On the relation between coherence semantics and multiplicative proof nets

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Abstract: It is known that (mix) proof nets admit a coherence semantics, computed as a set of experiments. We prove here the converse: a proof structure is shown to be a proof net whenever its set of experiments is a semantical object — a clique of the corresponding coherence space. Moreover the interpretation of atomic formulae can be restricted to a given coherent space with four tokens in its web. This is done by transforming `cut`-links into `tensor`-links.

Dealing directly with non-cut-free proof structure we characterise the deadlock freeness of the proof structure. These results are especially convenient for Abramsky's proof expressions, and are extended to the pomset calculus.

Key-words: Logic, proof theory, linear logic, proof nets. Denotational semantics.

(Résumé : tsvp)

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Sur le lien entre réseaux multiplicatifs et sémantique cohérente

Résumé : On sait qu'un réseau admet une sémantique cohérente calculée comme l'ensemble de ses expériences, et ce même en présence de la règle de mélange (mix). On démontre ici la réciproque: un préréseau est un réseau si et seulement si l'ensemble de ses expériences est un objet sémantique — une clique de l'espace cohérent correspondant. De plus l'interprétation des formules atomiques peut être restreinte à un unique espace cohérent fixé dont la trame a quatre points. On montre cela en transformant les liens `coupure` en liens `tenseur`. Si l'on traite directement des préréseaux avec coupures, alors on peut ainsi caractériser l'impossibilité qu'apparaisse une situation blocage dans le préréseau réduit. Ces résultats sont particulièrement opportuns pour les expressions de preuve d'Abramsky, et sont étendus au calcul ordonné.

Mots-clé : Logique, théorie de la démonstration, logique linéaire, réseaux de preuve. Sémantique dénotationnelle.

1 Introduction and reminder

We tried to be self-contained, in particular regarding experiments since they have not be taken up again since the original paper on linear logic [Gir87]. Nevertheless, in particular for proof nets, it may be worth to have a look at [Gir87, Tro92]. The next two subsections may be skipped by the reader familiar with proof nets (1.1) and experiments (1.2).

1.1 Multiplicative proof nets

We deal here with multiplicative proof nets [Gir87] with the *mix* rule [FR94]. We use a characterisation à la Danos-Regnier [DR90, Tro92], where proof structures are graphs. Following [FR94] we use the following:

Definition 1 *A path of a proof structure is said to be feasible whenever it does not contain the two edges of the same* `par`*-link.*

In this formalism, the *mix* proof nets are defined as the multiplicative proof structures such that any switching is an acyclic — but not necessarily connected — graph, i.e a forest, or as we prefer, as the multiplicative proof structures without any feasible cycle. They exactly correspond — see [FR94] for a proof — to the linear multiplicative sequent calculus enriched with the *mix* rule:

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\, mix$$

Throughout the paper, we assume all `axiom`-links to be $A \overline{\ } A^{\perp}$ with A atomic, as the η-expansion property for proof structures and nets allows.

1.2 Coherence semantics and experiments

The starting point of this note is the so-called *experiment* method of [Gir87], §3.17, §3.18 for computing the coherence semantics [Gir87, GLT88, Tro92] of a proof directly from the proof net.

Definition 2 *A coherence space A is a simple countable graph. The set of its vertices is denoted by $|A|$, and called the web of the coherence space; vertices are called tokens. Adjacency which is a symmetric and anti-reflexive relation in a simple graph is called strict coherence and "*x and y are adjacent in the coherence space A*" is written $x \frown y[A]$. The following shorthands are convenient:*

$x \subset y[A] : x = y \text{ or } x \frown y[A]$ — *coherent*

$x \asymp y[A] : \text{ not } x \frown y[A]$ — *incoherent*

$x \smile y[A] : x \neq y \text{ and not } (x \frown y[A])$ — *strictly incoherent*

The dual $A^\perp$ of a coherence space A is defined by its web $|A^\perp| = |A|$, and its strict coherence: $x \frown y[A^\perp]$ iff $x \smile y[A]$ — $A^\perp$ is the complement graph of A.

Definition 3 *An interpretation is a choice, for each propositional letter A of a coherence space (also denoted by A).*

Remember a binary connective $*$ is a functor which defines a new coherence space $A * B$ from two already built coherence spaces A and B. Therefore, an interpretation associates to each formulae a coherence space. A multiplicative (binary) connective is a connective $*$ such that the web $|A * B|$ is $|A| \times |B|$, and they are exactly three such connectives [Ret93b, Ret95]: $\otimes, \wp, <$ which are all associative, while only the two first ones are commutative.

$(x, y) \subset (x', y')[A \otimes B] \text{ iff } x \subset x'[A] \text{ and } y \subset y'[B]$

$(x, y) \frown (x', y')[A \wp B] \text{ iff } x \frown x'[A] \text{ or } y \frown y'[B]$

$(x, y) \frown (x', y')[A < B] \text{ iff } (x \frown x'[A] \text{ and } y = y') \text{ or } y \frown y'[B]$

Definition 4 *An experiment of a proof structure Π is a labelling of the nodes of the proof net — i.e. of the occurrences of formulae appearing in the proof net. The label of a node A is a token, say a, of the web $|A|$ of the coherence space A, and we write $a : A$ for this. An experiment is obtained as follows:*

- *for each axiom $A \overline{\quad} A^\perp$ we arbitrarily choose a* single *token $a \in |A| = |A^\perp|$ which is their* common *label:*

$$a : A \overline{\quad} a : A^\perp$$

and this completely determines the experiment.

- *these labels are spread all over the proof net, from the premises of links to their conclusions as follows.*

  Let $* \in \{\wp, \otimes\}$. *If the label of the left premise is* $u \in |A|$ *and the label of the right premise is* $v \in |B|$ *then the label of the conclusion* $A * B$ *is* (u, v) *— which belongs to* $|A * B| = |A| \times |B|$.

$$\begin{array}{ccc} x : A & & y : B \\ & \diagdown \quad \diagup & \\ & (x, y) : A * B & \end{array}$$

Definition 5 *The result of an experiment* $\mathcal{E}$ *is the tuple* $|\mathcal{E}| = (t_1, ..., t_n)$ *of the tokens* t_i *labelling the conclusion nodes:* $t_1 : F_1$, ... , *and* $t_n : F_n$.

An experiment is said to succeed whenever in each `cut`*-link the two labels are equal:*

$$t : F \_\_ t : F^{\perp}$$

Two experiments $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are said to be different experiments whenever there exists a node on which the two labels differ, i.e. whenever they use a different token for the same axiom — this does not mean that* $|\mathcal{E}_1| \neq |\mathcal{E}_2|$.

The semantics of a proof of $F_1, ..., F_n$ is a clique of the coherence space $F_1 \wp ... \wp F_n$ — i.e. a set of pairwise coherent tokens of the coherence space $F_1 \wp ... \wp F_n$. The semantics of a proof is invariant with respect to cut elimination — we are speaking of denotational semantics. Usually, it is computed by induction on the sequent calculus proof. The *experiments* method provides an alternative way to compute the semantics of a proof:

Theorem 1 ([Gir87], 3.18) *Let* Π *be a proof* net *with conclusions* $F_1, ..., F_n$. *Let* $\|\Pi\|$ *be the set of results of succeeding experiments of* Π *with respect to an interpretation. Then one has :*

- *If* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are two different experiments of* Π *then* $|\mathcal{E}_1| \frown |\mathcal{E}_2|$ *and therefore* $\|\Pi\|$ *is a clique of* $F_1 \wp ... \wp F_n$ *— notice that in the proof net case* $\mathcal{E}_1 \neq \mathcal{E}_2 \Rightarrow |\mathcal{E}_1| \neq |\mathcal{E}_2|$*)*
- *whenever* Π *reduces to* Π' *by cut-elimination, then* $\|\Pi\| = \|\Pi'\|$, *and since a normal proof net always possesses a non-trivial semantics, so does any proof net.*

We made few slight changes to the original presentation of [Gir87]:

- we spread labellings from axioms to conclusions and not the converse
- we define experiments for all proof structures and not simply for proof nets
- we are working with the mix rule
- we use a correctness criterion à la Danos-Regnier

The reader should not worry about that: Girard's original idea straightforwardly applies. Moreover, the proof for an even bigger calculus, implying theorem 1, is taken up again from [Ret93b, Ret95] in appendix.

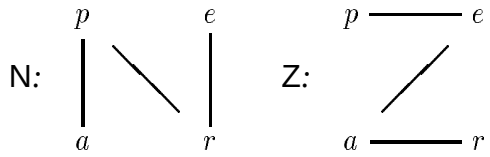
1.3 Contents of the paper

When proving the previous theorem, the argument makes such an intensive use of the correctness criterion, that we start thinking the converse is true. Noticing that:

- coherence spaces naturally interprets the *mix* rule
- experiments could be defined for proof structures as well
- as far as *correctness* is concerned `cut`-links may be viewed as `tensor`-links

we obtain the converse that we prove in section 2.

Definition 6 *Here are the two dual coherence spaces* [1] $\underline{\mathtt{N} = \mathtt{Z}^{\perp}}$ *and* $\underline{\mathtt{Z} = \mathtt{N}^{\perp}}$*:*



An $\underline{\mathtt{NZ}}$*-interpretation is an interpretation in which any atomic formula is interpreted as* $\mathtt{N}$ *or as* $\mathtt{Z}$*.*

[1] These are funny coherence spaces: $\mathtt{N}$ is linearly isomorphic to $\mathtt{Z} = \mathtt{N}^{\perp}$, and it is the smallest coherence space which is not definable from $\mathbf{0}$ and $\mathbf{1}$ with the binary connectives $\oplus, \&, \otimes, \wp, <$, nor with any kind of n-ary multiplicative connectives.

Theorem 2 (correctness)

Let Π *be a cut-free proof structure with conclusions* $F_1, .., F_n$,

let ∇ *be any* NZ*-interpretation,*

let $\mathcal{E}_1$ *be any* ∇*-experiment of* Π.

Π *is a proof net if and only if any other* ∇*-experiment* $\mathcal{E}_2$ *satisfies* $|\mathcal{E}_1| \supset |\mathcal{E}_2|[F_1 \wp F_2 ... \wp F_n]$ *— and therefore* $|\mathcal{E}_1| \frown |\mathcal{E}_2|[F_1 \wp F_2 ... \wp F_n]$*, since it is a cut-free proof structure.*

Our result also applies to non-cut-free proof structures, since a proof structure is a proof net if and only if the proof structure obtained by replacing the `cut`-links with `tensor`-links is a proof net too. Nevertheless it is worth looking directly at non-cut-free proof structures, since this direct study allows us to semantically characterise a property that we call deadlock freeness.

This property already appeared in the works of Lafont on interaction nets [Laf90, Laf94]; since we deal with well typed nets this is the only case of deadlock that may appear, and our notions agree. This property also appeared exactly as we define it in the work of Abramsky on proof expressions [Abr93] where it is called *acyclicity* — but this name is a bit misleading when dealing with proof nets.

Deadlock freeness corresponds to the absence of loop — cut on an axiom — in the *reduct*, i.e. to the possibility for the proof structure to interact with others.

Theorem 3 (deadlock freeness)

Let Π *be a proof structure with conclusions* $F_1, ..., F_n$,

let Π^* *be its reduct,*

let ∇ *be any* NZ*-interpretation,*

and let $\mathcal{E}_1$ *be any succeeding* ∇*-experiment of* Π.

Then

- Π *is deadlock free, i.e.* Π^* *is loop free if and only if any different succeeding* ∇*-experiment* $\mathcal{E}_2$ *of* Π *satisfies* $|\mathcal{E}_1| \neq |\mathcal{E}_2|$

- Π^* *consists in a proof net plus some — possibly no — loops if and only if any different succeeding* ∇*-experiment* $\mathcal{E}_2$ *of* Π *satisfies* $|\mathcal{E}_1| \supset |\mathcal{E}_2|[F_1 \wp ... \wp F_n]$
- Π^* *is a proof net if and only if any different succeeding* ∇*-experiment* $\mathcal{E}_2$ *of* Π *satisfies* $|\mathcal{E}_1| \frown |\mathcal{E}_2|[F_1 \wp ... \wp F_n]$

This tightens the relation between coherence semantics and multiplicative proof nets, and is a kind of completeness result for linear logic. These results therefore add a new facet to completeness results obtained via game theoretical semantics [AJ94], and totality[VdW89, VdW90, Loa94]. Although we a priori believed in this theorem, we were actually surprised that, in order to use it to decide whether a proof structure is a proof net — hence a proof — the interpretation does not need to vary, and can even be fixed (almost) at convenience. In particular, fixing any atomic formulae to be the same given finite coherent space $\mathtt{N}$, one gets an algorithm out of proof. This algorithm consists in finding among a finite number of experiments two incoherent ones — and moreover, one of the two can be arbitrarily fixed! Even though this algorithm is an exponential one and quadratic ones are already known for checking the correctness of a proof structure [Gir87, Dan90, Ret93a] , it is quite unusual to get an algorithm for a syntactical property from denotational semantics.

This property is quite useful for proof expressions of [Abr93], since it allows a *direct* (semantical) characterisation of the ones coming from proofs, and of the acyclic ones too. It should be noticed how close proof expressions and experiments are.

Finally,using similar arguments we show how our result also applies to the multiplicative calculus of pomset logic [Ret93b, Ret95].

The proofs are sometimes redundant or straightforward, but I prefer to give them in full details.

2 The semantical characterisation of correctness

Notation 1 *During this section:*

- ∇ *denotes a given but arbitrary* $\mathtt{NZ}$*-interpretation,*
- Π *denotes a* cut free *proof structure with conclusions* $F_1, ..., F_n$
- $\mathcal{E}_1$ *is a given but arbitrary* ∇*-experiment of* Π *— any experiment succeeds when there is no* `cut`*-link.*

Notation 2 *Given a node A of a proof structure, and two of its experiments $\mathcal{E}_1$ and $\mathcal{E}_2$, write $A{:}\frown$ for the two tokens t_1 and t_2 labelling the node A according to $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy $t_1 \frown t_2[A]$ — the same apply for $A{:}\smile$, $A{:}\asymp$, $A{:=}$, ...*

Proposition 1 *Let $A_1 \overline{\ } A_1^\perp$, $A_2 \overline{\ } A_2^\perp$ ... and $A_p \overline{\ } A_p^\perp$ be a family of axioms of Π (hence all the A_i are atomic), and let ϕ and ψ be two functions from $[1,p]$ to $\{A_1, A_1^\perp, A_2, A_2^\perp, ..., A_p, A_p^\perp\}$ such that $\{\psi(p), \phi(p)\} = \{A_p, {A_p}^\perp\}$.*

Then there always exists another ∇-experiment $\mathcal{E}_2 \neq \mathcal{E}_1$ such that $\phi(i){:}\frown \overline{\ } \psi(i){:}\smile$, and, for any axiom not in the family, $B{:=}\overline{\ } B^\perp{:=}$.

Proof: Assume the token for the axiom i according to $\mathcal{E}_1$ is $x_i \in |\mathtt{N}| = |\mathtt{Z}|$. Then in the interpretation of $\phi(i)$ which is $\mathtt{N}$ or $\mathtt{Z}$, there exists another token y_i such that $x_i \frown y_i[\phi(i)]$ and since $\psi(i) = \phi(i)^\perp$, we also have $x_i \smile y_i[\psi(i)]$. Indeed for any token x in $|\mathtt{N}| = |\mathtt{Z}|$ there exists another token y of $|\mathtt{N}| = |\mathtt{Z}|$ such that $x \frown y[\mathtt{N}]$ (resp. $x \frown y[\mathtt{Z}]$). ◇

2.1 Experiments and feasible paths

During this subsection, the proof structure Π is assumed to be a proof **net**[2].

Lemma 1 *Let X, Y be two conclusions of the proof net Π such that there exists a feasible path between X and Y, and let γ be such a path.*

Since X and Y are conclusions the feasible path necessarily uses some `axiom`*-links, and because it is a proof net all the successively met axioms are distinct.*

Assume that using this feasible path γ from X to Y the distinct axioms we met are: $A_1 \overline{\ } A_1^\perp$, $A_2 \overline{\ } A_2^\perp$ ... and $A_p \overline{\ } A_p^\perp$, from $\phi(i)$ to $\psi(i)$, where $\{\phi(i), \psi(i)\} = \{A_i, A_i^\perp\}$.

Proposition 1 provides another ∇-experiment $\mathcal{E}_2$ such that: $\phi(i){:}\frown \overline{\ } \psi(i){:}\smile$, and, for any other axiom, $B{:=}\overline{\ } B^\perp{:=}$.

Then, $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfy: $X{:}\frown$ and $Y{:}\smile$ while $Z{:}\asymp$ for any other conclusion Z.

[2] This is in fact not needed, as can be seen from section 5. Nevertheless it makes both the statement of the lemma and its proof easier to follow.

Proof: We proceed by induction, using the following fact: if a proof net is not a union of axioms, then it possesses a final `par`-link or a splitting `tensor`-link. The proof for the mix calculus follows from sequentialisation theorem in [FR94, Dan90], and is directly proved in [Ret93a].

(1) If the proof net is a union of axioms, then because of γ, X and Y are the two conclusions of the same axiom, and the result is obvious.

(2) If there is a final `par`-link, we arbitrarily choose one, and call Π' the proof net obtained by removing this final `par`-link.

(a) If X is its conclusion. Then the path γ makes use of one of the edge of the `par`-link. Call X' the corresponding premise, and γ' the restriction of γ to Π'. Therefore γ' uses the same axioms in the same order. We can apply the induction hypothesis to Π', X' and γ', and therefore we obtain $X'\!:\!\frown$ and $Y\!:\!\smile$ with $Z\!:\!\asymp$ for any other conclusion Z. From the coherence according to `par`, we obtain the result.

(b) If Y is its conclusion, we proceed similarly, noticing that $a' \smile b'[Y']$ and $a'' \asymp b''[Y'']$ implies $(a', a'') \smile (b', b'')[Y' \wp Y'' = Y]$.

(c) If none of X, Y is its conclusion, then γ does not use this link. So we apply induction hypothesis to Π', X, Y and γ, and the result immediately follows.

(3) If there is no final `par`-link, there exists a splitting `tensor`-link, and we arbitrarily choose one. Let Π' and Π'' be the two parts — we arbitrarily put totally disconnected parts of the proof net in one of these two parts. So we have a partition of the nodes: Π', Π'', and the `tensor`-link's conclusion.

(a) If X is this conclusion, say Y is in Π', and call X' the premise of X in Π'. Then necessarily γ starts with the edge $X - X'$, and call γ' the rest of γ which is necessarily included in Π'. We apply induction hypothesis to Π', X', Y and γ'. Noticing that all conclusions in Π'' are $Z\!:\!=$, the result is clear.

(b) If Y is this conclusion, we proceed similarly.

(c) If neither X nor Y is this conclusions, they either lie in the same part Π' or different parts, say $X \in \Pi'$ and $Y \in \Pi''$.

(i) If X, Y are in the same part, since the `tensor`-link is splitting then γ does not use it — otherwise there would exist a feasible cycle. So we apply induction hypothesis to Π', X, Y and γ, and the result follows — all conclusions are $Z\!:\!=$ in Π''.

(ii) If X is in Π' and Y in Π'', then γ uses the splitting `tensor`-link. Call U' its premise in Π', γ' the part of γ from X to U' (included in Π') and U'' its premise

in Π'', γ'' the part of γ from U'' to Y (included in Π''). We apply induction hypothesis to Π', X, U' with γ' and to Π'', U'' and Y with γ''. The result follows, since U':$\smile$ and U'':$\frown$ implies $U' \otimes U''$:$\smile$.

$\diamond$

2.2 Experiments and proof structures

We still assume Notation 1.

Lemma 2 *If the cut-free proof structure* Π *is not a proof net, then there exists another* ∇*-experiment* $\mathcal{E}_2 \neq \mathcal{E}_1$ *such that* $|\mathcal{E}_1| \smile |\mathcal{E}_2| [F_1 \wp \ldots \wp F_n]$

Proof: Remember that the proof structure is not a proof net whenever it possesses a feasible cycle, while the two experiments are not coherent in the `par` of its conclusions whenever all conclusions are Z:$\asymp$ one of them being W:$\smile$.

We here too proceed by induction on the number of links of the proof structure.

(1) Π can not be a union of axioms.

(2) If Π has a final `par`-link, then the proof structure obtained by removing this final `par`-link neither is a proof net. We apply induction hypothesis, and we are done.

(3) Otherwise Π possesses a final `tensor`-link.

(a) If the proof structure obtained by removing this final `tensor`-link is neither a proof net, we apply induction hypothesis, and we are done.

(b) Otherwise, the proof structure obtained by removing this final `tensor`-link is a proof net. Therefore this proof net contains a feasible path γ between its two premises, say X and Y. We apply the previous lemma, and thus we obtain another experiment $\mathcal{E}_2$ such that X:$\frown$ and Y:$\smile$ the other conclusions being Z:$\asymp$. This obviously provides another experiment $\mathcal{E}_2$ of Π such that $X \otimes Y$:$\smile$ and Z:$\asymp$ for any other conclusion.

$\diamond$

2.3 The characterisation

We can now easily deduce from our two lemmas the following theorem, which also applies for non-cut free proof net (by replacing `cut`-links with `tensor`-links):

Theorem 2 (correctness)

Let Π *be a cut-free proof structure with conclusions* $F_1, .., F_n$,

let ∇ *be any* NZ*-interpretation,*

let $\mathcal{E}_1$ *be any* ∇*-experiment of* Π.

Π *is a proof net if and only if any other* ∇*-experiment* $\mathcal{E}_2$ *satisfies* $|\mathcal{E}_1| \supset |\mathcal{E}_2|[F_1 \wp F_2 ... \wp F_n]$ *— and therefore* $|\mathcal{E}_1| \frown |\mathcal{E}_2|[F_1 \wp F_2 ... \wp F_n]$*, since it is a cut-free proof structure.*

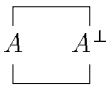
The direct implication was already known — it is theorem 1 — while the converse is the previous lemma. Notice this provides an algorithm to decide whether a proof structure is correct or not: arbitrarily choose a NZ-interpretation and a ∇-experiment $\mathcal{E}_1$, and then test whether the result $|\mathcal{E}_2|$ of each different experiment $\mathcal{E}_2$ is coherent with $|\mathcal{E}_1|$ in the coherence space $F_1 \wp .. \wp F_n$. Unfortunately, as such, it is not an efficient one, since they are $4^{\#ax}$ ∇-experiments. But it provides a semantical characterisation.

3 The semantical characterisation of deadlock freeness

We already told that the previous result applies to non-cut-free proof structure, turning `cut`-links into `tensor`-links. Nevertheless, a direct study of non-cut-free will enable us to semantically characterise the absence of *deadlock* in a proof structure and its reduct.

3.1 Remarks on the reduction of proof structures

Call loop the following (part of a) proof structure:



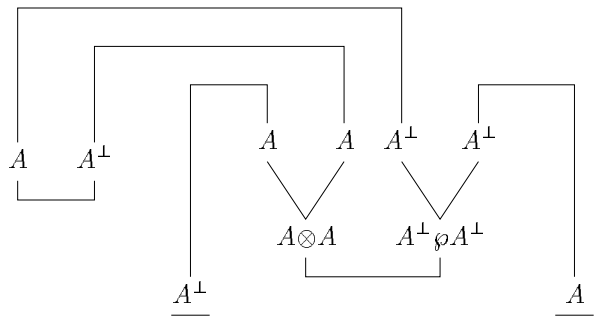
Proof structures also enjoy η expansion, so we can, without lost of generality, restrict our-selves to atomic `axiom`-links. Therefore, when one of the premise of a `cut`-link is the conclusion of an axiom, so is the other.

Cut-elimination steps may be defined for proof structures as well:

`par/tensor`**-case** as usual

`axiom/axiom`**-case** we also act as usual, unless the two involved axioms are the same axiom: in this case it is a loop, and there is no way to reduce this cut.

Firstly, notice that the reduct of proof structure which is not a proof net may be a proof net — because not any path is preserved under cut-elimination, but only the so called *persistent paths* which are the same as the *regular paths* [ADLR94]. The following example is a proof structure which is not a proof net, but reduces to a proof net — a single axiom linking the underlined $\underline{A^\perp}$ and $\underline{A}$ — we take the convention that $(A \wp B)^\perp = A^\perp \otimes B^\perp$ and not the convention $(A \wp B)^\perp = B^\perp \otimes A^\perp$.



The main thing to quote is that *this process preserves the set of succeeding experiments* and this is essential to this section.

This process is also terminating (the size decrease), and confluent (disjoint reductions).

The reduct consists in a cut free proof structure, plus, possibly, some loops.

Once the only remaining cuts are loops it is easily seen that the absence of a loop in the reduct is characterised by: any different experiments lead to different results. Indeed, changing the value of an axiom of a loop obviously does not change the result, while changing the value of some axiom belonging to the part which is a cut-free

proof structure obviously changes the result. Axioms of loops in the reduct where already in the proof structure before reduction, and, since the succeeding experiments of the original proof structure are the same as the ones of the reduct, it is clear that Π reduces to a loop free proof structure if and only if different experiments of Π have different results.

Finally notice that, in order to find two different experiments having the same result one of the two can be arbitrarily fixed.

3.2 The characterisation

Our result, together with the previous remarks, enable us to semantically characterise the possible reduct of a given proof structure without actually reducing it:

Theorem 3 (deadlock freeness)

Let Π *be a proof structure with conclusions* $F_1, ..., F_n$,

let Π^* *be its reduct,*

let ∇ *be any* NZ*-interpretation,*

and let $\mathcal{E}_1$ *be any succeeding* ∇*-experiment of* Π.

Then

- Π *is deadlock free, i.e.* Π^* *is loop free if and only if any different succeeding* ∇*-experiment* $\mathcal{E}_2$ *of* Π *satisfies* $|\mathcal{E}_1| \neq |\mathcal{E}_2|$
- Π^* *consists in a proof net plus some — possibly no — loops if and only if any different succeeding* ∇*-experiment* $\mathcal{E}_2$ *of* Π *satisfies* $|\mathcal{E}_1| \supset |\mathcal{E}_2|[F_1 \wp ... \wp F_n]$
- Π^* *is a proof net if and only if any different succeeding* ∇*-experiment* $\mathcal{E}_2$ *of* Π *satisfies* $|\mathcal{E}_1| \frown |\mathcal{E}_2|[F_1 \wp ... \wp F_n]$

As a proof structure which is not a proof net has the same semantics as its reduct, which may be a proof net, without turning `tensor`-links into `cut`-links there is no hope to semantically characterise the proof structures which are proof nets.

4 Application to Abramsky's Proof Expressions

We apply here our results to Abramsky's proof expressions [Abr93, Tro92]. Hitherto, the only way to check whether a proof expression comes from a proof is to translate it into a proof net and to apply one of the usual criterions. In the multiplicative case our previous results provide a direct characterisation which does not refer to proof nets, but to semantics, and which is still decidable.

Notice that proof expressions, once transformed in order to only use atomic axioms, closely correspond to experiments:

- variables may be viewed as ranging over the tokens of a coherence space,
- read both $t \otimes u$ and $t \wp u$ as (t, u)
- read any co-equation as an equation forcing the equality of some variables

Let ∇ be an interpretation, i.e. an assignment of a coherence space to each atomic type. A ∇-experiment $\mathcal{E}_1$ of a proof expression $\mathcal{P}$ simply consists in reading each variable x of $\mathcal{P}$ having an occurence of type A and one of type $A^\perp$ in $\mathcal{P}$ as a token of the coherence space corresponding to A via ∇ — since $|A| = |A^\perp|$ it is possible. The result $|\mathcal{E}_1|$ of an experiment $\mathcal{E}_1$ of $\mathcal{P}$ is simply the tuple of tokens corresponding to the tuple of terms of $\mathcal{P}$, once each variable of $\mathcal{P}$ is replaced with its corresponding token. A succeeding experiment $\mathcal{E}_1$ of $\mathcal{P}$ is an experiment for which each co-equation of $\mathcal{P}$ between two terms becomes a formal equality between two tokens, once each variable of $\mathcal{P}$ is replaced with its corresponding token.

If ∇ is an $\mathtt{NZ}$-experiment it is even simpler: the web of the atomic coherence space is always the four points set $|\mathtt{N}| = |\mathtt{Z}|$. Let us call a N-valuation the mapping of each variable of a proof expression to a token of $|\mathtt{N}| = |\mathtt{Z}|$. A N-valuation is a ∇-experiment for any $\mathtt{NZ}$-interpretation ∇.

As in the proof net case, if $\mathcal{P}$ is a proof expression, and if $\mathcal{P}_\otimes$ is the proof expression obtained from $\mathcal{P}$ by replacing each co-equation with the corresponding `tensor`-term, then either both $\mathcal{P}$ and $\mathcal{P}_\otimes$ come from a proof, or they both do not. Taking this into account, and reading again our theorem 2 we obtain:

Theorem 2 (correctness for proof expressions)

Let $\mathcal{P}$ be a proof expression with conclusions $F_1, .., F_n$, and let $\mathcal{P}_\otimes$ be the proof expression obtained from $\mathcal{P}$ by replacing each co-equation with the corresponding `tensor`*-terms, having the conclusions $F_1, .., F_n, K_1 \otimes {K_1}^\perp, .., K_p \otimes {K_p}^\perp$.*

Let ∇ be an arbitrary NZ*-interpretation, let $\mathcal{E}_1$ be a* N*-valuation of $\mathcal{P}$, and let $|\mathcal{E}_1|$ be its result according to $\mathcal{P}_\otimes$.*

The proof expression $\mathcal{P}$ comes from a proof if and only if any other N*-valuation $\mathcal{E}_2$ of $\mathcal{P}$ having the result $|\mathcal{E}_2|$ according to $\mathcal{P}_\otimes$ satifies:*

$$|\mathcal{E}_1| \supset |\mathcal{E}_2|[F_1 \wp .. \wp F_n \wp (K_1 \otimes K_1)^\perp \wp .. \wp (K_p \otimes K_p{}^\perp)]$$

[3]

The direct look at the proof expressions involving co-equations is although worth quoting, since deadlock freeness exactly corresponds to *acyclicity* of [Abr93]:

Theorem 3 (deadlock freeness or acyclicity for proof expressions)
Let $\mathcal{P}$ be a proof expression with conclusions $F_1, .., F_n$. Let ∇ be an arbitrary NZ*-interpretation, let $\mathcal{E}_1$ be a succeeding* N*-valuation of $\mathcal{P}$.*

- *The proof expression $\mathcal{P}$ is* acyclic*, i.e. reduces to a loop free proof expression if and only if any different succeeding* N*-valuation $\mathcal{E}_2$ of $\mathcal{P}$ satisfies $|\mathcal{E}_2| \neq |\mathcal{E}_1|$.*
- *The proof expression $\mathcal{P}$ reduces to a proof expression coming from a proof plus some — possibly no — loops if and only if any different succeeding* N*-valuation $\mathcal{E}_2$ of $\mathcal{P}$ satisfies $|\mathcal{E}_2| \supset |\mathcal{E}_1|[F_1 \wp ... \wp F_n]$.*
- *The proof expression $\mathcal{P}$ reduces to a proof expression coming from a proof if and only if any different succeeding* N*-valuation $\mathcal{E}_2$ of $\mathcal{P}$ satisfies $|\mathcal{E}_2| \frown |\mathcal{E}_1|[F_1 \wp ... \wp F_n]$*

[3]

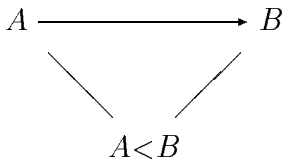
5 Extension to pomset logic

These results also apply to pomset logic [Ret93b, Ret95], for which an analogous of Girard's theorem (th. 1 of this paper) also holds — in appendix we take up again the proof of [Ret93b, Ret95].

[3] One can wonder where the interpretation ∇ is taken into account, since the variables are always mapped in $|\mathsf{N}| = |\mathsf{Z}|$ when ∇ is a NZ-interpretation. The interpretation ∇ is used when looking whether the results of the two experiments/N-valuations are coherent. In fact the standard notation is misleading, and it would be clearer to write $F_1^\nabla \wp .. \wp F_n^\nabla \wp (K_1^\nabla \otimes K_1{}^{\nabla\perp}) \wp .. \wp (K_p^\nabla \otimes K_p{}^{\nabla\perp})$ instead of $F_1 \wp .. \wp F_n \wp (K_1 \otimes K_1)^\perp \wp .. \wp (K_p \otimes K_p{}^\perp)$

5.1 Reminder on ordered proof nets

In this pomset calculus, the conclusions are partially ordered, and the connectives are $\otimes$, $\wp$ and the non-commutative $<$, the corresponding link being :



To be precise, we should say that the `cut`-link are viewed as *final* `tensor`-links, with dual premises: $X \otimes X^{\perp}$ [4].

The order on conclusions is represented by putting one arc from C_i to C_j whenever $C_i <_{\mathcal{I}} C_j$.[5]

Let us say arc for directed edge, and edge for undirected edge.

A feasible path of an ordered proof structure is a path, which does not use both edges of the same `par`- or `before` -link. We can always assume that a path, feasible or not does not use two consecutive arc of the order (which is transitive), and we always do so in the sequel.

The correctness criterion simply is: there is no feasible circuit (directed elementary cycle).

Remark 1 *Let* Π *be an ordered proof structure without any* `before` *-link, and whose order between conclusions is empty. Then* Π *is a usual mix proof structure.*

Let Π *be an ordered proof net without any* `before` *-link, and whose order between conclusions is empty. Then* Π *is a usual mix proof net.*

Here are two similar operations on orders needed in the proofs:

[4] In fact, this $X \otimes X^{\perp}$ is to be understood as $\exists X \; X \otimes X^{\perp} \sim \perp$, which does not modify the proof net. Since $\perp$ is the unit, it is clear that the proof net is a proof of the other conclusions with the restriction of the order to them.

[5] We could have written an arc from a conclusion C to a conclusion C' only when C' is a successor of C in $\mathcal{I}$, since the criterion would be the same, but it would make the following proofs more complicated.

Definition 7 *Let $\mathcal{I}$ be a partial order on a multiset of formulae $\Gamma, A \wp B$. Then $\underline{\mathcal{I}(A \sim B)}$ is the order on Γ, A, B defined by: $\mathcal{I}(A \sim B)$ restricted to Γ, A (resp. Γ, B) is $\mathcal{I}$ with $A \wp B := A$ (resp. $A \wp B := B$), and neither $A <_{\mathcal{I}(A \sim B)} B$ nor $B <_{\mathcal{I}(A \sim B)} A$.*

Let $\mathcal{I}$ be a partial order on a multiset of formulae $\Gamma, A < B$. Then, $\underline{\mathcal{I}(A \stackrel{<}{\sim} B)}$ is the order on Γ, A, B defined by: $\mathcal{I}(A \stackrel{<}{\sim} B)$ restricted to Γ, A (resp. Γ, B) is $\mathcal{I}$ with $A \wp B := A$ (resp. $A \wp B := B$), and $A <_{\mathcal{I}(A \stackrel{<}{\sim} B)} B$.

And here is the size of an ordered proof structure that we use in the proofs:

Definition 8 *Let Π be an ordered proof structure, call*
a *its number of* `axiom`*-links*
t *its number of* `tensor`*- and* `cut`*-links*
p *its number of* `par`*-links*
b *its number of* `before` *-links*
o *its number of order arcs*
Its size is defined as

$$\mathtt{a}(\mathtt{o}+1)3^{(2\mathtt{t})} + \mathtt{b} + \mathtt{p}$$

because we sometimes replace a `tensor`*-link by a* `before` *-link, and when replacing $\mathcal{I}$ with $\mathcal{I}(A \stackrel{<}{\sim} B)$ or with $\mathcal{I}(A \sim B)$ the number* o *becomes $(2 \times \mathtt{o}) + 1$ at worse. Some lexicographical size would work fine, but it is not actually needed.*

5.2 Coherence and experiments for the ordered proof nets

We already defined the coherence space for $<$ which is
$|A<B| = |A| \times |B|$
$(a,b) \frown (a',b')[A<B]$. iff . $\Big(a \frown a'[A] \text{ and } b = b'\Big)$ or $b \frown b'[B]$
Assuming the conclusions are C_i partially ordered by $\mathcal{I}$, the coherence space in which the semantics takes place is $\prod_{\mathcal{I}}(C_i)$:

$$\left|\prod_{\mathcal{I}}(C_i)\right| = |C_1| \times ... \times |C_n|$$

$$(c_1,...,c_n) \frown (c'_1,...,c'_n)[\prod_{\mathcal{I}} C_i] \text{ . iff . } \exists i \,.\, c_i \frown c'_i[C_i] \; and \; \Big(\forall j [C_j >_{\mathcal{I}} C_i \Rightarrow c_j = c'_j\Big)$$

All the definitions concerning experiments straightforwardly extend the ones we gave in the introduction for mix proof structures: fake conclusions $X \otimes X^{\perp}$ corresponding to cuts are *not* labelled, and for an experiment to succeed, we ask, as usual that the two labels of two the premises of any `cut`-link are equal. The result of the experiment consist in the tuple of the labels of the real conclusions — however the fake conclusions corresponding to cuts have no label.

5.3 Similar results for ordered proof structures

The proposition 1 still holds, since it only depends on the properties of `N`.

Although the argument itself is roughly the same, the analogous of lemma 1 looks more sophisticated, because we prove it here without assuming that the ordered proof structure is an ordered proof net. The remark 1 makes sure that that this refined lemma also holds for mix proof structure.

Lemma 1 (for pomset logic)

Let ∇ be any `NZ`-interpretation,

let Π be any cut free ordered proof structure,

let $\mathcal{E}_1$ be ∇ experiment of Π,

let X, Y be two of its conclusions such that there exists a feasible path from Y to X, such that neither the first nor the last edge of γ is an order arc,

and let γ be such a feasible path from Y to X using a minimal number of axiom edges.

Assume that using this feasible path γ from Y to X the distinct axioms we met are: $A_1 \overline{\ } A_1^{\perp}$, $A_2 \overline{\ } A_2^{\perp}$ ... and $A_p \overline{\ } A_p^{\perp}$, from $\psi(i)$ to $\phi(i)$, where $\{\phi(i), \psi(i)\} = \{A_i, A_i^{\perp}\}$.

Proposition 1 provides another ∇-experiment $\mathcal{E}_2$ such that: $\phi(i)$:$\frown$ $\overline{\ }$ $\psi(i)$:$\smile$, and, for any other axiom, B:=$\overline{\ }$$B^{\perp}$:=.

Then, according to the experiments $\mathcal{E}_1$ and $\mathcal{E}_2$ we have X:$\frown$ and Y:$\smile$ while for any other conclusion Z:$\frown$ there exists a conclusion $Z' >_{\mathcal{I}} Z$ such that Z':$\smile$.[6]

[6] If Π is a proof net, it means, as in Lemma of section 2, that X is the oly conclusion which makes the results of the two experiments coherents, while they are strictly incoherent in Y.

Proof: We proceed by induction on the size of Π previously defined.

If any conclusion different from X, Y is the conclusion of an axiom

Indeed, because γ is minimal, exactly one atom of X (resp. Y) is $\psi(1)$:$\frown$, (resp. $\phi(p)$:$\smile$) the others being C:=, and therefore X:$\frown$ (resp. Y:$\smile$). If Z:$\frown$, as Z is the conclusion of an axiom, it means that $Z = \phi(i)$. Following γ towards X, we necessarily use an arc of the order to another conclusion, which may not be Y because $\gamma$is minimal, nor X, because γ is minimal and does not end with an order arc. Therefore it goes to $Z' \neq X, Y, Z$, which is the conclusion of an axiom of γ, and hence $Z' = \psi(i+1)$:$\smile$, — and the order arc from Z to Z' means $Z' >_{\mathcal{I}} Z$.

Otherwise, let $T \neq X, Y$ be a non atomic conclusion

If $T = A \otimes B$, we consider one of the two proof structures obtained by replacing this final `tensor`-link with the `before` -link $A < B$ or $A > B$, the order on conclusions remaining the same. At least one of these two proof structures contains a feasible path from Y to X, and we assume it is Π', the one with $A < B$, the other case being symmetrical. We apply the induction hypothesis to Π' — `tensor`-links are counted twice more than the `before` -links. Now, let Z:$\frown$ be a conclusion of Π.

(1) If Z is a conclusion of Π' then there exists a conclusion Z' of Π' such that $Z' > Z$ and Z':$\smile$.

(a) If $Z' \neq A < B$ then Z' is a conclusion of Π and we are done.

(b) If $Z' = A < B$, then in Π we have $A \otimes B$:$\smile$, and thus $A \otimes B > Z$ and $A \otimes B$:$\smile$.

(2) If $Z = A \otimes B$, then in Π' we have $A < B$:$\frown$ and thus Π' contains a conclusion $Z' > (A < B)$ such that Z':$\smile$. But in Π we also have $Z' > A \otimes B$ and Z':$\smile$.

If $T = A \wp B$, we consider the proof structure Π' obtained from Π by removing this final `par`-link, and taking the order $\mathcal{I}' = \mathcal{I}(A \sim B)$. The path γ from Y to X in Π induces a path with the same properties in Π', we apply the induction hypothesis to Π' — the number of order arcs of Π' is at most twice the number of order arcs of Π. Now, let Z:$\frown$ be a conclusion of Π.

(1) If $Z \neq A \wp B$. Then there exits a conclusion Z' of Π' such that $Z' >_{\mathcal{I}'} Z$ and Z':$\smile$.

(a) If $Z' \neq A, B$ then Z' is a conclusion of Π such that Z':$\smile$ and $Z' >_{\mathcal{I}} Z$.

(b) If $Z' = A$:$\smile$

(i) and B:$\asymp$ then we have $A \wp B >_{\mathcal{I}} Z$ and $A \wp B$:$\smile$.

(ii) and $B{:}\frown$, then there exists a conclusion $Z''{:}\smile$ of Π' such that $Z'' >_{\mathcal{I}'} B$. Because of the definition of $\mathcal{I}'$ we have $Z'' \neq A$, — hence Z'' is a conclusion of Π; thus $Z'' >_{\mathcal{I}} Z$ and $Z'' : \smile$.

(c) If $Z' = B{:}\smile$, symmetrical to (b).

(2) If $Z = A \wp B$, then $A{:}\frown$ (or resp. $B{:}\frown$), and there exists a conclusion Z' of Π' such that $Z'{:}\smile$, $Z' >_{\mathcal{I}'} Z$. Because of the definition of $\mathcal{I}'$, we have $Z' \neq B$ (or resp. $Z' \neq A$), — hence Z' is also a conclusion of Π; thus $Z' >_{\mathcal{I}} Z$ and $Z'{:}\smile$.

If $T = A{<}B$, we consider the proof structure Π' obtained from Π by removing this final `par`-link, and taking the order $\mathcal{I}' = \mathcal{I}(A \overset{<}{\sim} B)$. The path γ from Y to X in Π induces a path with the same properties in Π'. We apply the induction hypothesis to Π' — the number of order arcs of Π' is at most one plus twice the number of order arcs of Π. Now, let $Z{:}\frown$ be a conclusion of Π.

(1) If $Z \neq A{<}B$, then there exists a conclusion Z' of Π' such that $Z' >_{\mathcal{I}'} Z$ and $Z'{:}\smile$.

(a) If $Z' \neq A, B$ then Z' is a conclusion of Π such that $Z'{:}\smile$ and $Z' >_{\mathcal{I}} Z$.

(b) If $Z' = A{:}\smile$

(i) and $B{:}\asymp$ then we have $(A{<}B) >_{\mathcal{I}} Z$ and $A{<}B{:}\smile$.

(ii) and $B{:}\frown$, then there exists a conclusion $Z''{:}\smile$ of Π' such that $Z'' >_{\mathcal{I}'} B$. Because of the definition of $\mathcal{I}'$ we have $Z'' \neq A$, and Z'' is a conclusion of Π such that $Z'' >_{\mathcal{I}} Z$ and $Z''{:}\smile$.

(c) If $Z' = B$ then $(A{<}B){:}\smile$ and $Z <_{\mathcal{I}} (A{<}B)$.

(2) If $Z = A{<}B$, then either

(a) $A{:}\frown \wedge B{:}=$ and there exists a conclusion $Z'{:}\smile$ of Π' such that $Z' >_{\mathcal{I}'} A$ and $Z'{:}\smile$. Because of $Z'{:}\smile$ it may not be B and it is therefore a conclusion of Π, and because of the definition of $\mathcal{I}'$ we have $Z' >_{\mathcal{I}'} B$ and thus $Z' >_{\mathcal{I}} Z$.

(b) or $B{:}\frown$, and there exists a conclusion Z' of Π' such that $Z' >_{\mathcal{I}'} B$. Therefore $Z' \neq A$ — hence Z' is also a conclusion of Π — and $Z' >_{\mathcal{I}} (A{<}B)$ and $Z'{:}\smile$. ◇

Lemma 2 (for pomset logic)

Let ∇ be any `NZ`*-interpretation,*

let Π be a cut-free ordered proof structure which is not a proof net, with conclusions C_i ordered by $\mathcal{I}$,

and let $\mathcal{E}_1$ be one of its ∇-experiments.

Then there exists another experiment $\mathcal{E}_2$ such that $|\mathcal{E}_1|^\smile|\mathcal{E}_2|[\prod_{\mathcal{I}} C_i]$.

Proof: We proceed by induction on the size of the proof net previously defined.

(1) If the order is not empty, we suppress one order arc between a conclusion X and one of its successors Y — but not the ones obtained by transitivity from it — and thus obtain another ordered proof structure Π' whose order is $\mathcal{I}'$.

(a) If it is not yet an ordered proof net, we apply induction hypothesis and we are done.

(b) If it is an ordered proof net, it means Π contains a path from Y to X, neither starting nor ending with an order arc — the transitivity of the order make sure that otherwise the proof structure could not be a proof net. We apply the previous lemma. We then have another experiment $\mathcal{E}_2$ such that C':$\smile$ and C:$\frown$, and such that for any other conclusion Z:$\frown$, there exists $Z' >_{\mathcal{I}'} Z$. Therefore $|\mathcal{E}_1|^\smile|\mathcal{E}_2|[\prod_{\mathcal{I}} C_i]$.

(2) If the order is empty,

(a) and there is a final `before` -link or `par`-link, we suppress it and replace $\mathcal{I}$ with, respectively $\mathcal{I}(A\sim B)$ or $\mathcal{I}(A\lesssim B)$, which is neither proof net. The induction hypothesis trivially gives the result.

(b) and there is a final `tensor`-link, let $A\otimes B$ be one of them. One of the two proof structures Π' and Π'' obtained by replacing respectively this final `tensor`-link $A\otimes B$ with the final `before` -link $A<B$ or with the final `before` -link $A>B$ is neither a proof net. We apply the induction hypothesis to it, and this gives the result since we have $A<B$:$\smile \Rightarrow A\otimes B$:$\smile$, and $A>B$:$\smile \Rightarrow A\otimes B$:$\smile$, while $A\otimes B$:= $\Leftrightarrow A<B$:= $\Leftrightarrow A>B$:=.

(c) and all links are `axiom`-links: this case can not happen, a family of axioms with no order is a proof net.

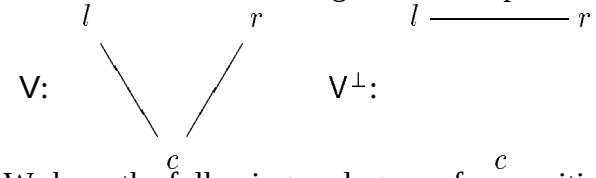
$\diamond$

As for the usual mix calculus, an ordered proof structure is an ordered proof net if an only if it is when looking at `cut`-links as `tensor`-links. Although `cut`-links are already pictured as `tensor`-links, it makes a difference for the succeeding experiments: in a `tensor`-link we do not ask for the labels of the premises to agree, and the final conclusion of the `tensor`-link corresponding to a cut, say $X\otimes X^\perp$, has a label which is part of the result of the experiment.

We thus obtain the semantical characterisation of correctness and of deadlock freeness for ordered proof structures, exactly as we did for the mix proof nets, i.e. theorems 2 and 3.

6 Variants and remarks

Let V and $\mathsf{V}^\perp$ be the following coherence space:



We have the following analogous of proposition 1:

Proposition 1 (bis) *Let ∇ be any $\mathsf{VV}^\perp$-interpretation, and let Π be a proof structure.*

Let $A_1 \overline{\ } A_1^\perp$, $A_2 \overline{\ } A_2^\perp$ ... and $A_p \overline{\ } A_p^\perp$ be a family of axioms of Π (hence all the A_i are atomic), and let ϕ and ψ be two functions from $[1,p]$ to $\{A_1, A_1^\perp, A_2 A_2^\perp, ..., A_p, A_p^\perp\}$ such that $\{\psi(p), \phi(p)\} = \{A_p, {A_p}^\perp\}$.

Then there always exist two ∇-experiments $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\phi(i)$:$\frown\overline{\ }\psi(i)$:$\smile$, and, for any axiom not in the family, B:=$\overline{\ }B^\perp$:=.

Proof: Since both V and $\mathsf{V}^\perp$ contain both a couple of strictly coherent tokens and a couple of strictly incoherent tokens, it is clear. ◇

From this we derive the same kind of results, where

> *For all NZ-interpretation ∇, for all proof structure (or net) Π, for all experiment $\mathcal{E}_1$ of Π there exists another experiment $\mathcal{E}_2$ such that ....*

is replaced with:

> *For all $\mathsf{VV}^\perp$-interpretation ∇, for all proof structure (or net) Π, there exist two experiments $\mathcal{E}_1$ and $\mathcal{E}_2$ of Π such that ....*

As we said this method does not give an efficient algorithm to test whether a proof structure is a proof net: the number of experiments to be looked at is $4^{\#ax}$ with proposition 1 and $9^{\#ax}$ with proposition 1 (bis). Nevertheless, once the feasible cycle is known, which is a quadratic algorithm, then finding two strictly incoherent experiments, using proposition 1 or proposition 1 bis, is immediate.

Appendix: proof of theorem 1 (for pomset logic)

We give here the proof of [Ret93b, Ret95] which extends Girard's argument to the ordered calculus, taking into account the slight changes quoted in the introduction. Because of remark 1 it also (re)prove Girard original theorem for usual mix proof nets. The sign $\circ$ indicates a case which never happen with the usual mix proof net.

Remark 2 ($\circ$) *If* $(a_1, ..., a_n)^\smile (a'_1, ..., a'_n) \left[\prod_{\mathcal{I}} A_i\right]$ *and* $a_k{}^\frown a'_k[A_k]$ *then* $\exists A_l >_{\mathcal{I}} A_k \;\; a_l{}^\smile a'_l[A_l]$.
(Obvious from the definition of the ordered product of coherence spaces.)

Lemma 3 (compatibility) *If two experiments differ somewhere in the proof net then they are strictly coherent.*

Proof: We shall assume that our to experiments differ somewhere in the proof net, and that they are not coherent in the ordered product of coherence spaces. Under these assumptions we shall build a feasible path, starting from the point where the two experiments differ. This path will be endlessly increased unless we exhibits a feasible circuit; however, since the proof net is finite both cases exhibit a feasible circuit, and this is a contradiction. While building the path, we shall also use a mark which is either "UP" or "DOWN" saying that the next edge to be used is above or below. This path will also follows the following principles:

- if the mark is UP the path ends on a formula $A{:}^\smile$
- if the mark is DOWN the path ends on a formula $A{:}^\frown$
- when the path goes down through a `par-` or `before` -link $A * B$ we have $A * B{:}^\frown$ and $X{:}^\frown$ where $X \in \{A, B\}$ is the premise that the path uses.
- when the path goes up through a `par-` or `before` -link we have $X{:}^\smile$ where $X \in \{A, B\}$ is the premise that the path uses, and $A * B{:}^\smile$.
- the path never ends in an unlabelled formula, i.e. the fake conclusion $X \otimes X^\perp$ of a `cut`-link.

We successively and patiently view all the possible endings and marks of our path, with the following conventions:

The expression we arrived in a link `link` with the mark UP (resp. DOWN), means that the end point of the path is $A : \smile$ (resp. $A : \frown$), that the link `link` is the one above $A : \smile$ (resp. below $A : \frown$). Therefore we can neither arrive in an axiom with the mark DOWN, nor in a conclusion with the mark UP — there always is some link above a conclusion — nor in a `cut`-link with the mark UP — because of the last principle.

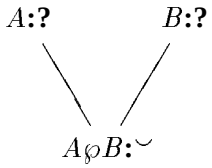
The afore mentionned principles are easily shown to be preserved while we extend the path, and we skipped that.

When a case does not say anything about the mark, it is that the extension of the path it defines does not change it.

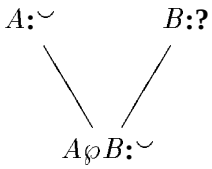
start Let X be the formula where the two experiments differs; we either have $X\mathbf{:}\frown$ or $X\mathbf{:}\smile$. In the first case we start with the mark DOWN, and in the second case with the mark UP.

the path ends in a `par`-link

with the mark UP Hence we have

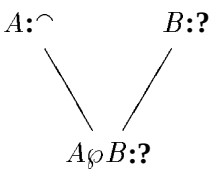


and therefore we have

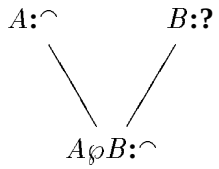


(or the symmetrical case $B\mathbf{:}\smile$). If our path already used the $A \wp B - B$ edge, because of the properties of our already built path, we used it upwards, and there is a feasible circuit. So we can extent our feasible path using the edge $A \wp B - A$, and it stills enjoys all the properties.

with the mark DOWN Assume we arrived via A (the case we arrived via B is symmetrical), hence



and therefore

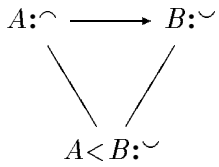


So if we used the $B - A \wp B$ edge we used it downwards, hence there is a feasible circuit: the part of the already built path starting from $A \wp B$ and leading to A together with the $A - A \wp B$ edge. Otherwise we can extent our feasible path using the edge $A \wp B - A$, and it stills enjoys all the properties.

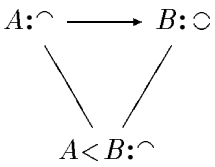
- **the path ends in a `before` -link**

  with the mark UP Similar to the case `par`-link with the mark UP.

  with the mark DOWN**, via the smaller premise** If we arrived via the A premise then either

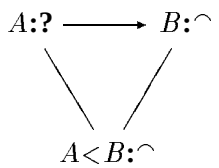


  or



  In the first case, we extend our path using the arc of the link and put the mark UP. In the second case, notice that if our path already used the $B - A < B$ edge it used it downwards; hence we have a feasible circuit using the path $B - A < B, \ldots A$ and the arc $A \to B$.

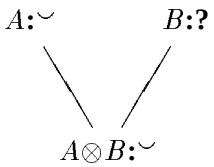
  with the mark DOWN**, via the bigger premise** If we arrived via the B premise then



If the path already used the $A - A<B$ edge it used it downwards; hence we have a feasible circuit using the $B - A<B$ edge. Otherwise we extend the path with the $B - A<B$ edge.

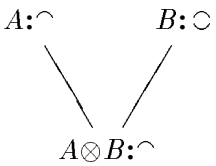
the path ends in a `tensor`-link

with the mark UP Hence we have

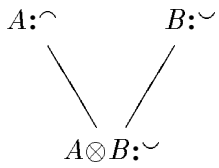


(or the symmetrical case B**:**$\smile$). Therefore we can extend our path using the $A\otimes B - A$ edge.

with the mark DOWN Assume we arrived via the premise $A : \frown$ (the case via the other premise $B : \frown$ is symmetrical). Hence we either have

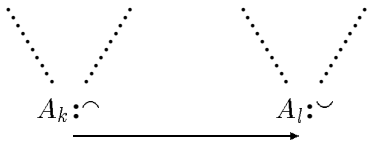


or



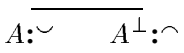
In the first case we extend the path using the $A - A\otimes B$ edge, and keep the mark DOWN and in the second we extend it using the two edges $A - A\otimes B - B$, and put the mark UP.

◦ **the path ends in a conclusion, with the mark** DOWN



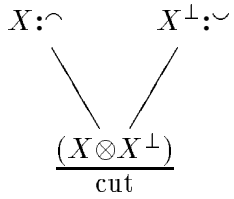
Since the path ends in a conclusion $A_k : \frown$, because of remark 2, as the two experiments are incoherent in the ordered product of coherence spaces there must be an arc leading to a conclusion A_l — and not a cut — where the two experiments are strictly incoherent. We extend the path with the corresponding arc, and put the mark UP.

the path ends in an `axiom`**-link with the mark** UP



The two experiments are strictly incoherent in this conclusion of the `axiom`-link We extend the path with the edge of the `axiom`-link where the two experiments are strictly coherent, and we put the mark DOWN.

the path ends in the premise of a `cut`**-link with the mark** DOWN



Hence the two experiments are strictly coherent. We extend the path with the two edges of the `cut`-link, ending in the other premise of the `cut`-link — because `cut`-links are pictured as `tensor`-links in this calculus we thus use two edges — and we put the mark UP. Indeed they are strictly incoherent in this other premise, because both experiments succeed.

The careful reader may wonder why we do not not need to use the arcs incident with a cut. This follows from the fact that cuts may be eliminated without changing the semantics of a proof net: otherwise the path we build could not be translated into the cut-free proof net. ◇

Lemma 4 *Any cut-free proof net has a non-trivial semantics.*

Proof: Any experiment succeeds when the proof net is cut-free; this provides numerous tuples in the semantics:

$$\prod_{A_i - A_i^{\perp} \in Ax} card(|A_i|)$$

◇

Lemma 5 *If a proof net* Π *reduces to* Π' *then they have the same semantics.*

Proof: We just need to check it holds when Π reduces to Π' using one elementary cut-elimination step. It is obvious that the succeeding experiments of Π and Π' are in a one-to-one correspondence. ◇

The two previous lemmas obviously entail the following

Theorem 1 (for pomset logic) *Let* Π *be a proof* net *with conclusions* $F_1, ..., F_n$. *Let* $\|\Pi\|$ *be the set of results of succeeding experiments of* Π *with respect to an interpretation. Then one has :*

- *If* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are two different experiments of* Π *then* $|\mathcal{E}_1| \frown |\mathcal{E}_2|$ *and therefore* $\|\Pi\|$ *is a clique of* $F_1 \wp ... \wp F_n$ *— notice that in the proof net case* $\mathcal{E}_1 \neq \mathcal{E}_2 \Rightarrow |\mathcal{E}_1| \neq |\mathcal{E}_2|$*)*
- *whenever* Π *reduces to* Π' *by cut-elimination, then* $\|\Pi\| = \|\Pi'\|$, *and since a normal proof net always possesses a non-trivial semantics, so does any proof net.*

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