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*On Transitions to Stationary States in  
a Maxwell-Landau-Lifchitz-Gilbert System*

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*Rapport  
de recherche*



## On Transitions to Stationary States in a Maxwell-Landau-Lifchitz-Gilbert System

Patrick Joly\*, Alexander Komech<sup>†</sup>, Olivier Vacus<sup>‡</sup>

Thème 4 — Simulation et optimisation  
de systèmes complexes  
Projet Ondes

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**Abstract:** In this paper we consider Maxwell's equations together with a dissipative non-linear magnetic law, the Landau-Lifchitz-Gilbert equation, and we study long time asymptotics of solutions in the 1D case in an infinite domain of propagation. We prove long-time convergence to zero of the electromagnetic field in a Fréchet topology defined by local energy seminorms: this corresponds to the local decay of energy. We then introduce that the set of stationary states for the Landau-Lifchitz-Gilbert equation and prove that it corresponds to the attractor set for the distribution of magnetization whose presence is one of the characteristics of ferromagnetic media.

**Key-words:** Maxwell's equations - Landau-Lifchitz-Gilbert law - local decay of energy - Liapunov theory - long-time asymptotics.

*(Résumé : tsvp)*

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# Comportement aux temps longs d'une couche ferromagnétique 1D

**Résumé :** Nous considérons dans ce rapport les équations de Maxwell couplées avec une loi de comportement magnétique dissipative et non linéaire, la loi de Landau-Lifchitz-Gilbert, et nous nous intéressons dans le cas monodimensionnel au comportement asymptotique des solutions pour les temps longs. Nous montrons la décroissance locale vers 0 du champ électromagnétique, résultat directement lié à la décroissance locale de l'énergie. Nous introduisons ensuite l'ensemble des états d'équilibre pour l'équation de Landau-Lifchitz-Gilbert, et nous montrons qu'il s'agit en fait d'un ensemble attracteur pour la distribution d'aimantation.

**Mots-clé :** Equations de Maxwell - loi de Landau-Lifchitz-Gilbert - décroissance locale de l'énergie - comportement asymptotique aux temps longs - théorie de Liapunov.

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# 1 Introduction

## 1.1 Definition of the problem

Ferromagnetic materials possess a spontaneous magnetization whose interaction with the magnetic field provides to this type of medium interesting absorbing properties with respect to electromagnetic waves. That is why the use of such materials as absorbing coatings for scatterers is of real importance for stealth technology. The present paper is a contribution to the mathematical theory of electromagnetic scattering by such objects. One of the main characteristics of ferromagnetic materials lies in the fact that their constitutive law, namely the relationship between the magnetic field  $H$  and the magnetization  $M$ , is non linear and non local with respect to time. This equation is the Landau-Lifschitz-Gilbert equation that can be written pointwise in the form:

$$\dot{M} = \gamma H_T \times M + \frac{\alpha}{|M|} M \times \dot{M} \quad (1.1)$$

where  $H_T$  is the total magnetic field defined as

$$H_T = H + H_s + H_a(M), \quad (1.2)$$

each of these contributions being defined as:

$$\left\{ \begin{array}{ll} H & \text{is the magnetic field;} \\ H_s = H_s(\mathbf{x}) & \text{is an exterior static field (given);} \\ H_a(M) = -K P(M) & \text{is a field of anisotropy.} \end{array} \right. \quad (1.3)$$

In (1.3),  $\mathbf{x} = (x, y, z)$  denotes the space variable,  $t$  denoting time;  $K$  denotes a positive coefficient, constant in time but that may depend on  $\mathbf{x}$ , and  $P(M)$  is the orthogonal projection in  $\mathbb{R}^3$  on the plane orthogonal to some unit vector  $\mathbf{p}$ , called the easy axis (this direction, which is linked to the cristalline structure of the material, may also depend on  $\mathbf{x}$ ):

$$P(M) = M - (p \cdot M)p. \quad (1.4)$$

Let us mention that in (1.1),  $\gamma$ , the gyromagnetic factor, is a universal constant while  $\alpha$ , the damping factor, is a phenomenologic coefficient which depends on  $\mathbf{x}$ . Therefore, for our applications, a propagation medium will be determined by  $H_s, \alpha, K$ , as function of  $\mathbf{x}$  (and by the initial distribution of magnetization  $M_0$ , see (1.7)).

Here, we are interested with the coupling of (1.1) with Maxwell's equations in some domain  $\Omega$  (typically an exterior domain, if one thinks of applications to scattering problems) with boundary  $\Gamma = \partial\Omega$ . We assume that space and time variables are scaled in such a way that the constants  $\gamma, \varepsilon_0$  (the electric permittivity) and  $\mu_0$  (the magnetic permeability) can be taken equal to 1. One then has to solve :

$$\left\{ \begin{array}{ll} \dot{E}(\mathbf{x}, t) - \text{curl } H(\mathbf{x}, t) = 0, \\ \dot{H}(\mathbf{x}, t) + \text{curl } E(\mathbf{x}, t) + \dot{M}(\mathbf{x}, t) = 0, & \mathbf{x} \in \Omega, t > 0. \\ \dot{M}(\mathbf{x}, t) = H_T(\mathbf{x}, t) \times M(\mathbf{x}, t) + \frac{\alpha}{|M(\mathbf{x}, t)|} M(\mathbf{x}, t) \times \dot{M}(\mathbf{x}, t), \end{array} \right. \quad (1.5)$$

with a perfectly conducting boundary condition on  $\Gamma$  with unit normal vector  $n$

$$E \times n |_{\Gamma} = 0, \quad (1.6)$$

and initial conditions on  $\mathbb{R}_-$ :

$$E(\mathbf{x}, t = 0) = E_0(\mathbf{x}) \quad , \quad H(\mathbf{x}, t = 0) = H_0(\mathbf{x}) \quad , \quad M(\mathbf{x}, t = 0) = M_0(\mathbf{x}) \quad (1.7)$$

From a mathematical point of view, even the existence and uniqueness result for system (1.5) appears to be a very difficult question (see [4],[5]). Another natural question that we wish to address here is the following one: is it possible to describe the asymptotic behavior of the solution of system (1.9) for large time? In the case of linear materials, the answer to this question has been known for a long time ([9], [10], [11]): the electric and magnetic fields tend locally to 0. This result is known as the local decay of energy. A more subtle question is an estimate of the rate of decay; this question is deeply related to the geometry of the obstacle ([18]).

In the case of non linear media, there are much fewer results in that direction (see [16], [17]). Our goal in this paper is to establish a result analogous to the local decay of energy in a simplified **1D model problem**. More precisely we assume that all the unknowns are functions of only one space variable  $x$  (i.e. we consider the propagation of plane waves). The curl operator is then defined by

$$\text{curl } H(x, t) = (0, -\frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x}), \quad \text{curl } E(x, t) = (0, -\frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x}). \quad (1.8)$$

We assume that the propagation medium is the half space  $x < 0$  and apply the perfectly conducting boundary condition at  $x = 0$  ( $e_x = (1, 0, 0)$ ):

$$E \times e_x = 0$$

We also assume that the support of the initial magnetization  $M_0$ , that defines the ferromagnetic layer (see 2.6), is compact:

$$\text{supp } M_0 \subset ]-a, 0]$$

Then, by a principle of reflection (or image principle), the analysis of (1.5), (1.6), (1.7) can be reduced to the analysis of the pure Cauchy problem on the whole line:

$$\left\{ \begin{array}{l} \dot{E}(x, t) - \text{curl } H(x, t) = 0, \\ \dot{H}(x, t) + \text{curl } E(x, t) + \dot{M}(x, t) = 0, \\ \dot{M}(x, t) = H_T(x, t) \times M(x, t) + \frac{\alpha}{|M(x, t)|} M(x, t) \times \dot{M}(x, t), \end{array} \quad x \in \mathbb{R}, \quad t > 0. \quad (1.9)$$

provided that the new initial data  $M_0, E_0, H_0$  are appropriate extensions of the original ones:

$$\left\{ \begin{array}{l} M_0(-x) = M_0(x) \\ E_0(-x) = \Pi_{\perp} E_0(x) \\ H_0(-x) = \Pi_{\parallel} H_0(x) \end{array} \quad (1.10)$$

where the operators  $\Pi_{\perp}$  and  $\Pi_{\parallel}$  are defined for any field  $A(A_x, A_y, A_z)$  by

$$\left| \begin{array}{l} \Pi_{\perp}(A_x, A_y, A_z) = (A_x, -A_y, -A_z) \\ \Pi_{\parallel}(A_x, A_y, A_z) = (-A_x, A_y, A_z) \end{array} \right. \quad (1.11)$$



**Remark 1.1** *Concerning the longitudinal components, equations (1.9) imply*

$$\begin{aligned} E_x(x, t) &= E_x^0(x), \\ H_x(x, t) + M_x(x, t) &= H_x^0(x) + M_x^0(x). \end{aligned} \quad (1.12)$$

*We see here that the component  $E_x$  is constant in time, while convergence results on  $M_x$  yield results on  $H_x$ .*

The outline of this article is as follows. In section 2, after having recalled some known results about weak and strong solutions of the 1D scattering problem, we state the main results of the paper, namely theorems A and A' for the local decay of energy (respectively for weak and strong solutions), and theorem B about long-time asymptotics of the magnetization  $M$ . Section 3 and 4 are respectively devoted to the proof of theorems A and A'. In section 5, we give some intermediate results about (LLG) equation seen as an ordinary differential equation; these results are preparatory for section 6, in which we prove theorem B. Finally, section 7 is devoted to some additional remarks and comments.

## 2 Statement of the main results

### 2.1 Overview of known results in the 1D case

As we said above, existence and uniqueness results for system (1.9) are very difficult in dimension 3. In the case of the one dimensional model, the problem is easier and can be handled via a fixed point Theorem. Let us summarize the main existence and uniqueness results from [7].

We introduce the phase space  $V$  for the system (1.9). Let  $L^p = [L^p(\mathbb{R})]^3$  for  $1 \leq p \leq \infty$ ,  $L^{2,\infty}$  be the Banach space  $L^2 \cap L^\infty$  with the norm

$$\|M\|_{L^{2,\infty}} = \|M\|_{L^2} + \|M\|_{L^\infty} \quad (2.1)$$

and  $H(\text{curl})$  denote the Hilbert space  $\{u \in L^2 / \text{curl } u \in L^2\}$  with the norm

$$\|u\|_{\text{curl}}^2 = \|\text{curl } u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (2.2)$$

**Definition 2.1** *Let  $V$  be the Banach space*

$$V = \{(E, H, M) \in H(\text{curl}) \times H(\text{curl}) \times L^{2,\infty}; H_x \in L^\infty(\mathbb{R})\}$$

*equipped with the norm*

$$\|(E, H, M)\|_V = \|E\|_{\text{curl}} + \|H\|_{\text{curl}} + \|M\|_{L^{2,\infty}} + \|H_x\|_{L^\infty(\mathbb{R})}. \quad (2.3)$$

**Remark 2.2** *The  $x$  component  $H_x$  of the magnetic field plays a particular role because of the particular form of the curl operator in the 1D case.*

**Definition 2.3** *A function  $Y(t) = (E(x, t), H(x, t), M(x, t)) \in C^0(0, \infty; V)$  is a global strong solution to the system (1.9) if*

$$(E, H) \in C^1(0, \infty; L^2) \cap C^0(0, \infty; H^1) \text{ and } M \in C^1(0, \infty; L^{2,\infty}) \cap C^2(0, \infty; L^2) \quad (2.4)$$

*and all equations (1.9) hold in the sense of distributions.*

One then shows the

**Theorem 2.4** *Let the following assumptions hold:*

$$\left\{ \begin{array}{l} \bullet \alpha(x), K(x) \in L^\infty(\mathbb{R}), p(x) \in L^\infty; \\ \bullet H_s(x) \in L^{2,\infty}, \\ \bullet (E^0(x), H^0(x), M^0(x)) \in V. \end{array} \right. \quad (2.5)$$

*Then the Cauchy problem (1.9) admits a unique global strong solution  $(E, H, M)$ , which moreover satisfies :*

$$|M(x, t)| = |M_0(x)| \quad \text{a.e. } x \in \mathbb{R}, \forall t \geq 0, \quad (2.6)$$

$$\frac{d}{dt} \mathcal{E}(E, H, M) + \int_{\mathbb{R}} \frac{\alpha}{|M|} |\dot{M}|^2 dx = 0, \quad (2.7)$$

where  $\mathcal{E}(E, H, M)$  denotes the electromagnetic energy defined by

$$\mathcal{E}(E, H, M) = \frac{1}{2} \int_{\mathbb{R}} \left[ |E|^2 + |H|^2 + K|P(M)|^2 + |H_s - M|^2 \right] dx. \quad (2.8)$$

**Remark 2.5** *For any strong solution, the electromagnetic energy is a function of class  $C^1$  with respect to time.*

One has to make precise the sense of the integral  $\frac{\alpha}{|M|} |\dot{M}|^2$ . In fact, from LLG equation, we have

$$\dot{M}(t) - \alpha \frac{M}{|M|} \times \dot{M}(t) = \gamma H_T(M(t)) \times M(t)$$

hence we deduce, via Pythagorus's Theorem, that

$$(1 + \alpha^2) |\dot{M}(t)|^2 = \gamma^2 |H_T(M) \times M|^2. \quad (2.9)$$

This observation leads to the following definition:

**Definition 2.6** *For strong solution  $(E, H, M)$  to system (1.9), we set*

$$\frac{\alpha}{|M|} |\dot{M}|^2 = \gamma^2 \frac{\alpha}{1 + \alpha^2} \frac{|H_T \times M|^2}{|M|}, \quad (2.10)$$

which is finite since the function  $M \mapsto \frac{|H_T \times M|^2}{|M|}$  can be continuously extended by 0 for  $M = 0$ . We have the estimate :

$$\int_{\mathbb{R}} \frac{\alpha}{|M|} |\dot{M}|^2 dx \leq \gamma^2 \int_{\mathbb{R}} \frac{\alpha}{1 + \alpha^2} |M| |H_T|^2 dx \quad (2.11)$$

which makes sense since one easily checks that for any time  $M \in (L^\infty)^3$  and  $H_T \in (L^2)^3$ .

**Proof.**

See [7] for a complete proof, We just explain below how to obtain the two estimates of Theorem 2.4. Concerning (2.6), the product of LLG equation with  $M$  shows that  $M \cdot \dot{M} = 0$ ; we deduce that

$$|M(x, t)| = |M_0(x)| \quad , \quad \text{a.e. } x \in \mathbb{R}, \forall t \geq 0.$$

For (2.7), from Maxwell's equations we get :

$$\begin{cases} \dot{E} \cdot E - \operatorname{curl} H \cdot E &= 0, \\ \dot{H} \cdot H + \operatorname{curl} E \cdot H &= -\dot{M} \cdot H. \end{cases} \quad (2.12)$$

Summing these two equalities and integrating over  $\mathbb{R}$  leads after integration by parts to the following identity :

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{R}} (|E|^2 + |H|^2) dx \right\} = - \int_{\mathbb{R}} \dot{M} \cdot H dx. \quad (2.13)$$

We now use the Landau-Lifschitz-Gilbert equation :

$$\dot{M} = \gamma H_T \times M + \frac{\alpha}{|M|} M \times \dot{M}. \quad (2.14)$$

Taking the scalar product with  $H_T$ , and using the notation  $(\cdot, \cdot, \cdot)$  for the mixed product in  $\mathbb{R}^3$ , we get

$$\dot{M} \cdot H_T = \frac{\alpha}{|M|} (M, \dot{M}, H_T).$$

Taking the scalar product of the Landau-Lifschitz-Gilbert equation by  $\dot{M}$  gives :

$$|\dot{M}|^2 = \gamma (\dot{M}, H_T, M).$$

Therefore, eliminating the mixed product :

$$\dot{M} \cdot H_T = \frac{\alpha}{\gamma |M|} |\dot{M}|^2. \quad (2.15)$$

(The meaning of the right hand side of this expression has to be understood in the sense we made precise in the definition 2.6.) Now, by definition of  $H_T$ , we have

$$\dot{M} \cdot H_T = \dot{M} \cdot H - KP(M) \cdot P(\dot{M}) + H_s \cdot \dot{M}.$$

Using the fact that  $|M(x, t)| = |M_0(x)|$ , we note that :

$$\begin{cases} H_s \cdot \dot{M} = -\frac{1}{2} \frac{\partial}{\partial t} |H_s - M|^2, \\ -KP(M) \cdot P(\dot{M}) = -\frac{1}{2} \frac{\partial}{\partial t} [K|P(M)|^2]. \end{cases} \quad (2.16)$$

Therefore, we have :

$$\dot{M} \cdot H_T = \dot{M} \cdot H - \frac{1}{2} \frac{\partial}{\partial t} [ |H_s - M|^2 + K|P(M)|^2 ] \quad (2.17)$$

that is to say, using (2.15) :

$$-\dot{M} \cdot H = -\frac{\alpha}{\gamma |M|} |\dot{M}|^2 - \frac{1}{2} \frac{\partial}{\partial t} [ |H_s - M|^2 + K|P(M)|^2 ]. \quad (2.18)$$

Plugging (2.18) into (2.13) leads to the energy identity :

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{R}} (|E|^2 + |H|^2 + |H_s - M|^2 + K|P(M)|^2) dx \right\} + \frac{1}{\gamma} \int_{\mathbb{R}} \frac{\alpha}{|M|} |\dot{M}|^2 dx = 0. \quad (2.19)$$

□

Thanks to a priori estimates (2.6) and (2.7), one is also able to obtain a theorem for weak solutions (i.e. less regular solutions) under weaker assumptions on the initial data:

**Definition 2.7** A function  $Y(t) = (E(x, t), H(x, t), M(x, t)) \in C^0(0, \infty; V)$  is a global weak solution to the system (1.9), if and only if

- (i)  $M \in C^1(0, \infty; L^{2, \infty})$
- (ii) For any  $(\varphi, \psi) \in V \times V$  where  $V$  is the space of test fields :

$$V = \left\{ \varphi \in C^1(H(\text{curl})) / \text{supp}(\varphi) \text{ is compact} \right\}$$

$$\begin{cases} \int \int (\dot{\varphi} \cdot E + \text{curl} \varphi \cdot H) dx dt = - \int E_0 \cdot \varphi(\cdot, 0) dx \\ \int \int (\dot{\psi} \cdot H - \text{curl} \psi \cdot E) dx dt = - \int H_0 \cdot \psi(\cdot, 0) dx - \int \int \dot{M} \cdot \psi dx dt, \end{cases} \quad (2.20)$$

- (iii) For almost every  $(x, t) \in \mathbb{R} \times \mathbb{R}^+$  :

$$\begin{cases} \dot{M}(x, t) = H_T(x, t) \times M(x, t) + \frac{\alpha}{|M(x, t)|} M(x, t) \times \dot{M}(x, t) \\ M(x, 0) = M_0(x) \end{cases} \quad (2.21)$$

**Theorem 2.8** Assume that

$$(E_0, H_0, M_0) \in (L^2)^3 \times (L^2)^3 \times (L^{2, \infty})^3$$

then system (1.9) admits a unique global weak solution which satisfies

$$\int_0^\infty \int_{\mathbb{R}} \frac{\alpha}{|M|} |\dot{M}|^2 dx dt < +\infty. \quad (2.22)$$

Moreover, a.e.  $x \in \mathbb{R}$ ,  $t \mapsto M(x, t)$  belongs to  $C^0(\mathbb{R})$ .

**Remark 2.9** From equation (2.22) and Fubini's theorem, we deduce that

$$\text{a.e. } x \in \mathbb{R}, \quad \frac{|\dot{M}(x, t)|^2}{|M_0(x)|} \in L^1(\mathbb{R}).$$

Therefore function  $t \mapsto M(x, t)$  is in  $H^1(\mathbb{R}_+) \subset C^0(\mathbb{R}_+)$ .

## 2.2 Main results of the paper

Our first main theorem concerns the convergence to 0 in Fréchet topology of the transverse components of the electromagnetic fields. For the sequel, we denote the transverse components  $E_{\parallel}(x, t) = (E_y(x, t), E_z(x, t))$ ,  $H_{\parallel}(x, t) = (H_y(x, t), H_z(x, t))$  and so on.

**Theorem A** Let all assumptions (2.5) hold. Let assume moreover that

$$M_0(x) = 0 \quad \text{for } |x| > a. \quad (2.23)$$

and

$$\exists \alpha_* > 0, \quad \text{a.e. } x \in [-a; a], \quad \alpha(x) \geq \alpha_*. \quad (2.24)$$

Then, for the global solution to the Cauchy problem (1.9),

(i) For almost every  $x \in \mathbb{R}$

$$\int_0^\infty (|E_\parallel(x, t)|^2 + |H_\parallel(x, t)|^2) dt < \infty. \quad (2.25)$$

(ii) For every  $R > 0$

$$\int_{|x| < R} (|E_\parallel(x, t)|^2 + |H_\parallel(x, t)|^2) dx \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (2.26)$$

Our second result is variant of Theorem A. We prove that, provided additional regularity assumptions on the initial data, we have convergence to zero of the transverse electromagnetic field, not only in the local energy norm, but also uniformly in any compact set.

**Theorem A'** *Let the assumptions of Theorem A hold (in particular  $(E_0, H_0) \in (H(\text{curl}))^2$ ). Then:*

$$\lim_{t \rightarrow +\infty} \int_{-a}^a |\dot{M}(x, t)|^2 dx = 0, \quad (2.27)$$

and, for every  $R > 0$ ,

$$\lim_{t \rightarrow +\infty} \int_{-R}^R \left( \left| \frac{\partial E_\parallel}{\partial x}(x, t) \right|^2 + \left| \frac{\partial H_\parallel}{\partial x}(x, t) \right|^2 \right) dx = 0, \quad (2.28)$$

$$\lim_{t \rightarrow +\infty} \sup_{|x| \leq R} \left\{ |E_\parallel(x, t)| + |H_\parallel(x, t)| \right\} = 0. \quad (2.29)$$

Our other results concern the asymptotic behavior of the magnetization  $M(x, t)$  (and thus of the longitudinal magnetic field). We first need to introduce a non degeneracy assumption. Let us define

$$\tilde{H}_T(x, M) = H_s + K(p \cdot M)p - (e_x \cdot M)e_x \quad (2.30)$$

where the dependence of  $\tilde{H}_T(x, M)$  with respect to  $x$  appears in the dependence of  $H_s$ ,  $K$  and  $p$ . We introduce the assumption, for any  $x$  in  $\mathbb{R}$ ,

$$(\mathcal{H}_x) : (\forall M \in \mathbb{R}^3, \exists \lambda(x, M) \in \mathbb{R}, \tilde{H}_T(M) = \lambda(x, M)e_x) \Rightarrow (\forall M \in \mathbb{R}^3, \lambda(x, M) \neq 0)$$

**Remark 2.10** *Assumption  $(\mathcal{H}_x)$  is not satisfied if and only if the two following properties are satisfied:*

(i)  $H_s$  and  $p$  are colinear to  $e_x$  ;

(ii) there exists  $M \in \mathbb{R}^3$  such that  $(H_s + (K - 1)M) \cdot e_x = 0$ .

For any  $x$ , we introduce the set

$$\mathcal{Z}(x, M_0) = \left\{ M \in \mathbb{R}^3 \text{ such that } \tilde{H}_T(x, M) \times M = 0 \text{ and } |M| = |M_0(x)| \right\}$$

This set will be identified in section 5 as the set of stationary states of some unperturbed (LLG) equation at point  $x$  that are possible limits for  $M(x, t)$  with the initial data  $M_0(x)$ . It will be proved to be a finite set containing between 2 and 6 elements (see Theorem 5.2).

Our first result for  $M$  concerns weak solutions:

**Theorem B** *Let the assumptions of Theorem A be satisfied. Assume that  $(\mathcal{H}_x)$  holds almost everywhere in  $x$ . Then*

$$M(x, t) \rightarrow \mathcal{Z}(x, M_0) \text{ as } t \rightarrow \infty \text{ for a.e. } x \in \mathbb{R}. \quad (2.31)$$

**Remark 2.11** *The set  $\mathcal{Z}(x, M_0)$  being discrete, (2.31) means that for almost every  $x$ , there exists  $M_\infty \in \mathcal{Z}(x, M_0)$  such that*

$$M(x, t) \rightarrow M_\infty \text{ as } t \rightarrow \infty$$

*In other words,*

$$\mathcal{M} = \{ M \in L^\infty([-a, a]) / \text{for a.e. } x \in [-a, a], M(x) = M_\infty \in \mathcal{Z}(x, M_0) \}$$

*is an infinite-dimensional attractor for  $M(x, t)$ .*

Our last result is a variant of Theorem B for strong solutions: it gives a result of uniform convergence for  $M(x, t)$ .

**Corollary 2.12** *Let the assumptions of Theorem A' be satisfied. Assume that  $(\mathcal{H}_x)$  holds everywhere. For every  $x$ , there exists  $M_\infty(x) \in \mathcal{Z}(x, M_0)$  such that*

$$\lim_{t \rightarrow \infty} |M(x, t) - M_\infty(x)| = 0. \quad (2.32)$$

**Remark 2.13** *We show with Theorem A' that the transverse magnetic field  $H_{\parallel}$  converges in time to 0 uniformly in space. It must be underlined that this property is not strong enough to ensure that the convergence of the magnetization  $M$  is also uniform in space. Section 7 will be devoted to a counter-example and some comments about this assertion.*

### 3 Proof of Theorem A: $L^2$ bounds of the transverse electromagnetic field for weak solutions

The two first equations of (1.9) read,

$$\begin{cases} \dot{E}_x = 0, & \dot{E}_y + \frac{\partial H_z}{\partial x} = 0, & \dot{E}_z - \frac{\partial H_y}{\partial x} = 0, \\ \dot{H}_x + \dot{M}_x = 0, & \dot{H}_y - \frac{\partial E_z}{\partial x} + \dot{M}_y = 0, & \dot{H}_z + \frac{\partial E_y}{\partial x} + \dot{M}_z = 0. \end{cases} \quad (3.1)$$

We easily deduce that :

$$\begin{cases} L_+(E_y + H_z) = - \dot{M}_z \\ L_-(E_y - H_z) = \dot{M}_z \\ L_+(E_z - H_y) = \dot{M}_y \\ L_-(E_z + H_y) = - \dot{M}_y \end{cases} \quad (3.2)$$

where we have introduced the two transport operators  $L_{\pm} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$ . Considering  $\dot{M}$  as known, we can solve explicitly (3.1) using the method of characteristics. After some algebraic manipulations, we

end up with the following formulas:

$$\left\{ \begin{array}{l} E_{\parallel}(x, t) = \frac{1}{2}\{E_{\parallel}^0(x+t) + E_{\parallel}^0(x-t)\} + \frac{J}{2}\{H_{\parallel}^0(x+t) - H_{\parallel}^0(x-t)\} \\ \quad - \frac{J}{2}\left\{\int_{\Gamma_+(x,t)} \dot{M}_{\parallel}(y, s)d\sigma - \int_{\Gamma_-(x,t)} \dot{M}_{\parallel}(y, s)d\sigma\right\} \\ H_{\parallel}(x, t) = \frac{1}{2}\{H_{\parallel}^0(x+t) + H_{\parallel}^0(x-t)\} - \frac{J}{2}\{E_{\parallel}^0(x+t) - E_{\parallel}^0(x-t)\} \\ \quad - \left\{\int_{\Gamma_+(x,t)} \dot{M}_{\parallel}(y, s)d\sigma + \int_{\Gamma_-(x,t)} \dot{M}_{\parallel}(y, s)d\sigma\right\} \end{array} \right. \quad (3.3)$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$\Gamma_{\pm}(x, t)$  is the curve  $\Gamma_{\pm}(x, t) = (x, t) - D_{\pm}$ , with  $D_{\pm} = \{(x, t)/x \pm t = 0\}$  (see figure 3.1), and  $\sigma$  the curvilinear abscissa along  $\Gamma_{\pm}(x, t)$ .

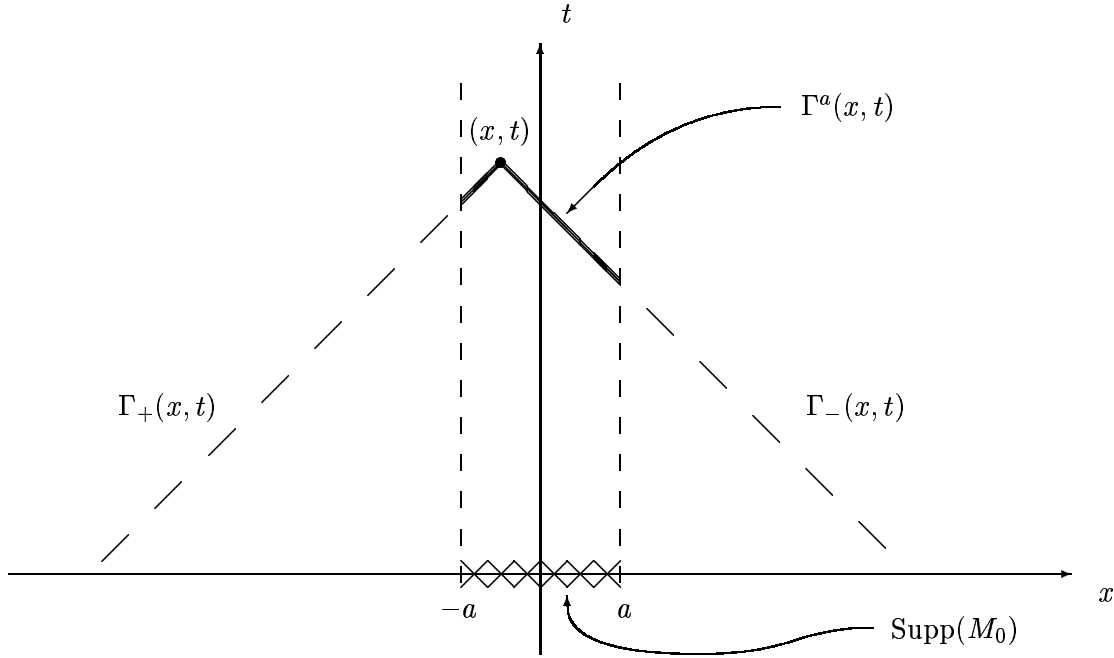


Figure 3.1: characteristics lines and the 1D layer.

Therefore, introducing:

$$F(x, t) = \begin{pmatrix} E_{\parallel}(x, t) \\ H_{\parallel}(x, t) \end{pmatrix},$$

we get (note that  $d\sigma = \sqrt{2}dy$ ,  $\|J\| = 1$  and  $|\dot{M}_{\parallel}| \leq |\dot{M}|$ ):

$$|F(x, t)| \leq \frac{\sqrt{2}}{2} \int_{\Gamma^a(x,t)} |\dot{M}(y, s)| dy + \frac{1}{2}\{|F^0(x+t)| + |F^0(x-t)|\} \quad (3.4)$$

with  $\Gamma^a(x, t) = \{(y, s) \in \Gamma_+(x, t) \cup \Gamma_-(x, t) : |y| < a\}$  (see figure 3.1). Then, the Cauchy-Schwarz inequality yields the estimate ( $\int_{\Gamma^a(x, t)} dy = 2a$ ):

$$|F(x, t)| \leq \sqrt{a} \left( \int_{\Gamma^a(x, t)} |\dot{M}(y, s)|^2 dy \right)^{\frac{1}{2}} + \frac{1}{2} \{|F^0(x+t)| + |F^0(x-t)|\} \quad (3.5)$$

Integrating this inequality between 0 and  $T$ , and noticing that:

$$K(x, T) \equiv \bigcup_{t=0}^T \Gamma^a(x, t)$$

is such that  $K(x, T) \subset [-a, a] \times [0, T]$ , we get the estimate:

$$\int_0^T |F(x, t)|^2 dt \leq 2a \int_0^T \left( \int_{|y|<a} |\dot{M}(y, s)|^2 dy \right) ds + \frac{1}{2} \sum_{\pm} \int_0^T |F^0(x \pm t)|^2 dt,$$

which yields, after integrating over  $[-a, a]$ :

$$\left| \int_0^T \int_{-a}^a |F(x, t)|^2 dx dt \leq 4a^2 \int_0^T \left( \int_{|y|<a} |\dot{M}(y, s)|^2 dy \right) ds + \frac{1}{2} \sum_{\pm} \int_0^T \int_{-a}^a |F^0(x \pm t)|^2 dt. \right.$$

Writing  $|\dot{M}|^2 = \frac{|M|}{\alpha} \cdot \frac{\alpha |\dot{M}|^2}{|M|}$  and using the conservation of the norm of  $M$  and the assumption (2.24) about the damping function  $\alpha(x)$ , we get

$$\left| \int_0^T \int_{-a}^a |F(x, t)|^2 dx dt \leq \frac{4a^2}{\alpha_*} \|M_0\|_{\infty} \int_0^T \left( \int_{|y|<a} \frac{\alpha}{|M|} |\dot{M}(y, s)|^2 dy \right) ds + \frac{1}{2} \sum_{\pm} \int_0^T \int_{-a}^a |F^0(x \pm t)|^2 dt. \right. \quad (3.6)$$

Therefore (2.5) and (2.22) imply (2.25). On the other hand, (3.4) can be rewritten:

$$\left| |F(x, t)| \leq \frac{\sqrt{2}}{2} \int_{\Gamma_a^+(x, t)} |\dot{M}(y, s)| ds + \frac{\sqrt{2}}{2} \int_{\Gamma_a^-(x, t)} |\dot{M}(y, s)| ds + \frac{1}{2} \{|F^0(x+t)| + |F^0(x-t)|\} \right.$$

where  $\Gamma_a^+ = \Gamma_+ \cap \Gamma^a$  and  $\Gamma_a^- = \Gamma_- \cap \Gamma^a$ . This yields, via the Cauchy Schwarz inequality and the fact that  $\left( \int_{\Gamma_a^{\pm}(x, t)} dt \leq a \right)$ :

$$\left| |F(x, t)| \leq \frac{\sqrt{2a}}{2} \left( \int_{\Gamma_a^+(x, t)} |\dot{M}(y, s)|^2 ds \right)^{\frac{1}{2}} + \frac{\sqrt{2a}}{2} \left( \int_{\Gamma_a^-(x, t)} |\dot{M}(y, s)|^2 ds \right)^{\frac{1}{2}} + \frac{1}{2} (|F^0(x+t)| + |F^0(x-t)|) \right.$$

Therefore

$$|F(x, t)|^2 \leq 2a \left( \int_{\Gamma_a^+(x, t)} |\dot{M}(y, s)|^2 ds + \int_{\Gamma_a^-(x, t)} |\dot{M}(y, s)|^2 ds \right) + |F^0(x+t)|^2 + |F^0(x-t)|^2 \quad (3.7)$$



After having remarked that, for  $t > R + a$ ,

$$\bigcup_{x=-R}^R \Gamma_a^\pm(x, t) \subset [-a, a] \times [t - (R + a), t]$$

we obtain:

$$\int_{-R}^R |F(x, t)|^2 dx \leq 4a \int_{t-(R+a)}^t \int_{-a}^a |\dot{M}(y, s)|^2 dy ds + \int_{-R}^R (|F^0(x+t)|^2 + |F^0(x-t)|^2) dx \quad (3.8)$$

which leads to

$$\left| \int_{-R}^R |F(x, t)|^2 dx \leq \frac{4a \|M_0\|_\infty}{\alpha_*} \int_{t-(R+a)}^t \int_{-a}^a \frac{\alpha}{|M|} |\dot{M}(y, s)|^2 dy ds \right. \\ \left. + \int_{-R-t}^{R-t} |F^0(x)|^2 dx + \int_{-R+t}^{R+t} |F^0(x)|^2 dx \right. \quad (3.9)$$

By (2.5) and (2.7),

$$\int_{-\infty}^{+\infty} |F^0(x)|^2 dx < +\infty \quad \text{and} \quad \int_0^\infty \int_{-a}^a \frac{\alpha}{|M|} |\dot{M}|^2 dy ds < +\infty.$$

Therefore equation (3.9) implies (2.26). This concludes the proof.  $\square$

## 4 Proof of Theorem A': uniform bounds of the transverse electromagnetic field for strong solutions

The proof of Theorem A' relies on a technical Lemma.

**Lemma 4.1** *The solution of the system satisfies*

$$\int_0^\infty \int_{-a}^a \frac{\alpha}{|M_0(x)|} |\ddot{M}(x, t)|^2 dx dt < \infty. \quad (4.1)$$

**Proof.**

We give a proof which supposes that the electromagnetic field  $(E, H)$  is slightly more regular in time than local strong solutions. However all the forthcoming assumptions can be justified using the method of differential quotients (see [7] for details).

Nevertheless our proof remains rather long and will be divided into 4 steps.

### Step 1: Estimates using the characteristics

In what follows, we keep the notation of the proof of Theorem A. In particular, we set:

$$F(x, t) = \begin{bmatrix} E_\parallel(x, t) \\ H_\parallel(x, t) \end{bmatrix} \in \mathbb{R}^4. \quad (4.2)$$

First, we note that, if we differentiate the system with respect to time, thanks to the linearity of Maxwell's equations, the relationship between  $\ddot{M}$  and  $\dot{E}$ ,  $\dot{H}$  is exactly the same as the one between  $\dot{M}$  and  $E$ ,  $H$ . Therefore, reproducing the computations in the proof of Theorem A leads to the estimate:

$$|\dot{F}(x, t)| \leq \sqrt{a} \left( \int_{\Gamma^a(x, t)} |\ddot{M}(y, s)| dy \right)^{\frac{1}{2}} + \frac{1}{2} \left\{ |\dot{F}^0(x+t)| + |\dot{F}^0(x-t)| \right\}. \quad (4.3)$$

Then, proceeding as in the proof of Theorem A, we easily get

$$\left| \int_0^T \int_{-a}^a |\dot{F}(x, t)|^2 dx dt \leq \frac{4a^2}{\alpha_*} \|M_0\|_\infty \int_0^T \left( \int_{|y|<a} \frac{\alpha}{|M|} |\ddot{M}(y, s)|^2 dy \right) ds \right. \\ \left. + \frac{1}{2} \sum_{\pm} \int_0^T \int_{-a}^a |\dot{F}^0(x \pm t)|^2 dt. \right. \quad (4.4)$$

## Step 2: Energy like estimates

We start from the linear Maxwell's equations after time derivation:

$$\begin{cases} \ddot{E} - \text{curl } \dot{H} = 0, \\ \ddot{H} + \text{curl } \dot{E} = -\ddot{M}. \end{cases} \quad (4.5)$$

We multiply the first equation of (4.5) by  $\dot{E}$ , the second by  $\dot{H}$  and add the two resulting equations. After integration over  $x$ , we easily get, using Green's formula and the fact that  $M$  is supported in  $[-a, a] \times \mathbb{R}^+$ :

$$\frac{d}{dt} \left( \frac{1}{2} \int (|\dot{E}|^2 + |\dot{H}|^2) dx \right) + \int_{-a}^a \dot{H} \ddot{M} dx = 0. \quad (4.6)$$

Now, we differentiate in time the LLG equation which leads to

$$\ddot{M} = \dot{H} \times M + K(p \cdot \dot{M})p \times M + H_T \times \dot{M} + \frac{\alpha}{|M|} M \times \ddot{M}, \quad (4.7)$$

using the fact that  $|M|$  is constant in time. Multiplying (4.7) successively by  $\dot{H}$  and  $\ddot{M}$  gives:

$$\begin{cases} \ddot{M} \cdot \dot{H} &= K(p \cdot \dot{M})(p, M \dot{M}) + (H_T, \dot{M}, \dot{H}) + \frac{\alpha}{|M|} (M, \ddot{M}, \dot{H}), \\ |\ddot{M}|^2 &= (\dot{H}, M, \ddot{M}) + K(p \cdot \dot{M})(p, M, \ddot{M}) + (H_T, \dot{M}, \ddot{M}). \end{cases} \quad (4.8)$$

By an adequate linear combination of these two equalities, we eliminate the mixed product  $(\dot{H}, M, \ddot{M})$ . This leads to

$$\ddot{M} \cdot \dot{H} = \frac{\alpha}{|M|} |\ddot{M}|^2 + (H_T, \dot{M}, \dot{H}) - \frac{\alpha}{|M|} (H_T, \dot{M}, \ddot{M}) \quad (4.9)$$

that we plug into (4.6) to obtain:

$$\frac{d}{dt} \left[ \frac{1}{2} \int (|\dot{H}|^2 + |\dot{E}|^2) dx \right] + \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dx = \int_{-a}^a \left\{ \frac{\alpha}{|M|} (H_T, \dot{M}, \ddot{M}) - (H_T, \dot{M}, \dot{H}) \right\} dx. \quad (4.10)$$

Let us use the bounds:

$$\begin{cases} |(H_T, \dot{M}, \ddot{M})| \leq \frac{1}{2} |\ddot{M}|^2 + \frac{1}{2} |H_T|^2 |\dot{M}|^2, \\ |(H_T, \dot{M}, \dot{H})| \leq \varepsilon |\dot{H}|^2 + \frac{1}{4\varepsilon} |H_T|^2 |\dot{M}|^2, \end{cases}$$

where  $\varepsilon$  is an arbitrary strictly positive real number. We then obtain:

$$\left| \begin{aligned} & \frac{d}{dt} \left( \frac{1}{2} \int (|\dot{H}|^2 + |\dot{E}|^2) dx \right) + \frac{1}{2} \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dx \\ & \leq \varepsilon \int_{-a}^a |\dot{H}|^2 dx + \int_{-a}^a \left( \frac{\alpha}{2|M|} + \frac{1}{4\varepsilon} \right) |H_T|^2 |\dot{M}|^2 dx, \end{aligned} \right.$$

that we can integrate between 0 and  $T$ , to obtain:

$$\left| \begin{aligned} & \frac{1}{2} \int (|\dot{H}(x, T)|^2 + |\dot{E}(x, T)|^2) dx + \frac{1}{2} \int_{-a}^a \int_0^T \frac{\alpha}{|M|} |\ddot{M}|^2 dx dt \leq \\ & \leq \frac{1}{2} \int (|\dot{H}(x, 0)|^2 + |\dot{E}(x, 0)|^2) dx + \varepsilon \int_{-a}^a \int_0^T |\dot{H}|^2 dx dt \\ & \quad + \int_{-a}^a \int_0^T \left( \frac{\alpha}{2|M|} + \frac{1}{4\varepsilon} \right) |H_T|^2 |\dot{M}|^2 dx dt. \end{aligned} \right. \quad (4.11)$$

### Step 3: Combination of the two estimates

We use (1.12) in remark 1.1 to write

$$\int_{-a}^a \int_0^T |\dot{H}|^2 dx dt \leq \int_{-a}^a \int_0^T |\dot{F}|^2 dx dt + \int_{-a}^a \int_0^T |\dot{M}_x|^2 dx dt.$$

We thus obtain:

$$\left| \begin{aligned} & \frac{1}{2} \int (|\dot{H}(x, T)|^2 + |\dot{E}(x, T)|^2) dx + \frac{1}{2} \int_{-a}^a \int_0^T \left( 1 - \frac{8\varepsilon a^2}{\alpha_*} \|M_0\|_\infty \right) \frac{\alpha}{|M|} |\ddot{M}|^2 dx dt \\ & \leq \frac{1}{2} \int (|\dot{H}(x, 0)|^2 + |\dot{E}(x, 0)|^2) dx + \frac{\varepsilon}{2} \sum_{\pm} \int_{-a}^a \int_0^T |\dot{F}^0(x \pm t)|^2 dx dt \\ & \quad + \int_{-a}^a \int_0^T \left( \frac{\alpha}{2|M|} + \frac{1}{4\varepsilon} \right) |H_T|^2 |\dot{M}|^2 dx dt + \varepsilon \int_{-a}^a \int_0^T |\dot{M}_x|^2 dx dt \end{aligned} \right.$$

We observe that

$$\int_{-a}^a \int_0^T |\dot{F}^0(x \pm t)|^2 dx dt \leq 2a \int (|\dot{H}(x, 0)|^2 + |\dot{E}(x, 0)|^2) dx$$

and we choose  $\varepsilon$  such that:

$$1 - \frac{8\varepsilon a^2}{\alpha_*} \|M_0\|_\infty = \frac{1}{2}. \quad (4.12)$$

Finally, if we set:

$$C_0 = \frac{1}{2} + 2a\varepsilon, \quad C_2 = \frac{1}{2} + \frac{\|M_0\|_\infty}{4\varepsilon\alpha_*}, \quad C_1 = \varepsilon \frac{\|M_0\|_\infty}{\alpha_*} \quad (4.13)$$

we obtain:

$$\left| \begin{aligned} & \frac{1}{2} \int (|\dot{H}(x, T)|^2 + |\dot{E}(x, T)|^2) dx + \frac{1}{4} \int_{-a}^a \int_0^T \frac{\alpha}{|M|} |\ddot{M}|^2 dx dt \\ & \leq C_0 \int (|\dot{H}(x, 0)|^2 + |\dot{E}(x, 0)|^2) dx + C_1 \int_{-a}^a \int_0^T \frac{\alpha}{|M|} |\dot{M}|^2 dx dt \\ & \quad + C_2 \int_{-a}^a \int_0^T \frac{\alpha}{|M|} |H_T|^2 |\dot{M}|^2 dx dt. \end{aligned} \right. \quad (4.14)$$

From (2.22), we already know that

$$\int_{-a}^a \int_0^T \frac{\alpha}{|M|} |\dot{M}|^2 dx dt < +\infty$$

while, because of the regularity assumptions on the initial data, one can see that

$$\int |\dot{E}(x, 0)|^2 dx = \int |\operatorname{curl} H_0|^2 dx < +\infty, \quad (4.15)$$

On the other hand

$$\int |\dot{H}(x, 0)|^2 dx \leq 2 \int |\operatorname{curl} E_0|^2 dx + 2 \int |\dot{M}(x, 0)|^2 dx, \quad (4.16)$$

and from the LLG equation one deduces that :

$$|\dot{M}(x, 0)|^2 = \frac{1}{1 + \alpha^2} |H_T(x, 0)|^2 |M(x, 0)|^2.$$

Therefore setting  $H_T^0(x) = H_T(x, 0)$ , we have:

$$\int |\dot{M}(x, 0)|^2 dx \leq \|M_0\|_{L^\infty}^2 \|H_T^0\|_{L^2}^2. \quad (4.17)$$

which is finite since, under the conditions on the data of the problem

$$H_T^0 = H_0 - KP(M_0) + H_s \in L^2, \quad (4.18)$$

We can conclude that

$$\int |\dot{H}(x, 0)|^2 dx \leq 2 \|\operatorname{curl} E_0\|_{L^2}^2 + 2 \|M_0\|_{L^\infty}^2 \|H_T^0\|_{L^2}^2. \quad (4.19)$$

Finally, if  $C$  denotes a constant which only depends on  $M_0, E_0, H_0, \alpha_*, K, H_s$  and  $a$ , we can write:

$$\left| \begin{aligned} & \frac{1}{2} \int (|\dot{H}(x, T)|^2 + |\dot{E}(x, T)|^2) dx + \frac{1}{4} \int_0^T \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dx dt \\ & \leq C + C_2 \int_0^T \int_{-a}^a \frac{\alpha}{|M|} |\dot{M}|^2 |H_T|^2 dx dt. \end{aligned} \right. \quad (4.20)$$

#### Step 4: A Gronwall type estimate

From the definition of  $H_T$ , we easily get the following bound for  $|H_T|^2$ :

$$|H_T|^2 \leq 3(1 + \|K\|_\infty^2) \left[ |H_s|^2 + |M|^2 + |H|^2 \right]$$

that implies, since  $H_x = -M_x$ ,

$$|H_T|^2 \leq 3(1 + \|K\|_\infty^2) (|H_s|^2 + 2|M|^2) + 3(1 + \|K\|_\infty^2) |H_\parallel|^2. \quad (4.21)$$

Using the interpolation inequality

$$\|H_\parallel\|_{L^\infty}^2 \leq \|H_\parallel\|_{L^2} \left\| \frac{\partial H_\parallel}{\partial x} \right\|_{L^2},$$

we can write

$$\|H_T\|_{L^\infty}^2 \leq 3(1 + \|K\|_\infty^2) (\|H_s\|_{L^\infty}^2 + 2\|M_0\|_{L^\infty}^2) + 3(1 + \|K\|_\infty^2) \|H_\parallel\|_{L^2} \left\| \frac{\partial H_\parallel}{\partial x} \right\|_{L^2}. \quad (4.22)$$

As, from Theorem A, we already know that  $\|H_\parallel\|^2$  remains bounded, from (4.20) and (4.22), we can write

$$\left| \begin{aligned} & \frac{1}{2} \int (|\dot{H}(x, T)|^2 + |\dot{E}(x, T)|^2) dx + \frac{1}{4} \int_0^T \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dx dt \\ & \leq C \left( 1 + \int_0^T \left( \int_{-a}^a \frac{\alpha}{|M|} |\dot{M}|^2 dx \right) \left\| \frac{\partial H_\parallel}{\partial x} \right\|_{L^2} dt \right) \end{aligned} \right. \quad (4.23)$$

where  $C$  denotes a new positive constant depending only on the data of the problem. Finally, using Maxwell's equations, we observe that:

$$\left\| \frac{\partial H_\parallel}{\partial x} \right\|_{L^2} = \|\dot{E}_\parallel\|_{L^2} . \quad (4.24)$$

Therefore, if we set:

$$\begin{cases} G(t) &= \int (|\dot{H}(x, t)|^2 + |\dot{E}(x, t)|^2) dx \\ m(t) &= \int_{-a}^a \frac{\alpha}{|M|} |\dot{M}|^2 dx , \end{cases} \quad (4.25)$$

we deduce from (4.23), using that  $\int_0^\infty \int_{-a}^a \frac{\alpha}{|M|} |\dot{M}|^2 dx dt < +\infty$ , the inequality ( $C$  denoting another constant)

$$G(t) \leq C \left( 1 + \int_0^t m(s) G(s)^{\frac{1}{2}} ds \right)$$

with  $m \in L^1(0, +\infty)$ . Therefore, using an appropriate generalization of Gronwall's Lemma, we get:

$$G(t) \leq \left( C^{\frac{1}{2}} + C \int_0^t m(s) ds \right)^2 . \quad (4.26)$$

As  $m \in L^1(0, +\infty)$ , this shows that  $G(t)$  remains bounded in time. On the other hand, from (4.23), we also deduce:

$$\int_0^T \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dx dt \leq C \left( 1 + \int_0^T G(s)^{\frac{1}{2}} m(s) ds \right) . \quad (4.27)$$

Setting  $G^* = \sup_{t \geq 0} G(t)$ , we thus obtain:

$$\int_0^T \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dx dt \leq C \left( 1 + (G^*)^{\frac{1}{2}} \|m\|_{L^1} \right) \quad (4.28)$$

which concludes the proof of the Lemma.  $\square$

### Proof of Theorem A'.

First note that the inequalities

$$\begin{cases} \int_{-a}^a |\dot{M}(x, t)|^2 dx \leq \frac{\|M_0\|_{L^\infty}}{\alpha_*} \int_{-a}^a \frac{\alpha}{|M|} |\dot{M}|^2 dx \\ \int_{-a}^a |\ddot{M}(x, t)|^2 dx \leq \frac{\|M_0\|_{L^\infty}}{\alpha_*} \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dx \end{cases}$$

joined to the results of Theorem A and Lemma 4.1 imply:

$$\int_0^\infty \int_{-a}^a (|\dot{M}|^2 + |\ddot{M}|^2) dx dt < +\infty ,$$

that is to say

$$\dot{M} \in H^1 \left( \mathbb{R}^+, L^2(-a, a) \right), \quad (4.29)$$

which implies

$$\lim_{t \rightarrow +\infty} \int_{-a}^a |\dot{M}(x, t)|^2 dx = 0. \quad (4.30)$$

Now, let us come back to the formula:

$$|\dot{F}(x, t)| \leq \frac{\sqrt{2}}{2} \int_{\Gamma_a(x, t)} |\dot{M}(y, s)| dy + \frac{1}{2} \left\{ |\dot{F}^0(x+t)| + |\dot{F}^0(x-t)| \right\}$$

By the same manipulations as in the proof of Theorem A (cf. the method we used to obtain (3.9)), we get:

$$\left| \int_{-R}^R |\dot{F}(x, t)|^2 dx \leq \frac{4a \|M_0\|_\infty}{\alpha_*} \int_{t-(R+a)}^t \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}(y, s)|^2 dy ds \right. \\ \left. + \int_{-R-t}^{R-t} |\dot{F}^0(x)| dx + \int_{-R+t}^{R+t} |\dot{F}^0(x)| dx \right. \quad (4.31)$$

Since  $\int_{-\infty}^{+\infty} |\dot{F}^0(x)|^2 dx < +\infty$  and  $\int_0^\infty \int_{-a}^a \frac{\alpha}{|M|} |\ddot{M}|^2 dy ds < +\infty$  (cf. Lemma 2.1), equation (4.31) implies that:

$$\lim_{R \rightarrow +\infty} \int_{-R}^R \left( |\dot{E}_\parallel|^2 + |\dot{H}_\parallel|^2 \right) dx = 0. \quad (4.32)$$

Now using Maxwell's equations we have:

$$\left| \int_{-R}^R \left| \frac{\partial E_\parallel}{\partial x} \right|^2 dx = \int_{-R}^R |\dot{H}_\parallel|^2 dx \right. \\ \left. \int_{-R}^R \left| \frac{\partial H_\parallel}{\partial x} \right|^2 dx \leq 2 \int_{-R}^R |\dot{E}_\parallel|^2 dx + 2 \int_{-a}^a |\dot{M}|^2 dx \right. \quad (4.33)$$

which, taking into account (4.30) and (4.32), lead to (2.28). Thanks to Theorem A, (2.29) is then a consequence of (2.28), since  $H^1(-R, R) \subset L^\infty(-R, R)$ . This concludes the proof of the Theorem.  $\square$

## 5 Transitions to stationary states in LLG equation

This section must be seen as an introduction to the second part of the paper, devoted to long-time asymptotics of the magnetization  $M$ .

### 5.1 Stationary states in LLG equation

We consider the solutions to the following non linear evolution equation, that we shall call the unperturbed (LLG) equation at point  $x$ :

$$\begin{cases} \dot{M}(x, t) = \tilde{H}_T(x, M(x, t)) \times M(x, t) + \frac{\alpha}{|M(x, t)|} M(x, t) \times \dot{M}(x, t) \\ M(x, t=0) = M_0(x), \end{cases} \quad (5.1)$$

where  $\tilde{H}_T(x, M(x, t))$  has been defined by (2.30).

**Remark 5.1** *One can easily check that the first equation of (5.1) can be rewritten:*

$$\dot{M}(x, t) = L(x, M(x, t)) \quad (5.2)$$

where we have defined

$$L(x, M) = \frac{1}{1 + \alpha^2} \left[ \tilde{H}_T(x, M) \times M + \frac{\alpha}{|M|} M \times (\tilde{H}_T(x, M) \times M) \right]. \quad (5.3)$$

We now introduce the set  $\mathcal{S}(x)$  of the stationary states for equation (5.1) as

$$\mathcal{S}(x) = \left\{ M_0 \in \mathbb{R}^3 / \dot{M}(x, t) = 0, \quad \forall t > 0 \right\}. \quad (5.4)$$

The main property of this set is that its intersection with any sphere  $\Sigma(R)$  is a discrete set.

**Theorem 5.2** *Under assumption  $(\mathcal{H}_x)$  (see section 2.2), the intersection of  $\mathcal{S}(x)$  with the sphere  $\Sigma(R)$  of radius  $R > 0$  is a set which contains at least 2 elements and at most 6.*

The interest of this result lies in the fact that, (see (2.9)),

$$|\dot{M}| = 0 \quad \Leftrightarrow \quad |\tilde{H}_T(x, M) \times M| = 0.$$

This shows that

$$\mathcal{Z}(x, M_0) = \mathcal{S}(x) \cap \Sigma(|M_0(x)|),$$

and we have

**Corollary 5.3** *The set  $\mathcal{Z}(x, M_0)$  defined in section 2.2 is a discrete set which contains at least 2 elements and at most 6.*

**Proof.**

We give a geometrical proof of Theorem 5.2: it simply consists in characterizing  $\mathcal{S}(x)$  in every possible case. To this aim, we start from

$$\tilde{H}_T(x, M) \times M = 0. \quad (5.5)$$

This equality leads to two different problems whether  $H_s$  is equal to 0 or not.

**Case 1:**  $H_s = 0$ .

If  $M$  belongs to  $\mathcal{S}(x)$ , we deduce from (5.5) that

$$\exists \lambda \in \mathbb{R} \quad , \quad K(p \cdot M)p - (e_x \cdot M)e_x = \lambda M. \quad (5.6)$$

which corresponds to a simple eigenvalue problem. Assumption  $(\mathcal{H}_x)$  ensures that  $p$  and  $e_x$  are not colinear and that  $K \neq 0$ . Thus the operator defined by  $M \mapsto K(p \cdot M)p - (e_x \cdot M)e_x$  is a real symmetric operator  $A$  ; in the basis  $(e_x, p, e_x \times p)$ , we have

$$A = \begin{bmatrix} -1 & -(e_x \cdot p) & 0 \\ K(e_x \cdot p) & K & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The operator  $A$  has 3 different real eigenvalues (0 of course, while the fact that the product of the two others is equal to  $K[(e_x \cdot p)^2 - 1] < 0$ , shows that they are non zero and with opposite sign). Thus there exist 3 different eigendirections  $(e_i)$  and  $\mathcal{S}(x)$  is made up of 3 lines passing through the center of the  $\Sigma(R)$ . There are exactly 6 stationary states such that  $|M_0| = R(\pm R e_i)$ .

**Case 2:**  $H_s \neq 0$ .

Let us denote  $u = e_x$ ,  $v = p$  and  $w = H_s/|H_s|$ . We shall distinguish three cases:

- Case 2,1 (general case):  $(u, v, w) \neq 0$ .

In the basis  $\{u, v, w\}$ , we denote  $M$  by  $(x, y, z)^t$ . Equality (5.5) then reads:

$$(|H_s| w + Ky v - x u) \times (x u + y v + z w) = 0,$$

which yields

$$|H_s| x w \times u + |H_s| y w \times v + K x y v \times u + K y z v \times w - x y u \times v - x z u \times w = 0.$$

As  $\{u \times v, u \times w, v \times w\}$  is also a basis of  $\mathbb{R}^3$ , we deduce that

$$\begin{cases} x (|H_s| + z) = 0 \\ y (|H_s| - Kz) = 0 \\ x y (K + 1) = 0 \end{cases} \quad (5.7)$$

This shows that  $\mathcal{S}(x)$  is made up of 2 or 3 lines:  $d_1 = \{x = 0, y = 0\}$ ,  $d_2 = \{y = 0, z = -|H_s|\}$  and, if  $K \neq 0$ ,  $d_3 = \{x = 0, z = |H_s|/K\}$ . We deduce that the intersection between  $\mathcal{S}(x)$  and a sphere of radius  $R$  has at least 2 points (because the line  $d_1$  passes through the point  $(0, 0, 0)$ ) and at most 6 points (because the intersection between a sphere and a line has at most 2 points). All the values between 2 and 6 are possible: it depends on  $R$  and on the distance of the lines  $d_2$  and  $d_3$  from the point  $(0, 0, 0)$ .

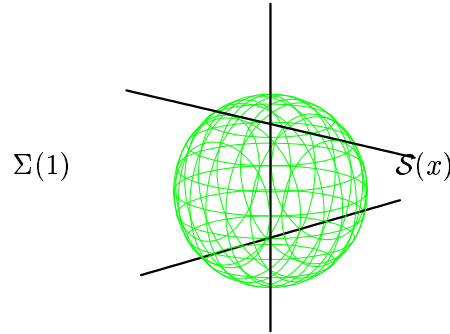


Figure 5.1: the unit sphere and the set  $\mathcal{S}(x)$  with 6 intersection points  
( $H_s = 0.5 e_z$ ,  $K = 0.7$ ,  $p = e_y$ ).

- Case 2,2 :  $(u, v, w) = 0$ ,  $(u, v, u \times v) \neq 0$ .

In the basis  $\{u, v, u \times v\}$ , we denote  $M$  by  $(x, y, z)^t$  and  $H_s$  by  $(h_u, h_v, 0)^t$ . The computations now leads to

$$\begin{cases} x (h_v + Ky) - y (h_u + x) = 0 \\ z (h_v + Ky) = 0 \\ z (h_u - x) = 0 \end{cases} \quad (5.8)$$



In this case, the set  $\mathcal{S}(x)$  is made up of a hyperbola ( $\{z = 0, (K + 1)xy + h_v x - h_u y = 0\}$ ) and, if  $K \neq 0$ , a line ( $\{x = h_u, y = -h_v/K\}$ ). As one of branches of the hyperbola passes through the point  $(0, 0, 0)$ , the conclusions are exactly the same as in the previous case.

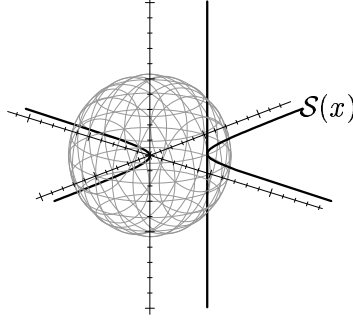


Figure 5.2: the unit sphere and the set  $\mathcal{S}(x)$  with 6 intersection points  
 $(H_s = 0.5(e_x + p), K = 0.5, p = e_y)$ .

- Case 2,3 :  $u, v$  and  $w$  are colinear.

In the canonic basis  $\{e_x, e_y, e_z\}$ , we have  $p = e_x$ ,  $H_s = |H_s|e_x$  and  $M = (x, y, z)^t$ . The computations now leads to

$$\begin{cases} y (|H_s| + Ky - x) = 0 \\ z (|H_s| + Ky - x) = 0 \end{cases} \quad (5.9)$$

As

$$(\mathcal{H}_x) \Leftrightarrow (|H_s| + Ky - x) \neq 0,$$

the set  $\mathcal{S}(x)$  reduces in this case to the line  $\{y = 0, z = 0\}$ , and there are always exactly 2 intersection points. □

## 5.2 Free transitions to stationary states

In this section we still consider the unperturbed case (i.e.  $H_{\parallel} = 0$ ). It is indeed easy to show in this case the convergence of  $M(x, t)$  to some element of  $\mathcal{Z}(x, M_0)$ . It is also possible to see which of these positions are stable and which are not (see remark 5.6.).

**Theorem 5.4** *Let  $\mathcal{Z}(x, M_0)$  be defined as section 2.2. The solution to system (5.1) satisfies*

$$\exists M_{\infty}(x) \in \mathcal{Z}(x, M_0) \quad \text{such that} \quad \lim_{t \rightarrow \infty} M(x, t) = M_{\infty}(x).$$

**Proof.**

Note that

$$\tilde{H}_T(x, M(x, t)) = H_s(x) + K(p \cdot M(x, t))p - (e_x \cdot M(x, t))e_x \quad (5.10)$$

can also be defined as:

$$\tilde{H}_T(x, M(x, t)) = -(\nabla_M V)(x, M(x, t)) \quad (5.11)$$

where we define

$$V(x, M) = -H_s(x) \cdot M(x, t) + \frac{1}{2}(e_x \cdot M(x, t))^2 + \frac{1}{2}|P(M)|^2, \quad \forall x \in \mathbb{R}, \forall M \in \mathbb{R}^3.$$

For this reason, we have

$$\tilde{H}_T(x, M(x, t)) \cdot \dot{M}(x, t) = -\frac{d}{dt}V(x, M(x, t)). \quad (5.12)$$

Besides, we deduce from equation (5.2) that

$$\tilde{H}_T(x, M(x, t)) \cdot \dot{M}(x, t) = \frac{\alpha}{1 + \alpha^2} \frac{1}{|M|} \left| \tilde{H}_T(x, M(x, t)) \times M(x, t) \right|^2. \quad (5.13)$$

Thus

$$\frac{d}{dt}V(x, M(x, t)) = g(x, M(x, t)), \quad (5.14)$$

where

$$g(x, M) = -\frac{\alpha}{1 + \alpha^2} \frac{1}{|M|} \left| \tilde{H}_T(x, M) \times M \right|^2, \quad \forall x \in \mathbb{R}, \forall M \in \mathbb{R}^3. \quad (5.15)$$

The main properties of this function  $g$  are respectively:

- (i)  $g(x, M) \leq 0$  ;
- (ii)  $(g(x, M) = 0 \text{ and } |M| = |M_0(x)|) \Leftrightarrow (M \in \mathcal{Z}(x, M_0))$  ;
- (iii)  $g(x, M) = -\frac{\alpha}{1 + \alpha^2} \frac{1}{|M|} |\nabla_M V(x, M) \times M|^2$ .

Therefore (5.14) means that the function  $M \rightarrow V(x, M)$  is a strict Liapunov function for the system (5.1). Classical results on dynamic systems then ensure the convergence to these stationary states (see for instance [6]).  $\square$

**Remark 5.5** Property (iii) of function  $g$  means in particular that the set of extremal points of  $V(x, M)$  on the sphere is included in  $\mathcal{Z}(x, M_0)$ .

Moreover, in the general case  $((e_x, p, H_s) \neq 0)$ , it can be shown that, if  $M \in \mathcal{Z}(x, M_0)$  is neither a maximum nor a minimum for  $V$ , then it is a saddlepoint for  $V$  (it can not be a local extremum).

**Remark 5.6** Although it is not the main objective of this paper, one could complete the analysis by results about the stability of the stationary points. Let us first recall the:

**Definition 5.7** A stationary state  $M_\infty$  is stable for the trajectory  $M(x, t)$  if and only if

$$\exists \mathcal{V}(M_\infty), \forall M_0 \in \mathcal{V}(M_\infty), \lim_{t \rightarrow \infty} M(x, t) = M_\infty, \quad (5.16)$$

where  $\mathcal{V}(M_\infty)$  is a neighborhood of  $M_\infty$  and  $M(x, t)$  is the solution to (5.1) associated to the initial data  $M_0$ . (Otherwise,  $M_\infty$  is said to be unstable.)

Two results can be now pointed out:

1. In the general case  $((e_x, p, H_s) \neq 0)$ , one can show that only the points  $M_\infty \in \mathcal{Z}(x, M_0)$  such that

$$V(M_\infty) = \min_{M \in S} V$$

are stable stationary states, while the others are unstable.

2. This is not necessarily true when  $(e_x, p, H_s) = 0$ .

As an illustration of these results and Theorem 5.2, we represent below two trajectories of the vector  $M(t)$  on the sphere of radius  $|M_0| = 1$ . These trajectories have been computed numerically and correspond to the following data :  $(H_s = 2e_z, K = 0)$  and  $(H_s = 0.5e_z, K = 0.7, p = e_y)$  respectively. One can check that the first case corresponds to 2 stationary states on the sphere while the second one corresponds to 6 states (they are all indicated by bold arrows). The initial position, indicated by a bold dot, is the same in both cases.

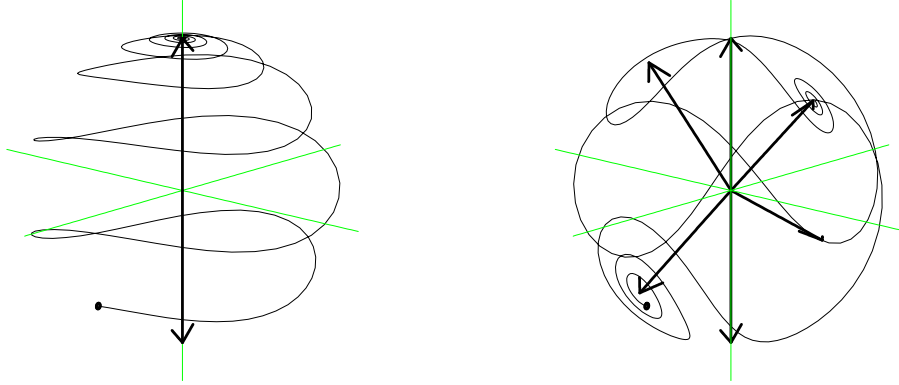


Figure 5.3: trajectories of  $M(t)$  with 2 and 6 stationary states.

## 6 Proof of Theorem B: attraction of $M$ for weak solutions

Using the notations of section 3, one can check that the (LLG) equation of (5.1) can be rewritten (see remark 5.1) in the form

$$\dot{M}(x, t) = L(x, M(x, t)) + R(x, t), \quad (6.1)$$

where

$$R(x, t) = \frac{1}{1 + \alpha^2} \left[ H_{\parallel}(x, t) \times M(x, t) + \frac{\alpha}{|M|} M(x, t) \times (H_{\parallel}(x, t) \times M(x, t)) \right]. \quad (6.2)$$

By definition of  $V$  and  $g$  (see section 5, proof of Theorem 4.3), one easily shows that

$$\frac{d}{dt} V(x, M(x, t)) = g(x, M(x, t)) + r(x, t), \quad t > 0, \quad (6.3)$$

where

$$r(x, t) = \tilde{H}_T(x, M(x, t)) \cdot R(x, t) - \frac{\alpha}{1 + \alpha^2} \frac{1}{|M|} |H_{\parallel}(x, t) \times M(x, t)|^2 \quad (6.4)$$

As a consequence of Theorems A and (2.4) (equation (2.6)), we see that  $R(x, M)$  satisfies

$$\int_0^{\infty} |R(x, M(x, t), H_{\parallel}(x, t))|^2 dt < \infty \text{ for a.e. } x \in \mathbb{R}, \quad (6.5)$$

and thus

$$\int_0^{\infty} |r(x, M(x, t), H_{\parallel}(x, t))| dt < \infty \text{ for a.e. } x \in \mathbb{R}. \quad (6.6)$$

Let us fix an  $x \in \mathbb{R}$  such that equations (6.1) to (6.6) hold. For this reason we shall write  $M(t)$ ,  $R(t)$  and  $r(t)$  instead of  $M(x, t)$ ,  $R(x, t)$  and  $r(x, t)$  respectively, and similarly  $g(M)$  and  $V(M)$  instead

of  $g(x, M)$  and  $V(x, M)$ . By theorem 2.8, we know that all these functions are, almost everywhere in  $x$ , continuous functions of time. In the sequel, we shall consider such an  $x$ . We shall also write  $\mathcal{Z}$  instead of  $\mathcal{Z}(x, M_0)$  and denote by  $S$  the sphere of radius  $|M_0(x)|$ .

The first step of the proof is the convergence of  $V(M(t))$ . Let  $V(\mathcal{Z}) = \{V_0, V_1, \dots, V_N\}$  with  $V_0 < V_1 < \dots < V_N$  (In particular, by remark 5.5,  $V_0$  is the minimum of  $V$  on  $S$  and  $V_N$  is the maximum). Let us introduce

$$V^+ = \limsup_{t \rightarrow \infty} V(M(t)) \in ]V_i, V_{i+1}] \quad \text{for some } i. \quad (6.7)$$

**Lemma 6.1** *Let  $V_{i+1}$  be defined by (6.7). Then we necessarily have :*

$$V(M(t)) \rightarrow V_{i+1} \text{ as } t \rightarrow \infty. \quad (6.8)$$

**Proof.**

We prove (6.8) by contradiction. Let us assume to the contrary that

$$V^- = \liminf_{t \rightarrow \infty} V(M(t)) < V_{i+1}.$$

Using also (6.7), we deduce that there exists  $\varepsilon > 0$  and  $V^0 \in \mathbb{R}$  such that

$$\max(V^-, V_i + \varepsilon) < V^0 < \min(V^+, V_{i+1} - \varepsilon) \quad (6.9)$$

Therefore, by continuity of  $V(M(t))$ , there exists a sequence  $t_k \rightarrow \infty$  such that

$$V(M(t_k)) = V^0. \quad (6.10)$$

If we prove that, for sufficiently large  $T$  and  $t > T$ ,

$$V(M(t)) \geq V_{i+1} - \varepsilon, \quad (6.11)$$

we will get a contradiction with (6.10) since  $t_k \rightarrow \infty$  and  $V^0 < V_{i+1} - \varepsilon$ . To prove (6.11), first note that the properties of  $g$  imply the existence of some  $\delta > 0$  such that

$$V(M) \in [V_i + \frac{\varepsilon}{2}, V_{i+1} - \frac{\varepsilon}{2}] \Rightarrow g(M) \geq \delta. \quad (6.12)$$

Let us introduce for each  $k$ , the set

$$\mathcal{E}_k = \{t > t_k : V(M(\tau)) \in [V_i + \frac{\varepsilon}{2}, V_{i+1} - \frac{\varepsilon}{2}] \text{ for } t_k < \tau < t\}, \quad (6.13)$$

It is easy to see that by construction  $\mathcal{E}_k$  is connected and that, taking into account (6.9) and (6.10) as well as the continuity of  $V(M(t))$ , it is not empty. One can also prove that this set is bounded. Indeed, let  $\tau \in \mathcal{E}_k$ , integrating (6.3) between  $t_k$  and  $\tau$ , we get

$$V(M(\tau)) - V(M(t_k)) = \int_{t_k}^{\tau} g(M(s)) ds + \int_{t_k}^{\tau} r(s) ds, \quad (6.14)$$

By definition of  $\mathcal{E}_k$ , we have

$$\forall t_k < s < \tau, \quad V(M(s)) \in [V_i + \frac{\varepsilon}{2}, V_{i+1} - \frac{\varepsilon}{2}]$$

Therefore, using (6.14) and (6.12), we get

$$V(M(\tau)) - V(M(t_k)) \geq \delta(\tau - t_k) - I_k, \quad (6.15)$$

and thus,

$$\tau_k \leq t_k + \frac{1}{\delta} \left( \max_{M \in S} V(M) - \min_{M \in S} V(M) + I_k \right) \quad (6.16)$$

where

$$I_k = \int_{t_k}^{\infty} |r(s)| ds. \quad (6.17)$$

therefore  $\mathcal{E}_k$  is bounded by

$$t_k + \frac{1}{\delta} \left( \max_{M \in S} V(M) - \min_{M \in S} V(M) + I_k \right) \quad (6.18)$$

Now, let us introduce

$$\tau_k = \sup \mathcal{E}_k < +\infty \quad (6.19)$$

By continuity,  $V(M(\tau_k))$  is equal to  $V_i + \frac{\varepsilon}{2}$  or  $V_{i+1} - \frac{\varepsilon}{2}$ . Let us show that  $V(M(\tau_k)) = V_{i+1} - \frac{\varepsilon}{2}$  at least for  $k$  large enough. Indeed, from (6.14), we deduce in particular that ( $g$  is positive and we take  $\tau = \tau_k$ ):

$$V(M(\tau_k)) - V(M(t_k)) \geq -I_k \quad (6.20)$$

$I_k$  tends to 0 when  $k$  tends to  $+\infty$  because of (6.6). Thus, there exists  $\bar{k} = \bar{k}(\varepsilon)$  such that, for  $k \geq \bar{k}$ ,

$$V(M(\tau_k)) - V(M(t_k)) \geq -\frac{\varepsilon}{2} \text{ for } t_k < \tau < \tau_k. \quad (6.21)$$

Therefore

$$V(M(\tau_k)) \geq V(M(t_k)) - \frac{\varepsilon}{2} = V_0 - \frac{\varepsilon}{2} > V_i + \frac{\varepsilon}{2}.$$

This proves that

$$k \geq \bar{k} \Rightarrow V(M(\tau_k)) = V_{i+1} - \frac{\varepsilon}{2}$$

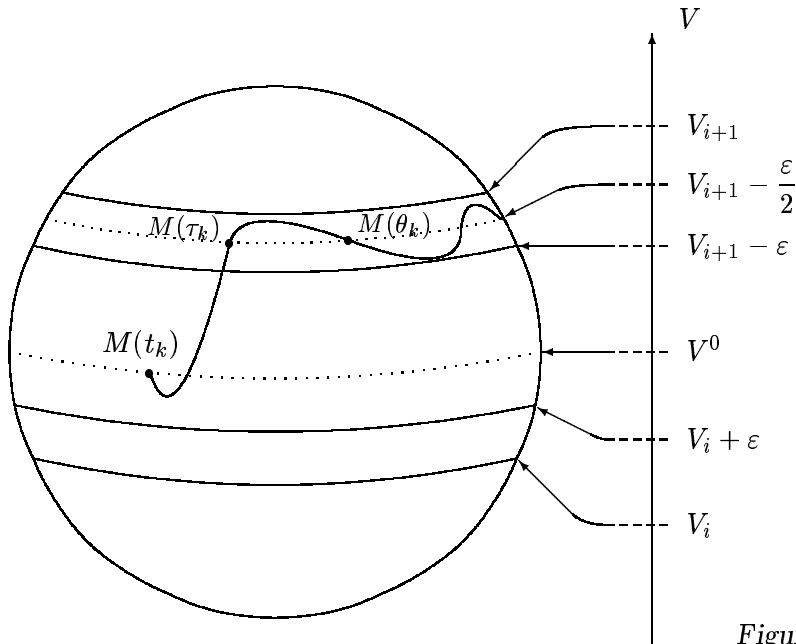


Figure 6.1

Now, we introduce the set

$$\mathcal{F}_k = \left\{ t \geq \tau_k : V(M(\tau)) \geq V_{i+1} - \frac{\varepsilon}{2} \text{ for } \tau_k \leq \tau \leq t \right\}. \quad (6.22)$$

Equations (6.9) and (6.10) ensure that  $\mathcal{F}_k$  is bounded for every  $k$ . Let us set

$$\theta_k = \max \mathcal{F}_k \quad (6.23)$$

By definition of  $\mathcal{F}_k$  and continuity of  $V(M(t))$ , we have

$$V(M(\theta_k)) = V_{i+1} - \frac{\varepsilon}{2}. \quad (6.24)$$

Integrating (6.14) between  $\theta_k$  and  $t > \theta_k$  and using once again the positivity of  $g$ , we get, for  $k \geq \bar{k}$

$$\forall t > \theta_k \quad V(M(\tau)) - V_{i+1} + \frac{\varepsilon}{2} \geq -I_k \geq -\frac{\varepsilon}{2}. \quad (6.25)$$

which proves (6.11) with  $T = \theta_{\bar{k}}$ .  $\square$

Now to prove Theorem B, let us assume by contradiction that the conclusion is not true. This means that there exists  $\rho > 0$  and a strictly increasing sequence  $t_k \rightarrow \infty$  such that (see fig. 6.2)

$$\text{dist}(M(t_k), \mathcal{Z}) \geq \rho. \quad (6.26)$$

We are going to prove that this contradicts Lemma 6.1.

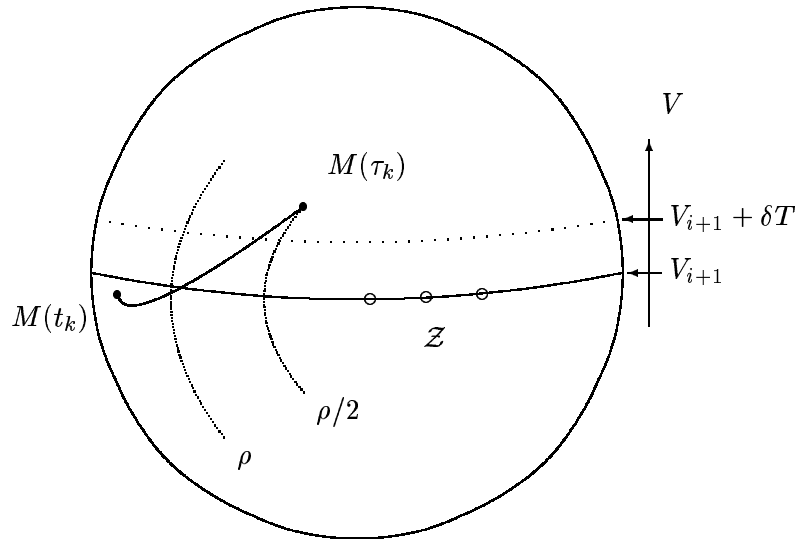


Figure 6.2

First, using the properties of  $g$ , we know that there exists  $\delta > 0$  such that

$$\text{dist}(M(t_k), \mathcal{Z}) \geq \frac{\rho}{2} \Rightarrow g(M) \geq \delta. \quad (6.27)$$

By continuity of  $M(t)$  and by (6.26), we know that the set

$$\mathcal{G}_k = \left\{ t > t_k / \tau \in ]t_k, t[ \Rightarrow \text{dist}(M(t_k), \mathcal{Z}) \geq \frac{\rho}{2} \right\}$$

is not empty. Let us check that, for  $k$  large enough,

$$]t_k, t_k + T] \subset \mathcal{G}_k \quad \text{where} \quad T = \frac{\rho}{8\omega}. \quad (6.28)$$

Let  $\tau_k = \sup \mathcal{G}_k$ . As the case  $\tau_k = +\infty$  is obvious ( $]t_k, t_k + T[ \subset ]t_k, +\infty[$ ), let us assume that  $\tau_k < +\infty$ . By continuity,  $\text{dist}(M(\tau_k), \mathcal{Z}) = \rho/2$ , and using (6.26),

$$|M(t_k) - M(\tau_k)| \geq \frac{\rho}{2}. \quad (6.29)$$

On the other hand, integrating (6.1) between  $t_k$  and  $\tau_k$  and using (6.5) and the Cauchy-Schwarz inequality, we get

$$|M(\tau_k) - M(t_k)| = \left| \int_{t_k}^{\tau_k} L(M(s))ds + \int_{t_k}^{\tau_k} R(s)ds \right| \leq \omega(\tau - t_k) + J_k(\tau_k - t_k)^{\frac{1}{2}}, \quad (6.30)$$

where

$$J_k^2 = \int_{t_k}^{\infty} |R(s)|^2 ds. \quad (6.31)$$

Regrouping (6.29) and (6.30), we get, using Young's inequality:

$$\frac{\rho}{2} \leq \omega(\tau_k - t_k) + J_k(\tau_k - t_k)^{\frac{1}{2}} \leq 2\omega(\tau_k - t_k) + \frac{J_k^2}{2\omega} \quad (6.32)$$

As  $J_k^2$  tends to 0 when  $k \rightarrow \infty$ , we have, for  $k \geq \tilde{k}$ ,  $\frac{J_k^2}{2\omega} < \frac{\rho}{4}$ . Therefore,

$$k \geq \tilde{k} \Rightarrow \frac{\rho}{4} \leq 2\omega(\tau_k - t_k) \Rightarrow (\tau_k - t_k) \geq T. \quad (6.33)$$

Now, integrating (6.3) between  $t_k$  and  $t_k + T$ , we get

$$V(M(t_k + T)) - V(M(t_k)) = \int_{t_k}^{t_k+T} g(M(s))ds + \int_{t_k}^{t_k+T} r(s)ds. \quad (6.34)$$

By (6.26) and definition of  $\mathcal{G}_k$ ,  $g(M(s)) \geq \delta$  for  $s \in [t_k, t_k + T]$ . Therefore,

$$V(M(t_k + T)) - V(M(t_k)) \geq \delta T - I_k, \quad (6.35)$$

where  $I_k = \int_{t_k}^{+\infty} |r(s)|ds$  tends to 0 when  $k \rightarrow +\infty$ . Therefore, for  $k$  large enough,

$$V(M(t_k + T)) - V(M(t_k)) \geq \frac{\delta T}{2} \quad (6.36)$$

which of course contradicts Lemma 6.1. □

## 7 About the attraction of $M$ for strong solutions

In this last section assumptions are those of Theorem A'. First note that the proof of Corollary 2.12 is obvious: indeed, the proof given in the previous section for weak solutions also applies to strong solutions and ensures the convergence of  $M(x, t)$  for every  $x \in \mathbb{R}$ .

One could think that the uniform convergence to 0 of the transverse field  $H_{\parallel}(x, t)$  would yield that the convergence of the magnetization distribution  $M(x, t)$  to  $M_{\infty}(x, t)$  is itself uniform. In fact, such a result is not obvious at all, and maybe not true. More precisely, we are going to prove, with the help of a suitable counter-example, that it is not possible to prove the uniform convergence of  $M(x, t)$  with the only assumption that the convergence of  $H_{\parallel}(x, t)$  is uniform.

To construct this counter-example, we denote by  $S$  the unit sphere, set  $M_\infty = e_z = (0, 0, 1)^t$ , and  $M_{-\infty} = -M_\infty$ . We shall need two simple lemmas which apply to the evolution equation

$$\begin{cases} \dot{M}(t) = \tilde{H}_T(M(t)) \times M(t) + \frac{\alpha}{|M(t)|} M(t) \times \dot{M}(t) \\ M(t=0) = M_0 \in S, \end{cases} \quad (7.1)$$

where  $\tilde{H}_T(M(t)) = -(M(t) \cdot e_x)e_x + 2e_z + H_{\parallel}(t)$ , which corresponds to  $H_s = 2e_z$  and  $K = 0$ . In such a case, one easily verifies that

$$\mathcal{Z} = \{M_\infty, M_{-\infty}\}.$$

**Lemma 7.1** *Assuming that  $H_{\parallel}(t) = 0$ , for any solution  $M(t)$  to system 7.1, and  $\forall M_0 \in S$ , we have*

$$(M_0 \neq M_{-\infty} V(M_0) < V_{max} = V(M_{-\infty})) \implies \left( \lim_{t \rightarrow \infty} M(t) = M_\infty \right) \quad (7.2)$$

where

$$\forall M \in S, \quad V(M) = -2M \cdot e_z + \frac{1}{2}|M \cdot e_x|^2.$$

**Remark 7.2** *This result means that  $M_{-\infty}$  can be a limit state if and only if  $M_0 = M_{-\infty}$  in which case the solution is stationary.*

**Proof.**

It is easy to check that

$$\max_{M \in S} V(M) = V_{max} = V(M_{-\infty}),$$

and

$$\min_{M \in S} V(M) = V_{min} = V(M_\infty).$$

This is true as soon as  $|H_s| > 1$  (see section 4). Moreover, for the same reason,  $V$  admits no other critical point on  $S$ . The lemma is then a direct consequence of the fact that  $V$  is a strict Liapunov function for the system (7.1).  $\square$

**Lemma 7.3** *Let  $\varepsilon \in ]0, 1]$ , and let us assume that  $H_{\parallel}(t) = \varepsilon e_y$  and  $M_0 = M_{-\infty}$ . Then,  $\forall T > 0$ , the solution to system (7.1) on  $[0, T]$  is such that*

$$V(M(T)) < V_{max} = V(M_{-\infty})$$

where  $V$  is as defined in Lemma 7.1.

**Proof.**

In this case, the total magnetic field is

$$\tilde{H}_T(M) = -(M(t) \cdot e_x)e_x + 2e_z + \varepsilon e_y.$$

The associated strict Liapunov function is

$$\forall M \in S, \quad V'(M) = \frac{1}{2}|M \cdot e_x|^2 - 2M \cdot e_z - \varepsilon M \cdot e_y, \quad (7.3)$$

which is such that

$$\max_{M \in S} V'(M) = V'_{max} = V'(M_{-\infty}) = V(M_{-\infty})$$



where  $V$  has been defined in lemma 7.1, and

$$\min_{M \in S} V'(M) = V'_{\min} = V'(M_\infty) = V(M_\infty).$$

Thus,

$$V'(M(T)) < V'_{\max} = V(M_\infty). \quad (7.4)$$

Let us define

$$\mathcal{C}' = \{M \in S \mid V'(M) = V'(M(T))\} \quad (7.5)$$

and

$$V_T = \max_{M \in \mathcal{C}'} V(M). \quad (7.6)$$

As  $V'_{\max} = V_{\max}$  is reached in  $M_\infty$  only, we see that

$$(V_T = V_{\max}) \implies (V'(M(T)) = V'_{\max}), \quad (7.7)$$

which is not true; whence the result, since

$$V'(M(T)) \leq V_T < V_{\max}. \quad (7.8)$$

□

We can now state

**Theorem 7.4** *Let  $\Omega = [0, 1]$  be a ferromagnetic layer defined by the initial distribution*

$$M_0(x) = M_{-\infty}, \quad \forall x \in [0, 1],$$

*$H_s = 2e_z$  and  $K = 0$ . Let us consider a transverse magnetic field defined by*

$$\left| \begin{aligned} H_{\parallel}(x, t) &= 0 \quad \forall x \notin [0, 1], \\ &= 0 \quad \forall t \notin \left[\frac{1}{x}, \frac{2}{x}\right], \\ &= x \quad \forall t \in \left[\frac{1}{x}, \frac{2}{x}\right]. \end{aligned} \right.$$

*Then,  $H_{\parallel}(x, t)$  converges uniformly to 0 when  $t$  goes to  $+\infty$ , however*

$$(i) \quad \lim_{t \rightarrow \infty} M(0, t) = M_{-\infty};$$

$$(ii) \quad \forall x \in ]0, 1], \quad \lim_{t \rightarrow \infty} M(x, t) = M_\infty;$$

*(iii)  $\forall \varepsilon \in [0, 2[$ ,  $\forall T > 0$ ,  $\exists x \in ]0, 1]$ ,  $\exists t > T$  such that*

$$|M(x, t) - \lim_{t \rightarrow \infty} M(x, t)| = |M(x, t) - M_\infty| = 2 > \varepsilon.$$

**Proof.**

First of all, it is easy to see that

$$\sup_{x \in ]0, 1]} H_{\parallel}(x, t) = \frac{2}{t} \quad (7.9)$$

which ensures that the convergence of  $H_{\parallel}$  to 0 in time is uniform in space.

Concerning the three assertions of the theorem, result (i) is clear since  $H_{\parallel}(0, t) = 0$  for all  $t \geq 0$ . For the two others, let us consider  $\varepsilon \in [0, 2[$  and  $T > 0$ . We choose  $x \in ]0, 1]$  such that  $1/x > T$ . Then, on the one hand,

$$\forall t \in [0, T], \quad H_{\parallel}(x, t) = 0 \quad \text{and} \quad M(x, t) = M_{-\infty}. \quad (7.10)$$

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On the other hand, by Lemma 7.3,

$$V \left( M \left( x, t = \frac{2}{x} \right) \right) < V_{\max} \quad (7.11)$$

and thus, by Lemma 7.1,

$$\lim_{t \rightarrow \infty} M(x, t) = M_{\infty} \quad (7.12)$$

□

**Remark 7.5** *The previous example is only a counter-example to the fact that the uniform convergence of  $M$  could be proven by using only the one of  $H_{\parallel}$ . It is not a counter-example to the uniform convergence of  $M$ : indeed, it is not a solution of the coupled system Maxwell-LLG.*

**Remark 7.6** *One might also believe that  $M(t)$  always converges to a stable stationary state. This is not so obvious, and maybe not true. In fact one can show that a “perturbation”  $H_{\parallel}(x, t)$  – that is to say a function  $C^1$  in time vanishing to 0 with  $t \rightarrow \infty$  – can lead the magnetization  $M$  from a stable position to an unstable one. Let us consider*

$$\begin{cases} \dot{\mu}(t) = -\mu(t) \times (H_T(\mu(t)) \times \mu(t)) \\ \mu(t=0) = \mu_0 \in S, \end{cases} \quad (7.13)$$

where  $H_T(\mu) = -(\mu \cdot e_x)e_x + 2e_z$ . The solution to this problem is such that

$$\forall \mu_0 \neq M_{\infty}, \quad \lim_{t \rightarrow \infty} \mu(t) = M_{-\infty}, \quad (7.14)$$

because

$$W(\mu) = 2e_z \cdot \mu - \frac{1}{2}|e_x \cdot \mu|^2 = -V(\mu)$$

is a strict Liapunov function for equation (7.13). Let us now define the function  $h(t)$  as

$$h(t) = \frac{1}{1 + \alpha^2} H_T(\mu(t)) \times \mu(t) - \frac{1 + \alpha + \alpha^2}{1 + \alpha^2} \mu(t) \times (H_T(\mu(t)) \times \mu(t)) \quad (7.15)$$

This function is such that

(i)  $h(t) \in C^1(\mathbb{R})$  because  $\mu(t) \in C^1(\mathbb{R})$ ;

(ii)  $h(t) \rightarrow 0$  when  $t \rightarrow \infty$  because  $\mu(t) \times H_T(\mu(t)) \rightarrow 0$  when  $t \rightarrow \infty$ ;

(iii) besides, computations lead to

$$(H_T(\mu(t)) + h(t)) \times \mu(t) + \alpha \mu(t) \times [(H_T(\mu(t)) + h(t)) \times \mu(t)] = -\mu(t) \times (H_T(\mu(t)) \times \mu(t)).$$

In other words, function  $\mu(t)$ , the solution to problem (7.13), is also the solution to problem (7.1) with the perturbation  $H_{\parallel}(t) = h(t)$ .

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