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# A characterization of Schur multipliers between character-automorphic Hardy spaces

D. Alpay and M. Mboup

## Abstract

We give a new characterization of character-automorphic Hardy spaces of order 2 and of their contractive multipliers in terms of de Branges Rovnyak spaces. Key tools in our arguments are analytic extension and a factorization result for matrix-valued analytic functions due to Leech.

**Keywords.** Character-automorphic functions, Hardy spaces, de Branges Rovnyak spaces, Schur multipliers.

**Mathematics Subject Classification (2000).** Primary: 30F35, 46E22. Secondary: 30B40

## 1 Introduction

Let  $\Gamma$  be a Fuchsian group of Möbius transformations of the unit disk  $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$  onto itself. For  $1 \leq p \leq \infty$  and for any character  $\alpha$  of  $\Gamma$ , we consider the spaces

$$\mathcal{H}_p^\alpha = \{f \in \mathcal{H}_p \mid f \circ \gamma = \alpha(\gamma)f, \quad \forall \gamma \in \Gamma\}.$$

These spaces are called character-automorphic Hardy spaces. A characterization of such spaces in terms of Poincaré theta series may be found in [15], [18], [13], [9]. In particular, Pommerenke showed in [15] that the series

$$f(z) = \frac{b_0(z)}{b'_0(z)} \sum_{\gamma \in \Gamma} \overline{\alpha(\gamma)} \theta(\gamma(z)) h(\gamma(z)) \frac{\gamma'(z)}{\gamma(z)} \quad (1.1)$$

defines a bounded linear operator from the classical Hardy space  $\mathcal{H}_p(\mathbb{D})$  into the subspace  $\mathcal{H}_p^\alpha(\mathbb{D})$ .

In the present paper we restrict ourselves to the case  $p = 2$ . We first give a characterization of the character-automorphic Hardy space  $\mathcal{H}_2^\alpha(\mathbb{D})$  in terms of an associated de Branges Rovnyak space of functions analytic in the open unit disk; see Theorem 3.2. We also characterize the contractive multipliers between  $\mathcal{H}_2^{\beta\alpha}(\mathbb{D})$  and  $\mathcal{H}_2^\alpha(\mathbb{D})$ , where  $\alpha$  and  $\beta$  two given characters; see Theorem 4.2. Our method is mainly based on analytic extension of positive kernels and factorization results from Nevanlinna-Pick interpolation theory.

## 2 A review on character-automorphic Hardy spaces

### 2.1 Fuchsian groups and automorphic functions

Let  $G$  be a group of linear transformations,  $T(z) = \frac{az+b}{cz+d}$ ,  $ad - bc = 1$ , in the complex plane and let  $\iota$  denotes the identity transformation. Two points  $z$  and  $z'$  in  $\mathbb{C}$  are said to be *congruent* with respect to  $G$ , if  $z' = T(z)$  for some  $T \in G$  and  $T \neq \iota$ . Two regions  $R, R' \subset \mathbb{C}$  are said to be  $G$ -congruent or  $G$ -equivalent if there exists a transformation  $T \neq \iota$  which sends  $R$  to  $R'$ . A region  $R$  which does not contain any two  $G$ -congruent points and such that the neighborhood of any point on the boundary contains  $G$ -congruent points of  $R$  is called a *fundamental region* for  $G$ . A *properly discontinuous* group is a group  $G$  having a fundamental region [10]. This amounts to saying that the identity transformation is isolated.

**Definition 2.1** *A Fuchsian group is a properly discontinuous group each of whose transformation maps  $\mathbb{D}$ ,  $\mathbb{T}$  and  $\mathbb{C} \setminus \overline{\mathbb{D}}$  onto themselves.*

A Fuchsian group  $\Gamma$  is said to be of convergence type (see *e.g.* [15]) if

$$\sum_{\gamma \in \Gamma} (1 - |\gamma(z)|^2) = (1 - |z|^2) \sum_{\gamma \in \Gamma} |\gamma'(z)| < \infty \quad z \in \mathbb{D}.$$

Then, the Green's function [15] of  $\Gamma$  with respect to a point  $\xi \in \mathbb{D}$  is defined as the Blaschke product

$$b_\xi(z) = \prod_{\gamma \in \Gamma} \frac{\gamma(\xi) - z}{1 - \overline{\gamma(\xi)}z} \frac{|\gamma(\xi)|}{\gamma(\xi)}. \quad (2.2)$$

It satisfies

$$b_\xi(\varphi(z)) = \mu_\xi(\varphi)b_\xi(z), \quad \forall \varphi \in \Gamma, \quad (2.3)$$

where  $\mu_\xi$  is the character of  $\Gamma$  associated with  $b_\xi(z)$ . A function satisfying the relation (2.3) is said to be *character-automorphic* with respect to  $\Gamma$  while a  $\Gamma$ -periodic function, as for example  $|b_\xi(z)| = |b_\xi(\varphi(z))|$ , is called *automorphic* with respect to  $\Gamma$ .

## 2.2 Spaces of character-automorphic functions

We now briefly mention the main properties pertaining to spaces of character-automorphic functions. The materials presented here are essentially borrowed from [15] and [19] (see also [11, 18]). Let  $\widehat{\Gamma}$  be the dual group of  $\Gamma$ , *i.e.* the group of (unimodular) characters. For an arbitrary character  $\alpha \in \widehat{\Gamma}$ , associate the subspaces of the classical space  $L_2(\mathbb{T})$

$$L_2^\alpha = \{f \in L_2 \mid f \circ \gamma = \alpha(\gamma)f, \forall \gamma \in \Gamma\}$$

$$\mathcal{H}_2^\alpha(\mathbb{D}) = L_2^\alpha \cap \mathcal{H}_2(\mathbb{D})$$

Let  $\Gamma$  be a Fuchsian group without elliptic and parabolic element. We say that  $\Gamma$  is of Widom type if, and only if, the derivative of  $b_0(z)$  is of bounded characteristic. In this case, Widom [20] has shown that the space  $\mathcal{H}_\infty^\alpha$  is not trivial for any character  $\alpha \in \widehat{\Gamma}$  and we have

**Theorem 2.2 (Pommerenke [15])** *Let  $\Gamma$  be of Widom type and let  $\theta(z)$  be the inner factor of  $b'_0(z)$ . If  $\alpha$  is any character of  $\Gamma$  and if  $h(z)$  is in  $\mathcal{H}_p(\mathbb{D})$ ,  $1 \leq p \leq \infty$ , then the function defined by (1.1) is in  $\mathcal{H}_p^\alpha(\mathbb{D})$  and*

$$\|f\|_p \leq \|h\|_p, \quad f(0) = \theta(0)h(0).$$

The Poincaré series [14] in (1.1) thus defines, in particular, a projection:  $P^\alpha : \theta\mathcal{H}_2(\mathbb{D}) \rightarrow \mathcal{H}_2^\alpha(\mathbb{D})$ . An important property of the space  $\mathcal{H}_2^\alpha(\mathbb{D})$  that we will need is that, point evaluation  $f \mapsto f(\xi)$ ,  $\xi \in \mathbb{D}$  is a bounded linear functional. The space therefore admits a reproducing kernel  $k^\alpha$ :

$$\langle f(z), k^\alpha(z, \xi) \rangle_{\mathcal{H}_2^\alpha(\mathbb{D})} = f(\xi), \quad \xi \in \mathbb{D} \quad \text{for all } f \in \mathcal{H}_2^\alpha(\mathbb{D})$$

$$\text{with } k^\alpha(z, \xi) \in \mathcal{H}_2^\alpha(\mathbb{D}), \quad \forall \xi \in \mathbb{D}$$

Since  $\mathcal{H}_2^\alpha(\mathbb{D}) \neq \{const\}$ , we have  $k^\alpha(\xi, \xi) = \|k^\alpha(\cdot, \xi)\|_{\mathcal{H}_2^\alpha(\mathbb{D})}^2 > 0$  for every  $\xi \in \mathbb{D}$ . In the sequel, the Green's function  $b_0(z)$  with respect to 0, will be denoted by  $b(z)$  for short.

Let  $\Gamma$  be a group of Widom type and let  $E$  be associated<sup>1</sup> to it in such a way that  $\overline{\mathbb{C}} \setminus E$  be equivalent to the Riemann surface  $\mathbb{D}/\Gamma$ , obtained by identifying  $\Gamma$ -congruent points. Then, there exists a universal covering map  $\mathfrak{z} : \mathbb{D} \rightarrow \overline{\mathbb{C}} \setminus E \simeq \mathbb{D}/\Gamma$  such that

- $\mathfrak{z}$  maps  $\mathbb{D}$  conformally onto  $\overline{\mathbb{C}} \setminus E$ ,
- $\mathfrak{z}$  is automorphic with respect to  $\Gamma$ :  $\mathfrak{z} \circ \gamma = \mathfrak{z}, \quad \forall \gamma \in \Gamma$
- and  $\mathfrak{z}(z_1) = \mathfrak{z}(z_2) \Rightarrow \exists \gamma \in \Gamma \mid z_1 = \gamma(z_2)$

In particular,  $\mathfrak{z}$  maps one-to-one the *normal fundamental domain* of  $\Gamma$  with respect to the origin,

$$\mathcal{F} = \{z \in \mathbb{D} : |\gamma'(z)| < 1 \text{ for all } \gamma \in \Gamma, \gamma \neq \iota\} \quad (2.4)$$

conformally onto some sub-domain of  $\overline{\mathbb{C}} \setminus E$ . We assume that  $\mathfrak{z}$  is normalized so that  $(\mathfrak{z}b)(0)$  is real and positive. In all the sequel, the character associated to the Green's function  $b(z)$  will be denoted by  $\mu$ . The starting point of the next section is the following result:

**Lemma 2.3** ([12]) *The reproducing kernel for the space  $\mathcal{H}_2^\alpha(\mathbb{D})$  has the form*

$$k^\alpha(z, \omega) = c(\alpha) \frac{\frac{k^{\alpha\mu}(z,0)}{b(z)} k^\alpha(\omega, 0)^* - \left(\frac{k^{\alpha\mu}(\omega,0)}{b(\omega)}\right)^* k^\alpha(z, 0)}{\mathfrak{z}(z) - \mathfrak{z}(\omega)^*} \quad (2.5)$$

where

$$c(\alpha) = \frac{\mathfrak{z}(0)b(0)}{k^{\alpha\mu}(0,0)} > 0. \quad (2.6)$$

### 3 An associated de Branges-Rovnyak space

In this section we give a characterization of the space  $\mathcal{H}_2^\alpha(\mathbb{D})$  in terms of an associated de Branges Rovnyak space. To begin, let

$$\Omega_+ = \{z \in \mathbb{D}; \operatorname{Im} \mathfrak{z}(z) > 0\}.$$

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<sup>1</sup>See [21] for an example of a construction of a group  $\Gamma$  associated with a finite union of disjoint arcs of the unit circle.

Setting

$$\begin{aligned} A^\alpha(z) &= \sqrt{\frac{c(\alpha)}{2}} \left( \frac{k^{\alpha\mu}(z, 0)}{b(z)} + ik^\alpha(z, 0) \right) \\ B^\alpha(z) &= \sqrt{\frac{c(\alpha)}{2}} \left( \frac{k^{\alpha\mu}(z, 0)}{b(z)} - ik^\alpha(z, 0) \right) \end{aligned}$$

we can rewrite the reproducing kernel  $k^\alpha$  as

$$k^\alpha(z, \omega) = \frac{2A^\alpha(z)}{1 - i\mathfrak{z}(z)} \frac{1 - S_\alpha(z)S_\alpha(w)^*}{1 - \sigma(z)\sigma(w)^*} \frac{2A^\alpha(w)^*}{1 + i\mathfrak{z}(w)^*} \quad (3.1)$$

where  $S_\alpha(z) = B^\alpha(z)/A^\alpha(z)$  and  $\sigma(z) = \frac{1+i\mathfrak{z}(z)}{1-i\mathfrak{z}(z)}$ . We note that the functions  $A^\alpha(z)$  and  $B^\alpha(z)$  are character-automorphic with the same character  $\alpha$  while  $S_\alpha$  and  $\sigma$  are automorphic functions. From now on, the notation  $f^\nu(z)$  will mean that the function  $f^\nu(z)$  is character-automorphic with the superscript  $\nu \in \widehat{\Gamma}$  being the associated character, and the notation  $f_\nu(z)$  will stand for a function depending on the character  $\nu$  (automorphic or not).

**Proposition 3.1** *There exists a Schur function  $\mathcal{S}_\alpha$  such that  $S_\alpha(z) = \mathcal{S}_\alpha(\sigma(z))$ .*

**Proof:** Since the kernel  $k^\alpha(z, \omega)$  is positive in  $\mathbb{D}$ , and hence in  $\Omega_+$ , it is clear that  $\frac{1-S_\alpha(z)S_\alpha(w)^*}{1-\sigma(z)\sigma(w)^*}$  is also positive in  $\Omega_+$ . Now, observe that the function  $\sigma$  maps  $\Omega_+ \cap \mathcal{F}$  into some subset  $\Delta \subset \mathbb{D}$  and this mapping is one-to-one. Let  $\varsigma$  be given by:  $(\varsigma \circ \sigma)(z) = z, \forall z \in \Omega_+ \cap \mathcal{F}$  (in particular this will also hold for any region congruent to  $\Omega_+ \cap \mathcal{F}$ ) and let the function  $\widetilde{\mathcal{S}}_\alpha$  be defined on  $\Delta$  by:

$$\widetilde{\mathcal{S}}_\alpha(\lambda) = (S_\alpha \circ \varsigma)(\lambda), \forall \lambda \in \Delta.$$

Then it comes that the kernel

$$\frac{1 - \widetilde{\mathcal{S}}_\alpha(\lambda)\widetilde{\mathcal{S}}_\alpha(\mu)^*}{1 - \lambda\mu^*}$$

is positive on  $\Delta$ . Now, this implies (see for instance [1, Theorem 2.6.5]) the existence of a unique extension of  $\widetilde{\mathcal{S}}_\alpha(\lambda)$ , analytic and contractive in all  $\mathbb{D}$ . We subsequently call  $\mathcal{S}_\alpha(\lambda)$  this extension, and denote by  $\mathcal{H}(\mathcal{S}_\alpha)$  the reproducing kernel Hilbert space with reproducing kernel

$$K_{\mathcal{S}_\alpha}(\lambda, \mu) = \frac{1 - \mathcal{S}_\alpha(\lambda)\mathcal{S}_\alpha(\mu)^*}{1 - \lambda\mu^*}$$

By construction, the equality

$$S_\alpha(z) = \mathcal{S}_\alpha(\sigma(z))$$

holds for all  $z \in \Omega_+ \cap \mathcal{F}$ . Since  $S_\alpha(z)$  is analytic in  $\mathbb{D}$ , it must also hold for all  $\mathbb{D}$ .  $\square$

In connection with the previous proposition and the next theorem, we recall that reproducing kernel Hilbert spaces  $\mathcal{H}(\mathcal{S})$  of functions analytic in the open unit disk and with a reproducing kernel of the form

$$\frac{1 - \mathcal{S}(\lambda)\mathcal{S}(\mu)^*}{1 - \lambda\mu^*}$$

were introduced and studied by de Branges and Rovnyak; see [5, Appendix], [6]. We also refer the reader to [8] and [1] for more information on these and on related spaces.

**Theorem 3.2** *The character-automorphic Hardy space  $\mathcal{H}_2^\alpha(\mathbb{D})$  can be described as*

$$\mathcal{H}_2^\alpha(\mathbb{D}) = \left\{ F(z) = \frac{\sqrt{2}A^\alpha(z)}{1 - i\mathfrak{z}(z)} f(\sigma(z)) ; f \in \mathcal{H}(\mathcal{S}_\alpha) \right\} \quad (3.2)$$

with the norm

$$\|F\|_{\mathcal{H}_2^\alpha(\mathbb{D})} = \|f\|_{\mathcal{H}(\mathcal{S}_\alpha)}.$$

**Proof:** Recall that the map which to  $F \in \mathcal{H}_2^\alpha(\mathbb{D})$  associates its restriction  $F|_{\Omega_+}$  to  $\Omega_+$  is an isometry from  $\mathcal{H}_2^\alpha(\mathbb{D})$  onto the reproducing kernel Hilbert space with reproducing  $k^\alpha(z, \omega)$  defined by (2.5), where  $z, \omega$  are now restricted to  $\Omega_+$ . We denote this last space by  $\mathcal{H}_2^\alpha(\mathbb{D})|_{\Omega_+}$ . By Proposition 3.1 and using (3.1) we see that the operator of multiplication by  $\frac{2A^\alpha(z)}{1 - i\mathfrak{z}(z)}$  is an isometry from the reproducing kernel Hilbert space  $\mathcal{H}$  with reproducing kernel

$$\frac{1 - \mathcal{S}_\alpha(\sigma(z))\mathcal{S}_\alpha(\sigma(w))^*}{1 - \sigma(z)\sigma(w)^*}$$

onto  $\mathcal{H}_2^\alpha(\mathbb{D})|_{\Omega_+}$ . Furthermore, the composition map by  $\sigma$  is an isometry from the de Branges Rovnyak space  $\mathcal{H}(\mathcal{S}_\alpha)$  onto  $\mathcal{H}$ . We have that

$$\mathcal{H} = \{f \circ \sigma ; f \in \mathcal{H}(\mathcal{S}_\alpha)\},$$

with norm  $\|f \circ \sigma\|_{\mathcal{H}} = \|f\|_{\mathcal{H}(\mathcal{S}_\alpha)}$ , as follows from the equalities

$$f(\sigma(\omega)) = \langle f(\cdot), K_{\mathcal{S}_\alpha}(\cdot, \sigma(\omega)) \rangle_{\mathcal{H}(\mathcal{S}_\alpha)} = \langle f \circ \sigma(\cdot), K_{\mathcal{S}_\alpha}(\sigma(\cdot), \sigma(\omega)) \rangle_{\mathcal{H}}$$

Thus the restrictions of the elements of  $\mathcal{H}_2^\alpha(\mathbb{D})$  to  $\Omega_+$  are of the form as in (3.2) for  $z$  restricted to  $\Omega_+$ . By analytic extension, the elements of  $\mathcal{H}_2^\alpha(\mathbb{D})$  have the same form in the whole of  $\mathbb{D}$ .  $\square$

## 4 Schur multipliers

A Schur function is a function analytic and contractive in the open unit disk. Equivalently, it is a function  $s$  such that the operator of multiplication by  $s$  is a contraction from the classical Hardy of the open unit disk into itself. This last definition is our starting point to define character-automorphic Schur multipliers.

**Definition 4.1** *A character-automorphic function  $s^\beta(z)$ , with character  $\beta$ , will be called a Schur multiplier if the operator of multiplication by  $s^\beta(z)$  is a contraction from  $\mathcal{H}_2^{\bar{\beta}\alpha}(\mathbb{D})$  into  $\mathcal{H}_2^\alpha(\mathbb{D})$ .*

Equivalently, the character-automorphic function  $s^\beta(z)$  is a Schur multiplier if and only if the kernel

$$K_{s^\beta}^\alpha(z, w) = k^\alpha(z, w) - s^\beta(z)s^\beta(w)^*k^{\bar{\beta}\alpha}(z, w) \quad (4.1)$$

is positive in  $\mathbb{D}$ . The kernel  $K_{s^\beta}^\alpha(z, w)$  is in particular positive in  $\Omega_+$ , and we will consider it in  $\Omega_+$ . We note that in view of [7, Lemma 2, p. 142], [4, Theorem 1.1.4, p. 10], the positivity of the analytic kernel  $K_{s^\beta}$  on  $\Omega_+$  implies its positivity on  $\mathbb{D}$ .

**Theorem 4.2** *A character-automorphic function  $s^\beta$  is a Schur multiplier if and only if there exists a  $\mathbb{C}^{2 \times 2}$ -matrix valued Schur function  $\Sigma(z)$  such that*

$$s^\beta(z) = \frac{A^\alpha(z)}{A^{\bar{\beta}\alpha}(z)} \frac{\Sigma_{12}(\sigma(z))}{1 - S_{\bar{\beta}\alpha}(z)\Sigma_{22}(\sigma(z))} \quad (4.2)$$

$$S_\alpha(z) = \Sigma_{11}(\sigma(z)) + \frac{\Sigma_{12}(\sigma(z))S_{\bar{\beta}\alpha}(z)\Sigma_{21}(\sigma(z))}{1 - S_{\bar{\beta}\alpha}(z)\Sigma_{22}(\sigma(z))} \quad (4.3)$$



**Proof:** The positivity of the kernel (4.1) in  $\Omega_+$  is equivalent to the positivity in  $\Omega_+$  of the kernel

$$K_{\mathcal{S}_\alpha}(\sigma(z), \sigma(\omega)) - T(z)T(\omega)^*K_{\mathcal{S}_{\bar{\beta}\alpha}}(\sigma(z), \sigma(\omega)),$$

where

$$T(z) = \frac{A^{\bar{\beta}\alpha}(z)}{A^\alpha(z)}s^\beta(z).$$

As in the proof of Proposition 3.1, we note that the function  $\sigma$  maps  $\Omega_+ \cap \mathcal{F}$  into some subset  $\Delta \subset \mathbb{D}$  and this mapping is one-to-one, and consider again the function  $\varsigma$  defined by:  $(\varsigma \circ \sigma)(z) = z$ ,  $\forall z \in \Omega_+ \cap \mathcal{F}$ . The kernel

$$K_{\mathcal{S}_\alpha}(\lambda, \mu) - T(\varsigma(\lambda))T(\varsigma(\mu))^*K_{\mathcal{S}_{\bar{\beta}\alpha}}(\lambda, \mu), \quad (4.4)$$

is positive in  $\Delta$ . We now show that  $T \circ \varsigma$  admits an analytic extension to  $\mathbb{D}$ . To that purpose we consider the linear relation in  $\mathcal{H}(\mathcal{S}_\alpha) \times \mathcal{H}(\mathcal{S}_{\bar{\beta}\alpha})$  spanned by the elements of the form

$$(K_{\mathcal{S}_\alpha}(\cdot, \omega), T(\varsigma(\omega))^*K_{\mathcal{S}_{\bar{\beta}\alpha}}(\cdot, \omega)), \quad \omega \in \Delta.$$

It is densely defined and it is contractive because of the positivity of the kernel (4.4) in  $\Delta$ . It is therefore the graph of a densely defined contraction  $\tilde{X}$ . We note its extension to  $\mathcal{H}(\mathcal{S}_\alpha)$  by  $X$ . For  $\omega \in \Delta$  and  $f \in \mathcal{H}(\mathcal{S}_\alpha)$  we have

$$\begin{aligned} (X^*f)(\omega) &= \langle X^*f, K_{\mathcal{S}_\alpha}(\cdot, \omega) \rangle_{\mathcal{H}(\mathcal{S}_\alpha)} \\ &= \langle f, T(\varsigma(\omega))^*K_{\mathcal{S}_{\bar{\beta}\alpha}}(\cdot, \omega) \rangle_{\mathcal{H}(\mathcal{S}_\alpha)} \\ &= T(\varsigma(\omega))f(\omega). \end{aligned}$$

Let  $f_0(\lambda) = K_{\mathcal{S}_{\bar{\beta}\alpha}}(\lambda, \omega_0)$  where  $\omega_0 \in \mathbb{D}$ . We have

$$T(\varsigma(\lambda)) = \frac{(X^*f_0)(\lambda)}{f_0(\lambda)}, \quad \lambda \in \Delta.$$

It follows that  $T \circ \varsigma$  has an analytic extension to  $\mathbb{D}$ , which we will denote by  $\mathcal{R}$ . Thus the kernel

$$K_{\mathcal{S}_\alpha}(\lambda, \mu) - \mathcal{R}(\lambda)\mathcal{R}(\mu)^*K_{\mathcal{S}_{\bar{\beta}\alpha}}(\lambda, \mu), \quad (4.5)$$

is analytic in  $\lambda$  and  $\mu^*$  in  $\mathbb{D}$ . Therefore it is still positive in  $\mathbb{D}$ ; see [7, Lemma 2, p. 142], [4, Theorem 1.1.4, p. 10]. By [2, Theorem 11.1, p. 61], a necessary

and sufficient condition for the kernel (4.5) to be positive is that there exists a  $\mathbb{C}^{2 \times 2}$ -matrix valued Schur function  $\Sigma(\lambda)$  such that

$$\mathcal{R}(\lambda) = \frac{\Sigma_{12}(\lambda)}{1 - \mathcal{S}_{\bar{\beta}\alpha}(\lambda)\Sigma_{22}(\lambda)} \quad (4.6)$$

$$\mathcal{S}_\alpha(\lambda) = \Sigma_{11}(\lambda) + \frac{\Sigma_{12}(\lambda)\mathcal{S}_{\bar{\beta}\alpha}(\lambda)\Sigma_{21}(\lambda)}{1 - \mathcal{S}_{\bar{\beta}\alpha}(\lambda)\Sigma_{22}(\lambda)}. \quad (4.7)$$

The above mentioned result from [2] stems from rewriting the positive kernel (4.5) as

$$\frac{\mathcal{A}(\lambda)\mathcal{A}(\mu)^* - \mathcal{B}(\lambda)\mathcal{B}(\mu)^*}{1 - \lambda\mu^*}$$

where

$$\begin{aligned} \mathcal{A}(\lambda) &= \begin{pmatrix} 1 & \mathcal{R}(\lambda)\mathcal{S}_{\bar{\beta}\alpha}(\lambda) \end{pmatrix} \\ \mathcal{B}(\lambda) &= \begin{pmatrix} \mathcal{S}_\alpha(\lambda) & \mathcal{R}(\lambda) \end{pmatrix}, \end{aligned}$$

and using a factorization result known as Leech's theorem, which insures the existence of a  $\mathbb{C}^{2 \times 2}$ -valued Schur function  $\Sigma$  such that

$$\mathcal{B}(z) = \mathcal{A}(z)\Sigma(z),$$

from which (4.6) follows.

This unpublished result of R.B. Leech has been proved using the commutant lifting theorem by M. Rosenblum; see [16, Theorem 2, p. 134] and [17, Example 1, p. 107]. Further discussions and applications can also be found in [3]. It can also be proved using tangential Nevanlinna-Pick interpolation and Montel's theorem.

Finally we replace in (4.6)  $\lambda$  by  $\sigma(z)$  where  $z \in \Omega_+ \cap \mathcal{F}$ . We obtain the formulas in the statement of the theorem for  $z \in \Omega_+ \cap \mathcal{F}$ , and hence for  $z \in \mathbb{D}$  by analytic extension.  $\square$

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