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# APPROXIMATE TRANSMISSION CONDITIONS THROUGH A WEAKLY OSCILLATING THIN LAYER

CLAIR POIGNARD

**ABSTRACT.** We study the behavior of the electro-quasistatic voltage potentials in a material composed by a bidimensional medium surrounded by a weakly oscillating thin layer and embedded in an ambient medium. We build approximate transmission conditions in order to replace the layer by these conditions on the boundary of the interior material. We deal with a weakly oscillating thin layer: the period of the oscillations is greater than the square root of the thinness. Our approach is essentially geometric and based on a suitable change of variable in the layer. This paper extends previous works [14], [15] of the former author, in which the layer had constant thickness.

## 1. INTRODUCTION

**1.1. Motivations.** The electromagnetic modelization of biological cells has become extremely important since several years, in particular in the biomedical research area. In the simple models of Fear, Stuchly or Foster and Schwan [10], [11], the biological cell is composed of a conducting cytoplasm surrounded by a thin insulating membrane. When exposed to an electric field, a potential difference is induced across the cell membrane. This transmembrane potential (TMP) may be of sufficient magnitude to be biologically significant. In particular, if it overcomes a threshold value, complex phenomenons as electroporation or electroporation may occur [19], [20]: some exterior molecules might be internalized inside the cell. These process hold great promises in oncology and gene therapy, particularly, to deliver drug molecules in cancer treatment.

This is the reason why several papers in the bioelectromagnetic research area deal with numerical modelizations of the cell (see for instance [13], [18],[17]) and with numerical computations of the TMP. Actually the main difficulties in the calculation of the TMP lie in the thinness of the membrane and in the high contrast between the electromagnetic parameters of the cytoplasm and the membrane.

In previous papers [16], [15], we proposed an asymptotic analysis to compute electromagnetic fields and particularly electric potentials in domains with thin layer of constant thickness. However, in certain circumstances, electric field imposed to the cell may destructure the membrane. This leads to a thin membrane with non constant thickness, and the calculation of the TMP is then quite more difficult. This is the reason why we present here an asymptotic analysis of the electro-quasistatic potentials in time-harmonic regime<sup>1</sup> in a biological cell with membrane of non constant thickness; we even allow that weak oscillations in the tangential variable may occur.

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<sup>1</sup>Roughly speaking, the electro-quasistatic equation in time-harmonic regime is the steady state voltage with complex parameters instead of real coefficients.

Since we are definitely interested in the calculation of the transmembrane potential, the asymptotic expansions of the steady state voltage potentials far from the thin layer, which have been developed by Ammari, Vogelius *et al.* (see for instance [3], [4], [5] or [7]) may not be used. Achdou *et al.* [2], Valentin *et al.* [12] or Abboud and Ammari [1] presented in previous papers how to build approximate boundary conditions for a highly oscillating thin membrane<sup>2</sup>, when a homogeneous Dirichlet boundary condition is imposed on the rough boundary. Here we are interested in approximate transmission conditions since our cell is embedded in an ambient, moreover, we deal with a weakly oscillating thin membrane. Once again previous analysis may not be applied.

Our asymptotic analysis is closed to those performed in [16]. Roughly speaking, it is based on a suitable change of variable in the membrane in order to write the explicit dependence of the studied differential operator in terms of the small parameter (the thinness of the membrane).

Throughout this paper, we consider a material composed of an interior domain surrounded by a thin membrane. This material, representing a biological cell, is embedded in an ambient medium. A voltage potential is imposed to the boundary of the ambient medium and we study the asymptotic behavior of the electroquasistatic voltage potential in the three domains (ambient medium, thin layer and cytoplasm) for the thinness of the membrane tending to zero. We build appropriate transmission conditions on the boundary of the cytoplasm in order to remove the thin layer from the problem. Actually, the influence of the layer is approached by these transmission conditions. To justify our asymptotic expansion, we estimate the  $L^2$ -error between the exact solution and the approximate solution.

This paper is structured as follows. In Section 2, we present the geometry of the problem and we perform our suitable change of variables. Then, in Section 3 we derive formally our asymptotics. This asymptotic expansion is then proved with error estimates, in Section 5.1.

**1.2. The studied problem.** Let  $\Omega$  be a smooth bounded bidimensional domain (see Fig. 1), composed of a smooth interior domain  $\mathcal{O}^i$  surrounded by a thin membrane  $\mathcal{O}_\delta$  and of an ambient medium  $\mathcal{O}_\delta^e$ , where the domain  $\overline{\mathcal{O}^i \cup \mathcal{O}_\delta}$  is embedded;

$$\Omega = \mathcal{O}^i \cup \mathcal{O}_\delta \cup \mathcal{O}_\delta^e.$$

We do not suppose that the thickness of the membrane is constant, we even allow that some parts of the membrane may be weakly oscillating.

Denote by  $\Gamma$  the boundary of  $\mathcal{O}^i$ , which is supposed to be smooth and by  $\Gamma_\delta^r$  the weakly oscillating (rough) boundary of  $\mathcal{O}_\delta$ . We suppose that the period of the oscillations is greater than the square root of the thin layer: this is the reason why the thin layer is said to be weakly oscillating.

Let  $\sigma_m$ ,  $\sigma_i$  and  $\sigma_e$  be three non null complex parameters with positive real part. Denote by  $\sigma_\delta$  the following piecewise constant functions

$$\forall x \in \Omega, \quad \sigma_\delta(x) = \begin{cases} \sigma_i, & \text{if } x \in \mathcal{O}^i, \\ \sigma_m, & \text{if } x \in \mathcal{O}_\delta, \\ \sigma_e, & \text{if } x \in \mathcal{O}_\delta^e. \end{cases}$$

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<sup>2</sup>These authors dealt with periods of the oscillations of the same length as the thickness of the membrane

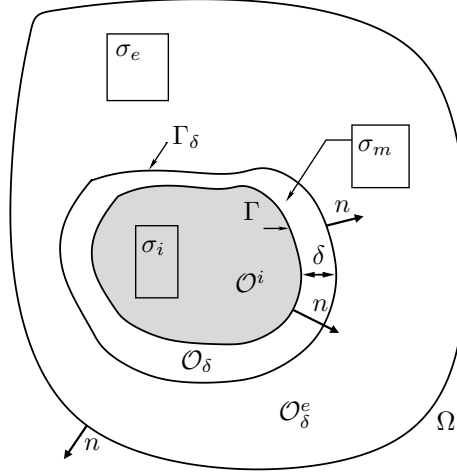


FIGURE 1. Geometric and dielectric data.

We would like to understand the behavior for  $\delta$  tending to zero of the solution  $V_\delta$  to the following problem :

$$\begin{aligned} (1) \quad & \nabla \cdot (\sigma_\delta \nabla V_\delta) = 0 \text{ in } \Omega, \\ (2) \quad & V_\delta|_{\partial\Omega} = \varphi \text{ on } \partial\Omega. \end{aligned}$$

The above function  $V_\delta$  is well-defined and belongs to  $H^{1+s}(\Omega)$  as soon as  $\varphi$  belongs to  $H^{1/2+s}(\partial\Omega)$ ,  $s \geq 0$ . Observe that  $V_\delta$  is continuous and its normal derivatives satisfy the following transmission conditions:

$$\begin{aligned} (3a) \quad & \sigma_m \partial_n V_\delta|_{\Gamma^+} = \sigma_i \partial_n V_\delta|_{\Gamma^-}, \\ (3b) \quad & \sigma_m \partial_n V_\delta|_{\Gamma_\delta^-} = \sigma_e \partial_n V_\delta|_{\Gamma_\delta^+}. \end{aligned}$$

We obtain in this paper, the approximate transmission conditions on  $\Gamma$ , which approaches the weakly oscillating thin layer, when the thickness of the membrane tends to zero.

**1.3. Main result.** The orientation of the boundary  $\Gamma$  of  $\mathcal{O}^i$  is the trigonometric orientation. To simplify, we suppose that the length of  $\Gamma$  equals 1. We denote by  $\mathbb{T}$  the flat torus:

$$\mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

Since  $\Gamma$  is smooth, we can parameterize it by a function  $\Psi$  of class  $\mathcal{C}^\infty$  from  $\mathbb{T}$  to  $\mathbb{R}^2$  satisfying:

$$(4) \quad \forall \theta \in \mathbb{T}, \quad |\Psi'(\theta)| = 1,$$

and therefore

$$(5) \quad \Gamma = \{\Psi(\theta), \theta \in \mathbb{T}\}.$$

Let  $f$  be a quasiperiodic smooth function defined on  $\mathbb{T} \times \mathbb{R}$  such that:

$$\begin{aligned} \forall (x, y) \in \mathbb{T} \times \mathbb{R}, \quad & f(x, y+1) = f(x, y), \\ \forall (x, y) \in \mathbb{T} \times \mathbb{R}, \quad & |f(x, y)| \leq M, \text{ for a given constant } M > 0. \end{aligned}$$

We suppose there exists  $\alpha \in [0, 1/2)$  such that the thin membrane is parameterized as follows:

$$\mathcal{O}_\delta = \left\{ \Psi(\theta) + \delta \eta f \left( \theta, \frac{\theta}{\delta^\alpha} \right) n(\theta), \quad (\eta, \theta) \in [0, 1] \times \mathbb{T} \right\}.$$

Here  $n(\theta)$  is the unitary exterior normal to  $\Gamma$  at the point  $\Psi(\theta)$ . If  $\alpha > 0$ , we denote by

$$\forall x \in \Gamma, \quad \bar{g}(x) = \left( \int_0^1 f(., y) dy \right) \circ \Psi^{-1}(x),$$

while for  $\alpha = 0$ ,  $\bar{g}$  denotes

$$\forall x \in \Gamma, \quad \bar{g} = f \circ \Psi^{-1}(x).$$

Suppose that  $\varphi$  belongs to  $H^{7/2}(\partial\Omega)$ , and let  $(v_0^e, v_0^i)$  satisfy:

$$(6a) \quad \begin{cases} \Delta v_0^e = 0, \text{ in } \mathcal{O}^e, \\ \Delta v_0^i = 0, \text{ in } \mathcal{O}^i, \end{cases}$$

with transmission conditions

$$(6b) \quad v_0^i|_{\Gamma^-} = v_0^e|_{\Gamma^+},$$

$$(6c) \quad \sigma_i \partial_n v_0^i|_{\Gamma^-} = \sigma_e \partial_n v_0^e|_{\Gamma^+},$$

and with the boundary condition

$$(6d) \quad v_0^e|_{\partial\Omega} = \varphi,$$

and define

$$\bar{v}_1 = \begin{cases} \bar{v}_1^e, & \text{in } \Omega \setminus \mathcal{O}^i, \\ \bar{v}_1^i, & \text{in } \mathcal{O}^i \end{cases}$$

where:

$$(7a) \quad \begin{cases} \Delta \bar{v}_1^e = 0, & \text{in } \Omega \setminus \mathcal{O}^i, \\ \Delta \bar{v}_1^i = 0, & \text{in } \mathcal{O}^i, \\ \bar{v}_1^e|_{\partial\Omega} = 0, \end{cases}$$

with the transmission conditions

$$(7b) \quad \sigma_i \partial_n \bar{v}_1^i|_{\Gamma^-} - \sigma_e \partial_n \bar{v}_1^e|_{\Gamma^+} = -(\sigma_e - \sigma_m) \partial_t (\bar{g} \partial_t v_0^i|_{\Gamma^-}),$$

$$(7c) \quad \bar{v}_1^i|_{\Gamma^-} - \bar{v}_1^e|_{\Gamma^+} = \bar{g} \left( \frac{1}{\sigma_e} - \frac{1}{\sigma_m} \right) \sigma_i \partial_n v_0^i|_{\Gamma^-}.$$

We have the following theorem.

**Theorem 1.1** (Main result). *Suppose that  $\alpha \in (0, 1/2)$ . The above functions  $(v_0^e, v_0^i)$  and  $(v_1^e, v_1^i)$  belong respectively to*

$$H^4(\mathcal{O}^e) \times H^4(\mathcal{O}^i), \quad \text{and to } H^3(\mathcal{O}^e) \times H^3(\mathcal{O}^i).$$

*Denote by  $\overline{V_{app}}$  the following function:*

$$(8) \quad \overline{V_{app}} = \begin{cases} v_0^i + \delta \bar{v}_1^i, & \text{in } \mathcal{O}^i, \\ v_0^e + \delta \bar{v}_1^e, & \text{in } \Omega \setminus \mathcal{O}^i. \end{cases}$$

Then we have the following estimates:

$$\|V_\delta - \bar{V}_{app}\|_{L^2(\mathcal{O}^i)} = O(\delta^{1+\alpha/2} + \delta^{3/2-\alpha}),$$

and for all bounded domain  $\omega$  such that  $\bar{\omega} \subset \mathcal{O}_\delta^e$ ,

$$\|V_\delta - \bar{V}_{app}\|_{L^2(\omega)} = O(\delta^{1+\alpha/2} + \delta^{3/2-\alpha}).$$

Without any oscillation on  $\Gamma_\delta^r$ , i.e. if  $\alpha = 0$ , we have the following estimates, which are more precise.

**Theorem 1.2** (Without oscillations). *Suppose that  $\Gamma_\delta^r$  has no oscillating parts, then the following estimates hold:*

$$\|V_\delta - \bar{V}_{app}\|_{L^2(\mathcal{O}^i)} = O(\delta^{3/2}),$$

and for all bounded domain  $\omega$  such that  $\bar{\omega} \subset \mathcal{O}_\delta^e$ ,

$$\|V_\delta - \bar{V}_{app}\|_{L^2(\omega)} = O(\delta^{3/2}).$$

**Remark 1.3.** According to the previous theorems, to approach the potential  $V_\delta$  defined by (1)–(2) with an accuracy of order  $o(\delta)$ , we first need the calculation of the terms  $(v_0^e, v_0^i)$  and then we obtain  $(v_1^e, v_1^i)$ . The approximate potential is then the sum  $(v_0^e + \delta v_1^e, v_0^i + \delta v_1^i)$ . In order to compute only one problem, it is classical to write approximated boundary conditions. We leave the reader showing that the potential  $\bar{v}_{app}$  satisfying the following problem:

$$\begin{aligned} \Delta \bar{v}_{app} &= 0, \text{ in } \Omega \setminus \mathcal{O}^i, \\ \Delta \bar{v}_{app} &= 0, \text{ in } \mathcal{O}^i, \\ \bar{v}_{app}|_{\partial\Omega} &= \varphi, \end{aligned}$$

with the following approximated boundary conditions deduced from (6)–(7):

$$\begin{aligned} \sigma_i \partial_n \bar{v}_{app}|_{\Gamma^-} - \sigma_e \partial_n \bar{v}_{app}|_{\Gamma^+} &= -\delta (\sigma_e - \sigma_m) \partial_t (\bar{g} \partial_t v_{app}|_{\Gamma^-}), \\ \bar{v}_{app}|_{\Gamma^-} - \bar{v}_{app}|_{\Gamma^+} &= \delta \bar{g} \left( \frac{1}{\sigma_e} - \frac{1}{\sigma_m} \right) \sigma_i \partial_n v_{app}|_{\Gamma^-}, \end{aligned}$$

is unique and satisfies the same estimates as  $(v_0^e + \delta v_1^e, v_0^i + \delta v_1^i)$  in each domain  $\mathcal{O}_\delta^e$  and  $\mathcal{O}^i$ .

To prove the estimates, it suffices to develop with respect to  $\delta$  the above potential  $\bar{v}_{app}$ . We easily see that the first two terms are exactly the above terms  $(v_0^e, v_0^i)$  and  $(v_1^e, v_1^i)$ , while the rest is  $o(\delta)$ . Existence and uniqueness of  $\bar{v}_{app}$  comes from classical arguments.

**Remark 1.4.** Observe that the transmission condition (7c) leads directly to an estimate of the TMP across the membrane, with an accuracy of order  $o(\delta)$ . Therefore, to obtain the transmembrane potential at the order 1, we only need to compute the electro-quasistatic potential without the thin membrane.

In this paper, we deal with a weakly oscillating membrane. However, in some modelization problems, it is interesting to deal with thin oscillating membrane, where the period of the oscillations as thin as the thickness of the membrane. The asymptotic expansion developed in this paper does not seem to be applicable, however, using a variational analysis, we may find the limit problem satisfied by the difference between the exact solution and the potential without membrane. These works are in progress and will be presented in a next paper [8].

## 2. GEOMETRY

Denote by  $\Phi$  the following map:

$$\forall (\eta, \theta) [0, 1] \times \mathbb{T}, \quad \Phi(\eta, \theta) = \Psi(\theta) + \delta f \left( \theta, \frac{\theta}{\delta^\alpha} \right) \eta n(\theta),$$

where  $\Psi$  is defined by (5) and let  $\kappa$  be the curvature of  $\Gamma$  in the curvilinear coordinate. Let  $\delta_0$  belong to  $(0, 1)$  such that:

$$(9) \quad \delta_0 < \frac{1}{\|\kappa f\|_\infty}.$$

Thus for all  $\delta$  in  $[0, \delta_0]$ , there exists an open interval  $I$  containing  $(0, 1)$  such that  $\Phi$  is a smooth diffeomorphism from  $I \times \mathbb{R}/\mathbb{Z}$  to its image, which is a neighborhood of the membrane. Let  $(g_{ij})_{i,j=1}^2$  be the matrix describing the Euclidean metric in  $(\eta, \theta)$ -coordinates

$$(10) \quad g_{11} d\eta^2 + g_{22} d\theta^2 + 2g_{12} d\eta d\theta.$$

The coefficients  $(g_{ij})_{i,j=1,2}$  are defined by (see for example Dubrovin *et al.* [9]):

$$g_{11} = \langle \partial_\eta \Phi, \partial_\eta \Phi \rangle, \quad g_{22} = \langle \partial_\theta \Phi, \partial_\theta \Phi \rangle, \quad g_{12} = g_{21} = \langle \partial_\eta \Phi, \partial_\theta \Phi \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the Euclidean scalar product of  $\mathbb{R}^2$ .

Denote by  $\nabla_\alpha \cdot f$  and  $\nabla_\alpha^2 \cdot f$  the following quantities:

$$\begin{aligned} \nabla_\alpha \cdot f(\theta, \theta/\delta^\alpha) &= \partial_x f|_{(\theta, \theta/\delta^\alpha)} + \frac{1}{\delta^\alpha} \partial_y f|_{(\theta, \theta/\delta^\alpha)}, \\ \nabla_\alpha^2 \cdot f(\theta, \theta/\delta^\alpha) &= \partial_x^2 f|_{(\theta, \theta/\delta^\alpha)} + \frac{2}{\delta^\alpha} \partial_x \partial_y f|_{(\theta, \theta/\delta^\alpha)} + \frac{1}{\delta^{2\alpha}} \partial_y^2 f|_{(\theta, \theta/\delta^\alpha)}. \end{aligned}$$

We have

$$(11) \quad g_{11} = \left( \delta f(\theta, \theta/\delta^\alpha) \right)^2,$$

$$(12) \quad g_{22} = \left( 1 + \delta f(\theta, \theta/\delta^\alpha) \eta \kappa(\theta) \right)^2 + \left( \eta \delta \nabla_\alpha \cdot f(\theta, \theta/\delta^\alpha) \right)^2,$$

$$(13) \quad g_{12} = g_{21} = \eta \delta^2 f(\theta, \theta/\delta^\alpha) \nabla_\alpha \cdot f(\theta, \theta/\delta^\alpha);$$

and let  $|G|$  be:

$$(14) \quad |G| = g_{11} g_{22} - g_{12}^2 = \delta^2 f^2(\theta, \theta/\delta^\alpha) \left( 1 + \eta \kappa(\theta) \delta f(\theta, \theta/\delta^\alpha) \right)^2.$$

Write now Laplacian operator in  $(\eta, \theta)$ -coordinates. Using the formulae of differential geometry<sup>3</sup> (see for instance Appendix of [16]), we have:

$$(15) \quad \Delta_{\eta, \theta} = \frac{1}{\sqrt{|G|}} \left\{ \partial_\eta \left( \frac{1}{\sqrt{|G|}} (g_{22} \partial_\eta - g_{12} \partial_\theta) \right) + \partial_\theta \left( \frac{1}{\sqrt{|G|}} (-g_{12} \partial_\eta + g_{11} \partial_\theta) \right) \right\}.$$

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<sup>3</sup>Without any knowledge in differential geometry area but simply using the change of variables  $(x, y) \rightarrow (\eta, \theta)$ , the above expression of the Laplacian might also be derived by tedious calculations.

Therefore we have

$$\begin{aligned}
 |G|\Delta_{\eta,\theta} &= g_{22}\partial_\eta^2 + g_{11}\partial_\theta^2 - 2g_{12}\partial_\eta\partial_\theta \\
 &+ \left( \partial_\eta g_{22} - \partial_\theta g_{12} + g_{12} \frac{\partial_\theta |G|}{2|G|} - g_{22} \frac{\partial_\eta |G|}{2|G|} \right) \partial_\eta \\
 &+ \left( \partial_\theta g_{11} - \partial_\eta g_{12} + g_{12} \frac{\partial_\eta |G|}{2|G|} - g_{11} \frac{\partial_\theta |G|}{2|G|} \right) \partial_\theta.
 \end{aligned}
 \tag{16}$$

On the boundary  $\Gamma$ , the normal derivative equals:

$$\partial_n|_\Gamma \circ \Phi|_{\eta=0} = \frac{1}{\delta f(\theta, \theta/\delta^\alpha)} \partial_\eta \Big|_{\eta=0},$$

and on the oscillating boundary  $\Gamma_\delta^r$ ,

$$\partial_n|_{\Gamma_\delta^r} \circ \Phi|_{\eta=1} = \frac{1}{\sqrt{|G|}} \left\{ \sqrt{g_{22}} \partial_\eta - \frac{g_{12}}{\sqrt{g_{22}}} \partial_\theta \right\} \Big|_{\eta=1}.$$

**Remark 2.1.** *In this paper we suppose that  $\alpha$  belongs to  $[0, 1/2)$ . Observe that if  $\alpha$  belongs to  $[1/2, 1)$ , the multiplication by the above function  $\nabla_\alpha^2 f$  (or equivalently the multiplication by  $(\nabla_\alpha f)^2$ ) is of order  $-1$  with respect to the power of  $\delta$ . This case will be consider later on.*

### 3. FORMAL ASYMPTOTICS

This section is devoted to derive formally the asymptotic expansion of  $V$ .

Let us set our ansatz:

$$(17a) \quad v^e = v_0^e + \delta v_1^e + \dots,$$

$$(17b) \quad v^i = v_0^i + \delta v_1^i + \dots,$$

$$(17c) \quad v^m = v_0^m + \delta v_1^m + \delta^2 v_2^m + \dots$$

*A priori*, each functions in the above equalities depends on  $\theta/\delta^\alpha$ , but we expect that this dependency will be explicit. For all  $k \geq 0$ , we set:

$$\begin{aligned}
 \Delta v_k^e &= 0, \text{ in } \mathcal{O}^e, \\
 \Delta v_k^i &= 0, \text{ in } \mathcal{O}^i, \\
 v_k^e|_{\partial\Omega} &= \delta_{0,k} \varphi, \text{ on } \partial\Omega.
 \end{aligned}$$

We order the terms of the same power of  $\delta$  in the expressions of  $\Delta_{\eta,\theta}$  and of the normal derivatives. We emphasize that for all  $k \geq 0$ ,  $v_k^m$  depends *a priori* on  $\eta$ ,  $\theta/\delta^\alpha$  and  $\theta$ . Using the following equalities:

$$\begin{aligned}
 \frac{\partial_\eta |G|}{2|G|} &= \frac{\delta \kappa(\theta) f(\theta, \theta/\delta^\alpha)}{1 + \eta \kappa(\theta) \delta f(\theta, \theta/\delta^\alpha)}, \\
 \frac{\partial_\theta |G|}{2|G|} &= \frac{\nabla_\alpha \cdot f(\theta, \theta/\delta^\alpha)}{f(\theta, \theta/\delta^\alpha)} + \eta \delta \frac{\kappa(\theta) \nabla_\alpha \cdot f(\theta, \theta/\delta^\alpha) + \kappa'(\theta) f(\theta, \theta/\delta^\alpha)}{1 + \eta \kappa(\theta) \delta f(\theta, \theta/\delta^\alpha)},
 \end{aligned}$$



we infer, with the notation  $\tau = (\theta, \theta/\delta^\alpha)$ ,

$$\begin{aligned}
 |G|\Delta_{\eta,\theta} &= \partial_\eta^2 + \delta\kappa(\theta)f(\tau) \left( 2\eta\partial_\eta^2 + \partial_\eta \right) \\
 &+ \delta^2 \left\{ \left( \eta^2 (\nabla_\alpha \cdot f(\tau))^2 + (\eta\kappa(\theta)f(\tau))^2 \right) \partial_\eta^2 + f^2(\tau) \partial_\theta^2 \right. \\
 (18) \quad &- 2\eta f(\tau) \nabla_\alpha \cdot f(\tau) \partial_\theta \partial_\eta + \eta \left( \kappa^2(\theta)f^2(\tau) + 2(\nabla_\alpha \cdot f(\tau))^2 \right. \\
 &\left. \left. - f(\tau) \nabla_\alpha^2 \cdot f(\tau) \right) \partial_\eta \right\} \\
 &+ \delta^3 R(\eta, \tau),
 \end{aligned}$$

where  $R$  is an operator of order 1 on  $[0, 1] \times \mathbb{T}$  defined by:

$$(19) \quad R(\eta, \tau) = \frac{\eta f(\tau)}{1 + \kappa(\theta)\eta\delta f(\tau)} \left\{ \eta \left( \kappa'(\theta) \nabla_\alpha f - f^2(\tau) \kappa^3(\theta) \right) \partial_\eta - \kappa'(\theta) f^2(\tau) \partial_\theta \right\}.$$

We also have:

$$\begin{aligned}
 \partial_n|_{\Gamma_\delta^r} \circ \Phi|_{\eta=1} &= \frac{1}{\delta f(\tau)} \left\{ \partial_\eta \right. \\
 &\left. + \delta^2 \nabla_\alpha \cdot f(\tau) \left( \frac{\nabla_\alpha \cdot f(\tau)}{2} \partial_\eta - f(\tau) \partial_\theta \right) + T(\tau) \right\} \Big|_{\eta=1},
 \end{aligned}$$

where  $T$  is the following differential operator defined on  $\mathbb{T}$  by:

$$\begin{aligned}
 T(\tau) &= \left( \sqrt{1 + \delta^2 (\nabla_\alpha \cdot f(\tau))^2 / (1 + \kappa\delta f(\tau))^2} - 1 - \delta^2 (\nabla_\alpha \cdot f(\tau))^2 / 2 \right) \partial_\eta \\
 &- \delta^2 f(\tau) \nabla_\alpha \cdot f(\tau) \left\{ \frac{1}{(1 + \kappa\delta f(\tau))^2 \sqrt{1 + \left( \frac{\delta \nabla_\alpha \cdot f(\tau)}{1 + \kappa\delta f(\tau)} \right)^2}} - 1 \right\} \partial_\theta.
 \end{aligned}$$

To obtain the approximated transmission conditions on the smooth boundary  $\Gamma$ , we need to extend *formally*  $v^e$  in the membrane. Using a Taylor expansion in the variable  $\eta$ , we obtain,

$$v^e \circ \Phi|_{\eta=1} = v_0^e \circ \Phi|_{\eta=0} + \delta \left( v_1^e \circ \Phi|_{\eta=0} + f(\tau) \partial_n v_0^e \circ \Phi|_{\eta=0} \right) + \dots,$$

and similarly, denoting by  $\partial_t$  the tangential derivative on  $\Gamma$  we have

$$\begin{aligned}
 \partial_n v^e \circ \Phi|_{\eta=1} &= \partial_n v_0^e \circ \Phi|_{\eta=0} + \delta \left( \partial_n v_1^e \circ \Phi|_{\eta=0} \right. \\
 &\left. - \nabla_\alpha \cdot f(\tau) \partial_t v_0^e \circ \Phi|_{\eta=0} + f(\tau) \partial_n^2 v_0^e \circ \Phi|_{\eta=0} \right) + \dots
 \end{aligned}$$

To identify the first two terms of the asymptotic expansion of the steady state potentials, we need few notations. Let  $\mathfrak{K}$  be the curvature of  $\Gamma$  in Euclidean coordinates:

$$\forall x \in \Gamma, \quad \mathfrak{K}(x) = \kappa \circ \Psi^{-1}(x),$$

and define  $g_{\delta,\alpha}$  on  $\Gamma$  as follows:

$$\forall \theta \in \mathbb{T}, \quad g_{\delta,\alpha} \circ \Psi(\theta) = f\left(\theta, \frac{\theta}{\delta^\alpha}\right),$$

therefore, for all  $\theta \in \mathbb{T}$ ,

$$\partial_t g_{\delta,\alpha} \circ \Psi(\theta) = \nabla_\alpha \cdot f\left(\theta, \frac{\theta}{\delta^\alpha}\right).$$

Now, we just have to identify the term with the same power of  $\delta$ .

- 0<sup>th</sup> order term. We have necessarily:

$$\partial_\eta^2 v_0^m = 0,$$

and  $\partial_\eta v_0^m|_{\eta=0} = 0$ , hence  $v_0^m = v_0^m(\theta, \theta/\delta^\alpha)$  and therefore,  $v_0^e|_{\Gamma^+} = v_0^i|_{\Gamma^-}$ .

- First order term. Using  $\partial_\eta v_0^m \equiv 0$ , we obtain:

$$\partial_\eta^2 v_1^m = 0,$$

hence  $\partial_\eta v_1^m$  is constant with respect to  $\eta$ . According to transmission conditions (3), by identification, we necessarily have:

$$\sigma_m \partial_\eta v_1^m|_{\eta=0} = f(\tau) \sigma_i \partial_n v_0^i|_{\Gamma^-} \circ \Psi,$$

and

$$\sigma_m \partial_\eta v_1^m|_{\eta=1} = f(\tau) \sigma_e \partial_n v_0^e|_{\Gamma^+} \circ \Psi,$$

we infer :

$$\sigma_e \partial_n v_0^e|_{\Gamma^+} = \sigma_i \partial_n v_0^i|_{\Gamma^-},$$

which determine uniquely  $v_0^e$  and  $v_0^i$ . Moreover, we have  $v_0^m = v_0^i \circ \Psi$ , thus  $v_0^m$  depends only on  $\theta$ , and

$$\partial_\eta v_1^m = f(\theta, \theta/\delta^\alpha) \frac{\sigma_i}{\sigma_m} \partial_n v_0^i|_{\Gamma^-} \circ \Psi.$$

- Second order term. Using the previous equality, we obtain:

$$\partial_\eta^2 v_2^m + \kappa f \partial_\eta v_1^m + f^2 \partial_\theta^2 v_0^m = 0,$$

hence  $\partial_\eta v_2^m$  equals:

$$(20) \quad \begin{aligned} \partial_\eta v_2^m = & -\eta f^2(\theta, \theta/\delta^\alpha) \left( \kappa \frac{\sigma_i}{\sigma_m} \partial_n v_0^i|_{\Gamma^-} \circ \Psi + \partial_t^2 v_0^i|_{\Gamma^-} \circ \Psi \right) \\ & + f(\theta, \theta/\delta^\alpha) \frac{\sigma_i}{\sigma_m} \partial_n v_1^i|_{\Gamma^-} \circ \Psi. \end{aligned}$$

Since we have:

$$\begin{aligned} \frac{1}{f(\tau)} (\partial_\eta v_2^m|_{\eta=1} - f(\tau) \nabla_\alpha \cdot f(\tau) \partial_t v_0^i|_{\Gamma^-}) &= \frac{\sigma_e}{\sigma_m} \left( \partial_n v_1^e|_{\Gamma^+} - \nabla_\alpha \cdot f(\tau) \partial_t v_0^e|_{\Gamma^+} \right. \\ &\quad \left. + f(\tau) \partial_n^2 v_0^e|_{\Gamma^+} \right), \end{aligned}$$

and since

$$\partial_n^2 v_0^e|_{\Gamma^+} = -\partial_t^2 v_0^e|_{\Gamma^+} - \mathfrak{K} \partial_n v_0^e|_{\Gamma^+},$$

we infer easily the transmission conditions between  $v_1^e$  and  $v_1^i$ :

$$\sigma_e \partial_n v_1^e|_{\Gamma^+} - \sigma_i \partial_n v_1^i|_{\Gamma^-} = (\sigma_e - \sigma_m) \partial_t (g_{\delta, \alpha} \partial_t v_0^i|_{\Gamma^-}),$$

and

$$v_1^e|_{\Gamma^+} - v_1^i|_{\Gamma^-} = -g_{\delta, \alpha} \left( \frac{1}{\sigma_e} - \frac{1}{\sigma_m} \right) \sigma_i \partial_n v_0^i|_{\Gamma^-}.$$

Moreover, in the membrane, we have

$$v_1^m = \eta f(\theta/\delta^\alpha) \frac{\sigma_i}{\sigma_m} \partial_n v_0^i|_{\Gamma^-} \circ \Psi + v_1^i|_{\Gamma^-} \circ \Psi.$$

Therefore, formally, we have built the first two terms of the solution  $v$  to problem (1)–(2). We summarize here our formal results.

- *The  $0^{th}$ -order coefficients.* The electric potentials  $v_0^e$  and  $v_0^c$  are defined by (6). In the membrane, the field  $v_0^m$  equals:

$$(21) \quad \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad v_0^m = v_0^i|_{\Gamma^-} \circ \Psi.$$

- *The first order coefficients.* The potentials  $v_1^e$  and  $v_1^i$  are solution to the following problem in  $\Omega$ :

$$(22a) \quad \begin{cases} \Delta v_1^e = 0, \text{ in } \Omega \setminus \mathcal{O}^i, \\ \Delta v_1^i = 0, \text{ in } \mathcal{O}^i, \\ v_1^e|_{\partial\Omega} = 0, \end{cases}$$

with the transmission conditions

$$(22b) \quad \sigma_i \partial_n v_1^i|_{\Gamma^-} - \sigma_e \partial_n v_1^e|_{\Gamma^+} = -(\sigma_e - \sigma_m) \partial_t (g_{\delta, \alpha} \partial_t v_0^i|_{\Gamma^-}),$$

$$(22c) \quad v_1^i|_{\Gamma^-} - v_1^e|_{\Gamma^+} = g_{\delta, \alpha} \left( \frac{1}{\sigma_e} - \frac{1}{\sigma_m} \right) \sigma_i \partial_n v_0^i|_{\Gamma^-}.$$

In the membrane, we have:

$$(23) \quad \forall (\eta, \theta) \in [0, 1] \times \mathbb{T}, \quad v_1^m = \eta f(\theta, \theta/\delta^\alpha) \frac{\sigma_i}{\sigma_m} \partial_n v_0^i|_{\Gamma^-} \circ \Psi + v_1^i|_{\Gamma^-} \circ \Psi.$$

Observe that  $v_1$  depends on  $\delta/\alpha$ , thus we have to find the limit problem of  $v_1$ . Remember the expression of  $\partial_\eta v_2^m$  given in (20):

$$\begin{aligned} \partial_\eta v_2^m = & -\eta f^2(\theta, \theta/\delta^\alpha) \left( \kappa \frac{\sigma_i}{\sigma_m} \partial_n v_0^i|_{\Gamma^-} \circ \Psi + \partial_t^2 v_0^i|_{\Gamma^-} \circ \Psi \right) \\ & + f(\theta, \theta/\delta^\alpha) \frac{\sigma_i}{\sigma_m} \partial_n v_1^i|_{\Gamma^-} \circ \Psi. \end{aligned}$$

Let us give now useful estimates for  $(v_0^i, v_0^e)$  and  $(v_1^i, v_1^e)$ .

**Lemma 3.1.** *Let  $k_0 \in \mathbb{N}^*$  and  $\varphi$  belong to  $H^{k_0-1/2}(\partial\Omega)$ . Then we have the following regularity:*

$$\begin{aligned} (v_0^e, v_0^i) &\in H^{k_0}(\Omega \setminus \mathcal{O}^i) \times H^k(\mathcal{O}^i), \\ (v_1^e, v_1^i) &\in H^{k_0-1}(\Omega \setminus \mathcal{O}^i) \times H^3(\mathcal{O}^i), \\ v_0^m &\in \mathcal{C}^\infty([0, 1], H^{k_0-1/2}(\mathbb{T})), \\ v_1^m &\in \mathcal{C}^\infty([0, 1], H^{k_0-3/2}(\mathbb{T})), \\ \partial_\eta v_2^m &\in \mathcal{C}^\infty([0, 1], H^{k_0-3/2}(\mathbb{T})). \end{aligned}$$

Moreover there exists a constant  $C > 0$   $\delta$ -independent such that for  $l = 0, \dots, k_0$ ,

$$\begin{aligned} \|v_0^i\|_{H^l(\mathcal{O}^i)} &\leq C |\varphi|_{H^{k_0-1/2}(\partial\Omega)}, \\ \|v_0^e\|_{H^l(\Omega \setminus \mathcal{O}^i)} &\leq C |\varphi|_{H^{k_0-1/2}(\partial\Omega)}, \\ \sup_{\eta \in [0, 1]} |v_0^m|_{H^{l-1/2}(\mathbb{T})} &\leq C |\varphi|_{H^{k_0-1/2}(\partial\Omega)}, \end{aligned}$$

and for  $l = 1, \dots, k_0 - 1$ ,

$$\begin{aligned} \|v_1^i\|_{H^l(\mathcal{O}^i)} &\leq C \delta^{-l\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)}, \\ \|v_1^e\|_{H^l(\Omega \setminus \mathcal{O}^i)} &\leq C \delta^{-l\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)}, \\ \sup_{\eta \in [0, 1]} |v_1^m|_{H^{l-3/2}(\mathbb{T})} &\leq C \delta^{-l\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)}, \\ \sup_{\eta \in [0, 1]} |\partial_\eta v_2^m|_{H^{l-3/2}(\mathbb{T})} &\leq C \delta^{-l\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)}. \end{aligned}$$

*Proof.* The estimates involving the  $0^{\text{th}}$ -order terms are well-known, since these terms do not take the thin oscillating layer into account. The estimates on  $v_1^m$  and  $\partial_\eta v_2^m$  easily come from the estimates on  $v_1^e$  and  $v_1^i$ . Therefore we just have to prove the results for  $v_1^e$  and  $v_1^i$ .

Observe that  $(v_1^e, v_1^i)$  are defined in a domain without oscillation. Denote by  $F_1$  and  $F_2$  the two following functions

$$F_1 = -(\sigma_e - \sigma_m) \partial_t (g_{\delta, \alpha} \partial_t v_0^i|_{\Gamma^-}),$$

and

$$F_2 = g_{\delta, \alpha} \left( \frac{1}{\sigma_e} - \frac{1}{\sigma_m} \right) \sigma_i \partial_n v_0^i|_{\Gamma^-}.$$

Obviously, we have the following estimates, for  $l = 0, \dots, k_0 - 1$ :

$$|F_1|_{H^{l-1/2}(\partial\mathcal{O}^i)} \leq C \delta^{-(l+1)\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)},$$

and

$$|F_2|_{H^{l+1/2}(\partial\mathcal{O}^i)} \leq C \delta^{-(l+1)\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)}.$$

Since  $(v_1^e, v_1^i)$  satisfies:

$$\begin{cases} \Delta v_1^e = 0, & \text{in } \Omega \setminus \mathcal{O}^i, \\ \Delta v_1^i = 0, & \text{in } \mathcal{O}^i, \\ v_1^e|_{\partial\Omega} = 0, \end{cases}$$

with the transmission conditions

$$\begin{aligned}\sigma_i \partial_n v_1^i|_{\Gamma^-} - \sigma_e \partial_n v_1^e|_{\Gamma^+} &= F_1, \\ v_1^i|_{\Gamma^-} - v_1^e|_{\Gamma^+} &= F_2,\end{aligned}$$

using a well-known extension of Lemma 0.1 of [12] (see also [2]), we infer easily the following estimates, for  $l = 1, \dots, k_0 - 1$ :

$$\begin{aligned}\|v_1^i\|_{H^l(\mathcal{O}^i)} &\leq C\delta^{-l\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)}, \\ \|v_1^e\|_{H^l(\Omega \setminus \mathcal{O}^i)} &\leq C\delta^{-l\alpha} |\varphi|_{H^{k_0-1/2}(\partial\Omega)},\end{aligned}$$

hence the lemma.  $\square$

#### 4. LIMIT PROBLEM OF $v_1$

If  $\alpha = 0$ , the functions  $v_1$  and  $\bar{v}_1$  are equal, since no oscillation occurs on  $\Gamma_\delta^r$ . For  $\alpha \in (0, 1/2)$ , the following lemma ensures that the solution  $\bar{v}_1$  to problem (7) is the  $L^2$ -limit of  $v_1$ , for  $\delta$  tending to zero.

**Lemma 4.1.** *Suppose  $\alpha \in (0, 1/2)$  and let  $\varphi$  belong to  $H^{k_0-1/2}(\partial\Omega)$ ,  $k_0 \geq 2$ . Denote by:*

$$v_1 = (v_1^e, v_1^i), \text{ and } \bar{v}_1 = (\bar{v}_1^e, \bar{v}_1^i)$$

*the respective solutions to problem (22) and (7). Then,*

$$\|v_1 - \bar{v}_1\|_{L^2(\Omega)} = O(\delta^{\alpha/2}), \quad \text{for } \delta \text{ tending to zero.}$$

**Remark 4.2.** *To prove of this lemma, we need a precise bound of the function  $g_{\delta,\alpha} - \bar{g}$  in  $H^{-1}(\Gamma)$ . Actually, there exists a  $\delta$ -independent positive constant  $C$  such that*

$$(24) \quad \|g_{\delta,\alpha} - \bar{g}\|_{H^{-1}(\Gamma)} \leq C\delta^\alpha.$$

*Since the change of variable  $x = \Psi(\theta)$  does not involve the small parameter  $\delta^\alpha$ , we just have to prove the above estimate for the function*

$$\theta \in \mathbb{T} \longrightarrow f(\theta, \theta/\delta^\alpha) - \int_0^1 f(\theta, y) dy.$$

*Moreover, by a density argument, we suppose that  $f(x, y) = f_1(x)f_2(y)$ , where  $f_1$  and  $f_2$  are 1-periodic functions. Therefore,*

$$f(\theta, \theta/\delta^\alpha) - \int_0^1 f(\theta, y) dy = f_1(\theta)g_2(\theta/\delta^\alpha),$$

*where  $g_2(y) = f_2(y) - \int_0^1 f_2(s) ds$ . The following function  $\mathcal{P}_f$  defined for  $\theta \in [0, 1]$ :*

$$\mathcal{P}_f(\theta) = \int_0^\theta f_1(t)g_2(t/\delta^\alpha) dt - \theta \int_0^1 f_1(t)g_2(t/\delta^\alpha) dt,$$

*and extended by periodicity to  $\mathbb{T}$  is a periodic primitive of*

$$\theta \in \mathbb{T} \longrightarrow f(\theta, \theta/\delta^\alpha) - \int_0^1 f(\theta, y) dy,$$

*and a simple calculation shows that*

$$\|\mathcal{P}_f\|_{L^2(\mathbb{T})} \leq C\delta^\alpha,$$

*from which we infer (24).*

We are now ready to proof the above lemma.

*Proof of Lemma 4.1.* Denote by  $\tilde{\sigma}$ , the function:

$$\tilde{\sigma} = \begin{cases} \sigma_i, & \text{in } \mathcal{O}^i, \\ \sigma_e, & \text{in } \Omega \setminus \mathcal{O}^i. \end{cases}$$

For all  $\psi \in L^2(\Omega)$ , denote by  $\phi$  the solution in  $H_0^1(\Omega)$  to the following problem:

$$\nabla \cdot (\tilde{\sigma} \nabla \phi) = \psi, \quad \text{in } \Omega.$$

Let  $w_1$  equal  $w_1 = v_1 - \bar{v}_1$ . This function satisfies:

$$(25a) \quad \begin{cases} \Delta w_1 = 0, & \text{in } \Omega \setminus \mathcal{O}^i, \\ \Delta w_1 = 0, & \text{in } \mathcal{O}^i, \\ w_1|_{\partial\Omega} = 0, \end{cases}$$

with the transmission conditions

$$(25b) \quad \sigma_i \partial_n w_1|_{\Gamma^-} - \sigma_e \partial_n w_1|_{\Gamma^+} = -(\sigma_e - \sigma_m) \partial_t ((g_{\delta, \alpha} - \bar{g}) \partial_t v_0^i|_{\Gamma^-}),$$

$$(25c) \quad w_1|_{\Gamma^-} - v_1|_{\Gamma^+} = (g_{\delta, \alpha} - \bar{g}) \left( \frac{1}{\sigma_e} - \frac{1}{\sigma_m} \right) \sigma_i \partial_n v_0^i|_{\Gamma^-}.$$

Multiplication of (25) by  $\phi$ , and integration by parts now gives:

$$(26) \quad \begin{aligned} \int_{\Omega} w_1(x) \psi(x) \, dx &= \int_{\Gamma} (g_{\delta, \alpha}(s) - \bar{g}(s)) \left( (\sigma_e - \sigma_m) \partial_t v_0^i|_{\Gamma^-}(s) \partial_t \phi|_{\Gamma^-}(s) \right. \\ &\quad \left. + \left( \frac{1}{\sigma_e} - \frac{1}{\sigma_m} \right) \sigma_i^2 \partial_n v_0^i|_{\Gamma^-}(s) \partial_n \phi|_{\Gamma^-}(s) \right) \, ds. \end{aligned}$$

Observe that the following bounds hold, for a  $\delta$ -independent positive constant  $C$ :

$$\begin{aligned} \|g_{\delta, \alpha} - \bar{g}\|_{L^2(\Gamma)} &\leq C, \\ \|g_{\delta, \alpha} - \bar{g}\|_{H^{-1}(\Gamma)} &\leq C\delta^\alpha, \end{aligned}$$

according to (24). Using the well-known interpolation theorem, it follows

$$\|g_{\delta, \alpha} - \bar{g}\|_{H^{-1/2}(\Gamma)} \leq C\delta^{\alpha/2}.$$

From (26), there exists a  $\delta$ -independent constant  $C$  such that :

$$\begin{aligned} \left| \int_{\Omega} w_1(x) \psi(x) \, dx \right| &\leq C\delta^{\alpha/2} \|\phi\|_{H^2(\Omega)}, \\ &\leq C\delta^{\alpha/2} \|\psi\|_{L^2(\Omega)}. \end{aligned}$$

hence the lemma.  $\square$

## 5. ERROR ESTIMATES

The following theorem justify our asymptotic expansion.

**Theorem 5.1.** *We suppose that the boundary data  $\varphi$  belongs to  $H^{7/2}(\partial\Omega)$ . Denote by  $V_{app}$  the following function:*

$$V_{app} = \begin{cases} v_0^e + \delta v_1^e, & \text{in } \mathcal{O}_\delta^e \\ v_0^i + \delta v_1^i, & \text{in } \mathcal{O}^i \\ \left( v_0^m + \delta v_1^m + \delta^2 \int_0^\eta \partial_\eta v_2^m(s, \tau) ds \right) \circ \Phi^{-1}, & \text{in } \mathcal{O}^m. \end{cases}$$

Then, we have

$$\|V - V_{app}\|_{L^2(\Omega)} = O(\delta^{3/2-\alpha}), \quad \text{uniformly for } \delta \text{ tending to zero.}$$

**Remark 5.2.** *Since  $\alpha$  belongs to  $[0, 1/2)$ , Theorem 5.1 combined with Lemma 4.1 leads obviously to Theorem 1.1.*

*If the curve  $\Gamma_\delta^r$  has no oscillating part, which means that  $\alpha = 0$ , the above function  $v_1$  and  $\overline{v_1}$  are equal. And the above theorem with  $\alpha = 0$  is exactly Theorem 1.2.*

*Proof.* It is convenient to set

$$v_{app}^m = v_0^m + \delta v_1^m + \delta^2 \int_0^\eta \partial_\eta v_2^m(s, \tau) ds.$$

Observe that

$$\partial_\eta v_{app}^m(\eta, \theta) = \delta f(\theta, \theta/\delta^\alpha) \frac{\sigma_i}{\sigma_m} \partial_n v_0^i|_{\Gamma^-} \circ \Psi + \delta^2 \partial_\eta v_2^m(\eta, \theta).$$

Define

$$W = \frac{V_\delta - V_{app}}{\delta}.$$

In each domain  $\mathcal{O}_\delta^e$  and  $\mathcal{O}^i$ ,  $W$  is a harmonic function. In  $\mathcal{O}_\delta$ , according to (18) and the definition of  $V_{app}$  we have:

$$\begin{aligned} \Delta W = & \frac{\delta^2}{|G|} \left\{ \kappa(\theta) f(\tau) \left( 2\eta \partial_\eta^2 v_2^m + \partial_\eta v_2^m \right) + \delta \left\{ \left( \eta^2 (\nabla_\alpha \cdot f(\tau))^2 \right. \right. \right. \\ & + \left. \left. \left( \eta \kappa(\theta) f(\tau) \right)^2 \right) \partial_\eta^2 v_2^m + f^2(\tau) \partial_\theta^2 v_2^m \right. \\ (27) \quad & - 2\eta f(\tau) \nabla_\alpha \cdot f(\tau) \partial_\theta \partial_\eta v_2^m + \eta \left( \kappa^2(\theta) f^2(\tau) + 2(\nabla_\alpha \cdot f(\tau))^2 \right. \\ & \left. \left. \left. - f(\tau) \nabla_\alpha^2 \cdot f(\tau) \right) \partial_\eta v_2^m \right\} + R(\eta, \tau) v_{app}^m \right\} \circ \Phi^{-1}, \end{aligned}$$

According to (19):

$$\begin{aligned} R(\eta, \tau) v_{app}^m = & \frac{\eta f(\tau)}{1 + \kappa(\theta) \eta \delta f(\tau)} \left\{ \eta \left( \kappa'(\theta) \nabla_\alpha f - f^2(\tau) \kappa^3(\theta) \right) \partial_\eta v_{app}^m \right. \\ & \left. - \kappa'(\theta) f^2(\tau) \partial_\theta v_{app}^m \right\}. \end{aligned}$$

Denote by  $\gamma \circ \Phi^{-1}$  the right hand side of (27). Using Lemma 3.1 we infer:

$$\sup_{\eta \in [0,1]} |\gamma|_{H^{-1/2}(\mathbb{T})} \leq C \delta^{-\alpha} |\varphi|_{H^{7/2}(\partial\Omega)}$$

We have for all  $\phi \in H^1(\mathcal{O}_\delta)$ ,

$$\begin{aligned} \left| \int_{\mathcal{O}_\delta} \gamma \circ \Phi^{-1}(x) \phi(x) dx \right| &= \left| \int_0^1 \int_0^{2\pi} \sqrt{|G|} \gamma(\eta, \theta) \phi \circ \Phi(\eta, \theta) d\eta d\theta \right|, \\ &\leq C \delta^{1/2} \sup_{\eta \in [0,1]} |\gamma|_{H^{-1/2}(\mathbb{T})} \|\phi\|_{H^1(\mathcal{O}_\delta)}, \\ &\leq C \delta^{1/2-\alpha} |\varphi|_{H^{7/2}(\partial\Omega)} \cdot \|\phi\|_{H^1(\mathcal{O}_\delta)} \end{aligned}$$

Observe that  $W$  satisfies the following transmission conditions on  $\Gamma$ :

$$\begin{aligned} W|_{\Gamma^+} - W|_{\Gamma^-} &= 0, \\ \sigma_m \partial_n W|_{\Gamma^+} - \sigma_i \partial_n W|_{\Gamma^-} &= 0, \end{aligned}$$

and on  $\Gamma_\delta^r$ ,

$$\begin{aligned} W|_{\Gamma_\delta^{r+}} - W|_{\Gamma_\delta^{r-}} &= \frac{1}{\delta} \left( V_{app}|_{\Gamma_\delta^{r-}} - V_{app}|_{\Gamma_\delta^{r+}} \right), \\ \sigma_e \partial_n W|_{\Gamma_\delta^{r+}} - \sigma_m \partial_n W|_{\Gamma_\delta^{r-}} &= \frac{1}{\delta} \left( \sigma_m \partial_n V_{app}|_{\Gamma_\delta^{r-}} - \sigma_e \partial_n V_{app}|_{\Gamma_\delta^{r+}} \right). \end{aligned}$$

Using a Taylor expansion with respect to the normal variable  $\eta$ , we explicit the terms  $(v_j^e|_{\Gamma_\delta^{r+}})_{j=0,1}$  and  $(\partial_n v_j^e|_{\Gamma_\delta^{r+}})_{j=0,1}$  in terms of  $(v_j^e|_{\Gamma^+})_{j=0,1}$ ,  $(\partial_n v_j^e|_{\Gamma^+})_{j=0,1}$  and  $(\partial_t v_j^e|_{\Gamma^+})_{j=0,1}$ . Actually, in local coordinates  $(\eta, \theta)$ , we have:

$$\begin{aligned} (28) \quad & \left( W|_{\Gamma_\delta^{r+}} - W|_{\Gamma_\delta^{r-}} \right) \circ \Phi|_{\eta=1} = \delta f(\theta, \theta/\delta^\alpha) \int_0^1 (1-s) \left( \partial_\eta (v_1^e \circ \Phi)(s, \theta) \right. \\ & \left. + \frac{1-s}{2} f(\theta, \theta/\delta^\alpha) \partial_\eta^2 (v_0^e \circ \Phi)(s, \theta) \right) ds, \end{aligned}$$

$$\begin{aligned} (29) \quad & \left( \sigma_e \partial_n W|_{\Gamma_\delta^{r+}} - \sigma_m \partial_n W|_{\Gamma_\delta^{r-}} \right) \circ \Phi|_{\eta=1} = -\sigma_e \delta \left( \int_0^1 (1-s) f(\tau) \left( \partial_n^2 v_1^e \right. \right. \\ & \left. \left. + \frac{(1-s)f^2(\tau)}{2} \partial_n^3 v_0^e \right) \circ \Phi(s, \theta) ds \right. \\ & \left. - \nabla_\alpha \cdot f(\tau) \left( \partial_t v_1^e \circ \Phi|_{\eta=1} \right. \right. \\ & \left. \left. + \int_0^1 \partial_n \partial_t v_0^e \circ \Phi ds \right) \right. \\ & \left. + \nabla_\alpha \cdot f(\tau) \left( \nabla_\alpha \cdot f(\tau) \partial_n^2 v_0^e \circ \Phi|_{\eta=1} \right. \right. \\ & \left. \left. + \kappa(\theta) f(\tau) \partial_t v_0^e \circ \Phi|_{\eta=1} \right) \right) ds \\ & + \frac{\sigma_m}{f(\tau)} \left\{ \nabla_\alpha \cdot f(\tau) \left( \frac{\nabla_\alpha \cdot f(\tau)}{2} \partial_\eta v_{app}^m \right. \right. \\ & \left. \left. - f(\tau) \partial_\theta v_{app}^m \right) + T(\tau) v_{app}^m \right\} \Big|_{\eta=1}. \end{aligned}$$



Observe that since  $\Phi(1, \theta) = \Psi(\theta) + \delta f(\theta/\delta^\alpha)n(\theta)$ , the following estimates hold

$$\begin{aligned} \|W|_{\Gamma_\delta^{r+}} - W|_{\Gamma_\delta^{r-}}\|_{H^1(\Gamma_\delta^r)} &\leq C_\Gamma(1 + \delta^{1-\alpha}) \left\| \left( W|_{\Gamma_\delta^{r+}} \right. \right. \\ &\quad \left. \left. - W|_{\Gamma_\delta^{r-}} \right) \circ \Phi|_{\eta=1} \right\|_{H^1(\mathbb{T})}, \end{aligned}$$

and similarly

$$\begin{aligned} \left\| \sigma_e \partial_n W|_{\Gamma_\delta^{r+}} - \sigma_m \partial_n W|_{\Gamma_\delta^{r-}} \right\|_{L^2(\Gamma_\delta^r)} &\leq C_\Gamma \left\| \left( \sigma_e \partial_n W|_{\Gamma_\delta^{r+}} \right. \right. \\ &\quad \left. \left. - \sigma_m \partial_n W|_{\Gamma_\delta^{r-}} \right) \circ \Phi|_{\eta=1} \right\|_{L^2(\mathbb{T})}. \end{aligned}$$

Therefore, from the right-hand side of (28)–(29) and using the definitions of  $(v_j^e)_{j=0,1}$  and  $(v_j^m)_{j=0,1}$ , we infer easily with the help of Lemma 3.1 that there exists two functions  $b_1$  and  $b_2$  :

$$\begin{aligned} \|b_1\|_{H^1(\Gamma_\delta^r)} &\leq C |\varphi|_{H^{7/2}(\partial\Omega)}, \\ \|b_2\|_{L^2(\Gamma_\delta^r)} &\leq C |\varphi|_{H^{7/2}(\partial\Omega)}. \end{aligned}$$

and

$$\begin{aligned} W|_{\Gamma_\delta^{r+}} - W|_{\Gamma_\delta^{r-}} &= \delta^{1-2\alpha} b_1, \\ \sigma_e \partial_n W|_{\Gamma_\delta^{r+}} - \sigma_m \partial_n W|_{\Gamma_\delta^{r-}} &= \delta^{1-2\alpha} b_2. \end{aligned}$$

Then we have shown that  $W$  satisfies:

$$\begin{aligned} (30a) \quad & \Delta W = 0, \quad \text{in } \mathcal{O}_\delta^e \cup \mathcal{O}^i, \\ (30b) \quad & \Delta W = \gamma \circ \Phi^{-1}, \quad \text{in } \mathcal{O}_\delta, \\ (30c) \quad & W|_{\Gamma^+} - W|_{\Gamma^-} = 0, \\ (30d) \quad & \sigma_m \partial_n W|_{\Gamma^+} - \sigma_i \partial_n W|_{\Gamma^-} = 0, \\ (30e) \quad & W|_{\Gamma_\delta^{r+}} - W|_{\Gamma_\delta^{r-}} = \delta^{1-2\alpha} b_1, \\ (30f) \quad & \sigma_e \partial_n W|_{\Gamma_\delta^{r+}} - \sigma_m \partial_n W|_{\Gamma_\delta^{r-}} = \delta^{1-2\alpha} b_2, \\ (30g) \quad & W|_{\partial\Omega} = 0. \end{aligned}$$

For all  $\psi \in L^2(\Omega)$ , define  $\phi \in H_0^1(\Omega)$  by:

$$\nabla \cdot (\sigma \nabla \phi) = \psi, \quad \text{in } \Omega.$$

It is well-known that

$$\|\phi\|_{H^1(\Omega)} \leq C \|\psi\|_{L^2(\Omega)}.$$

Moreover, since the amplitude of the oscillations of  $\Gamma_\delta^r$  are the square of their period, the result of Bonder *et al.* [6] leads to the following estimates for an  $\delta$ -independent constant  $C$ :

$$\begin{aligned} |\phi|_{L^2(\Gamma_\delta^r)} &\leq C \|\psi\|_{L^2(\Omega)}, \\ |\partial_n \phi|_{H^{-1}(\Gamma_\delta^r)} &\leq C \|\psi\|_{L^2(\Omega)}. \end{aligned}$$

Then, multiplying (30) by  $\phi$  and integrating by parts, there exists a constant  $C$  such that:

$$\left| \int_{\Omega} W(x)\psi(x) \, dx \right| \leq C\delta^{1/2-\alpha} |\varphi|_{H^{7/2}(\partial\Omega)} \|\psi\|_{L^2(\Omega)}.$$

Hence we infer

$$\|W\|_{L^2(\Omega)} \leq C\delta^{1/2-\alpha} |\varphi|_{H^{7/2}(\partial\Omega)}.$$

Since  $\alpha < 1/2$ , the theorem is shown.  $\square$

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