition of a space $\mathcal{A}$ of functions with values in $\mathbb{R}^3$ and a norm

$$\|(x,y,z)\|_\alpha = \max\left\{\frac{|x_t-x_s|}{(t-s)^\alpha}, \frac{|y_t-y_s|}{(t-s)^\alpha}, \frac{\left|z_t-z_s-\frac{1}{2}\begin{vmatrix} x_s-x_0 & y_t-y_s \\ y_s-x_0 & y_t-y_s\end{vmatrix}\right|}{(t-s)^{2\alpha}},\right\}$$

[1]Projet-Team TOSCA, Institut Elie Cartan Nancy (Nancy-Université, CNRS, INRIA), Boulevard des Aiguillettes B.P. 239 F-54506 Vandœuvre-lès-Nancy, France. E-mail: `Antoine.Lejay@iecn.u-nancy.fr`

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- How to lift a path in $\mathcal{C}^\alpha([0,T],\mathbb{R}^2)$ into $\mathcal{A}^\alpha$ when $\alpha < 1/2$? Uniqueness of such a lift.
- How to lift a path in $\mathcal{C}^\alpha([0,T],\mathbb{R}^2)$ into $\mathcal{A}^\delta$ when $\alpha > 1/2$ and $\delta < 1/2$.
- Construction on approximation of sequences of such lifts.
- Consequences on the integral of differential forms.
- The Doss-Sussmann theorem for integrating controlled differential equations.
- Integration of a controlled differential equation of type $\mathrm{d}z_t = Az_t\,\mathrm{d}x_t + Bz_t\,\mathrm{d}y_t$ when A and B are matrices in the Lie algebra of the Heisenberg group: a closed form.

☞ The material from this lecture is taken from [5] which has been strongly inspired by [10] for the use the Green-Riemann formula. Instead of using approximations of paths with loops, P. Friz and N. Victoir propose to use sub-Riemannian geodesics, which is the core of their approach on Rough Differential Equations [6, 20]. The Doss-Sussmann theorem [13, 14] is classical and has lead to several results (See for example [34]). The result on linear controlled differential equations follows some standard results in control theory. The references [7, 8] contains an interesting insight on this theory.

Lecture 3: The rough path theory

- Definition of a rough path and the algebraic structure.
- Integration of a differential form along a rough path.
- Results on existence and uniqueness of solutions of rough differential equations.
- On the possible applications of the theory.
- How to construct a rough path?
- Smooth rough paths, geometric and non geometric rough paths.
- Case of the Brownian motion: Stratonovich integrals and Itô integrals.
- Case of the fractional Brownian motion.

☞ The content of this lecture is more classical and is taken from [1–5] and [22] for the local Lipschitz continuity of the Itô map.

Selected bibliography

Books and surveys

[1] T.J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 215–310.

[2] T. Lyons and Z. Qian, *System Control and Rough Paths*, Oxford Mathematical Monographs, Oxford University Press, 2002.

[3] A. Lejay, *An introduction to rough paths*, Séminaire de probabilités XXXVII, Lecture Notes in Mathematics, vol. 1832, Springer-Verlag, 2003, pp. 1–59.

[4] T. Lyons, M. Caruana, and T. Lévy, *Differential Equations Driven by Rough Paths*, École d'été des probabilités de Saint-Flour XXXIV — 2004 (J. Picard, ed.), Lecture Notes in Math., vol. 1908, Springer, Berlin, 2007.

[5] A. Lejay, *Yet another introduction to rough paths* (2009). To appear in *Séminaire de probabilités*, Lecture Notes in Mathematics, Springer-Verlag.

[6] P. Friz and N. Victoir, *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications*, Cambridge University Press, 2009.

The following references allows one to better understand the algebraic and geometric structure used the in theory of rough paths.

[7] F. Baudoin, *An introduction to the geometry of stochastic flows*, Imperial College Press, London, 2004.

☞ This in not a book about rough paths, but it gives some nice insight about the algebraic and geometric structure used the the theory of rough paths.

[8] S. Blanes, F. Casas, J.A. Oteo, and J. Ros, *The Magnus expansion and some of its applications*, Phys. reports **470** (2009), 151–238.

Alternative views on the theory

[9] M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140.

☞ The point of view of M. Gubinelli focuses on an algebraic version of the lemma that gives a rough path from an almost rough paths.

[10] D. Feyel and A. de La Pradelle, *Curvilinear integrals along enriched paths*, Electron. J. Probab. **11** (2006), no. 35, 860–892.

☞ The point of view of D. Feyel and A. de La Pradelle is mainly based on the Green-Riemann formula.

[11] Y. Hu and D. Nualart, *Rough path analysis via fractional calculus*, Trans. Amer. Math. Soc. **361** (2009), no. 5, 2689–2718.

☞ In this article, the authors use fractional calculus to develop a notion of stochastic integral in the spirit of rough path theory.

On the general theory: main articles

[12] L.C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. **67** (1936), 251–282.

☞ The original article where the so-called Young integral is constructed.

[13] H. Doss, *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré Sect. B (N.S.) **13** (1977), no. 2, 99–125.

[14] H.J. Sussmann, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Probability **6** (1978), no. 1, 19–41.

☞ These two articles [13, 14] presents a way to solve a controlled differential equations in dimension one or when the vector fields commute.

[15] P. Friz, *Continuity of the Itô-Map for Hölder rough paths with applications to the support theorem in Hölder norm*, Probability and Partial Differential Equations in Modern Applied Mathematics, IMA Volumes in Mathematics and its Applications, vol. 140, Springer, 2005, pp. 117–135.

☞ This article show that one may use a more precise norm than the p-variation norm and show a support theorem.

[16] A. Lejay and N. Victoir, *On (p, q)-rough paths*, J. Differential Equations **225** (2006), no. 1, 103–133.

☞ This article defines the notion of (p, q)-rough path which allows one to define a rough differential equation with a drift. Also, it brings a proof of the existence of a solution to some rough differential equation when the vector field is not regular enough using a Brower fixed point theorem and bring some precisions on the difference between geometric and non geometric rough paths.

[17] P. Friz and N. Victoir, *A Note on the Notion of Geometric Rough Paths*, Probab. Theory Related Fields **136** (2006), no. 3, 395–416.

☞ This article deals with the notion of geometric rough paths and correct some results in [1].

[18] A.M. Davie, *Differential Equations Driven by Rough Signals: an Approach via Discrete Approximation*, Appl. Math. Res. Express. AMRX **2** (2007), Art. ID abm009, 40.

☞ This article presents an alternative notion of solution to rough differential equation and prove the convergence of the Euler scheme. It also gives counterexamples to the uniqueness when the coefficients are not smooth enough.

[19] T. Lyons and N. Victoir, *An Extension Theorem to Rough Paths*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 5, 835–847.

☞ This article is about the way to lift a Hölder continuous path into a rough path. See also [23].

[20] P. Friz and N. Victoir, *Euler Estimates of Rough Differential Equations*, J. Differential Equations **244** (2008), no. 2, 388–412.

☞ The authors develop another view on Rough Differential Equation using sub-Riemanniann geodesics, which borrows some ideas from [18].

[21] D. Feyel, A. de La Pradelle, and G. Mokobodzki, *A non-commutative sewing lemma*, Electron. Commun. Probab. **13** (2008), 24–34.

☞ This article presents a nice and elegant proof of the transformation of an almost rough path into a rough path.

[22] A. Lejay, *On rough differential equations*, Electron. J. Probab. **14** (2009), no. 12, 341–364.

☞ This article deal with the case of unbounded vector field. Yet it does not cover the case of vector field with linear growth.

[23] J. Unterberger, *An explicit rough path construction for continuous paths with arbitrary Hölder exponent* (2009), available at `arxiv:0903.2716`.

☞ This article is about the way to lift a Hölder continuous path into a rough path. Its content is completely different from [19].

[24] A. Lejay, *Controlled differential equations as Young integrals: a simple approach* (2009), available at `hal:inria-00402397`.

☞ This article covers many results (existence, uniqueness, continuity, ...) on rough differential equations driven by paths of finite p-variations with $p < 2$.

SDE driven by fractional Brownian motion: survey articles and books

A lot of work has been done for developing a theory of Stochastic Differential Equations driven by a fractional Brownian motion, and several approaches are possible (see [25, 26]). Rough path is only one of these construction.

[25] L. Coutin, *An introduction to (stochastic) calculus with respect fo fractional Brownian motion*, Séminaire de Probabilités XL, 2007, pp. 3–65.

[26] Yu. Mishura, *Stochastic calculus for fractional Brownian motion and related processes*, Lecture Notes in Mathematics, vol. 1929, Springer-Verlag, Berlin, 2008.

On fractional Brownian motion: theory

[27] L. Coutin and Z. Qian, *Stochastic analysis, rough path analysis and fractional Brownian motions*, Probab. Theory Related Fields **122** (2002), no. 1, 108–140.

☞ This article shows existence of iterated integrals for the fBM with $H > 1/4$ and the non existence of such integrals when $H < 1/4$ as a limit of piecewise linear approximations of the path. To overcome this difficulty, other constructions have been provided to cover this case [40, 41, 44].

[28] A. Millet and M. Sanz-Solé, *Large deviations for rough paths of the fractional Brownian motion*, Ann. Inst. H. Poincaré Probab. Statist. **42** (2006), no. 2, 245–271.

☞ This article shows a large deviation principle for the fBM.

[29] F. Baudoin and L. Coutin, *Operators associated with stochastic differential equations driven by fractional Brownian motions*, Stochastic Proces. Appl. **117** (2007), no. 5, 550–574.

[30] ______, *Self-similarity and fractional Brownian motions on Lie groups*, Electron. J. Probab. **13** (2008), no. 38, 1120–1139.

☞ These articles [29, 30] study the properties in small time of the semi-group associated to a SDE driven by a fBM using some algebraic properties on controlled differential equations and differential equations on Lie group (see for example [7]).

[31] F. Baudoin and M. Hairer, *A version of Hörmander's theorem for the fractional Brownian motion*, Probab. Theory Related Fields **139** (2007), no. 3-4, 373–395.

[32] A. Millet and M. Sanz-Solé, *Approximation of rough paths of fractional Brownian motion*, Seminar on Stochastic Analysis, Random Fields and Applications V, Progr. Probab., vol. 59, Birkhäuser, Basel, 2008, pp. 275–303.

[33] L. Coutin, P. Friz, and N. Victoir, *Good Rough Path Sequences and Applications to Anticipating & Fractional Stochastic Calculus*, Ann. Probab. **35** (2007), no. 3, 1172–1193.

☞ This article shows that the fBM solves some anticipative stochastic differential equation.

[34] A. Neuenkirch and I. Nourdin, *Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion*, J. Theoret. Probab. **20** (2007), no. 4, 871–899.

☞ The Doss-Sussmann theory is used to study the rate of convergence of several schemes.

[35] S. Tindel and J. Unterberger, *The rough path associated to the multidimensional analytic fbm with any Hurst parameter* (2008), available at `hal:hal-00327355`.

☞ This article shows the existence of a rough path can be constructed for any value of H by using an *analytical rough path* proposed by J. Unterberger [41].

[36] A. Neuenkirch, I. Nourdin, and S. Tindel, *Delay equations driven by rough paths*, Electron. J. Probab. **13** (2008), no. 67, 2031–2068.

[37] S. Tindel and I. Torrecilla, *Some differential systems driven by a fBm with Hurst parameter greater than 1/4* (2009), available at `arxiv:0901.2010`.

☞ These articles [37, 48] study some delay equations driven by fBM.

[38] A. Deya and S. Tindel, *Rough Volterra equations 1: the algebraic integration setting* (2008), available at `arxiv:0809.2000`.

[39] ———, *Rough Volterra equations 2: convolutional generalized integrals* (2008), available at `arxiv:0810.1824`.

☞ These articles [38, 39] develops a rough path theory for Volterra equations and gives examples of applications to the fBM.

[40] J. Unterberger, *Stochastic calculus for fractional Brownian motion with Hurst exponent $H > \frac{1}{4}$: a rough path method by analytic extension*, Ann. Probab. **37** (2009), no. 2, 565–614.

[41] ———, *A Lévy area by Fourier normal ordering for multidimensional fractional Brownian motion with small Hurst index* (2009), available at `arxiv:0906.1406`.

[42] ———, *An explicit rough path construction for continuous paths with arbitrary Hölder exponent by Fourier normal ordering* (2009), available at `arxiv:0906.2716`.

☞ These articles [40–42] show that one may construct a rough path above a fBM. It uses techniques from renormalization theory and contruct indeed an *analytic rough path* that projects onto a process defined on the complex plane. See also [35].

[43] ———, *Moment estimates for solutions of linear stochastic differential equations driven by analytic fractional Brownian motion* (2009), available at `arxiv:0905.0782`.

[44] D. Nualart and S. Tindel, *A construction of the rough path above fractional Brownian motion using Volterra's representation* (2009), available at `arxiv:0909.1307`.

☞ This article shows how to construct a rough path above a fBM for any Hurst index using a construction alternative to the one based on the analytic fBM [40, 41].

On fractional Brownian motion: application and numerical analysis

[45] R. Marty, *Asymptotic behavior of differential equations driven by periodic and random processes with slowly decaying correlations*, ESAIM Probab. Stat. **9** (2005), 165–184.

☞ This article is an exemple of the use of the continuity theorem in the study of the asymptotic behavior of some random process.

[46] G. Pagès and A. Sellami, *Convergence of multi-dimensional quantized SDE's* (2008), available at `arxiv:0801.0726`.

☞ This article develops a way of approximating a solution of some SDE using quantization (*i.e.*, the replacement of a random variable by a discrete one) of the coefficients in the Karhunen-Loève decomposition. It may then be used for any Gaussian process.

[47] X. Bardina, I. Nourdin, C. Rovira, and S. Tindel, *Weak approximation of a fractional SDE* (2007), available at `arxiv:0790.0805`.

☞ This article studies an approximation of a fractional SDE when the fBM is approximated by a Kac-Stroock approximation.

[48] A. Neuenkirch, S. Tindel, and J. Unterberger, *Discretizing the fractional Levy area* (2009), available at `arxiv:0902.0497`.

[49] A. Neuenkirch, I. Nourdin, A. Rößler, and S. Tindel, *Trees and asymptotic developments for fractional stochastic differential equations*, Ann. Inst. H. Poincaré Probab. Statist. **45** (2009), no. 1, 157–174.

On fractional Brownian motion: toward Stochastic Partial Differential Equations

We present here some results on SPDE related where fractional noise is used as an example. Recently, other constructions have been proposed.

[50] M. Gubinelli, A. Lejay, and S. Tindel, *Young integrals and SPDEs*, Potential Anal. **25** (2006), no. 4, 307–326.

☞ This article was a first attempt to deal with Stochastic Partial Differential Equations driven by rough paths. Here, a notion of mild solution is developped which can be used for fractional noise with enough regularity.

[51] L. Quer-Sardanyons and S. Tindel, *The 1-d stochastic wave equation driven by a fractional Brownian motion*, Stochastic Process. Appl. **117** (2007), no. 10, 1448–1472.

☞ The case of a wave equation is treated.

[52] M. Gubinelli and S. Tinde, *Rough evolution equation* (2008), available at `arvix:0803.0552`.

☞ This article develops a notion of SPDE using the theory of rough paths and proposed as an example a SPDE driven by a space-time fractional Brownian motion.

On Gaussian processes

[53] P. Friz and N. Victoir, *Differential Equations Driven by Gaussian Signals I* (2007), available at `arxiv:0707.0313`.

[54] P. Friz, N. Victoir, P. Friz, and N. Victoir, *Differential Equations Driven by Gaussian Signals II* (2007), available at `arxiv:0711.0668`.

☞ These articles [53,54] relates the regularity of a Gaussian process to the covariance function and construct a rough path using the Karhunen-Loève decomposition.

[55] L. Coutin and N. Victoir, *Enhanced Gaussian Processes and Applications*, ESAIM Probab. Stat. **13** (2009), 247–269, available at `doi:10.1051/ps:2008007`.

☞ This article also uses the Karhunen-Loève decomposition to construct an approximation of some Gaussian process and shows results of Wong-Zakai type.