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# *Normal Cone Approximation and Offset Shape Isotopy*

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Thème SYM



*Rapport  
de recherche*



## Normal Cone Approximation and Offset Shape Isotopy

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**Abstract:** This work addresses the problem of the approximation of the normals of the offsets of general compact sets in euclidean spaces. It is proven that for general sampling conditions, it is possible to approximate the gradient vector field of the distance to general compact sets. These conditions involve the  $\mu$ -reach of the compact set, a recently introduced notion of feature size. As a consequence, we provide a sampling condition that is sufficient to ensure the correctness up to isotopy of a reconstruction given by an offset of the sampling. We also provide a notion of normal cone to general compact sets which is stable under perturbation.

**Key-words:** Distance Function, Medial Axis, geometric approximation, normal cone

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## **Approximation des cônes normaux et isotopie des offsets de formes**

**Résumé :** Ce travail aborde le problème de l'approximation des offsets des sous-ensembles compacts des espaces euclidiens. On prouve que sous des conditions d'échantillonnage générales, il est possible d'approximer le gradient de la fonction distance à un ensemble compact. Ces conditions la notion récemment introduite de  $\mu$ -reach. Ce résultat permet de fournir une condition d'échantillonnage de formes suffisante pour assurer que la reconstruction obtenue en considérant un offset de l'échantillon est isotope à la forme considérée. On introduit également une notion de cône normal stable par perturbation des compacts.

**Mots-clés :** Fonction distance, Axe médian, Approximation géométrique, cône normal

## 1 Introduction

**Motivation.** Let  $K'$  be a finite set of points measured, with some accuracy, on a physical object  $K$ . Given  $K'$  as *input*, is it possible to infer some reliable information on first order properties such as tangent planes or sharp edges, of the boundary of  $K$ ? We consider here the case when the approximation  $K'$  of  $K$  has an error bounded for the Hausdorff distance. In other words we only assume that  $d_H(K, K') < \varepsilon$  which means that any point of  $K'$  lies within a distance  $\varepsilon$  of some point of  $K$  and symmetrically, any point of  $K$  lies within a distance  $\varepsilon$  of a point of  $K'$ . The question is of primary interest in surface reconstruction applications. More generally, in the context of geometric processing, we would like to be able to extrapolate to a large class of non smooth compact sets, including finite points samples and meshes, the usual notions of tangent plane or normal cones.

**Previous work on smooth manifolds.** When  $K'$  is sampled exactly:  $K' \subset K$ , on a smooth boundary, it has been proved [2, 3], that the normals to  $K$  can be estimated from the *poles*: for each point sample  $q \in K'$ , its pole is the Voronoi vertex farthest from  $q$  on the boundary of the Voronoi cell of  $q$ . In [14] this Voronoi based approach has been extended to the approximation of normals and feature lines from *noisy* sampling of a smooth manifold by considering only the poles corresponding to sufficiently large Delaunay balls.

**Reconstruction of “sufficiently regular” non-smooth objects from sampling.** In [7, 5], the authors have considered the problem of recovering the topology of a compact set  $K$  given a sampling  $K'$  without any smoothness assumption on  $K$ .

In the same manner as the resolution power of a microscope constraints the minimal size of observable details, any topological feature (such as a connected component or a tunnel for example) of a compact set  $K$  which would be small with respect to  $\varepsilon$  can certainly not be “reliability detected” from the knowledge of a sample  $K'$  with Hausdorff distance bounded by  $\varepsilon$ . A realistic measure of the topology should consider only the “topological information observable at the scale  $\varepsilon$ ”: in the context of [7, 5], this has lead to consider topological features which are stable under sufficiently large offsets. Note that *topological persistence* [9] is an algebraic counterpart of this notion of stable topology.

The problem of the reconstruction, from a set of measure points, of a geometric numerical model carrying the same topology as the sampled object has been addressed previously for smooth manifolds ([1, 22]), for which the sampling condition is related to the distance to the medial axis of  $K$ . The main contribution of [5] is to give a sampling condition for non-smooth objects, through the notion of *critical function* which encodes the regularity of the compact set boundary at different “scales”.

When it is reasonable to assume some regularity conditions on the object’s boundary, which can be formally expressed through lower bounds on the critical function, it is possible to recover the object’s topology from a sufficiently dense and accurate sampling. In contrast, if we make no assumption about the regularity of the measured object  $K$ , it is still possible to decide some guaranteed topological information, not about the object  $K$  itself of course, but on offsets of  $K$ .

**Contributions.** The aim of the present work is to apply the previous approach, which has been successful for the retrieval of topological information, to the determination, beyond the topology, of generalized tangency informations, which include tangents planes or sharp edges detection. Note that classical “exact” definitions of first order geometric informations such as tangent planes on surfaces, are not preserved in general by Hausdorff approximations. In other words, they are in general broken by arbitrary small perturbations (small for Hausdorff distance) of the object boundary. For example, a finite set sampled “near” the boundary of a smooth shape “contains” some information about the shape boundary tangency, but has no tangent plane in the usual sense. Still if one consider a  $d$ -offset of the point sample, that is a union of spheres of radius  $d$  centered on the points, the tangent planes to the offset boundary may bring some meaningful tangency informations about the initial shape. Following this simple idea and using properties of the distance function to compact sets developed in [5] we propose to introduce “stable” quantities that extend usual exact first order differential quantities. These informations are preserved by small Hausdorff distance perturbation of the object: from this perspective, they can be “really observed” and carry more reality than their classical “exact and ideal” counterpart. These stable informations are generalization of first order differential properties of surfaces. They apply to a large class of compact sets, which suggest applications for meshes and point clouds modeling. For smooth manifolds, our quantities coincide, in the limit, with usual definitions of first order tangent affine manifold.

**Outline.** Section 2 gives the necessary background notions on the distance function and its generalized gradient.

Section 3 and in particular theorem 3.2 gives a first stability property of the generalized gradient with respect to perturbations of the compact sets bounded in Hausdorff distance. This property bounds the maximal angular deviation between the gradient of the distance functions to two compact sets  $K$  and  $K'$ . An important consequence of this theorem is theorem 4.2 which asserts the isotopy between the offsets of the compact set and its sampling with almost the same sampling conditions as in the main theorem in [5].

Section 5 introduces a stability theorem on the Clarke Gradient of the distance function. The stable quantity is a kind of “interval Clarke’s Gradient”: to be more precise, it is the convex hull of the union of the values taken by the Clarke gradient in a ball. From this stability theorem, one introduces (section 6), a *normal cone at a given scale*, which is a stable generalization of first order differential properties, defined at any point on or near a compact set.

## 2 Definitions and background on Distance Functions

We are using the following notations in the sequel of the paper. Given  $X \subset \mathbb{R}^n$ , one denotes by  $X^c$  the complement of  $X$ , by  $\overline{X}$  its closure and by  $\partial X$  the boundary of  $X$ . Given  $A \subset \mathbb{R}^n$ ,  $co(A)$  denotes the convex hull of  $A$ .

The *distance function*  $R_K$  of a compact subset  $K$  of  $\mathbb{R}^n$  associates to each point  $x \in \mathbb{R}^n$  its distance to  $K$ :

$$x \mapsto R_K(x) = \min_{y \in K} d(x, y)$$

where  $d(x, y)$  denotes the euclidean distance between  $x$  and  $y$ . Conversely, this function characterizes completely the compact set  $K$  since  $K = \{x \in \mathbb{R}^n \mid R_K(x) = 0\}$ . Note that  $R_K$  is 1-Lipschitz. The *Hausdorff distance*  $d_H(K, K')$  between two compact sets  $K$  and  $K'$  in  $\mathbb{R}^n$  is the minimum number  $r$  such that  $K \subset K'_r$  and  $K' \subset K_r$ . It is not difficult to check that the Hausdorff distance between two compact sets is the maximum difference between the distance functions associated with the compact sets:

$$d_H(K, K') = \sup_{x \in \mathbb{R}^n} |R_K(x) - R_{K'}(x)|$$

Given  $K$  and  $K'$  be two homeomorphic compact subset of  $\mathbb{R}^n$ , let

$$\mathcal{F} = \{f : K \rightarrow K' : f \text{ is an homeomorphism}\}$$

be the set of all homeomorphisms between  $K$  and  $K'$ . Given such a homeomorphism  $f$ ,  $\sup_{x \in K} d(x, f(x))$  is the maximum displacement of the points of  $K$  by  $f$ . The *Fréchet distance* between  $K$  and  $K'$  is the infimum of this maximum displacement among all the homeomorphisms. It is defined by

$$d_F(K, K') = \inf_{f \in \mathcal{F}} \sup_{x \in K} d(x, f(x)).$$

It is a classical exercise to check that the Fréchet distance satisfies the properties defining a distance and that one always has  $d_H(S, S') \leq d_F(S, S')$ .

Given a compact subset  $K$  of  $\mathbb{R}^n$ , the *medial axis*  $\mathcal{M}(K)$  of  $K$  is the set of points in  $\mathbb{R}^n \setminus K$  that have at least two closest points on  $K$ . The minimal distance between  $K$  and  $\mathcal{M}(K)$  is called, according to Federer, the *reach* of  $K$  and is denoted  $\text{reach}(K)$ .

## 2.1 The gradient and its flow.

The distance function  $R_K$  is not differentiable on  $\mathcal{M}(K)$ . However, it is possible [21] to define a *generalized gradient* vector field  $\nabla_K : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that coincides with the usual gradient of  $R_K$  at points where  $R_K$  is differentiable. For any point  $x \in \mathbb{R}^n \setminus K$ , we denote by  $\Gamma_K(x)$  the set of points in  $K$  closest to  $x$  (figure 1):

$$\Gamma_K(x) = \{y \in K \mid d(x, y) = d(x, K)\}$$

Note that  $\Gamma_K(x)$  is a non empty compact set. The function  $x \mapsto \Gamma_K(x)$  is upper semi-continuous (see [21] Lemma 4.6, also [11] 2.1.4 for the same definition of semi-continuity p.29):

$$\forall x, \forall r > 0, \exists \alpha > 0, \|y - x\| \leq \alpha \Rightarrow \Gamma_K(y) \subset \{z : d(z, \Gamma_K(x)) \leq r\} \quad (1)$$

There is a unique smallest closed ball  $\sigma_K(x)$  enclosing  $\Gamma_K(x)$  (cf. figure 1). We denote by  $\theta_K(x)$  the center of  $\sigma_K(x)$  and by  $\mathcal{F}_K(x)$  its radius.  $\theta_K(x)$  can equivalently be defined as the point on the convex hull of  $\Gamma_K(x)$  nearest to  $x$ . For  $x \in \mathbb{R}^n \setminus K$ , the generalized gradient  $\nabla_K(x)$  is defined as follows:

$$\nabla_K(x) = \frac{x - \theta_K(x)}{R_K(x)}$$



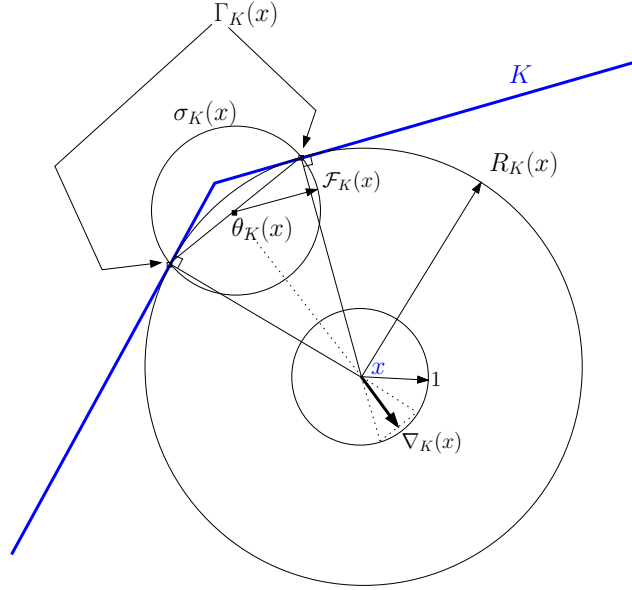


Figure 1: A 2-dimensional example with 2 closest points.

It is natural to set  $\nabla_K(x) = 0$  for  $x \in K$ . For  $x \in \mathbb{R}^n \setminus K$ , one has the following relation [21]:

$$\|\nabla_K(x)\|^2 = 1 - \frac{\mathcal{F}_K(x)^2}{R_K(x)^2}$$

Equivalently,  $\|\nabla_K(x)\|$  is the cosine of the (half) angle of the smallest cone with apex  $x$  that contains  $\Gamma_K(x)$ . As an immediate consequence, one has the following lemma.

**Lemma 2.1** *Let  $K \subset \mathbb{R}^n$  be a compact set. For any  $x \in \mathbb{R}^n$ ,*

$$\|\nabla_K(x)\| \geq \sup_{y, y' \in \Gamma_K(x)} \cos \frac{(\vec{xy}, \vec{xy}')}{2}$$

The map  $x \mapsto \|\nabla_K(x)\|$  is lower semicontinuous [21]. Although  $\nabla_K$  is not continuous, it is shown in [21] that Euler schemes using  $\nabla_K$  converges uniformly, when the integration step decreases, toward a continuous flow  $\mathfrak{C} : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . The integral line of this flow starting at a point  $x \in \mathbb{R}^n$  can be parameterized by arc length  $s \mapsto \mathfrak{C}(t(s), x)$ . It is possible to express the value of  $R_K$  at the point  $\mathfrak{C}(t(l), x)$  by integration along the integral line with length  $l$  downstream the point  $x$ :

$$R_K(\mathfrak{C}(t(l), x)) = R_K(x) + \int_0^l \|\nabla_K(\mathfrak{C}(t(s), x))\| ds \quad (2)$$

It is proved in [21] that the functions  $\mathcal{F}_K$  and  $R_K$  are increasing along the trajectories of the flow. In the particular case where  $K$  is a finite set, various notions of flows related to this one have been independently introduced by H. Edelsbrunner [15], J. Giesen and al. [18] and R. Chaine [4] using Voronoï diagrams.

## 2.2 Critical point theory for distance functions.

The *critical points* of  $R_K$  are defined as the points  $x$  for which  $\nabla_K(x) = 0$ . Equivalently, a point  $x$  is a critical point if and only if it lies in the convex hull of  $\Gamma_K(x)$ . When  $K$  is finite, this last definition means that critical points are precisely the intersections of Delaunay  $k$ -dimensional simplices with their dual  $(n - k)$ -dimensional Voronoï facets [18]. Note that this notion of critical point is the same as the one considered in the setting of non smooth analysis [11] and Riemannian geometry [10, 19]. The topology of the offsets  $R_K^{-1}(a)$ ,  $a > 0$  of a compact set  $K$  are closely related to the critical values of  $R_K$ . The next proposition shows that it can change only at critical values.

**Theorem 2.2 (isotopy lemma)** [19] *If  $0 < a < b$  are such that  $R_K^{-1}([a, b])$  does not contain any critical point of  $R_K$ , then all the level sets  $R_K^{-1}(d)$ ,  $d \in [a, b]$ , are isotopic topological manifolds and  $R_K^{-1}([a, b])$  is homeomorphic to  $R_K^{-1}(a) \times [a, b]$ .*

Recall that an *isotopy* between two manifolds  $S$  and  $S'$  is a continuous map  $F : S \times [0, 1] \rightarrow \mathbb{R}^n$  such that  $F(\cdot, 0)$  is the identity of  $S$ ,  $F(S, 1) = S'$ , and for each  $t \in [0, 1]$ ,  $F(\cdot, t)$  is a homeomorphism onto its image. An *ambient isotopy* between  $S$  and  $S'$  is a continuous map  $F : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$  such that  $F(\cdot, 0)$  is the identity of  $\mathbb{R}^n$ ,  $F(S, 1) = S'$ , and for each  $t \in [0, 1]$ ,  $F(\cdot, t)$  is a homeomorphism of  $\mathbb{R}^n$ . Restricting an ambient isotopy between  $S$  and  $S'$  to  $S \times [0, 1]$  thus yields an isotopy between them. It is actually true that if there exists an isotopy between  $S$  and  $S'$ , then there is an ambient isotopy between them [Hi].

The *weak feature size* of  $K$ , or  $\text{wfs}(K)$ , is defined as the infimum of the positive critical values of  $R_K$ . Equivalently it is the minimum distance between  $K$  and the set of critical points of  $R_K$ . Notice that  $\text{wfs}(K)$  may be equal to 0. Nevertheless,  $\text{wfs}(K)$  is non zero for a large class of compact sets including polyhedrons and piecewise analytic sets (see [6, 7]). As an immediate consequence of previous proposition, one deduces that the distance level sets  $R_K^{-1}(d)$  are all isotopic for  $0 < d < \text{wfs}(K)$ .

## 2.3 The critical function and the $\mu$ -reach

The results of this paper rely strongly on the notions of  $\mu$ -critical point, critical function and  $\mu$ -reach, introduced in [5].

**Definition 2.3 ( $\mu$ -critical point)** *A  $\mu$ -critical point  $x$  of the compact set  $K$  is a point at which the norm of the gradient  $\nabla_K$  does not exceed  $\mu$ :  $\|\nabla_K(x)\| \leq \mu$ .*

The most important property of  $\mu$ -critical points is their stability with respect to Hausdorff perturbations of  $K$  proved in [5].

**Theorem 2.4 (critical point stability theorem)** *Let  $K$  and  $K'$  be two compact subsets of  $\mathbb{R}^n$  and  $d_H(K, K') \leq \varepsilon$ . For any  $\mu$ -critical point  $x$  of  $K$ , there is a  $(2\sqrt{\varepsilon/R_K(x)} + \mu)$ -critical point of  $K'$  at distance at most  $2\sqrt{\varepsilon R_K(x)}$  from  $x$ .*

**Definition 2.5 (critical function)** *Given a compact set  $K \subset \mathbb{R}^n$ , its critical function  $\chi_K : (0, +\infty) \rightarrow \mathbb{R}_+$  is the real function defined by:*

$$\chi_K(d) = \inf_{R_K^{-1}(d)} \|\nabla_K\|$$

Figure 2 shows the respective critical functions of a square in 3-space and of a sampling of it. We note that the infimum can be replaced by a minimum since  $\|\nabla_K\|$  is lower semi-continuous and  $R_K^{-1}(d)$  is compact. It also results from the compactness of  $R_K^{-1}(d)$  that  $d \mapsto \chi_K(d)$  is lower semi-continuous. The critical function is in some sense “stable” with respect to small (measured by Hausdorff distance) perturbations of a compact set, precisely [5]:

**Theorem 2.6 (critical function stability theorem)** *Let  $K$  and  $K'$  be two compact subsets of  $\mathbb{R}^n$  and  $d_H(K, K') \leq \varepsilon$ . For all  $d \geq 0$ , we have:*

$$\inf\{\chi_{K'}(u) \mid u \in I(d, \varepsilon)\} \leq \chi_K(d) + 2\sqrt{\frac{\varepsilon}{d}}$$

where  $I(d, \varepsilon) = [d - \varepsilon, d + 2\chi_K(d)\sqrt{\varepsilon d} + 3\varepsilon]$

Theorem 2.6 claim can be read as  $\chi_K(d) \geq \inf\{\chi_{K'}(u) \mid u \in I(d, \varepsilon)\} - 2\sqrt{\frac{\varepsilon}{d}}$  and says that the knowledge of a lower bound on the critical function of a compact set  $K'$  gives a lower bound on the critical function of “nearby” (for Hausdorff distance) compact sets  $K$ . In particular, if a set  $K'$  of measured points is known to lie within some Hausdorff distance of a physical object represented by the unknown compact set  $K$ , the critical function of  $K'$  gives, by theorem 2.6, a lower bound on the critical function of the partially known physical object  $K$ . Note that as explained in [5], starting from the Voronoi complex of the sample, the computation of the critical function of a finite sample is straightforward. This stability of the critical function with respect to small perturbations of the object in Hausdorff distance makes it realistic with respect to physical interactions – it does not rely on unmeasurable quantities – but also robust with respect to numerical computations because, by backward error analysis, the impact of rounding errors on the evaluation of the critical function can be controlled.

The  $\mu$ -reach of a compact set  $K$  is the maximal offset value  $d$  for which  $\chi_K(d') \geq \mu$  for  $d' < d$ . More precisely, it is defined by:

$$r_\mu(K) = \inf\{d \mid \chi_K(d) < \mu\}$$

Closely related to the  $\mu$ -reach and the critical point stability theorem is the following result [5] that will be used in section 4.

**Theorem 2.7 (critical values separation theorem)** *Let  $K$  and  $K'$  be two compact subsets of  $\mathbb{R}^n$ ,  $\varepsilon$  be the Hausdorff distance between  $K$  and  $K'$ , and  $\mu$  be a non-negative number. The distance function  $R_{K'}$  has no critical values in the interval  $]4\varepsilon/\mu^2, r_\mu(K) - 3\varepsilon[$ .*

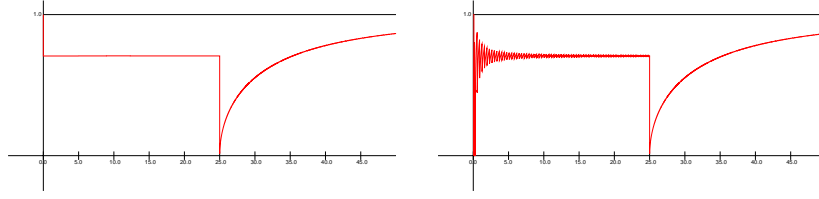


Figure 2: Critical function of a square embedded in  $\mathbb{R}^3$  with side length 50 (left), and of a sampling of that square (right).

These previous notions allow to define a sampling condition for compact sets that lead to a reconstruction theorem [5]. Given two non-negative real numbers  $\kappa$  and  $\mu$ , we say that a compact  $K \subset \mathbb{R}^n$  is a  $(\kappa, \mu)$ -approximation of a compact  $K' \subset \mathbb{R}^n$  if the Hausdorff distance between  $K$  and  $K'$  does not exceed  $\kappa$  times the  $\mu$ -reach of  $K'$ .

**Theorem 2.8 (Reconstruction theorem)** *Let  $K' \subset \mathbb{R}^n$  be a  $(\kappa, \mu)$ -approximation of a compact set  $K$ . If*

$$\kappa < \frac{\mu^2}{5\mu^2 + 12}$$

*then the complement of  $R_K'^{-1}([0, \alpha])$  is homotopy equivalent to the complement of  $K$ , and  $R_K'^{-1}([0, \alpha])$  is homotopy equivalent to  $R_K^{-1}([0, \eta])$  for sufficiently small  $\eta$ , provided that*

$$\frac{4d_H(K, K')}{\mu^2} \leq \alpha < r_\mu(K) - 3d_H(K, K')$$

In the following of the paper, we prove that under similar condition, one can improve this result by comparing the topology of the level sets of  $R_K$  and  $R_{K'}$  up to isotopy.

### 3 A first stability property of the gradient

In this section one deduces results on the stability of the gradient of distance functions from the stability theorem for  $\mu$ -critical points. In the following, given two compact sets  $K$  and  $K'$ , for any  $x \in \mathbb{R}^n$ , one denotes by  $\tilde{\Gamma}_{K'}(x)$  the projection of  $\Gamma_{K'}(x)$  on the sphere  $\mathbb{S}(x, R_K(x))$ :  $\tilde{y} \in \tilde{\Gamma}_{K'}(x)$  if and only if there exists  $y \in \Gamma_{K'}(x)$  such that  $\tilde{y}$  is the intersection of the half-line  $[xy)$  with the sphere  $\mathbb{S}(x, R_K(x))$ .

**Theorem 3.1** *Let  $K, K' \subset \mathbb{R}^n$  be two compact sets and let  $\varepsilon > 0$  be such that  $d_H(K, K') < \varepsilon$ . If  $x \in \mathbb{R}^n$  is a  $\mu$ -critical point of  $K''_x = K \cup \tilde{\Gamma}_{K'}(x)$  then there exists a  $(\mu + 2\sqrt{\frac{2\varepsilon}{R_K(x)}})$ -critical point of  $K$  at distance at most  $2\sqrt{2\varepsilon R_K(x)}$  from  $x$ .*

PROOF. Let  $x \in \mathbb{R}^n$  and let  $K'' := K''_x$ . Since  $d_H(K, K') < \varepsilon$ , one has  $d_H(K, K'') < 2\varepsilon$ . One obtains immediately from the critical point stability theorem applied to  $K, K''$  and  $x$  that there exists a  $(\mu + 2\sqrt{\frac{2\varepsilon}{R_{K''}(x)}})$ -critical point of  $K$  at distance at most  $2\sqrt{2\varepsilon R_{K''}(x)}$  from  $x$ . It suffices to note that  $R_K(x) = R_{K''}(x)$  to conclude the proof.  $\square$

As a consequence of theorem 3.1, one obtains a bound on the angle between the vector fields  $\nabla_K$  and  $\nabla_{K'}$  of two near compact sets.

**Theorem 3.2** *Let  $K, K' \subset \mathbb{R}^n$  be two compact sets and let  $\varepsilon > 0$  be such that  $d_H(K, K') < \varepsilon$ . Given  $\mu > 0$ , if  $x \in \mathbb{R}^n$  is such that  $\|\nabla_K(y)\| > \mu$  for any  $y \in \mathbb{B}(x, 2\sqrt{2\varepsilon R_K(x)})$ , then for any  $y \in \Gamma_K(x)$  and any  $y' \in \Gamma_{K'}(x)$ ,*

$$\cos \frac{\langle \vec{xy}, \vec{xy}' \rangle}{2} \geq \mu - 2\sqrt{\frac{2\varepsilon}{R_K(x)}} \quad (3)$$

Moreover, if  $K''_x = K \cup \tilde{\Gamma}_{K'}(x)$ , then

$$\|\nabla_{K''_x}(x)\| \geq \mu - 2\sqrt{\frac{2\varepsilon}{R_K(x)}} \quad (4)$$

PROOF. Let  $\theta$  be the angle between  $\vec{xy}$  and  $\vec{xy}'$  and let  $\tilde{y}' \in \tilde{\Gamma}_{K'}(x)$  be the projection of  $y'$  on the sphere  $\mathbb{S}(x, R_K(x))$ . Since the convex hull of  $y$  and  $\tilde{y}'$  is contained in the convex hull of  $\Gamma_K(x) \cup \tilde{\Gamma}_{K'}(x)$ ,  $x$  is a  $\cos \frac{\theta}{2}$ -critical point of  $K''_x = K \cup \tilde{\Gamma}_{K'}(x)$  (see figure 3). The first inequality of the theorem follows immediately from critical point stability theorem. Now, remark that this proof remains valid if  $y$  and  $y'$  are any two points in the convex hull of  $\Gamma_K(x) \cup \tilde{\Gamma}_{K'}(x)$ . The second inequality thus follows from lemma 2.1.  $\square$

Recall that the direction of the vector  $\nabla_K(x)$  (resp.  $\nabla_{K'}(x)$ ) is contained in the convex hull of the directions defined by the segments joining  $x$  to the points of  $\Gamma_K(x)$  (resp.  $\Gamma_{K'}(x)$ ). So, the previous theorem immediately leads to the following result.

**corollary 3.3** *Let  $K, K' \subset \mathbb{R}^n$  be two compact sets and let  $\varepsilon > 0$  be such that  $d_H(K, K') < \varepsilon$ . Given  $\mu > 0$ , if  $x \in \mathbb{R}^n$  is such that  $\|\nabla_K(y)\| > \mu$  for any  $y \in \mathbb{B}(x, 2\sqrt{2\varepsilon R_K(x)})$ , then*

$$\cos \frac{\langle \nabla_K(x), \nabla_{K'}(x) \rangle}{2} \geq \mu - 2\sqrt{\frac{2\varepsilon}{R_K(x)}} \quad (5)$$

The bound of the corollary is tight: there are some examples where the cosine of angle between  $\nabla_K(x)$  and  $\nabla_{K'}(x)$  is of order  $1 - O(\sqrt{\varepsilon})$ . Let  $K$  be the circle of center  $O \in \mathbb{R}^2$  and radius 1 and let  $O'$  be a point such that  $d(O, O') = 2\sqrt{\varepsilon}$ . The circle of center  $O'$  and radius  $(1 - 2\sqrt{\varepsilon} + \varepsilon)$  meets  $K$  in two points  $A$  and  $B$ . Let  $K'$  be the boundary of the union of the disc of center  $O$  and

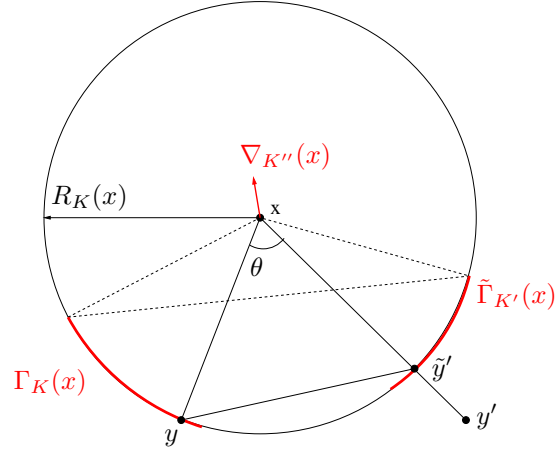


Figure 3: Proof of theorem 3.2

radius 1 with angular area of radius  $1 + \varepsilon$  and delimited by the half-lines  $OA$  and  $OB$  (see figure 4). The vector field  $\nabla_K$  is continuous in a neighborhood of  $O'$  and  $\nabla_K(O')$  is collinear to  $O'O$ . Along the segment  $[O'A]$ ,  $\nabla_{K'}$  is collinear to  $AO'$  and makes an angle  $\beta$  with  $OO'$ . An easy computation leads to  $\cos \beta = 1 - \frac{1}{2}\sqrt{\varepsilon} + O(\varepsilon)$ . As a consequence, if  $x \in [O'A]$  is chosen sufficiently near from  $O'$ , then it satisfies the hypothesis of the previous theorem and

$$\cos \frac{(\nabla_K(x), \nabla_{K'}(x))}{2} = 1 - O\left(\sqrt{\frac{\varepsilon}{R_K(x)}}\right)$$

## 4 Isotopy between offsets

We are now able to use the stability properties of the gradient established in the previous section to compare the topology of distance level sets of two near compact sets. Let  $K, K' \subset \mathbb{R}^n$  be two compact sets and let  $\varepsilon > 0$  be such that  $d_H(K, K') < \varepsilon$ .

**Lemma 4.1** *Let  $a > 0$  be such that for any  $x \in R_K^{-1}([a - \varepsilon, a + \varepsilon])$ ,  $\|\nabla_{K''_x}(x)\| \neq 0$  where  $K''_x = K \cup \tilde{\Gamma}_{K'}(x)$ . Then  $R_K^{-1}(a)$  and  $R_{K'}^{-1}(a)$  are isotopic hypersurfaces. Moreover, if*

$$\nu = \inf\{\|\nabla_{K''_x}(x)\| : x \in R_K^{-1}([a - \varepsilon, a + \varepsilon])\} > 0$$

*then the Frechet distance between  $R_K^{-1}(a)$  and  $R_{K'}^{-1}(a)$  is bounded by  $\frac{\varepsilon}{\nu}$ .*

Note that the condition of the lemma is equivalent to the fact that 0 is not contained in the convex hull of the union of the Clarke gradients of  $R_{K'}$  and  $R_K$ , or equivalently  $x \notin \text{co}(\Gamma_K(x) \cup \tilde{\Gamma}_{K'}(x))$ .

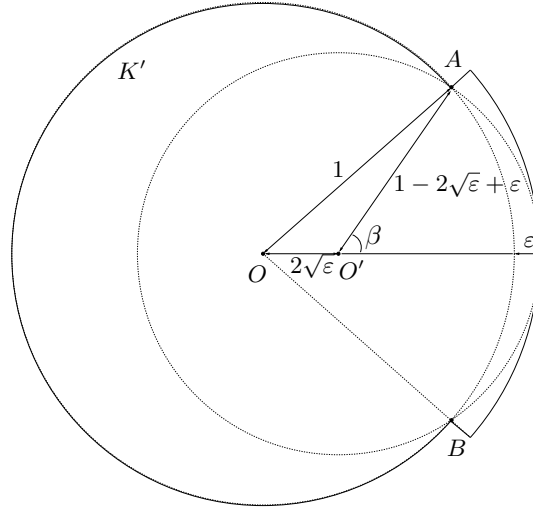


Figure 4: An example showing the tightness of the bound of corollary 3.3

PROOF. The proof of the lemma is based upon a classical techniques in differential geometry: one constructs a  $C^\infty$  vector field which is “transverse” to the level sets of  $R_K$  and  $R_{K'}$  in  $A := R_K^{-1}([a - \varepsilon, a + \varepsilon])$  and that allows to realize an isotopy between  $R_K^{-1}(a)$  and  $R_{K'}^{-1}(a)$ .

Let  $x \in A$  and let  $v(x) = \nabla_{K'_x}(x) \neq 0$ . Since  $y \rightarrow \Gamma_K(y)$  and  $y \rightarrow \Gamma_{K'}(y)$  are upper semi-continuous (see [11]2.1.4 for a definition), there exist  $\delta_0, \delta_1 > 0$  such that

$$\begin{aligned} R_K\left(y + t \frac{v(x)}{\|v(x)\|}\right) &\geq R_K(y) + t \frac{\|v(x)\|}{2} \\ R_{K'}\left(y + t \frac{v(x)}{\|v(x)\|}\right) &\geq R_{K'}(y) + t \frac{\|v(x)\|}{2} \end{aligned}$$

for any  $y \in \mathbb{B}(x, \delta_0)$  and any  $t \in (-\delta_0, \delta_0)$ . Since  $A$  is compact, it is covered by a finite set of balls  $\mathbb{B}(x_i, \delta_0(x_i))$ ,  $i = 1 \dots p$ . Using a  $C^\infty$  partition of unity associated to this covering and the constant vector field  $v(x_i)$  on each  $\mathbb{B}(x_i, \delta_0(x_i))$ , one constructs a  $C^\infty$  vector field  $X$  on  $A$  such that for any trajectory  $\phi(x, t)$  of  $X$  in  $A$ , one has

$$R_K(\phi(x, t)) \geq R_K(x) + t\nu \quad (6)$$

$$R_{K'}(\phi(x, t)) \geq R_{K'}(x) + t\nu \quad (7)$$

with  $\nu = \min_{i=1 \dots p} \frac{\|v(x_i)\|}{2}$ . It follows immediately that any trajectory  $t \rightarrow \phi(x, t)$  issued from  $R_K^{-1}(a - \varepsilon)$  meets  $R_K^{-1}(a + \varepsilon)$  and  $R_{K'}$  is strictly increasing along this trajectory. Moreover, since  $\|v(x_i)\| < 1$  for all  $i = 1, \dots, p$ ,  $\|X\| < 1$  and the length of the trajectory between  $x$  and  $\phi(x, t)$  is bounded by  $|t|$ . It follows from inequality (7) that the length of any trajectory between  $R_K^{-1}(a)$  and  $R_K^{-1}(a - \varepsilon)$  or  $R_K^{-1}(a + \varepsilon)$  is bounded by  $\frac{\varepsilon}{\nu}$ .

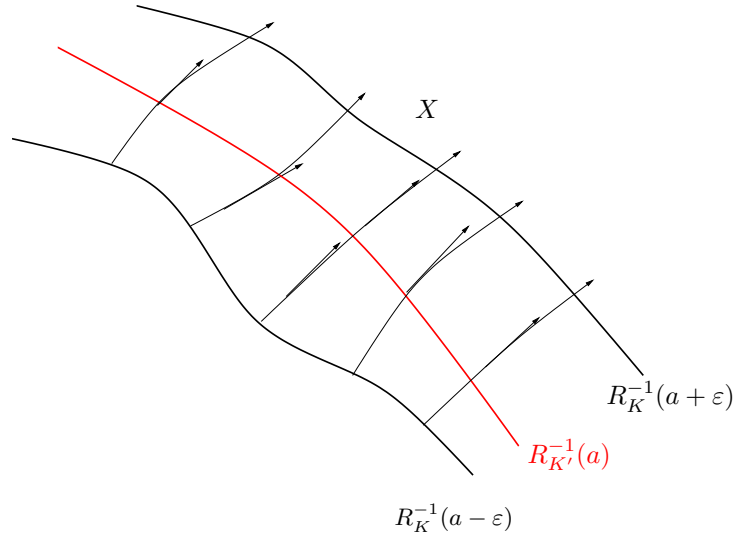


Figure 5: Proof of lemma 2

Now, since  $\|R_K - R_{K'}\| < \varepsilon$ ,  $R_{K'}$  is smaller than  $a$  on  $R_K^{-1}(a - \varepsilon)$  and bigger than  $a$  on  $R_K^{-1}(a + \varepsilon)$  (see figure 5). So,  $R_{K'}^{-1}(a)$  is contained in  $A$  and it separates the two boundary components  $R_K^{-1}(a - \varepsilon)$  and  $R_K^{-1}(a + \varepsilon)$  of  $A$ . As a consequence for each  $x \in R_K^{-1}(a)$ , the trajectory  $t \rightarrow \phi(x, t)$  intersects  $R_{K'}^{-1}(a)$  in exactly one point  $f(x) = \phi(x, t_x)$  (note that  $t_x$  may be negative). The map  $x \rightarrow f(x)$  defines a continuous bijection between  $R_K^{-1}(a)$  and  $R_{K'}^{-1}(a)$  and the flow of  $X$  allows to define an isotopy between these two hypersurfaces. The distance between  $x$  and  $f(x)$  is bounded by the length of the trajectory between  $x$  and  $f(x)$ . So,  $d(x, f(x)) < \frac{\varepsilon}{\nu}$  and the Frechet distance between  $R_K^{-1}(a)$  and  $R_{K'}^{-1}(a)$  is bounded by  $\frac{\varepsilon}{\nu}$ .  $\square$

The following results provide a sufficient condition involving the critical function for two compact sets to have isotopic offsets.

**Theorem 4.2 (level sets isotopy theorem)** *Let  $K, K' \subset \mathbb{R}^n$  be two compact sets such that  $d_H(K, K') < \varepsilon$  for some  $\varepsilon > 0$ . If  $a > 0$  is such that  $\chi_K > \gamma + 2\sqrt{\frac{2\varepsilon}{a-\varepsilon}}$  on the interval  $[a - \varepsilon - 2\sqrt{2\varepsilon(a + \varepsilon)}, a + \varepsilon + 2\sqrt{2\varepsilon(a + \varepsilon)}]$  for some constant  $\gamma > 0$  then  $R_K^{-1}(a)$  and  $R_{K'}^{-1}(a)$  are isotopic hypersurfaces. Moreover the Frechet distance between these two hypersurfaces is bounded by  $\frac{\varepsilon}{\gamma}$ .*

PROOF. From lemma 4.1, one just has to show that  $x \notin \text{co}(\Gamma_K(x) \cup \tilde{\Gamma}_{K'}(x))$  for any  $x \in A = R_K^{-1}([a - \varepsilon, a + \varepsilon])$ . Suppose this is not the case for some  $x \in A$ . It follows from theorem 3.1 that there exists a  $(2\sqrt{\frac{2\varepsilon}{R_K(x)}})$ -critical point  $y$  of  $K$  at distance at most  $2\sqrt{2\varepsilon R_K(x)}$  from  $x$ . Since  $a - \varepsilon \leq R_K(x) \leq a + \varepsilon$ ,  $y$  is a  $(2\sqrt{\frac{2\varepsilon}{a-\varepsilon}})$ -critical point of  $K$  at distance at most  $2\sqrt{2\varepsilon(a + \varepsilon)}$  from



x. Moreover

$$\begin{aligned} R_K(y) &\leq R_K(x) + 2\sqrt{2\varepsilon(a + \varepsilon)} \\ &\leq a + \varepsilon + 2\sqrt{2\varepsilon(a + \varepsilon)} \end{aligned}$$

and in the same way  $R_K(y) \geq a - \varepsilon - 2\sqrt{2\varepsilon(a + \varepsilon)}$ . These two last inequalities contradict the hypothesis of the theorem. The second part of the theorem follows from the second part of the lemma 4.1 and the second part of the theorem 3.2.  $\square$

The previous theorem can be restated in terms of  $(\kappa, \mu)$ -approximations to give the following result.

**Theorem 4.3 (isotopic reconstruction theorem)** *Let  $K \subset \mathbb{R}^n$  be a compact set such that  $r_\mu(K) > 0$  for some  $\mu > 0$ . Let  $K'$  be a  $(\kappa, \mu)$ -approximation of  $K$  where*

$$\kappa < \min\left(\frac{\sqrt{5}}{2} - 1, \frac{\mu^2}{16 + 2\mu^2}\right)$$

and let  $d, d'$  be such that

$$0 < d < \text{wfs}(K) \text{ and } \frac{4\kappa r_\mu}{\mu^2} \leq d' < r_\mu(K) - 3\kappa r_\mu$$

Then the level set  $R_{K'}^{-1}(d')$  is isotopic to the level set  $R_K^{-1}(d)$ .

PROOF. Let  $r_\mu = r_\mu(K)$ ,  $a = r_\mu/2$  and  $\varepsilon = \kappa r_\mu$ . It follows from isotopy lemma 2.2 that  $R_K^{-1}(d)$  is isotopic to  $R_K^{-1}(a)$ . It follows from the separation of the critical values theorem and from the isotopy lemma 2.2 that  $R_{K'}^{-1}(d')$  is isotopic to  $R_{K'}^{-1}(a)$ . Using that  $\chi_K > \mu$  on  $(0, r_\mu)$  and  $\kappa < \frac{\mu^2}{16 + 2\mu^2}$  one easily checks that

$$\chi_K > 2\sqrt{\frac{2\varepsilon}{a - \varepsilon}}$$

on the interval  $(0, r_\mu)$ . Theorem 4.2 allow to conclude the proof provided that the interval with center  $a$  and half-length  $\varepsilon + 2\sqrt{2\varepsilon(a + \varepsilon)}$  is included in the interval  $(0, r_\mu)$ . This last condition is equivalent to

$$\kappa r_\mu + 2\sqrt{2\kappa r_\mu\left(\frac{r_\mu}{2} + \kappa r_\mu\right)} < \frac{r_\mu}{2}$$

or, after division by  $r_\mu$ ,

$$2\kappa + 4\sqrt{\kappa(1 + 2\kappa)} < 1$$

This is satisfied as soon as  $\kappa < \frac{\sqrt{5}}{2} - 1$ .  $\square$

## 5 A second stability property of the gradient

In this section we consider the Clarke's generalized gradient  $\partial R_K$  of the distance function [11] and prove a stability theorem of  $\partial R_K$  with respect to Hausdorff distance perturbation of the compact  $K$ . Because  $\partial R_K$  carry more information than the generalized gradient  $\nabla_K$ , we expect this stability property to allow to "extract" more geometric informations about a compact set  $K$  from a Hausdorff approximation of it.

For a set  $E$  and a number  $r \geq 0$ , we denote by  $E^r$  the set  $E^r = \{z : d(z, E) \leq r\}$ .

### 5.1 Clarke's gradient of the distance function

Instead of the usual definition of Clarke gradient we use the following characterization. For  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  we denote by  $\Omega_f$  the set of point where  $f$  fails to be differentiable and, for  $x \notin \Omega_f$ , we denote  $\frac{\partial f}{\partial X}(x)$  the usual gradient of the function  $f$  at  $x$ .

**Theorem 5.1 (F.H. Clarke, adapted from [11], section 2.5.1)** *Let  $f$  be Lipschitz near  $x$ , then:*

$$\partial f(x) = \text{co} \left\{ \lim_{x_i \rightarrow x} \frac{\partial f}{\partial X}(x_i), x_i \notin \Omega_f \right\}$$

Rephrasing [11], the above characterization means the following. Consider any sequence  $x_i$  converging to  $x$  with  $f$  differentiable at each  $x_i$  and such that the usual gradient  $\frac{\partial f}{\partial X}(x_i)$  converges; then  $\partial f(x)$  is the convex hull of such limit points.

([11], section 2.5.6) gives a characterization of  $\partial R_K(x)$  for  $x \in K$ . However, because our stability property is meaningful for  $x \notin K$  only, we first prove the following characterization of  $\partial R_K$  for  $x \in K^c$ .

For  $x \in K^c$  and  $\rho > 0$  we introduce the notations  $\tilde{G}_K(x)$ ,  $G_K(x)$  and  $G_K(x, \rho)$ :

$$\begin{aligned} \tilde{G}_K(x) &= \left\{ \frac{x-z}{R_K(x)}, z \in \Gamma_K(x) \right\} \\ G_K(x) &= \text{co} \left( \tilde{G}_K(x) \right) \\ G_K(x, \rho) &= \text{co} \left( \bigcup_{\|y-x\| \leq \rho} G_K(y) \right) \end{aligned}$$

**Lemma 5.2** *If  $x \in K^c$ , one has:*

$$\partial R_K(x) = G_K(x)$$

PROOF.

We first prove  $G_K(x) \subset \partial R_K(x)$ . For that we use the Lemma 5.3 below.

**Lemma 5.3** *If  $x \in K^c$  and  $v \in \tilde{G}_K(x)$  then for any  $z$  on the open line segment  $(x, x - R_K(x)v)$ ,  $R_K$  is differentiable at  $z$  and:*

$$\frac{\partial R_K}{\partial X}(z) = v$$

PROOF. [proof of Lemma 5.3] From the definition of  $\tilde{G}_K(x)$ , one has  $v \in \tilde{G}_K(x) \Rightarrow x_v = x - R_K(x)v \in \Gamma_K(x)$ . Let us denote by  $B_{(x,r)}$  and  $B_{(x,r)}^\circ$  respectively the closed and open balls centered at  $x$  with radius  $r$  and let  $R^{\max}$  be such that  $K \subset B_{(x,R^{\max})}$ . We consider the two compact sets  $K^+ = \{x_v\}$  and  $K^- = B_{(x,R^{\max})} \setminus B_{(x,R_K(x))}^\circ$  one has:

$$K^+ \subset K \subset K^-$$

which entails:

$$R_{K^-} \leq R_K \leq R_{K^+} \quad (8)$$

On another hand,  $R_{K^-}$  and  $R_{K^+}$  have simple radial expressions which gives us that:

$$R_{K^-}(z) = R_K(z) = R_{K^+}(z) \quad (9)$$

$R_{K^-}$  and  $R_{K^+}$  are differentiable in  $z$  and an easy computation shows that

$$\frac{\partial R_{K^-}}{\partial X}(z) = \frac{\partial R_{K^+}}{\partial X}(z) = v$$

this together with equations (8) and (9) entails that  $R_K$  is differentiable at  $z$  and:

$$\frac{\partial R_K}{\partial X}(z) = v$$

□

Now let  $v \in \tilde{G}_K(x)$ . Lemma 5.3 entails that, for any positive integer number  $n$ , there exists  $x_n \in B_{(x, \frac{1}{n})}$  such that:

$$\frac{\partial R_K}{\partial X}(x_n) = v$$

this together with the characterization of theorem 5.1 implies that  $v \in \partial R_K(x)$ . We have proved that  $\tilde{G}_K(x) \subset \partial R_K(x)$  and, because  $\partial R_K(x)$  is convex, it entails  $G_K(x) \subset \partial R_K(x)$ .

We prove now  $\partial R_K(x) \subset G_K(x)$ . As seen in section 2.1, equation (1), the function  $x \mapsto \Gamma_K(x)$  is upper semi-continuous. When  $R_K(x) > 0$ ,  $G_K(x)$  is the image of  $\Gamma_K(x)$  by a simple continuous transformation which allows easily to derive the following Lemma.

**Lemma 5.4**  $G_K$  is upper semi-continuous in  $K^c$ , in other words:

$$\forall x \in K^c, \forall r > 0, \exists \alpha > 0, \|y - x\| \leq \alpha \Rightarrow G_K(y) \subset G_K(x)^r$$

Let us consider a vector  $v$  such that there exists a sequence of points  $x_i$  which as in theorem 5.1, are such that  $\lim_{n \rightarrow \infty} x_i = x$ ,  $R_K$  is differentiable at each  $x_i$  and

$$\lim_{n \rightarrow \infty} \frac{\partial R_K}{\partial X}(x_i) = v \quad (10)$$

Let us consider  $\varepsilon > 0$ . From Lemma 5.4, there is  $\alpha > 0$  such that:

$$\|y - x\| \leq \alpha \Rightarrow G_K(y) \subset G_K(x)^{\frac{\varepsilon}{2}} \quad (11)$$

From (10), there is  $x_k$  such that:

$$\|x_k - x\| \leq \alpha \quad \text{and} \quad \left\| \frac{\partial R_K}{\partial X}(x_k) - v \right\| < \frac{\varepsilon}{2} \quad (12)$$

From ([11] 2.5.4)  $R_K$  differentiable at  $x_k$  entails that  $\Gamma_K(x_k)$  is a single point and, if we denote  $\{y_k\} = \Gamma_K(x_k)$  and  $v_k = \frac{x_k - y_k}{R_K(x_k)}$  one has  $v_k = \frac{\partial R_K}{\partial X}(x_k)$ , which gives, with (12):

$$\|v_k - v\| < \frac{\varepsilon}{2}$$

From another hand one has from (11):

$$\{v_k\} = G_K(x_k) \subset G_K(x)^{\frac{\varepsilon}{2}}$$

which entails:

$$\{v\} \subset G_K(x)^\varepsilon$$

Because this inclusion holds for any  $\varepsilon > 0$  and  $G_K(x)$  is closed, it entails

$$v \in G_K(x)$$

Because  $G_K(x)$  is convex and  $\partial R_K(x)$  is defined in theorem 5.1 as the convex hull of all such  $v$ , we get  $\partial R_K(x) \subset G_K(x)$ .  $\square$

## 5.2 Stability of $\partial R_K$

We consider again two compact subsets of  $\mathbb{R}^n$ ,  $K$  and  $K'$  which are “close” to each other for the Hausdorff distance:  $d_H(K, K') \leq \varepsilon$ .

Let  $x$  be a point in  $K'^c$ . For any  $w' \in G_{K'}(x)$ , the point  $z' = x - R_{K'}(x)w'$  is in  $\Gamma_{K'}(x)$  and therefore in  $K'$ . One has then, for any  $y \in \mathbb{R}^n$ :

$$\begin{aligned} R_{K'}(y)^2 &\leq (y - z')^2 \\ &= (x - z')^2 + 2 \langle x - z', y - x \rangle \\ &\quad + (y - x)^2 \\ &\leq R_{K'}(x)^2 + 2 \langle w', y - x \rangle R_{K'}(x) \\ &\quad + (y - x)^2 \end{aligned}$$

which gives, for any  $w' \in G_{K'}(x)$

$$R_{K'}(y) - R_{K'}(x) \leq R_{K'}(x) \left( \sqrt{1 + \frac{2}{R_{K'}(x)} \langle w', y - x \rangle + \frac{(y - x)^2}{R_{K'}(x)^2}} - 1 \right)$$

And, from  $\sqrt{1+\alpha} \leq 1 + \frac{\alpha}{2}$ :

$$R_{K'}(y) - R_{K'}(x) \leq \langle w', y - x \rangle + \frac{(x - y)^2}{2R_{K'}(x)} \quad (13)$$

Lemma 5.2 says that  $\partial R_K(x) = G_K(x)$  which allows to use the following *mean value theorem* theorem which holds in general for Clarke gradients:

**Theorem 5.5 (Lebourg [11] 2.3.7)** *Let  $x$  and  $y$  be points in  $X$ , and suppose that  $f$  is Lipschitz in an open set containing the line segment  $[x, y]$ . Then there exists a point  $w \in (x, y)$  such that:*

$$f(y) - f(x) \in \langle \partial f(w), y - x \rangle$$

Let  $\rho > 0$  and  $x$  such that  $R_K(x) \geq \rho$ , applying theorem 5.5 to the function  $R_K$  gives:

$\forall y \in B(x, \rho)$ ,  $\exists w \in G_K(x, \rho)$  such that:

$$R_K(y) - R_K(x) = \langle w, y - x \rangle$$

Using  $R_K(y) - R_K(x) \leq R_{K'}(y) - R_{K'}(x) + 2\varepsilon$  and equation (13) we get:  $\forall y \in B(x, \rho)$  there is  $w \in G_K(x, \rho)$  such that for any  $w' \in G_{K'}(x)$ :

$$\langle w, y - x \rangle \leq \langle w', y - x \rangle + \frac{(x - y)^2}{2R_{K'}(x)} + 2\varepsilon$$

or:

$$\langle w' - w, x - y \rangle \leq \frac{(x - y)^2}{2R_{K'}(x)} + 2\varepsilon$$

Assuming now  $\rho = \|y - x\|$ , we consider the unit vector  $u = -\frac{y-x}{\rho}$ , which gives the following property:  $\forall u, \|u\| = 1, \forall w' \in G_{K'}(x)$  there is  $w \in G_K(x, \rho)$  such that:

$$\langle w' - w, u \rangle \leq \frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho} \quad (14)$$

This property, which hold for any unit vector  $u$  gives in fact a relation between the support functions of the compact sets  $G_{K'}(x)$  and  $G_K(x, \rho)$ .

Let  $w' \in G_{K'}(x)$  such that  $w' \notin G_K(x, \rho)$  and let  $w'' \in G_K(x, \rho)$  be its unique nearest point in the convex set  $G_K(x, \rho)$ :

$$d(w', w'') = d(w', G_K(x, \rho))$$

let us consider the unit vector  $u^* = \frac{1}{\|w' - w''\|}(w' - w'')$ . Because  $G_K(x, \rho)$  is convex,  $\forall w \in G_K(x, \rho)$ , one has:

$$\langle w, u^* \rangle \leq \langle w'', u^* \rangle$$

or equivalently:

$$\langle w - w'', u^* \rangle \leq 0$$

adding member to member with (14), for  $u = u^*$ , gives:

$$\langle w' - w'', u^* \rangle \leq \frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho}$$

that is, for any  $w' \in G_{K'}(x) \setminus G_K(x, \rho)$ , there is  $w'' \in G_K(x, \rho)$  such that:

$$\|w' - w''\| \leq \frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho}$$

which proves the following.

**Theorem 5.6** *For any  $x$  such that  $R_K(x) \geq \rho$ , one has:*

$$G_{K'}(x) \subset G_K(x, \rho)^{\frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho}}$$

## 6 Application to normal approximation

Based on the results from the previous section, we now introduce a scale-dependent notion of normal cone that allows to infer first order information from finite approximations of compact sets, even in the non-smooth case.

**Definition 6.1** *The normal cone at scale  $(r, l)$  of a compact  $K$  at the point  $p \in \mathbb{R}^n$  is defined as:*

$$N_K^{r,l}(p) = \text{co}\left\{ \frac{\vec{xq}}{\|\vec{xq}\|} \mid d(x, p) \leq r, d(x, q) \geq l, d(x, q) = d(x, K) \right\}$$

It is not difficult to check that  $\lim_{r \rightarrow 0} N_K^{r,0}(p)$  coincides with the normal cone  $N_K(p)$  in the sense of Clarke (see [11] p.51 and proposition 2.5.7 p. 68). Now, Theorem 5.6 gives, taking  $\rho = 2\sqrt{\varepsilon r}$  as in [5]:

**Lemma 6.2** *Let  $K$  and  $K'$  be two compacts with Hausdorff distance  $\varepsilon$  and let  $\eta = \varepsilon + 2\sqrt{\varepsilon r}$ . We have for all  $0 \leq l \leq r$ :*

$$N_{K'}^{r,l}(p) \subset N_K^{r+\eta, l-\eta}(p)^{2\sqrt{\varepsilon/l}}$$

This lemma directly implies the following one:

**Lemma 6.3** *Let  $K_n$  be a sequence of compact sets converging to a compact  $K$  for the Hausdorff distance. Let  $p \in \mathbb{R}^n$  and  $0 \leq l \leq r$  be such that  $N_K^{r,l}(p)$  is continuous at  $(r, l)$ . Then,  $N_{K_n}^{r,l}(p)$  converges to  $N_K^{r,l}(p)$ . In other words,  $N_K^{r,l}(p)$  is continuous at  $K$ .*

Since our notion of normal cone is stable under Hausdorff approximation, it can be inferred from finite approximations of compact sets. It now remains to pick suitable values for the parameters  $r$  and  $l$ .

Let us first consider the case where  $K$  has positive reach. We then have that  $N_K^{r,r}(p)$  coincides with Clarke's normal cone  $N_K(p)$  whenever  $0 < r < r_1(K)$ . The problem is that  $N_K^{r,r}(p)$  is not right continuous in the second variable at  $(r, r)$  since  $N_K^{r,l}(p)$  is empty whenever  $l > r$ . However, it is not difficult to prove that the function is continuous at any point  $(r, l)$  with  $l < r < r_1(K)$ . Hence, by the lemma above,  $N_K^{r,\lambda r}(p)$  is a good estimate of  $N_K^{r,r}(p)$  for any  $\lambda < 1$ , provided that  $\varepsilon$  is small enough. Moreover, when  $\lambda$  tends to 1,  $N_K^{r,\lambda r}(p)$  tends to  $N_K^{r,r}(p) = N_K(p)$ . Hence  $N_K^{r,\lambda r}(p)$  is a reliable way to obtain an estimate of  $N_K(p)$  when  $\lambda \rightarrow 1$  and  $\varepsilon \rightarrow 0$ . We note that unlike Dey's method for normal estimation in noisy smooth surfaces, this estimator provides a normal cone at every point. Besides, it can also deal with non necessarily smooth convex sets.

Now, if  $K$  is a (non necessarily convex) polyhedron, let  $r(p)$  be the distance from  $p$  to the closest (closed) face of  $K$  not containing  $p$ . Also, let  $\mu = \inf\{\|\nabla_K(x)\| \mid x \in B(p, r(p)/2)\}$ . We have that for  $l \leq r\sqrt{1-\mu^2}$  and  $r < r(p)/2$ ,  $N_K^{r,l}(p) = N_K(p)$ . Hence  $N_K^{r,l}(p)$  is a good estimate of  $N_K(p)$  for such a choice of  $r$  and  $l$ . More precisely, this estimator has precision  $O(\sqrt{\varepsilon/l})$ . Finally, we note that the critical function of  $K'$  might prove useful to automate the choice of the parameters  $r$  and  $l$ .

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