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► **To cite this version:**

Cécile Amblard, Stéphane Girard. A new symmetric extension of FGM copulas. [Research Report] 2007, pp.20. inria-00134433v1

HAL Id: inria-00134433

<https://inria.hal.science/inria-00134433v1>

Submitted on 2 Mar 2007 (v1), last revised 8 Feb 2008 (v2)

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A new symmetric extension of FGM copulas

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Abstract

We propose a new family of copulas generalizing the Farlie-Gumbel-Morgenstern family and generated by two univariate functions. The main feature of this family is two permit the modeling of high positive dependence. In particular, it is established that the correlation range is $[-3/4, 1]$ and that the upper tail dependence coefficient can reach any value in $[0, 1]$. Some examples of parametric subfamilies are provided.

Keywords: Copulas, semiparametric family, measures of association, positive dependence.

AMS Subject classifications: Primary 62H05, secondary 62H20.

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1 Introduction

A bivariate copula defined on the unit square $I^2 =: [0, 1]^2$ is a bivariate cumulative distribution function (cdf) with univariate uniform I margins. Sklar's Theorem [28] states that any bivariate distribution with cdf H and marginal cdf F and G can be written $H(x, y) = C(F(x), G(y))$, where C is a copula. This result justifies the use of copulas for building bivariate distributions.

One of the most popular parametric family of copulas is the Farlie-Gumbel-Morgenstern (FGM) family defined when $\theta \in [-1, 1]$ by

$$C_{\theta}^{\text{FGM}}(u, v) = uv + \theta u(1 - u)v(1 - v), \quad (1.1)$$

and studied in [6, 9, 17]. A well-known limitation to this family is that it does not allow the modeling of large dependences since the correlation range ρ between the marginal distributions is limited to $\rho \in [-1/3, 1/3]$. Basing on this remark, more general copulas have been introduced in 1960 by Sarmanov [25],

$$C_{\theta, \phi, \psi}^{\text{Sarmanov}}(u, v) = uv + \theta \phi(u)\psi(v), \quad (1.2)$$

an re-discovered in 2004 by Rodríguez-Lallena and Úbeda-Flores [24]. See [13] for an extension of this model. Properties of Sarmanov copulas are studied in [15, 26]. Unfortunately, characterization of admissible parameters θ and functions ϕ and ψ is not tractable to obtain closed-form bounds on the corresponding correlation coefficient ρ . Thus, several parametric sub-families of (1.2) were introduced. In [20], it is remarked that copulas with quadratic section [22] are not able to model large dependences. Copulas with cubic sections are thus introduced, with the conclusion that copulas with higher order polynomial sections would increase the dependence degrees but simultaneously the complexity of the model. In [10], two kernel extensions of FGM copulas are studied

$$C_{\theta, \gamma}^{\text{HK1}}(u, v) = uv + \theta u(1 - u)^{\gamma}v(1 - v)^{\gamma}, \quad (1.3)$$

for $\gamma \geq 1$ and

$$C_{\theta, \gamma}^{\text{HK2}}(u, v) = uv + \theta u(1 - u^{\gamma})v(1 - v^{\gamma}), \quad (1.4)$$

for $\gamma \geq 1/2$. It is shown that the positive correlation between the marginal distributions can be increased up to approximately 0.39 while the maximal negative correlation remains $-1/3$.

Another similar extension is

$$C_{\theta,p,q}^{\text{LX}}(u,v) = uv + \theta u^p(1-u)^q v^p(1-v)^q, \quad (1.5)$$

see [14]. Copulas (1.3) and (1.4) are particular cases of Bairamov-Kotz family [3] defined by

$$C_{\theta,p,q,n}^{\text{BK}}(u,v) = u^p v^p [1 + \theta(1-u)^q(1-v)^q]^n, \quad (1.6)$$

and with associated correlation coefficient $\rho \in [-0.48, 0.501594]$. Moreover, it has been remarked in [11] that dependence degrees arbitrarily close to ± 1 cannot be obtained with polynomial functions of fixed degree. An alternative approach to generalize the FGM family of copula is to consider the semi-parametric family of symmetric copulas defined by

$$C_{\theta,\phi}^{\text{SP}}(u,v) = uv + \theta\phi(u)\phi(v), \quad (1.7)$$

with $\theta \in [-1, 1]$. It was first introduced in [23], and extensively studied in [1, 2]. Clearly, this family also includes the FGM copulas (1.1) (which contains all copulas with both horizontal and vertical quadratic sections [22]), the parametric family of symmetric copulas with cubic sections proposed in [20], equation (4.4), and kernel families (1.3), (1.4) and (1.5). It can be shown that, for a properly chosen function ϕ , the range of the correlation coefficient is extended to $[-3/4, 3/4]$, whereas the upper tail dependence coefficient is always null. We refer to [7] for a very interesting method for constructing admissible functions ϕ .

In this paper, we propose an extension of the $C_{\theta,\phi}^{\text{SP}}$ family where θ is a univariate function. This modification allows the modeling of strong positive dependences since this new family can take into account the extremal case of positive functional dependence between two random variables. Moreover, arbitrary upper tail dependence coefficients can be reached in $[0, 1]$. The new family is described in Section 2 and the associated Spearman's Rho and upper tail dependence coefficient are studied in Section 3. Section 4 is dedicated to the dependence properties of this new family of copulas. Finally, some examples of copulas taken in this family are provided in Section 5. Lemmas are postponed to the appendix.

2 Definition and basic properties

We consider the family of functions defined on I^2 by:

$$C_{\theta,\phi}(u, v) = uv + \theta[\max(u, v)]\phi(u)\phi(v), \quad (2.1)$$

where ϕ and θ are two $I \rightarrow \mathbb{R}$ continuously differentiable functions. We moreover assume that ϕ vanishes at most on isolated points of I . The following theorem gives sufficient and necessary conditions on ϕ and θ to ensure that $C_{\theta,\phi}$ is a copula.

Theorem 1 $C_{\theta,\phi}$ is a copula if and only if ϕ and θ satisfy the following conditions :

- (a) $\phi(0) = 0$,
- (b) $\phi(1)\theta(1) = 0$,
- (c) $\phi'(u)(\theta\phi)'(v) \geq -1$ for all $0 < u \leq v < 1$,
- (d) $\theta'(u) \leq 0$ for all $u \in \overset{\circ}{I} = (0, 1)$.

Proof: Recall that, C is a copula if and only if the following properties hold:

- (P1) $C(u, 0) = C(0, v) = 0, \forall (u, v) \in I^2$,
- (P2) $C(u, 1) = u$ and $C(1, v) = v, \forall (u, v) \in I^2$,
- (P3) $\Delta(u_1, u_2, v_1, v_2) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \forall (u_1, u_2, v_1, v_2) \in I^4$,
such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

The proof involves four steps.

- First, it is clear that (P1) \Leftrightarrow (a) and (P2) \Leftrightarrow (b).
- Second, we show that (P3) \Rightarrow (c). To this end, consider $0 < u_1 < u_2 \leq v_1 < v_2 < 1$. In this case, $\Delta(u_1, u_2, v_1, v_2)$ can be rewritten as

$$\Delta(u_1, u_2, v_1, v_2) = (u_2 - u_1)(v_2 - v_1) + (\phi(u_2) - \phi(u_1))(\theta(v_2)\phi(v_2) - \theta(v_1)\phi(v_1)), \quad (2.2)$$

and thus $\Delta(u_1, u_2, v_1, v_2) \geq 0$ implies

$$\frac{\phi(u_2) - \phi(u_1)}{u_2 - u_1} \frac{\theta(v_2)\phi(v_2) - \theta(v_1)\phi(v_1)}{v_2 - v_1} \geq -1.$$

Letting $u_1 \xrightarrow{<} u_2$ and $v_1 \xrightarrow{<} v_2$ in the previous inequality yields **(c)**.

– Similarly, we now show that **(P3)** \Rightarrow **(d)**. Taking $0 < u < v < 1$, we have

$$\begin{aligned}\Delta(u, v, u, v) &= (v - u)^2 + \theta(v)\phi^2(v) + \theta(u)\phi^2(u) - 2\phi(u)\phi(v)\theta(v), \\ &= (v - u) \left[(v - u) + \theta(v)\phi(v) \frac{\phi(v) - \phi(u)}{v - u} - \phi(u) \frac{(\theta\phi)(v) - (\theta\phi)(u)}{v - u} \right].\end{aligned}$$

Letting $u \xrightarrow{<} v$ in the inequality $\Delta(u, v, u, v) \geq 0$ yields

$$\theta(v)\phi(v)\phi'(v) - \phi(v)(\theta\phi)'(v) \geq 0,$$

which is equivalent to $\phi^2(v)\theta'(v) \leq 0$. This implies that $\theta'(v) \leq 0$ for all $v \in I$ such that $\phi(v) \neq 0$. Since ϕ vanishes only on isolated points and θ is continuous, **(d)** is proved.

– Finally, it remains to prove that **(c, d)** \Rightarrow **(P3)**. Let $(u_1, u_2, v_1, v_2) \in I^4$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let us denote by R the rectangle $[u_1, u_2] \times [v_1, v_2]$, by T the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$, and by \bar{T} the triangle $\bar{T} = I^2 \setminus T$. Suppose **(c, d)** hold and let us prove that $\Delta(R) =: \Delta(u_1, u_2, v_1, v_2) \geq 0$. Three cases are considered. (i) If $R \subset T$, *i.e.* $u_2 \leq v_1$ then $\Delta(u_1, u_2, v_1, v_2)$ can be written as in (2.2) and the mean value theorem entails that there exist $u \in (u_1, u_2)$ and $v \in (v_1, v_2)$ such that

$$\begin{aligned}\Delta(u_1, u_2, v_1, v_2) &= (u_2 - u_1)(v_2 - v_1) \left[1 + \frac{\phi(u_2) - \phi(u_1)}{u_2 - u_1} \frac{\theta(v_2)\phi(v_2) - \theta(v_1)\phi(v_1)}{v_2 - v_1} \right] \\ &= (u_2 - u_1)(v_2 - v_1) [1 + \phi'(u)(\theta\phi)'(v)] \geq 0,\end{aligned}$$

as a consequence of **(c)**. (ii) If $R \subset \bar{T}$, then symmetry considerations show that $\Delta(u_1, u_2, v_1, v_2) = \Delta(v_1, v_2, u_1, u_2) \geq 0$ from the case (i). (iii) If $R \cap T \neq \emptyset$, then R can be decomposed as disjoint rectangles $R = R_1 \cup R_2 \cup R_3$ such that $R_1 \subset T$, $R_2 \subset T$ or $R_2 \subset \bar{T}$ and R_3 is a square of the form $[u, v] \times [u, v]$. Thus, $\Delta(R) = \Delta(R_1) + \Delta(R_2) + \Delta(R_3)$ and (i, ii) entails that $\Delta(R_1) \geq 0$ and $\Delta(R_2) \geq 0$. Let us focus on $\Delta(R_3)$:

$$\begin{aligned}\Delta(u, v, u, v) &= (v - u)^2 + \theta(v)\phi(v)[\phi(v) - \phi(u)] - \phi(u)[\theta(v)\phi(v) - \theta(u)\phi(u)] \\ &= (v - u)^2 + \int_u^v [\theta(v)\phi(v)\phi'(t) - \phi(u)(\theta\phi)'(t)] dt.\end{aligned}$$

Note that **(c)** implies that for all $0 < t < v < 1$,

$$\phi'(t) \int_t^v (\theta\phi)'(y) dy \geq t - v,$$

and thus $\phi'(t)(\theta\phi)(v) \geq t - v + \theta(t)\phi(t)\phi'(t)$. Similarly, **(c)** shows that for all $0 < u < t < 1$,

$$(\theta\phi)'(t) \int_u^t \phi'(x)dx \geq u - t,$$

and thus $-(\theta\phi)'(t)\phi(u) \geq u - t - (\theta\phi)'(t)\phi(t)$. It follows that

$$\Delta(u, v, u, v) \geq \int_u^v [(\theta\phi)(t)\phi'(t) - \phi(t)(\theta\phi)'(t)]dt = - \int_u^v \theta'(t)\phi^2(t)dt \geq 0$$

under condition **(d)**. As a conclusion, $\Delta(R) \geq 0$ and **(P3)** is proved. ■

Note that **(b)** is true if $\phi(1) = 0$ or $\theta(1) = 0$. We refer to Section 5 for a detailed study of the corresponding sub-families. Although the copula $C_{\theta,\phi}$ has full support I^2 , the following proposition shows that, in general, it is neither absolutely continuous, nor singular.

Proposition 1 *The copula $C_{\theta,\phi}$ has both absolutely continuous and singular components $A_{\theta,\phi}$ and $S_{\theta,\phi}$, respectively, given by*

$$A_{\theta,\phi}(u, v) = uv + \theta(\max(u, v))\phi(u)\phi(v) + \int_0^{\min(u,v)} (\theta'\phi^2)(t)dt,$$

and

$$S_{\theta,\phi}(u, v) = - \int_0^{\min(u,v)} (\theta'\phi^2)(t)dt.$$

Proof: The absolutely continuous component of $C_{\theta,\phi}$ is given by

$$A_{\theta,\phi}(u, v) = \int_0^u \int_0^v \frac{\partial^2}{\partial s \partial t} C_{\theta,\phi}(s, t) dt ds,$$

with

$$\frac{\partial^2}{\partial s \partial t} C_{\theta,\phi}(s, t) = 1 + (\theta\phi)'(\max(s, t))\phi'(\min(s, t)).$$

Assume for instance $v \geq u$. Then, $A_{\theta,\phi}$ can be written as

$$\begin{aligned} A_{\theta,\phi}(u, v) &= uv + \int_0^u \int_0^s (\theta\phi)'(s)\phi'(t) dt ds + \int_0^u \int_s^v (\theta\phi)'(t)\phi'(s) dt ds \\ &= uv + (\theta\phi)(v)\phi(u) + \int_0^u (\theta'\phi^2)(s) ds \\ &= C_{\theta,\phi}(u, v) + \int_0^u (\theta'\phi^2)(s) ds, \end{aligned}$$

and the conclusion follows. The case $v < u$ is similar. ■

Thus, the mass of the singular component is concentrated on the first diagonal of the square I^2 .

Denoting by (U, V) a uniform random pair on I^2 with copula $C_{\theta, \phi}$, we have

$$\mathbb{P}(U = V) = - \int_0^1 (\theta' \phi^2)(t) dt.$$

Besides, the copula $C_{\theta, \phi}$ has no singular component if and only if θ is a constant function. This case is described more precisely in Section 5.

3 Measures of association

In the next two sections, we note (X, Y) a random pair with joint cdf H , copula C and margins F and G . The case $C = C_{\theta, \phi}$ is explicitly precised.

3.1 Spearman's Rho

Three invariant to strictly increasing function measures of association between the components of the random pair (X, Y) are usually considered: the normalized volume between graphs of H and FG [27], the Kendall's Tau [21], paragraph 5.1.1, and the Spearman's Rho [21], paragraph 5.1.2, which is the probability of concordance minus the probability of discordance of two random pairs with joint cdf H and FG . Here, we focus on this latter measure, which can also be interpreted as the correlation coefficient between the two uniform margins. An interesting point is that the Spearman's Rho can be written only in terms of the copula C :

$$\rho = 12 \int_0^1 \int_0^1 C(u, v) dudv - 3,$$

and coincides with the correlation range between the uniform marginal distributions. In the case of a copula generated by (2.1), it can be expressed thanks to the functions ϕ and θ .

Proposition 2 *Let (X, Y) be a random pair with copula $C_{\theta, \phi}$. The Spearman's Rho is given by*

$$\rho_{\theta, \phi} = 12 \left[\Phi^2(1)\theta(1) - \int_0^1 \Phi^2(t)\theta'(t)dt \right],$$

where $\Phi(t) = \int_0^t \phi(x)dx$.

Proof: Clearly, the Spearman's Rho can be expanded as

$$\rho_{\theta,\phi} = 12 \left[\int_0^1 \int_v^1 \phi(u)\phi(v)\theta(u)dudv + \int_0^1 \int_0^v \phi(u)\phi(v)\theta(v)dudv \right].$$

An integration by parts shows that the two terms are equal and thus,

$$\begin{aligned} \rho_{\theta,\phi} &= 24 \int_0^1 \left(\phi(v)\theta(v) \int_0^v \phi(u)du \right) dv \\ &= 24 \int_0^1 \theta(v)\phi(v)\Phi(v)dv \\ &= 12 \left[\Phi^2(1)\theta(1) - \int_0^1 \Phi^2(t)\theta'(t)dt \right], \end{aligned}$$

by a new integration by parts. ■

3.2 Tail dependence coefficient

The upper tail dependence can be quantified by the upper tail dependence coefficient [12], paragraph 2.1.10, defined as:

$$\lambda = \lim_{t \rightarrow 1} \mathbb{P}(F(X) > t | G(Y) > t).$$

Again, this coefficient can be written only in terms of the copula C (see [21], Theorem 5.4.2):

$$\lambda = \lim_{u \rightarrow 1} \frac{\bar{C}(u, u)}{1 - u},$$

where \bar{C} is the survival copula, *i.e.* $\bar{C}(u, v) = 1 - u - v + C(u, v)$. In our family, the following simplified expression can be obtained:

Proposition 3 *Let (X, Y) be a random pair with copula $C_{\theta,\phi}$. The tail dependence coefficient is: $\lambda_{\theta,\phi} = -\phi^2(1)\theta'(1)$.*

Proof: Clearly, the tail dependence coefficient can be simplified as

$$\lambda_{\theta,\phi} = \lim_{u \rightarrow 1} \frac{\phi^2(u)\theta(u)}{1 - u}.$$

Taking into account of **(b)** yields

$$\lambda_{\theta,\phi} = - \lim_{u \rightarrow 1} \frac{\phi^2(1)\theta(1) - \phi^2(u)\theta(u)}{1 - u} = -(\phi^2\theta)'(1) = -\phi^2(1)\theta'(1),$$

and the result is proved. ■

4 Concepts of dependence

In this section, for the sake of simplicity, we assume that X and Y are exchangeable. Several concepts of positive dependence have been introduced. X and Y are

- Positively Quadrant Dependent (PQD) if

$$\forall (x, y) \in \mathbb{R}^2, P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y). \quad (4.1)$$

- Left Tail Decreasing (LTD) if $P(Y \leq y|X \leq x)$ is non-increasing in x for all y .
- Right Tail Increasing (RTI) if $P(Y > y|X > x)$ is non-decreasing in x for all y .
- Stochastically Increasing (SI) if $P(Y > y|X = x)$ is non-decreasing in x for all y .
- Left Corner Set Decreasing (LCS D) if $P(X \leq x, Y \leq y|X \leq x', Y \leq y')$ is non-increasing in x' and y' for all x and y .
- Right Corner Set Increasing (RCSI) if $P(X > x, Y > y|X > x', Y > y')$ is non-decreasing in x' and y' for all x and y .

The links between the concepts of positive dependence enumerated above [16, 18] are illustrated in Figure 1. Concepts of negative dependence can be similarly defined. Focusing on the $C_{\theta, \phi}$ family, note that, if θ is the null function on I then $C_{\theta, \phi} = \Pi$, the independent copula. Therefore, in the following, we suppose not to be in this case and we thus introduce

$$v^* = \sup\{v \in I; \theta(v) \neq 0\}.$$

The point v^* , which can be seen as the endpoint of θ , plays a central role in the dependence properties of the copula $C_{\theta, \phi}$, see Theorem 2 below.

Theorem 2 *Let (X, Y) a random pair with copula $C_{\theta, \phi}$. X and Y are*

- (i) *PQD if and only if $\theta(u) \geq 0$ for all $u \in I$ and ϕ has a constant sign on $[0, v^*]$.*
- (ii) *LTD if and only if $\theta(u) \geq 0$ for all $u \in I$ and*

- either $\{u \rightarrow \phi(u)/u \text{ is non-increasing and } \phi(u) \geq 0\}$ for all $u \in [0, v^*]$,
- or $\{u \rightarrow \phi(u)/u \text{ is non-decreasing and } \phi(u) \leq 0\}$ for all $u \in [0, v^*]$.

(iii) RTI if and only if $\theta(u) \geq 0$ for all $u \in I$ and $u \rightarrow \phi(u)/u$ and $u \rightarrow \theta(u)\phi(u)/u$ are either both non-increasing or both non-decreasing.

(iv) LCSD if and only if they are LTD.

(v) RCSI if and only if they are RTI.

Proof: (i) The pair (X, Y) is PQD if and only if the uniform I-margins pair (U, V) with cdf $C_{\theta, \phi}$ is PQD. Thus, condition (4.1) can be rewritten as

$$\theta[\max(u, v)]\phi(u)\phi(v) \geq 0, \quad \forall (u, v) \in I^2. \quad (4.2)$$

Suppose first that (X, Y) is PQD. Considering $u = v$ in (4.2) shows that $\theta(u)\phi^2(u) \geq 0$ for all $u \in I$. Since ϕ vanishes at most on isolated points, $\theta(u) \geq 0$ almost everywhere on I . Recalling that θ is continuous on I , we have $\theta(u) \geq 0$ for all $u \in I$. Moreover, from (d), θ is non-increasing on I , and consequently $\theta(t) > 0$ for all $t \in [0, v^*]$. Thus, for all $(u, v) \in [0, v^*]^2$, $\theta(\max(u, v)) > 0$ and condition (4.2) yields that $\phi(u)\phi(v) \geq 0$ which implies that ϕ has a constant sign on $[0, v^*]$. Conversely, suppose $\theta(u) \geq 0$ for all $u \in I$ and ϕ has a constant sign on $[0, v^*]$. For symmetry reasons, it suffices to verify condition (4.2) for $0 \leq u \leq v \leq 1$. In this case $\theta(v)\phi(u)\phi(v) = 0$ if $v \geq v^*$ and $\theta(v)\phi(u)\phi(v) \geq 0$ otherwise.

(ii) Since $C_{\theta, \phi}$ is symmetric, the necessary and sufficient conditions given by Theorem 5.2.5 in [21], simply reduce to the unique condition: $u \rightarrow C_{\theta, \phi}(u, v)/u$ is non-increasing for all $v \in I$. In our case, this necessary and sufficient condition can be rewritten as

$$u \rightarrow \theta[\max(u, v)]\phi(v)\phi(u)/u \text{ is non-increasing for all } v \in [0, v^*], \quad (4.3)$$

since $\theta[\max(u, v)] = 0$ if $v \in [v^*, 1]$. Suppose (X, Y) is LTD. Then (X, Y) is also PQD and, from (i), it follows that $\theta(u) \geq 0$ for all $u \in I$ and ϕ has a constant sign on $[0, v^*]$. Assume for example that $\phi(u) \geq 0$ for all $u \in [0, v^*]$. Condition (4.3) implies that $u \rightarrow \phi(u)/u$ is non-increasing on $[0, v]$ for all $v \in [0, v^*]$ which means that $u \rightarrow \phi(u)/u$ is non-increasing on $[0, v^*]$. The case

$\phi(u) \leq 0$ for all $u \in [0, v^*]$ is similar. Conversely, assume for instance that $u \rightarrow \phi(u)/u$ is non-increasing and $\phi(u) \geq 0$ for all $u \in [0, v^*]$, together with $\theta(u) \geq 0$ for all $u \in I$. Lemma 1(ii) entails that $u \rightarrow \theta(u)\phi(u)/u$ is non-increasing on $[0, v^*]$, and (4.3) is straightforwardly true.

(iii) Similarly to (ii), the necessary and sufficient condition can be rewritten as

$$u \rightarrow \theta[\max(u, v)]\phi(v)\phi(u)/(1-u) \text{ is non-decreasing for all } v \in [0, v^*]. \quad (4.4)$$

Supposing that (X, Y) is RTI also implies that (X, Y) is PQD and consequently $\theta(u) \geq 0$ for all $u \in I$ and ϕ has a constant sign on $[0, v^*]$. Assuming for example that $\phi(u) \geq 0$ for all $u \in [0, v^*]$, condition (4.4) implies that $u \rightarrow \phi(u)/(1-u)$ is non-decreasing on $[0, v]$ and that $u \rightarrow \theta(u)\phi(u)/(1-u)$ is non-decreasing on $[v, v^*]$, for all $v \in [0, v^*]$. Thus, $u \rightarrow \phi(u)/(1-u)$ and $u \rightarrow \theta(u)\phi(u)/(1-u)$ are both non-decreasing on $[0, v^*]$. Conversely, assume $\theta(u) \geq 0$ for all $u \in I$, and that $u \rightarrow \phi(u)/(1-u)$ and $u \rightarrow \theta(u)\phi(u)/(1-u)$ are non-decreasing on $[0, v^*]$. From Lemma 1(i), ϕ is non-negative on $[0, v^*]$, and (4.4) is clearly true.

(iv) It is well-known that (X, Y) LCS_D implies (X, Y) LTD. Let us prove that the converse result is also true in the $C_{\theta, \phi}$ family. In view of Corollary 5.2.17 in [21], a necessary and sufficient condition for X and Y LCS_D is: $C_{\theta, \phi}$ is a totally positive function of order 2, *i.e.* for all $(u_1, u_2, v_1, v_2) \in I^4$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, we have

$$C_{\theta, \phi}(u_1, v_1)C_{\theta, \phi}(u_2, v_2) - C_{\theta, \phi}(u_1, v_2)C_{\theta, \phi}(u_2, v_1) \geq 0. \quad (4.5)$$

Suppose (X, Y) is LTD. Following (ii), one can assume that $u \rightarrow \phi(u)/u$ is non-increasing and $\phi(u) \geq 0$ for all $u \in [0, v^*]$, together with $\theta(u) \geq 0$ for all $u \in I$. Lemma 1(ii) entails that $u \rightarrow \theta(u)\phi(u)/u$ is non-increasing on $[0, v^*]$. Four cases have to be considered to prove (4.5):

– If $0 \leq u_1 \leq u_2 \leq v_1 \leq v_2 \leq 1$, condition (4.5) reduces to $A_1 \geq 0$, where we have defined

$$A_1 =: \left[\frac{\theta(v_1)\phi(v_1)}{v_1} - \frac{\theta(v_2)\phi(v_2)}{v_2} \right] \left[\frac{\phi(u_1)}{u_1} - \frac{\phi(u_2)}{u_2} \right].$$

If $u_2 \leq v^*$, Lemma 2 yields $A_1 \geq 0$. Otherwise, $u_2 \geq v^*$ implies $v_1 \geq v^*$ and Lemma 2 yields $A_1 = 0$.

– If $0 \leq u_1 \leq v_1 \leq u_2 \leq v_2 \leq 1$, condition (4.5) can be rewritten $A_2 \geq 0$, with

$$A_2 =: A_1 + C_{\theta, \phi}(u_1, v_2)[\theta(v_1) - \theta(u_2)] \frac{\phi(u_2)\phi(v_1)}{u_1 u_2 v_1 v_2}.$$

If $u_2 \leq v^*$, then $\phi(u_2)\phi(v_1) \geq 0$ and Lemma 2 yields $A_1 \geq 0$. Consequently, $A_2 \geq 0$.

If $v_1 \leq v^* \leq u_2$, then $\theta(u_2) = \theta(v_2) = 0$ and A_2 reduces to

$$A_2 = \frac{\theta(v_1)\phi(v_1)\phi(u_1)}{u_1v_1} \geq 0.$$

Finally, if $v^* \leq v_1$, then $A_2 = 0$.

– If $0 \leq u_1 \leq v_1 \leq v_2 \leq u_2 \leq 1$, condition (4.5) can be rewritten $A_3 \geq 0$, with

$$\begin{aligned} A_3 &=: A_1 + \frac{\phi(u_2)}{u_2} \left[\frac{\phi(v_1)}{v_1} \frac{C_{\theta,\phi}(u_1, v_2)}{u_1v_2} (\theta(v_1) - \theta(u_2)) + \frac{\phi(v_2)}{v_2} \frac{C_{\theta,\phi}(u_1, v_1)}{u_1v_1} (\theta(u_2) - \theta(v_2)) \right] \\ &= A_1 + \frac{\phi(u_2)}{u_2} \left[(\theta(v_1) - \theta(u_2)) \left(\frac{\phi(v_1)}{v_1} - \frac{\phi(v_2)}{v_2} \right) \right. \\ &\quad \left. + (\theta(v_1) - \theta(v_2)) \frac{\phi(v_2)}{v_2} \left(1 + \theta(u_2) \frac{\phi(u_1)}{u_1} \frac{\phi(v_1)}{v_1} \right) \right]. \end{aligned}$$

If $u_2 \leq v^*$, then Lemma 2 yields $A_1 \geq 0$ and all the above differences are non-negative. Consequently, $A_3 \geq 0$.

If $v_2 \leq v^* \leq u_2$, then $\theta(u_2) = 0$ and A_3 reduces to

$$A_3 = \left(\frac{\theta(v_1)\phi(v_1)}{v_1} - \frac{\theta(v_2)\phi(v_2)}{v_2} \right) \frac{\phi(u_1)}{u_1} \geq 0.$$

If $v_1 \leq v^* \leq v_2$, then $\theta(u_2) = \theta(v_2) = 0$ and A_3 reduces to

$$A_3 = \frac{\theta(v_1)\phi(v_1)}{v_1} \frac{\phi(u_1)}{u_1} \geq 0.$$

Finally, if $v^* \leq v_1$, then $A_3 = 0$.

– The situation $0 \leq v_1 \leq v_2 \leq u_1 \leq u_2 \leq 1$ is equivalent to $0 \leq u_1 \leq u_2 \leq v_1 \leq v_2 \leq 1$ since the considered copulas are symmetric.

(v) The proof is similar to (iv). ■

As a consequence, the links between the concepts of positive dependence illustrated in Figure 1 can be simplified in our family as presented in Figure 2.

5 Sub-families and examples

Recall that (b) is true if $\phi(1) = 0$ or $\theta(1) = 0$. The corresponding sub-families are now studied in details and examples of copulas in each sub-family are given.

5.1 The case $\theta(1) = 0$

Let us focus on the sub-family of $C_{\theta,\phi}$ defined by conditions **(a)**, **(b1)**, **(c)** and **(d)**, where

(b1) $\theta(1) = 0$,

First, note that **(b1, d)** implies that θ is non negative on I . From Proposition 2 and **(b1)** the Spearman's Rho is given by

$$\rho_{\theta,\phi} = -12 \int_0^1 \Phi^2(t)\theta'(t)dt, \quad (5.1)$$

and **(d)** entails that, in this sub-family, $\rho_{\theta,\phi} \geq 0$. Second, we focus on copulas generated by univariate cdf and defined by

$$C_{\bar{K}^{-1},Id}(u,v) = uv[1 + \bar{K}^{-1}(\max(u,v))],$$

where K is a cdf strictly increasing on \mathbb{R}^+ , \bar{K} is the associated survival function, k is the associated point distribution function and Id is the identity function. The following corollary provides sufficient and necessary conditions on K to ensure that $C_{\bar{K}^{-1},Id}$ is a copula.

Corollary 1 $C_{\bar{K}^{-1},Id}$ is a copula if and only if, for all $t \geq 0$,

$$\frac{k(t)}{\bar{K}(t)} \geq \frac{1}{1+t}. \quad (5.2)$$

Proof: Condition **(a)** is verified since $\phi(x) = x$. Besides, $K(0) = 0$ is equivalent to condition **(b)**. Condition **(c)** is equivalent to

$$\frac{x}{k(\bar{K}^{-1}(x))} - \bar{K}^{-1}(x) \leq 1,$$

for all $x \in I$, which can be rewritten as

$$\frac{\bar{K}(t)}{k(t)} - t \leq 1,$$

for all $t \geq 0$ by introducing $t = \bar{K}^{-1}(x)$. The conclusion follows from Theorem 1. ■

As a consequence of condition (5.2), one can easily show that necessarily, $\bar{K}(x) \geq 1/(1+x)$. Let us also note that, from (5.1), the Spearman's Rho can be rewritten in terms of K as

$$\rho_{\bar{K}^{-1},Id} = 3 \int_0^{+\infty} \bar{K}^4(t)dt. \quad (5.3)$$

– A first example of copula belonging to this sub-family is the Cuadras-Augé copula [4]:

$$C_{\alpha}^{\text{CA}}(u, v) = \min(u, v)^{\alpha} (uv)^{1-\alpha} = M^{\alpha}(u, v) \Pi^{1-\alpha}(u, v),$$

where $\alpha \in [0, 1]$, M is the Fréchet upper bound defined by $M(u, v) = \min(u, v)$ and Π is the product copula $\Pi(u, v) = uv$. The copula C_{α}^{CA} can be interpreted as the weighted geometric mean of M and Π . It is generated by the $C_{\bar{K}^{-1}, Id}$ family with $\bar{K}(x) = (1+x)^{-1/\alpha}$, which is the survival function of a Pareto distribution with shape parameter $1/\alpha$ (see for instance Table 1.2.6 in [5]). The associated Spearman's Rho given by (5.3) is $\rho_{\alpha}^{\text{CA}} = 3\alpha/(4-\alpha)$ and the upper tail dependence coefficient is $\lambda_{\alpha}^{\text{CA}} = \alpha$. Basing on these remarks, we can state the following result:

Proposition 4 *Suppose $C_{\theta, \phi}$ is a copula and $\theta(1) = 0$. Thus, $0 \leq \rho_{\theta, \phi} \leq 1$ and $0 \leq \lambda_{\theta, \phi} \leq 1$, and these bounds are reached within the sub-family.*

– Another similar example is the family (B11), introduced in [12], page 148:

$$C_{\sigma}^{\text{B11}}(u, v) = \sigma \min(u, v) + (1-\sigma)uv = \sigma M(u, v) + (1-\sigma)\Pi(u, v),$$

where $\sigma \in [0, 1]$. The copula C_{σ}^{B11} can be interpreted as the weighted arithmetic mean of M and Π . It is generated by the $C_{\bar{K}^{-1}, Id}$ family with $\bar{K}(x) = (1+x/\sigma)^{-1}$, which is the survival function of a Pareto distribution with scale parameter σ (see for instance Table 1.2.6 in [5]). The associated Spearman's Rho and upper tail dependence coefficient are $\rho_{\sigma}^{\text{B11}} = \lambda_{\sigma}^{\text{B11}} = \sigma$. Note that, for all $\alpha \in [0, 1]$, one always has $\rho_{\alpha}^{\text{B11}} \geq \rho_{\alpha}^{\text{CA}}$. Since both Cuadras-Augé and (B11) copulas are indexed by a single parameter they do not allow the pair (ρ, λ) to reach arbitrary values in $[0, 1]^2$.

– To overcome this limitation, it is natural to consider $\bar{K}(x) = (1+x/\sigma)^{-1/\alpha}$, which is the survival function of a Pareto distribution with shape parameter $1/\alpha$ and scale parameter σ , $\alpha \in (0, 1)$, $\alpha\sigma \in (0, 1)$. The associated Spearman's Rho given by (5.3) is $\rho_{\alpha, \sigma}^{\text{Pareto}} = 3\alpha\sigma/(4-\alpha)$ and the upper tail dependence coefficient $\lambda_{\alpha, \sigma}^{\text{Pareto}} = \alpha\sigma$. Thus, the $C_{\alpha, \beta}^{\text{Pareto}}$ copula allows to obtain any value in $(0, 1)^2$ for the pair (ρ, λ) with the following choice of parameters: $\alpha(\lambda, \rho) = 4 - \lambda/\rho$ and $\sigma(\lambda, \rho) = (\rho\lambda)/(4\rho - 3\lambda)$.

– Finally, note that choosing K as the cdf of the uniform distribution on $[0, \alpha]$, $\alpha \leq 1$ gives rise to the family of copulas

$$C_{\alpha}^{\text{Uniform}}(u, v) = uv(1 + \alpha \min(1 - u, 1 - v)),$$

introduced in [10], Section 1, and with associated Spearman's Rho $\rho_{\alpha}^{\text{Uniform}} = 3\alpha/5$ and upper tail dependence coefficient $\lambda_{\alpha}^{\text{Uniform}} = \alpha$.

5.2 The case $\phi(1) = 0$

Here, we focus on the sub-family of $C_{\theta, \phi}$ defined by conditions **(a)**, **(b2)**, **(c)** and **(d)**, where

(b2) $\phi(1) = 0$,

Note that **(b2)** implies that the upper tail dependence coefficient is always null in this sub-family.

This sub-family encompasses the semiparametric family of copulas with constant function θ defined in (1.7). Consequently, this sub-family also includes the FGM family (1.1), the parametric family of symmetric copulas with cubic sections proposed in [20], equation (4.4), both kernel families (1.3) and (1.4) introduced in [10], and the PQD copulas (1.5) introduced in [14].

From Proposition 2, in the subfamily of $C_{\theta, \phi}$ constrained by **(b2)**, the following lower bound for the Spearman's Rho holds:

$$\rho_{\theta, \phi} \geq 12\Phi^2(1)\theta(1),$$

where the right-hand term can be interpreted as the Spearman's Rho associated to the copula (1.7) with constant function $\theta(\cdot) = \theta(1)$. Since, in this particular case, the Spearman's Rho is lower bounded by $-3/4$ (see [1], Proposition 2), the following result can be given:

Proposition 5 *Suppose $C_{\theta, \phi}$ is a copula and $\phi(1) = 0$. Thus, $\rho_{\theta, \phi} = 0$ and $\lambda_{\theta, \phi} \geq -3/4$, and this bound is reached within the subfamily.*

Collecting this result and Proposition 4, we are now in position to provide the bounds for the general family (2.1).

Proposition 6 *Suppose $C_{\theta, \phi}$ is a copula. Thus, $0 \leq \rho_{\theta, \phi} \leq 1$ and $-3/4 \leq \lambda_{\theta, \phi} \leq 1$, and these bounds are reached within the family.*

Besides, Proposition 1 entails that the copulas (1.7) are the only ones which are absolutely continuous. Thus, we can conclude that, in the general $C_{\theta,\phi}$ family, the absolute continuity is incompatible with the upper tail dependence.

Remark. It is of course possible to build copulas such that $\phi(1) = 0$ and θ is a non constant function. As an example, consider the function $\phi(x) = x(1-x)$ which generates the FGM family of copulas. Condition (c) implies that $\theta(x) \leq 1/x$ for all $0 < x \leq 1$. Let us consider the extreme case $\theta(x) = 1/x$. The copula writes

$$C(u, v) = \Pi(u, v) + (1 - u)(1 - v)M(u, v),$$

and the associated Spearman's Rho is $\rho = 3/5$ which is much larger than the maximum value ($\rho = 1/3$) in the FGM family.

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Appendix: Auxiliary lemmas

Lemma 1 Assume $C_{\theta,\phi}$ is a copula.

- (i) If $u \rightarrow \phi(u)/(1-u)$ is non-decreasing (resp. non-increasing) on $[0, v^*]$ then $\phi(u) \geq 0$ (resp. ≤ 0) for all $u \in [0, v^*]$.

If, moreover, $\theta(u) \geq 0$ for all $u \in I$, then

(ii) If $\phi(u) \leq 0$ (resp. ≥ 0) for all $u \in J \subset I$ and $u \rightarrow \phi(u)/u$ is non-decreasing (resp. non-increasing) on J then $u \rightarrow \theta(u)\phi(u)/u$ is non-decreasing (resp. non-increasing) on J .

Proof:

(i) Assume $u \rightarrow \phi(u)/(1-u)$ is non-decreasing on $[0, v^*]$. Then, $\forall u \in [0, v^*]$, $\phi(u)/(1-u) \geq \phi(0) = 0$ because of (a). Therefore, $\phi(u)/(1-u)$ is non-negative on $[0, v^*]$ and the conclusion follows.

(ii) Remark that $[\theta(u)\phi(u)/u]' = \theta'(u)\phi(u)/u + \theta(u)[\phi(u)/u]'$. Thus, if $\phi(u) \leq 0$, $\theta(u) \geq 0$ and $u \rightarrow \phi(u)/u$ is non-decreasing, (d) implies that $[\theta(u)\phi(u)/u]' \geq 0$ for all $u \in J$. ■

Lemma 2 Assume $C_{\theta, \phi}$ is a copula, $\theta(u) \geq 0$ for all $u \in I$ and

(i) either $\{u \rightarrow \phi(u)/u \text{ is non-increasing and } \phi(u) \geq 0\}$ for all $u \in [0, v^*]$,

(ii) or $\{u \rightarrow \phi(u)/u \text{ is non-decreasing and } \phi(u) \leq 0\}$ for all $u \in [0, v^*]$.

Let $(u_1, u_2, v_1, v_2) \in I^4$ such that $u_1 \leq u_2$, $v_1 \leq v_2$ and introduce

$$A_1 =: \left[\frac{\theta(v_1)\phi(v_1)}{v_1} - \frac{\theta(v_2)\phi(v_2)}{v_2} \right] \left[\frac{\phi(u_1)}{u_1} - \frac{\phi(u_2)}{u_2} \right].$$

Then, $u_2 \leq v^*$ entails $A_1 \geq 0$ and $v^* \leq v_1$ entails $A_1 = 0$.

Proof: Assume (i) hold, situation (ii) is similar. First, remark that $u \rightarrow \theta(u)\phi(u)/u$ is non-increasing on the whole I interval, since this function is non-negative and non-increasing on $[0, v^*]$ (Lemma 1(ii)) and vanishes on $[v^*, 1]$. Therefore,

$$\frac{\theta(v_1)\phi(v_1)}{v_1} - \frac{\theta(v_2)\phi(v_2)}{v_2} \geq 0$$

in all cases. Now, if $u_2 \leq v^*$, then $u \rightarrow \phi(u)/u$ is non-increasing on the considered interval and the conclusion follows. It $v^* \leq v_1$, then $\theta(v_1) = \theta(v_2) = 0$ and the result is proved. ■

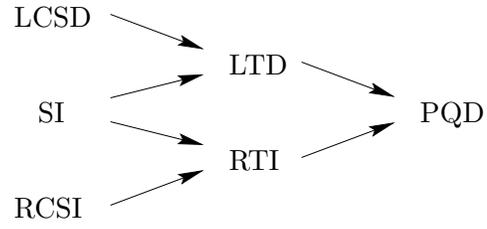


Figure 1: Implications among dependence properties in the general case.

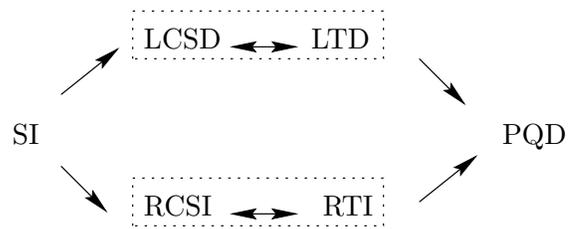


Figure 2: Implications among dependence properties within the $C_{\theta, \phi}$ family.