Handsome Proof-nets: R&B-Graphs, Perfect Matchings and Series-parallel Graphs

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Abstract: The main interest of this paper is to provide proof-nets, a proof syntax which identify proofs with the same meaning, with a standard graph-theoretical description. More precisely, we give two such descriptions which both view proof-nets as graphs endowed with a perfect matching, and in both cases the graphs corresponding to proofs are recognized by a simple correctness criterion (with and without the so-called mix rule). The first description may be viewed as a graph-theoretical reformulation of usual proof-nets; nevertheless it allows us to recover various results on proof-nets as the corollaries of a single graph theoretical result. The second description, inspired from the first one, is more innovative: a proof-structure simply consists in the set of its axioms — the perfect matching — plus one single series-parallel graph (a.k.a cograph) which encodes the whole syntactical forest of the sequent. Unlike other approaches, every such graph, with out any further specification, corresponds to a proof structure, and this description identify proof structures which only differ up to the commutativity or associativity of the connectives, or because final disjunctions have been performed or not. Thus these proof-nets are even closer to the proofs themselves than usual proof-nets.

Key-words: Proof theory; linear logic. Graph theory

(Résumé : tsvp)

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De beaux réseaux: graphes R&B, couplages parfaits et graphes séries-parallèles .

Résumé : L'intérêt principal de cet article est de donner une description dans la théorie des graphes usuelle des réseaux de démonstration — une syntaxe qui identifie les démonstrations de même signification. Plus précisément nous donnons deux telles descriptions, qui considèrent l'une et l'autre qu'un réseau est un graphe muni d'un couplage parfait. La première description peut se voir comme une reformulation des réseaux usuels dans la théorie des graphes; cela permet néanmoins de déduire divers résultats sur les réseaux de démonstration d'un unique résultat sur les graphes munis d'un couplage parfait. Inspirée de cette première description, la seconde est plus novatrice. Un préréseau consiste en l'ensembles de ses axiomes — le couplage parfait — plus un unique graphe série-parallèle (ou cographe) qui représente la totalité des arbres syntaxiques du séquent démontré. A la différence d'autres approches, toute telle structure, sans aucune propriété supplémentaire, décrit un préréseau. On quotiente ainsi les prérśeaux par l'associativité et la commutativité des connecteurs, et par la présence ou non des disjonctions finales. Ces réseaux sont donc encore plus proches des démonstrations elles-mêmes que les réseaux usuels.

Mots-clé : Théorie de la démonstration; logique linéaire. Théorie des graphes.

Presentation

This paper introduces two new ways of looking at proof structures and nets, and their correctness criteria. Our basic tool for describing proof-nets is edge-bicoloured graph, that we call R&B-graphs: one of the colours, B, defines a perfect matching or 1-factor of the graph, — a standard topic in graph theory: a matching B is a set of pairwise non-adjacent edges, and it is said to be perfect whenever each vertex is incident to an edge of B. An edge not in B is in R. We then consider æ-cycles — **a**lternate **e**lementary cycles — i.e. the even cycles with edges alternatively in B and in R, whose vertices are pairwise distinct.

In the first of our two approaches, the connectives are directly encoded in the R&B-graph. The criterion is the absence of æ-cycle, and the further restriction corresponding to the absence of MIX is the existence of an æ-path between each pair of distinct vertices. We prove a theorem related to one by Kotzig [12] which characterizes the R&B-graph without æ-cycles as an inductively defined class of R&B-graphs which recursively contain a B-bridge. Using our theorem, we obtain a B-bridge, and this is actually enough for establishing sequentialisation. We then consider two mappings of a proof-net into an R&B-graph without æ-cycle. This enables us to obtain from the same graph theoretical theorem the existence of a splitting *tensor* link (sequentialisation à la Girard, [10]) and the existence of a section or splitting *par* link (sequentialisation à la Danos-Regnier, [7, 8]).

The second approach, inspired by the first, is a more abstract representation of proof-structures and nets. A proof structure is still an R&B-graph, but Ris asked to be a series-parallel graph — this inductive class of graphs is rather famous, see e.g. [15]. The perfect matching B encodes the axiom links, and the single series-parallel graph R encodes the whole of the syntactical forest of the sequent. Still the criterion is still very simple: any æ-cycle should contain a chord, and the absence of MIX corresponds to the fact that each pair of distinct vertices are joined by a chordless æ-path. One of the important fact which we are proud of is that *any* SP-R&B-graph is a proof structure, i.e. we do not need to specify the degree of the vertices, or the bricks they are made of, etc. Moreover, this presentation identifies proof structures/nets which only differ because of the commutativity and associativity of the connectives, or because final *par*s have been or not performed. So we push further the research program associated with proof-net theory which is to get as close as possible from the proof itself, ignoring as much as we can the syntactical "bureaucracy", and

at the same time we provided proof structures and nets with a neat mathematical definition.

As usual in this kind of study, we ignore the cut-rules and links, viewing them as *tensor* rules or links, but we say a word about cut-elimination in the conclusion.

The combinatorial proofs are more developed than the logical ones, for which we assume some familiarity.

Organization of the paper The results are easy to state, but their proofs are rather technical, and purely combinatorial. As other people from the linear logic community may not be as interested as I am in combinatorics I postponed the combinatorial proofs to the latter sections, admitting the results in the first ones devoted to proof-nets. So the results regarding proof-nets may be read without getting deeper into the combinatorial aspects.

Section 1 gives the precise terminology that I use.

Section 2 introduces R&B-proof-nets and structures, and the correctness criterion. We translate proofs of the sequent calculus into R&B-proof nets, show that their translations fulfills the criterion, and give a direct proof of sequentialisation — relying on a combinatorial lemma to be proved in section 4. We also deduce from this result two hitherto unrelated proofs of sequentialisation, namely the original one by Girard and the one of Danos.

Section 3 defines the SP-R&B-proof structures and nets and gives the correctness criterion. We translate proofs of the sequent calculus into R&B-proof nets, show that their translations fulfills the criterion. We also prove that the absence of MIX exactly corresponds to an easy supplementary requirement. But the sequentialisation is proved later on: unlike R&B-proof-nets, no natural intermediate lemma suggests itself — this result is already a purely combinatorial one.

Section 4 gives the proof of the lemma needed for the sequentialisation of R&B-proof-nets, which will be needed for the sequentialisation of SP-R&B-proof-nets as well.

Section 5 proves that the criterion we gave for SP-R&B-proof-nets is actually equivalent to a property which seems much stronger, namely the one we need for sequentialisation of SP-R&B-proof-nets.

Section 6 contains the lemmata for proving the sequentialisation of SP-R&B-proof-nets. They concern two transformations, one being the inverse of the other.These lemmata also give information on the structural properties of these SP-R&B-graphs.

Section 7 achieves the proof of the sequentialisation for SP-R&B-proof-net.

Section 8 explains how we came up with this approach, and how an SP-R&B-graph can be gradually turned into a R&B-proof-net and vice versa. These transformations, together with the lemmata of section 6, provide an alternative proof of the sequentialisation for SP-R&B-proof-nets which is sketched.

Shorthands I used some shorthand notation. Although I introduce them in the text, let me give them right now:

R&B	regular and bold, red and blue[1]
æ	alternate elementary
SP	series-parallel

1 Graph theoretical not(at)ions

I recall the basic terminology that I use, because there are a lot of little variations that can be puzzling sometimes — I mostly follow [13].

Set theoretical notations We use $A \oplus B$ to denote the multiset union of two (multi)sets A and B, and $A \cup B$ to denote the union of two sets. We use $A \uplus B$ to denote the set union of A and B and to stress that A and B are sets, which, furthermore are *disjoint*. The cardinality of a (multi)set A is denoted by $|A|$ — in case of a multiset the number of occurrences of an element is taken into account: $|\{a, a, b, c\}| = 4$. The restriction of a relation R on V to a subset V' is denoted $R|_{V'}$.

[1] It is convenient that these two current colours have the same initial in French and in English. Besides, in a black and white paper these letters can be read as Regular and Bold. Finally, I find the similarity with Rythm'n Blues pleasing.

Graphs A **graph** $\mathsf{G} = (V; \mathrm{R})$ consists of a finite non-empty set V of elements called **vertices** written $a, b, \ldots, u, v, \ldots$ and of a multi-set R of unordered pairs of vertices called **edges**. An edge is written xy — thus $xy = yx$ — possibly with an integral index when there are multiple xy edges. We do not allow edges of the form xx. A graph is said to be **simple** whenever there are no multiple edges — i.e. in case the multi-set of edges is a set.

If a graph has only one vertex, it may not contain any edge, and we denote by ONE the class of all these trivial (simple) graphs $(\{x\}; \emptyset)$.

A bijective function f mapping the vertices of G onto the vertices of H such that both f and f^{-1} preserve the number of edges joining each pair of vertices is called an **isomorphism**, and when there exists an isomorphism from G to H, G and H are said to be isomorphic.

If there is an edge xy in a graph G, xy is said to join vertices x and y, to be be **incident** with vertices x and y, and vertices x and y are said to be **adjacent**. Two edges which share a vertex are also said two be **adjacent**. A set of edges is said to be **independent** if no two edges are adjacent. Given a set of edges R the R-neighbors are the vertices adjacent to a and are denoted by $R(a)$ — as the considered graphs contain no loop, $a \notin R(a)$.

Two vertices are said to be **twins** whenever $R(a) \setminus \{b\} = R(b) \setminus \{b\}$. Thus there are two kinds of twins, namely twins a and b for which $ab \in R$ and twins for which $ab \notin R$.

The **degree** of a vertex x is the number of edges incident to x. In case the degree of x is one, the vertex and its unique incident edge are said to be be **pendant**.

A **path** is an alternating sequence of vertices and edges, beginning and ending with vertices, two consecutive items being incident. A path is said to **join** its first and last vertices. If all vertices are distinct the path is said to be **elementary**, and if all edges are distinct the path is said to be **simple**. The **length** $\lg(\mathcal{P})$ of a path is the number of occurrences of edges in it. A **cycle** is a path of length at least two whose end vertices are equal. A cycle is said to be **elementary** if all its vertices are distinct but the first and last. If the cycle is $\mathcal{C} = x_0, \ldots, x_n, x_0$, the operations on the indices have to be understood modulo $n = \lg(\mathcal{C})$.

A **chord** of a cycle (resp. path) is an edge joining two vertices of the cycle (resp. path), but not in the cycle (resp. path).

If G is a graph and H is also a graph the vertices and edges of which are vertices and edges of G, H is said to be a **subgraph** of G. If H is a subgraph of G and if every

edge joining two vertices of H which lies in G also lies in H, we call H an **induced** subgraph of G. Given a graph $\mathsf{G} = (V; \mathrm{R})$ and $V' \subset V$ the induced subgraph of a graph $\mathsf{G} = (V; R)$ on $V' \subset V$ is denoted by $\mathsf{G}\big|_{V'}$. Given a graph H, a graph G having no induced subgraph isomorphic to H is said to be H**-free**.

A graph is **connected** if every two vertices are joined by a path. The maximal connected induced subgraphs of G are called its **components**. Given a graph $\mathsf{G} = (V; R)$ a (multi)set of edges $R' \subset R$ is called and **edge-cut-set** of G whenever $\mathsf{G}' = (V; R')$ has more components than G. If the edge cut set consists of one edge, this edge is said to be an **bridge** of G. A graph is said to be k-edge-connected whenever ones needs to remove at least k edges in order to disconnect it. The maximal k edge connected induced subgraphs of a given graph are called its k-edge-connected components. As soon as $k \geq 2$, an bridge never belong to k-edge-connected component of a graph.

Complement graph, series and parallel compositions Given a *simple* graph $\mathsf{G} = (V; \mathrm{R})$ its **complement graph** is $\mathsf{G}^c = (V; \mathrm{R}^c)$ where $xy \in \mathrm{R}^c$ iff $x \neq y$ and $xy \notin \mathrm{R}$. Given two simple graphs, $\mathsf{G} = (V; \mathrm{R})$ and $\mathsf{G}' = (V'; \mathrm{R}')$ with $V \cap V' = \emptyset$, their **parallel composition** or sum or disjoint union $\mathsf{G} + \mathsf{G}'$ is $(V \uplus V'; \mathrm{R} \uplus \mathrm{R}')$. Their **series composition** $\mathsf{G} * \mathsf{G}'$ is $(\mathsf{G}^c + \mathsf{G}'^c)^c$ which may also be defined as $(V \uplus V'; \mathrm{R} \uplus \mathrm{R}' \uplus \{xx' | x \in V\ x' \in V'\})$. The class of **series parallel graphs**, or cographs, is the smallest class of (simple) graphs containing ONE and closed under series and parallel compositions — or, if ones prefers, under complement and disjoint union. Notice that both $*$ and $+$ are commutative and associative.

Complete bipartite graphs, bipartite cuts Given two disjoint sets of vertices V and V' (which may be considered as graphs with no edges) the simple graph $V * V' = (V; \emptyset) * (V'; \emptyset)$ is called the **complete bipartite graph** with parts V and V'. By extension, given sets of vertices $A, B, C \ldots$, we sometimes use $A*B*C \ldots$ to denote the edges of $A * B * C \ldots$. Beware that this set of edges is not empty as long as two of these sets are not empty. When V is a one element set $V = \{v\}$ we omit the curly brackets: $a * V = \{ax / x \in V\}$, and $a * b = \{ab\}$. Let us call a **bipartite cut** of a graph G a complete bipartite subgraph $A * B$ of G such that a vertex of A and a vertex of B are not anymore connected in G minus the edges of $A * B$. Notice that "$A * B$ is a bipartite cut" is strictly stronger than "$A * B$ is an edge cut set".

Matchings, æ-paths, æ-cycles A set of edges in a graph G is called a **matching** if no two edges are adjacent. A matching is said to be **perfect** if every vertex is incident to an edge of the matching.

Given a graph G and a matching B, a path p is said to be **alternating** if the edges of p are alternately in B and not in B.

Given a graph and a matching, an alternating elementary path will be written an æ-**path**. An alternating elementary cycle of odd length is called an æ-**loop**. An alternating elementary cycle of even length is called an æ-**cycle**. Observe that the chord of an æ-cycle never is a B-edge since no two B-edges are adjacent.

Nicknames for simple graphs up to isomorphism are very convenient:

- C_i: the cycle with i vertices
- P_i: the path with i vertices
- K_i: the complete graph on i vertices
- $\mathsf{K}_{i,j}$: the complete bipartite graph, one part having i vertices and the other j vertices

2 R&B-proof-nets

2.1 R&B-graphs

An R&B-**graph** G is a triple $(V; B, R)$ with:

- V is an even set of vertices
- (V, B) is a simple graph
- (V, R) is a simple graph too
- B is a perfect matching of the **underlying (multi)graph** $\underline{\mathsf{G}} = (V; \mathrm{B} \oplus \mathrm{R})$

To paraphrase this formal definition, an R&B-graph G is an edge bicoloured (multi) graph, without any loop, the colour of each edge being B or R, such that the B-edges define a perfect matching (or a covering of the vertices by a set of non adjacent edges), and the R-edges be whatever they want, provided they are not parallel (i.e. not twice the R-edge ab). Throughout the paper we picture them so,

denoting B-edges by Bold (or Blue) edges and R-edges by Regular (or Red) edges. Notice that if the underlying graph $\underline{\mathsf{G}}$ is not simple, it is"almost´´ simple: it may only contain edges of multiplicity two: one occurrence being in R and the other in B.

An R&B-graph is said to be **æ-cycle free** whenever is contains no æ-cycle. It is said to be **critically æ-cycle free** whenever it is æ-cycle free and there is an æ-path between each pair of distinct vertices.

Remark.— In an R&B-graph, because R is a set, an alternate path is completely determined by its sequence of vertices — unless it is the æ-cycle of length two a, b, a, in case $ab \in \mathsf{R} \cap \mathsf{B}$, and you wish to know which edge you used first!

Remark.— In a æ-cycle free R&B-graph, the underlying graph is a simple graph, i.e. $\mathsf{B} \cap \mathsf{R} = \emptyset$.

Remark.— An æ-cycle free R&B-graph can contain alternate cycles, as may be observed in the example of a R&B-proof-net given on page 11.

A standard breadth search algorithm show that the existence of an æ-path between a pair of vertices may be checked checked in $O(v^2)$, where v is the number of vertices, and thus that the property of being (critically) æ-cycle free or not is checked in $O(v^3)$. Actually, for the particular case of R&B-proof structures, to be defined next, these questions may be answered in $O(v^2)$.

2.2 From sequent calculus to R&B-proof-nets

This definition of proof structures and nets is rather conventional: namely they are defined by links, or by subformula trees. Nevertheless it already brought in [3, 14] a very simple characterisation of the proof-nets corresponding to the non-commutative calculi of Yetter[19] and Abrusci[1, 2].[2]

Given a set of propositional variables $\mathcal{V}$, let us call **atoms** or atomic formulae the formulae of $\mathcal{N} = \mathcal{V} \uplus \mathcal{V}^{\perp}$. Because of the de Morgan laws, the grammar of multiplicative formulae over a set of propositional variables $\mathcal{V}$ may be given by:

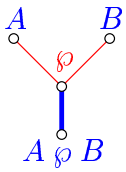
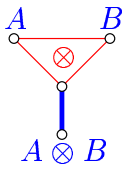
[2]This recently obtained characterisation does not refer to the embeddings in the plane, even for non cut-free proof-nets. This is pleasing, since in these non-commutative calculi, a criterion valid for non cut-free proof-nets too, and not referring to embeddings in the plane usually cause troubles unless —and this is very unsatisfying —different proof-nets correspond to the same proof of the sequent calculus.

$$\mathcal{M} ::= \mathcal{N} | \mathcal{M} \otimes \mathcal{M} | \mathcal{M} \wp \mathcal{M}$$

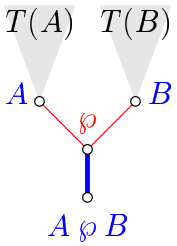
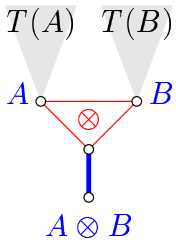
Here is the sequent calculus, where $\sigma(\Gamma)$ denotes a permutation of Γ:

MLL				MIX
$\frac{}{\vdash a, a^\perp} ax.$	$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)} exch.$	$\frac{\vdash A, B, \Gamma}{\vdash A \wp B, \Gamma} \wp$	$\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \otimes$	$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$ MIX

Let us defines the **links** as the following edge bicoloured graphs:

Links			
Name	`axiom`	*par*	*tensor*
Premises	none	A and B	A and B
R&B-graph			
Conclusions	$a^\perp$ and a	$A \wp B$	$A \otimes B$

The R&B-**tree** $T(C)$ of a formula C is defined inductively as follows:

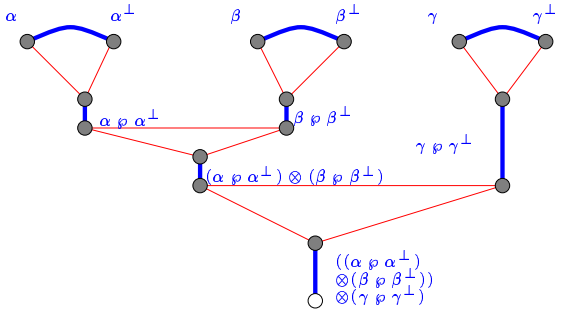
Formula C	$a \in \mathcal{P} \uplus \mathcal{P}^\perp$	$A \wp B$	$A \otimes B$
R&B-tree $T(C)$			

A R&B **proof structure** is a R&B-graph such that there exists a partition of its edges such that each part together with its incident vertices is isomorphic to an R&B-link — in such a way that the labels of the labeled vertices unify. Pendant vertices are called the **conclusions** of the R&B proof structure.

An alternative definition is to say that an R&B-proof structure consists of the R&B-trees of some formulae $C_1, \ldots, C_n$, together with a matching of B-edges joining each atom to a dual atom.

The R&B-proof-structures which are æ-cycle free are called **MIX R&B-proof-nets**, and the one which are critically æ-cycle free are called R&B-**proof-net**.

Proof of the sequent calculus are mapped into R&B-proof-structures as usual, and here is an example of an R&B-proof-structure, which is a R&B-proof-net:



Theorem 1 (R&B-sequentialisation) *Let Π be a (resp.*MIX*)* R&B*-proof-net with conclusion Γ, then there exists a proof π of $\vdash \Gamma$ in the Sequent calculus* MLL *(resp.* MLL+MIX*) which translates into Π, and which is reconstructed in quadratic time.*

The following parts of this theorem are easily established:

Proposition 2 *The proof-net Π corresponding to π always is æ-cycle free.*

Proof: Straightforward induction. Also observe that the *par* and *tensor* and MIX rules correspond to a sequence of the constructions given in section 4 which only lead to æ-cycle free R&B-graphs. □

Proposition 3 *The proof-net Π corresponding to π contains an* æ*-path between any pair of vertices if and only if π does not use* MIX*.*

Proof: First, a straightforward induction shows that whenever π uses MIX then Π contains two vertices which are not joined by any æ-path. Second, a similar straightforward induction shows that whenever π does not use MIX, there is an æ-path between any pair of vertices. □

2.3 From R&B-proof-nets to sequent calculus

Assume we have the following result, which is a part of lemma 12 to be established in section 4, page 25:

Proposition 4 *Every æ-cycle free* R&B*-graph contains a* B*-bridge.*

Then we are able to prove sequentialisation of an R&B-proof-net (in quadratic time w.r.t. the size of the conclusion) and to unify various techniques which hitherto seemed unrelated.

2.4 A direct sequentialisation

The following proof makes use of non-atomic axioms, even if the proof-net we are sequentializing does not contain any non atomic axiom. In case one wants to avoid them, the proof is easily adapted by considering proof-nets with hypotheses F_i and, correspondingly extra-logical axioms $\vdash F_i$ in the sequent calculus. Of course, they disappear when the proof of the sequent calculus is completely reconstructed.
Proof: [of Theorem 1] Because of proposition 3 it is enough to show that for every æ-cycle free proof-structure there exists a proof structure of MLL+MIX which translates into Π. We then compute the complexity of the algorithm.

Of course, if there is a final *par* link, we are done, just applying the induction hypothesis.

The key point is of course proposition 4, which asserts that there exists at least one B-bridge in Π. Actually we need a little more than this, i.e. a non-pendant B-bridge: in order to apply the induction hypothesis, we need to part Π into two *smaller* proof-structures. So we apply proposition 4, not to Π itself, but to Π^- defined as Π minus the pendant B-edges and their adjacent R-edges. As we only suppressed the B-edges which are the conclusions of *tensor* links and the pendant axioms, the B-bridge of Π^- provided by proposition 4 is a B-bridge in Π as well.

If Π^- is empty, Π consists in a family of proof-nets which are either an axiom, a *tensor* or a *par* between two axioms, and its sequentialisation is easy — remember we a priori allow MIX since its exact influence has been cleared up once and for all in proposition 3.

If Π^- is not empty, we have a non-pendant B-bridge in Π which may either be:

1. an axiom $A, A^\perp$

2. or a B-edge inside a subformula tree the bottom vertex being labeled with a formula A.

1. In the first case let us:

   (a) suppress the axiom-B-edge $A, A^\perp$

   (b) add two new vertices respectively labeled with A_0 and $A_0^\perp$ — the 0 subscript has no meaning, it is just to distinguish the new vertex labeled with A or $A^\perp$ from the original vertex labeled with A or $A^\perp$.

   (c) add two new B-edges $A, A_0^\perp$ and $A_0, A^\perp$

   Since $A, A^\perp$ is a B-bridge in Π we have split Π into two smaller æ-cycle free R&B-proof-structures: Π_1 with conclusions $\Gamma_1, A_0^\perp$ and Π_2 with conclusions Γ_2, A_0, with $\Gamma_1 \uplus \Gamma_2 = \Gamma$, the conclusions of Π.

   By induction hypothesis, we have a proof π_1 (resp. π_2) of the sequent calculus corresponding to Π_1 (resp. Π_2) with conclusions $\Gamma_1, A_0^\perp$ (resp. Γ_2, A_0). The proof π_1 contains an axiom $\vdash A, A_0^\perp$, and has the conclusion $\Gamma_1, A_0^\perp$. Obviously this A_0 can be traced from the axiom to the conclusion sequent: in a multiplicative calculus, nothing may happen to it. The proof corresponding to Π is obtained as follows: replace $A_0^\perp$ with Γ_2 in π_1: this yields a proof of $\vdash \Gamma_1, \Gamma_2$ i.e. $\vdash \Gamma$ under the hypothesis $\vdash A_0, \Gamma_1$. Now, replace the hypothesis with π_2: this yields a proof of conclusion $\vdash \Gamma$ corresponding to Π.

2. In the second case, let us:

   (a) suppress the B-edge incident to A

   (b) add two fresh vertices one labeled with A, the other with $A^\perp$

   (c) add two B-edges, one from the vertex of Π labeled with A to the fresh vertex labeled with $A^\perp$, and the other one from the B-isolated vertex of Π to the fresh vertex labeled with A.

   Thus we have split Π into two smaller æ-cycle free R&B-proof-nets, and we proceed exactly as above — but this time the rôles of the two parts are not symmetrical. Indeed we still have an axiom $A, A^\perp$ but the A conclusion of Π_2 is not the conclusion of an axiom.

Let v be the number of vertices of an R&B-proof-structure, and let $e = b + r$ be its number of edges. Clearly $v \leq 2e$ and $e \leq 3/2v$ (the degree of each vertex is less than 3). Let a be the number of atoms of the conclusion(s): one has $a \leq v$ and $v \leq 3a$. Thus $O(a) = O(v) = O(e)$.

Observe that by construction if there is an R-bridge xy, in an R&B-graph, then this R-edge may not be one of a *tensor* link (which lies in a $\mathtt{C}_3$) so it is the R-edge of a *par* link, and the B-edge above is a B-bridge.

As all bridges in a graph may be found in $O(e)$ (algorithm of Tarjan, cf. e.g. [11]) which is equal to $O(v)$ in our case, the previous procedure works in $O(e^2)$ ($= O(v^2) = O(a^2)$ in our case).

In case there is an æ-cycle, it stops — because there is no bridge — so it simultaneously checks the correctness in $O(v^2)$. □

2.5 Proving sequentialisation à la Girard and à la Danos

The proof of this subsection are simply sketched: the only point is to show how these other techniques may be viewed as corollaries of the same graph theoretical lemma.

The existence of a splitting *tensor* In [10], the key point in the sequentialisation is to show that whenever there is no final *par* link, there is a splitting *tensor* link, which in our formalism is a final *tensor* link, the two premises of which are B-bridges — but if one is a B-bridge, so is the other.

Let us prove sequentialisation this may, and extend it to MLL+MIX — which is a trivial extension, because of proposition 3.

Consider the following R&B-graph $\Pi_\otimes$:

vertices: premises of *tensor* links — for each *tensor* $A \otimes B$ two vertices A and B

B**-edges** a B-edge between any pair of premises of the same times link

R**-edges** put an R-edge between the vertex A of a *tensor* link $A \otimes B$ and the vertex A' of a *tensor* link $A' \otimes B'$ whenever there is an atom of A and an atom of A' which are linked by an axiom link.

It is easily observed that there is an æ-cycle in Π if and only if there is one in $\Pi_\otimes$. So when Π is æ-cycle free, so is $\Pi_\otimes$, and thus $\Pi_\otimes$ contains a B-bridge. But a B-bridge in $\Pi_\otimes$ is precisely a splitting *tensor*.

Such a *tensor* may obviously be taken to be, in the sequent calculus proof, a final *tensor* rule, and thus sequentialisation is achieved.

The existence of a "section" or splitting *par* In his thesis [7, Ch. 7] Danos considered the following way to sequentialize a proof-net, possibly involving MIX. Find a "section" or splitting *par*, that is to say a *par* link whose edges (isomorphic to $\mathtt{K}_{1,2}$) are a bipartite cut in the proof-net à la Danos-Regnier. In our formalism this corresponds to a par link whose two R-edges are a bipartite cut, or, equivalently whose conclusion B-edge is a B-bridge.

This way of sequential-Ising proof-nets relies on the notion of block, introduced in his thesis too [7, Ch. 7] and used in [9] as well. A block is a maximal connected part of a proof-net the links of which are not *par* links. These parts are easily sequentialized by induction on the number of links they contain.

Consider now the following R&B-graph $\Pi_\wp$:

vertices: of two kinds (A, B), $A \wp B$ for each *par* link $A \wp B$

B**-edges** a B-edge between (A, B) and $A \wp B$ whenever A and B are the premises of $A \wp B$.

R**-edges** put an R-edge between the vertex (A, B) and $A' \wp B'$ whenever (A and $A' \wp B'$ lie in the same block or A' and $A' \wp B'$ lie in the same block.

Since Π is æ-cycle free, then $\Pi_\wp$ is æ-cycle free. For instance, there may not be an R-edge parallel to a B-edge since it would mean that the conclusion and one premise are in the same block: this would clearly correspond to an æ-cycle in Π.

Notice that $\Pi_\wp$ is empty if and only if Π contains no *par* link, and that in this case Π is itself a block whose sequentialisation is obvious.

Otherwise, by proposition 4, $\Pi_\wp$ contains a B-bridge, which is a section or splitting *par* link. Once such a section is found, we proceed as we did in the direct sequentialisation of R&B-proof-nets in order to reconstruct the proof. Indeed this is particular case of the proof we gave, when the B-bridge is the conclusion of a *par* link.

3 SP-R&B-proof-nets

3.1 SP-graphs and SP-terms

Let us define, in a straightforward manner, a set T of linear terms for denoting SP-graphs. Each of these SP-**terms** $\mathtt{t}$ comes out of the following definition together with a domain $\mathrm{dom}(\mathtt{t})$

- $\forall x \quad x \in \mathsf{T}, \quad \mathrm{dom}(x) = \{x\}$
- $\forall \mathtt{t}, \mathtt{t}' \in \mathsf{T}$ such that $\mathrm{dom}(\mathtt{t}) \cap \mathrm{dom}(\mathtt{t}') = \emptyset$

$$\mathtt{t} + \mathtt{t}' \in \mathsf{T} \quad \mathrm{dom}(\mathtt{t} + \mathtt{t}') = \mathrm{dom}(\mathtt{t}) \uplus \mathrm{dom}(\mathtt{t}')$$
$$\mathtt{t} * \mathtt{t}' \in \mathsf{T} \quad \mathrm{dom}(\mathtt{t} * \mathtt{t}') = \mathrm{dom}(\mathtt{t}) \uplus \mathrm{dom}(\mathtt{t}')$$

We can now state the well-known (see e.g. [15]) properties of SP-graphs and SP-terms:

Proposition 5 *Here is a summary of the (beautiful) properties of* SP*-graphs:*

1. *A series parallel graph maybe written as a linear term over is set of vertices, and this term is unique up to the associativity and commutativity of the two laws $*$ and $+$.*
2. *A simple graph is series parallel if and only if it the induced subgraph on 4 vertices never is isomorphic to $\mathtt{P}_4$.*
3. *Each induced subgraph of a series parallel graph is itself series parallel.*
4. *Either G or G^c is disconnected.*
5. *A series parallel graph always contains at least a pair of twins.*

Proof: For 1, it is clear that there is a surjection which maps an SP-term $\mathtt{t}$ onto an SP-graphs the vertices of which are $\mathrm{dom}(\mathtt{t})$. The SP-terms up to associativity and commutativity of $*, +$ are n-ary trees whose leaves are vertices, and whose internal nodes are either $*, +$ in such a way that each branch is an alternating sequence

of $*, +$. Clearly, there is an edge between two vertices in the corresponding SP-graph if and only if and only if these two vertices meets on a $*$. This shows that whenever two SP-terms describe the same SP-graph then they are equivalent up to the associativity and commutativity of $*, +$.

For 2, it is clear by induction that there is no induced P_4 in an SP-graph. For the converse, show by induction on the number of vertices that when G is P_4-free, either G or G^c is disconnected, the result follows by induction.

Items 3 and 4 are deduced from 2, and the last item 5 is deduced from 1.

For more details, see e.g. [15]. □

3.2 Multiplicative formulae as SP terms and graphs

A multiplicative formula, of the grammar $\mathcal{M} ::= \mathcal{N} | \mathcal{M} \otimes \mathcal{M} | \mathcal{M} \parr \mathcal{M}$ is easily turned into an SP-term over its atoms by replacing every $\otimes$ with $*$, and every $\parr$ with $+$. Thus every multiplicative formula is turned into an SP-graph. A less formal way to define the SP-graph associated with a multiplicative formula is to say that there is an edge between two atoms a and b if and only if they meet on a $\otimes$. Hence two formulae which only differ up to the associativity and commutativity of the connectives are interpreted as the *same* SP-graph.

3.3 SP-R&B-graphs and chords

An SP-R&B-**graph** is an R&B-graph $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$ such that the simple graph $(V; \mathrm{R})$ is an SP-graph. It is said to be **chorded** whenever each æ-cycle contains a chord. It is said to be **critically chorded** whenever any pair of distinct vertices are joined by a chordless æ-path.

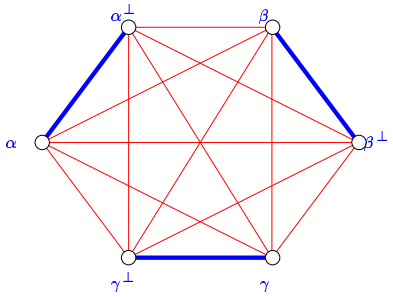
Remark.— The underlying graph of a chorded SP-R&B-graph is is a simple graph — $\mathrm{B} \cap \mathrm{R} = \emptyset$. Indeed, if $ab \in \mathrm{R} \cap \mathrm{B}$ the æ-cycle a, b, a can not contain a chord, i.e. an R-edge not in the cycle since there is at most one R-edge joining a and b.

3.4 SP-R&B-proof structures and nets

Here comes a definition which pleases me: an SP-R&B-**proof-structure** is *any* SP-R&B-graph, i.e. any graph endowed with a perfect matching such that the com-

plement of the perfect matching is a series-parallel graph. Indeed this gives these objects a simple and structured graph theoretical definition. In order to write the SP-graphs, we shall use SP-terms, but one must be aware that it is just a typographically convenient notation, the intended meaning being the SP-graph it denotes. The SP-R&B-proof structures which are chorded will are called **MIX**-SP-R&B-**proof-nets**, and the ones which are critically chorded are called SP-R&B-proof-net.

Here is the SP-R&B-proof-net corresponding to the R&B-proof-net given in section 2.



As this conception of proof structure does not make any distinction according to whether the final $\wp$ are performed or not, we ought to specify, during their inductive definition where the conclusions are. A choice of the conclusions is a partition of the connected components of R — each conclusion corresponding to an element of this partition, i.e. to the union of several connected component of R. In order to denote such a partition of the connected components of R, which are gathered by $+$ operations in the SP-term, we use brackets $\langle , \rangle$.

Therefore we will map a proof of $\vdash A_1, \ldots, A_n$ into an SP-R&B-proof-net $(V; \mathrm{B}, \mathrm{R})$ where

1. V is the set of the occurrences of atoms in $\vdash A_1, \ldots, A_n$
2. B is a set of B-edges xy with $x = y^\perp$ — axioms
3. R is the SP-graph corresponding to the SP-term $\langle \mathtt{t}_{A_1} \rangle + \cdots + \langle \mathtt{t}_{A_n} \rangle$ where $\langle$ and $\rangle$ have no meaning, and $\mathtt{t}_{A_i}$ is the SP-term associated with A_i.

- If π is an axiom $\vdash a, a^\perp$ than $\Pi = (\{a, a^\perp\}, \{a, a^\perp\}, \langle a \rangle + \langle a^\perp \rangle)$
- If π is obtained by a *par* rule applied to a proof π_1 of $\vdash A, B, C_1, \ldots, C_n$ leading to $\vdash A \wp B, C_1, \ldots, C_n$ than we had, by induction hypothesis an

SP-R&B-proof-net

$$\Pi_1 = (V; \mathrm{B}, \langle \mathsf{t}_A \rangle + \langle \mathsf{t}_B \rangle + \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle)$$

Then the SP-R&B-proof-net associated with π is

$$\Pi = (V; \mathrm{B}, \langle \mathsf{t}_A + \mathsf{t}_B \rangle + \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle)$$

as $\mathsf{t}_{A \wp B} = \mathsf{t}_A + \mathsf{t}_B$ we can write Π as

$$\Pi = (V; \mathrm{B}, \langle \mathsf{t}_{A \wp B} \rangle + \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle)$$

Notice that the SP-R&B-graphs corresponding to the sequent calculus proofs π_1 and π are the same.

- If π is obtained by a *tensor* rule between a proof π_1 of $\vdash A, C_1, \cdots, C_n$ and a proof π_2 of $\vdash B, C'_1, \cdots, C'_p$, leading to $\vdash A \otimes B, C_1, \cdots, C_n, C'_1, \cdots, C'_p$ then we had, by induction hypothesis:

$$\Pi_1 = (V_1; \mathrm{B}_1, \langle \mathsf{t}_A \rangle + \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle)$$
$$\Pi_2 = (V_2; \mathrm{B}_2, \langle \mathsf{t}_B \rangle + \langle \mathsf{t}_{C'_1} \rangle + \cdots + \langle \mathsf{t}_{C'_p} \rangle)$$

Then π is defined as:

$$\Pi = (V_1 \uplus V_2; \mathrm{B}_1 \uplus \mathrm{B}_2, \langle \mathsf{t}_A * \mathsf{t}_B \rangle + \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle + \langle \mathsf{t}_{C'_1} \rangle + \cdots + \langle \mathsf{t}_{C'_p} \rangle)$$

as $\mathsf{t}_A * \mathsf{t}_B = \mathsf{t}_{A \otimes B}$ we can write Π as:

$$\Pi = (V_1 \uplus V_2; \mathrm{B}_1 \uplus \mathrm{B}_2, \langle \mathsf{t}_{A \otimes B} \rangle + \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle + \langle \mathsf{t}_{C'_1} \rangle + \cdots + \langle \mathsf{t}_{C'_p} \rangle)$$

- If π is obtained by a MIX rule between a proof π_1 of $\vdash C_1, \cdots, C_n$ and a proof π_2 of $\vdash C'_1, \cdots, C'_n$, leading to $\vdash C_1, \cdots, C_n, C'_1, \cdots, C'_n$ then we had, by induction hypothesis:

$$\Pi_1 = (V_1; \mathrm{B}_1, \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle)$$
$$\Pi_2 = (V_2; \mathrm{B}_2, \langle \mathsf{t}_{C'_1} \rangle + \cdots + \langle \mathsf{t}_{C'_p} \rangle)$$

  then

$$\Pi = (V_1 \uplus V_2; \mathrm{B}_1 \uplus \mathrm{B}_2, \langle \mathsf{t}_{C_1} \rangle + \cdots + \langle \mathsf{t}_{C_n} \rangle + \langle \mathsf{t}_{C'_1} \rangle + \cdots + \langle \mathsf{t}_{C'_p} \rangle)$$

As usual, for instance in semantics, once the proof-net is build, let us immediately forget the brackets $\langle , \rangle$, and the term notation for the SP-graphs: this way, more proofs gets identified, which is very much in the spirit of proof-nets.

3.5 Criterion, sequentialisation

The correctness criterion for these SP-R&B-proof-nets is extremely simple:

Theorem 6 *An* SP-R&B*-graph is a* MIX*-proof-net, i.e. corresponds to a proof of* MLL+MIX *if and only if it is chorded, i.e. if every* æ*-cycle contains a chord*

An SP-R&B*-graph is a proof-net, i.e. corresponds to a proof of* MLL *if and only if it is critically chorded, i.e. if every* æ*-cycle contains a chord and if every two distinct vertices are joined by a chordless* æ*-path.*

In this section we only prove the two following propositions:

Proposition 7 *If* Π *is an* SP-R&B*-graph corresponding to a proof* π *of the sequent calculus then each* æ*-cycle contains a chord, i.e.* Π *is chorded.*

Proposition 8 *The* SP-R&B*-proof-net* Π *associated with a proof* π *of* MLL+MIX *is critically chorded, i.e. contains a chordless* æ*-path between any pair of vertices, if and only if* π *does not use* MIX*.*

The rest of the paper is devoted to the remaining point to prove, because it requires a heavier combinatorial apparatus. It will be established in theorem 27, but we can already state it:

Proposition 9 *Every chorded* SP-R&B*-graph is the translation of a proof of* MLL+MIX. *Moreover checking the criterion and reconstructing the proof may be done in quadratic time w.r.t. the number of vertices or atoms.*

Observe that the propositions 7 and 8 only concerns the SP-R&B-graph, which is not modified by the *par* rule.

Proof: [of Proposition 7] We proceed by induction on the proof of the sequent calculus, and only the *tensor* rule needs to be investigated. The *tensor* rule links two independent chorded SP-R&B-graph by a complete bipartite graph which is a bipartite cut. As to be chorded is preserved under restriction the only thing to check is that an æ-cycle which contains vertices in the two SP-R&B-graph contains a chord. As the complete bipartite graph is a bipartite cut between the two chorded SP-R&B-graph, the cycle must contain two R-edges, one in each direction, of the complete bipartite R-graph, say x_ix_{i+1} and x_jx_{j+1} with $x_i, x_{j+1} \in V_1$ and $x_jx_{i+1} \in V_2$. As the æ-cycle is elementary, these four vertices are all distinct. As $V_1 * V_2 \subset \mathrm{R}$, we have $x_ix_{j+1} \in \mathrm{R}$, which is chord of the æ-cycle. □

Proof: [of Proposition 8] **If MIX is not used, then the** SP-R&B**-graph is critically chorded.** The only thing to show is that the *tensor* rule preserve the property of being critically chorded. So we have to prove that any two vertices are joined by a chordless æ-path, assuming it holds for the two parts, Π_1 and Π_2. If they both belong to Π_1 or Π_2, it is the induction hypothesis, because Π_1 and Π_2 are induced subgraphs of Π.

If $x \in A$ and $y \in B$ there is an R-edge between x and y, so we are done. Consider now the case where $x \in V \setminus A$ and $y \in V' \setminus B$. Let us choose any vertex $a \in A$ and $b \in B$. By induction hypothesis we know that there exists æ-paths $\mathcal{P}$ joining x and a and $\mathcal{P}'$ joining b and y. Now let us consider the first vertex u of $\mathcal{P}$ in A and the last vertex v of $\mathcal{P}'$ in B. Necessarily the edge before u on $\mathcal{P}$ is a B-edge because A is the union of some connected components of R, and, symmetrically the edge of $\mathcal{P}'$ is a B-edge. So we can consider the æ-path from x to y obtained by following $\mathcal{P}$ up to u, followed by the R-edge uv and then by the part of $\mathcal{P}'$ from v to y. This æ-path is chordless. Indeed, there may not be any R-edge

between u and $x \ldots u$ and vy are chordless, since they are parts of two chordless æ-paths. Moreover, there may neither be any R-edge between a vertex of $x \ldots u$ and u, nor between v a vertex of $v \ldots y$, because A and B are the unions of connected components of R. Finally, observe there may not exists any other R-edge than uv between $x \ldots u$ and $v \ldots y$ because there is no R-edge between $V \setminus A$ and $V' \setminus B$. So x and y are joined by a chordless æ-path.

If MIX has been used, then there exist two vertices which are not joined by a chordless æ-path. When MIX is used last, it is obvious, just take one vertex in a connected component and the other in the other. The only thing to prove is that the *tensor* rule preserves the existence of two vertices not joined by any chordless æ-path, since the *tensor* rule introduces æ-paths between vertices previously not joined by any æ-path. The two vertices that are not joined belong, w.r.t. *tensor* to the same SP-R&B-graph G. As the freshly introduced R-edges defines a bipartite cut, any æ-path joining x and y must use at least two of them, and being elementary, they are not adjacent. Let xx' and yy' be two of these R-edges ($x, y \in A$ and $x'y' \in B$). By construction there is an R-edge x, y' which is a chord of the æ-path. □

4 Key result for sequentializing R&B-proof-nets

The purpose of this section is to establish the theorem on which relies our proof of sequentialisation for R&B-proof-nets.

Let us define an inductive class of R&B-graphs, R&B-OK.

1. A single B-edge is in R&B-OK.

   $\boxed{\forall x, y \quad \mathsf{G}_{xy} = (\{x, y\}; xy, \emptyset) \in \text{R\&B-OK}}$

2. Given two disjoint R&B-graphs in R&B-OK, $\mathsf{G}_1 = (V_1; \mathrm{B}_2, \mathrm{R}_2)$ and $\mathsf{G}_2 = (V_2; \mathrm{B}_2, \mathrm{R}_2)$, with $V_1 \cap V_2 = \emptyset$, the following R&B-graphs are in R&B-OK too.

   (a) Their disjoint union.

   $\boxed{(V_1 \uplus V_2; \mathrm{B}_1 \uplus \mathrm{B}_2, \mathrm{R}_1 \uplus \mathrm{R}_2) \in \text{R\&B-OK}}$

   (b) Every connection of G_1 and G_2 through a new B-bridge.

For all $x_1, x_2 \notin V_1 \uplus V_2$,
for all V_1', V_2' such that $\emptyset \neq V_1' \subset V_1$ and $\emptyset \neq V_2' \subset V_2$
the R&B-graph:
$$\left(\begin{array}{l} V_1 \uplus V_2 \uplus \{x_1, x_2\}; \\ \mathrm{B}_1 \uplus \mathrm{B}_2 \uplus \{x_1 x_2\}, \\ \mathrm{R}_1 \uplus \mathrm{R}_2 \uplus (V_1' * x_1) \uplus (x_2 * V_2') \end{array}\right)$$
is in R&B-OK too.

3. The addition of a pendant B-edge to an R&B-graph $\mathsf{G}_1 = (V_1; \mathrm{B}_1, \mathrm{R}_1)$ of R&B-OK is in R&B-OKtoo.

For all $x_1, x_2 \notin V_1$,
for all V_1' such that $\emptyset \neq V_1' \subset V_1$
the R&B-graph:
$$\left(\begin{array}{l} V_1 \uplus \{x_1, x_2\}; \\ \mathrm{B}_1 \uplus \{x_1 x_2\}, \\ \mathrm{R}_1 \uplus (V_1' * x_1) \end{array}\right)$$
is in R&B-OK too.

Remark.— The inductive definition of the R&B-proof-nets according to the sequent calculus are instances of these constructions. Indeed the axiom corresponds to 1, the *tensor* corresponds to 2(b) followed by 3, and *par* corresponds to 3.

Theorem 10 *Given an* R&B*-graph* $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$ *the following properties are equivalent:*

1. $\mathsf{G} \in$ R&B-OK
2. B *is the unique perfect matching of the underlying graph* $\underline{\mathsf{G}} = (V; \mathrm{B} \oplus \mathrm{R})$.
3. G *is æ-cycle free, i.e. contains no* æ*-cycle.*

More over if G *is known to be æ-cycle free which may be checked in cubic time w.r.t.* $|V|$*, the inductive definition of* G *as a member of* R&B-OK *is reconstructed in cubic time w.r.t.* $|V|$.

This requires two lemmata, which involve a graph transformation, called the **contraction of an** æ**-loop**. Let $\mathsf{G} = (V; B, R)$ an R&B-graph, and ℓ be an æ-loop on v, i.e. an æ-path of odd length whose end vertices are the same vertex v — thus

the two edges of ℓ incident to v are R-edges. Contracting the loop ℓ consists in identifying all its vertices with v (quotient graph), and suppressing the vv R-edge if any. [3] The graph resulting from such a transformation is denoted by $\mathsf{G} - \ell$. More formally if $\ell = v, v_1, v_2, \ldots, v_{2l}, v$, then

$$\mathsf{G} - \ell = (V'; \mathrm{B}', \mathrm{R}') \quad \text{with} \quad \left| \begin{array}{l} V' = V \setminus \{v_1, \ldots, v_{2l}\} \\ \mathrm{B}' = \mathrm{B} \setminus \{v_1v_2, v_3v_4, \ldots, v_{2l-1}v_{2l}\} \\ \mathrm{R}' = \mathrm{R}\big|_{V'} \cup \{vw | w \neq v \text{ and } \exists i \in [1, 2l]\ wv_i \in \mathrm{R}\} \end{array} \right.$$

Lemma 11 *Here are the useful properties of the contraction of an* æ*-loop:*

1. $\mathsf{G} - \ell$ *is an* R&B*-graph*
2. *If there exists an* æ*-path between two vertices of* $\mathsf{G} - \ell$ *then there exists one in* G *too, with endings of the same colour.*
3. *If* G *is æ-cycle free, so is* $\mathsf{G} - \ell$*.*
4. *Given a* B*-edge* xy *common to* G *and* $\mathsf{G} - \ell$ *this* B*-edge* xy *is a* B*-bridge of* G *if and only if it is a* B*-bridge of* $\mathsf{G} - \ell$*.*

Proof:

1. Indeed we suppressed the possibility of a vv edge, and as R' is defined in terms of *set union* (and not multiset union) $(V'; \mathrm{R}')$ is a simple graph; on the other hand still exactly one B-edge is incident to each vertex.
2. If the æ-path $\mathcal{P}$ in $\mathsf{G} - \ell$ does not use any of the new edges vw, than it is itself an æ-path in G. Otherwise, assume it makes use of an R edge vw with $v_iw \in \mathrm{R}'$. Firstly observe that because $\mathcal{P}$ is an æ-path, only a single R-edge of $\mathcal{P}$ may be incident to v, and, a fortiori only a single R-edge of $\mathcal{P}$ may not be in R. This R-edge may be replaced with an æ-path whose endings are both in R; this path which consists in ℓ-edges but the last one v_iw; if i is

[3] On some particular non-æ-cycle free R&B-graphs, the suppression of the possible vv edge may delay the algorithm we present later on. We do so because it reduces the number of cases and allow us not to consider R&B-graphs with R-loops. However this does not affect the worst case complexity of the forthcoming algorithm.

even then this æ-path to replace vw is $v, v_1, v_2, \ldots, v_i, w$ and if i is odd it is $v, v_n, v_{n-1}, \ldots, v_i, w$. When replacing the R-edge vw of the æ-path in $\mathsf{G} - \ell$ with the just defined æ-path p in G, there is no risk of getting a non-elementary path in G, since none of the v_i lie in V.

3. Because of the previous remark, if there was an æ-path from some x to itself there would be one in G.

4. Notice the loop and v belong to the same 2-edge-connected component of the underlying graph $\underline{\mathsf{G}}$.

□

The following lemma has itself a meaning when applied to proof-nets. Indeed, considering the R&B-graph $\Pi_\otimes$ defined in subsection 2.5, it shows that any vertex is joined by an æ-path to a splitting times, even when the proof(net) uses MIX.

Lemma 12 *Given an* R&B*-graph* $\mathsf{G} = (V; B, R)$*, and a vertex* $x_0 \in V$*, there exists an* æ*-path from* x *to an* æ*-cycle, or to a* B*-bridge.*

In particular if G *is* æ*-cycle free, then there exists a* B*-bridge.*

Proof: We extend an æ-path $\mathcal{P}$ starting with the unique B-edge incident to x_0, using the following algorithm, which stops when it finds one of the two wanted configurations. An easy induction on the number of B-edges proves its termination.

1. When ς ends on an R-edge $x_{2i+1}x_{2i+2}$, we can only extend the path with the *unique* B-edge incident to x_{2i+2}, say $x_{2i+2}x_{2i+3}$. The path is still elementary. Indeed, for all vertices x_k with $k \leq 2i+1$ their incident B-edge is already in the ς, thus one would have $x_{2i+2} = x_l$ with $l < 2i+1$ which conflicts with ς being elementary.

2. When ending on a B-edge x_{2i}, x_{2i+1},

2.1 if there is no R-edge incident to x_{2i+1}, we are done: this B-edge is a B-bridge.

2.2. Otherwise we randomly choose an R-edge x_{2i+1}, x_{2i+2} extending the path.

2.2.1. If it is still elementary, we extend the æ-path.

2.2.2. If this path is no more elementary, i.e. if $x_{2i+2} = x_k$ with $k \leq 2i+1$

2.2.2.1. if k is even, we have an æ-cycle, and an æ-path from x_0 to this elementary cycle,

2.2.2.2. if k is odd, say $k = 2l + 1$ we have an æ-loop ℓ on the end vertex $x_{2l+1} = x_{2i+2}$ of the æ-path. In this latter case we contract this æ-loop on x_{2l+1}, namely:

$$\ell = x_{2l+1}, x_{2l+2}, \ldots, x_{2i+1}, x_{2i+2} = x_{2l+1}$$

We proceed with $\mathsf{G} - \ell$ and x_0 which is still a vertex of $G - \ell$, since $0 < 2l + 1$. As $\mathsf{G} - \ell$ has at least one B-edge less, while a wanted configuration in $\mathsf{G} - \ell$ yields a similar configuration in G by lemma 11 (2). Hence, by induction on the number of B-edges we are done. □

Now we can come back to the proof of the theorem:

Proof: [of Theorem 10]

$\neg(1) \to \neg(2)$ Assume G contains an æ-cycle $\mathcal{C}$. Every edge incident to a vertex of $\mathcal{C}$ but not in $\mathcal{C}$ is an R-edge. Exchanging the colours of the edges of $\mathcal{C}$, we obtain another perfect matching of $\underline{\mathsf{G}}$ — notice that the fact that an æ-cycle is *elementary* is necessary: otherwise we could obtain adjacent B-edges.

$\neg(2) \to \neg(1)$ Assume $\underline{\mathsf{G}}$ is also the underlying graph of $\mathsf{G}' = (V; \mathrm{B}', \mathrm{R}')$ with, for instance, $x_0x_1 \in \mathrm{B} \setminus \mathrm{B}'$ — or the converse, the question being symmetrical.

We extend a path of $\underline{\mathsf{G}}$ starting with x_0x_1 which will be an æ-path both in G and G': the $(2p+1)^{\text{th}}$ edge is in B but not in B′, hence in R′ and the $2p^{\text{th}}$ edge is in B′ but not in B, hence in R.

Assume the path already built is of odd length: its last edge $x_{2i}x_{2i+1}$ is in R′; since B′ is a *perfect* matching of G', there must be a (unique) edge $x_{2i+1}x_{2i+2}$ in B′ incident to x_{2i+1}. Because $x_{2i}x_{2i+1}$ is in B, while $x_{2i}x_{2i+1}$ and are incident, and B is a (perfect) matching $x_{2i+1}x_{2i+2}$ is in R. When the path already built is of even length, the argument is symmetrical.

Since $\underline{\mathsf{G}}$ is finite, we meet a vertex x again, and thus we found an æ-path from a vertex to itself. The first and last edge may not be of the same colour in either of the R&B-graph G and G': they would be both in B or both in B′ while B and B′ are (perfect) matching.

Therefore this æ-path is an æ-cycle (both in G and G').

$(3) \to (1)$ Straightforward induction.

$(1) \to (3)$ This is obtained by iterating lemma 12, and thus gives the inductive definition of G, using 3 if the B-bridge is pendant and 2(b) otherwise. [4] □

[4] We could have use $(2) \to (3)$, which is known [12]. In the literature it is deduced from diffi cult results: "cathedral structure theorem" or Tütte's theorem [13, 5]. That is the reason why we gave our

5 An alternative definition of chorded SP-R&B-graphs: bow-ties

Here we introduce a property of SP-R&B-graphs, being bow-tied, which seems, at first sight, much stronger than being chorded. In fact, an SP-R&B-graph is bow-tied if and only if it is chorded, but this apparently stronger property is the one we need in order to prove sequentialisation.

Given an æ-cycle $\mathcal{C} = x_1, \ldots, x_n, x_1$ of an (SP)-R&B-graph, let us call a **bow-tie** any pair of R-edges $x_i x_{i+1}, x_j x_{j+1}$ such that:

1. $x_i x_j, x_{i+1} x_{j+1} \in \mathrm{R}$
   — in other words $\mathrm{R}\big|_{x_i, x_{i+1}, x_j, x_{j+1}} \subset \{x_i, x_{j+1}\} * \{x_{i+1}, x_j\} \simeq \mathsf{K}_{2,2}$
2. $x_i x_{j+1} \notin \mathrm{R}$ or $x_{i+1} x_j \notin \mathrm{R}$
   — so $\mathrm{R}\big|_{x_i, x_{i+1}, x_j, x_{j+1}} \not\simeq \mathsf{K}_4$.

An SP-R&B-graph is said to be **bow-tied** whenever each æ-cycle contains a bow-tie.

Given an SP-R&B-graph G and an æ-cycle $\mathcal{C}$ of G let us denote by $\mathsf{Chord}(\mathsf{G}, \mathcal{C})$ the fact that $\mathcal{C}$ contains a chord (in G), and by $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C})$ the fact that $\mathcal{C}$ contains a bow-tie (in G).

Given an R&B-graph $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$ and a set of B-edges $x_i x_i' \in \mathrm{B}$ the **restriction** of G to $V' = V \setminus \{x_i, x_i'\}$ is $\mathsf{G}' = (V'; \mathrm{B}\big|_{V'}, \mathrm{R}\big|_{V'})$. An R&B-graph is said to be an **induced R&B-subgraph** of an R&B-graph $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$ whenever there exists B-edges $x_i x_i' \in \mathrm{B}$ such that G' is the restriction of G to $V' = V \setminus \{x_i, x_i'\}$. Notice that if G is an SP-R&B-graph, all it induced subgraphs or restrictions are SP-R&B-graphs as well, because of proposition 5 .

We first will state the useful but obvious:

Proposition 13 *Let* G *be an (*SP-*)*R&B*-graph, and let* G' *be an* induced R&B*-subgraph of* G*. Let* $\mathcal{C}$ *be an* æ*-cycle of* G *the vertices of which lie in* V'*. Then* $\mathcal{C}$ *is an* æ*-cycle of* G' *as well and* $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C}) \Leftrightarrow \mathsf{Bowtie}(\mathsf{G}', \mathcal{C})$*. Thus, whenever an (*SP-*)*R&B*-graph is bow-tied, so are its induced (*SP-*)*R&B*-subgraphs.*

own (independent) direct proof of $(1) \to (3)$ —which yields $(2) \to (3)$ together with $(2) \to (1)$; furthermore our proof is elementary and yields a cubic algorithmic.

The same holds for if one replaces "bow-tied " with "chorded".

Finally every induced R&B*-subgraph of an æ-cycle free* R&B*-graph is æ-cycle free too.*

Lemma 14 *Given an* SP-R&B*-graph* $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$*, the following statements are equivalent:*

1. G *is bow-tied — each of its* æ*-cycle contains a bow-tie.*
2. G *is chorded— each of its* æ*-cycle contains a chord.*

Proof: Obviously 1 implies 2, and in order to prove 2 implies 1, we have to establish the following, where æ-c(G) denotes the set of the æ-cycles of G:

$$\forall \mathsf{G} \in \text{SP}-\text{R\&B} \quad \forall \mathcal{C} \in \text{æ-c}(\mathsf{G}) \quad [\,\forall \mathcal{C}' \in \text{æ-c}(\mathsf{G})\ \mathsf{Chord}(\mathsf{G}, \mathcal{C}')\,] \quad \Rightarrow \quad \mathsf{Bowtie}(\mathsf{G}, \mathcal{C})\,]$$

We proceed by induction on $(|V|, \lg(\mathcal{C}))$, letting $\mathcal{C}$ be $x_0, \ldots, x_n, x_0$.

Basic cases The basic case $|V| = 2\ \lg(\mathcal{C}) = 2$ is silly: either there is no æ-cycle, or there is one a, b, a and the premise of the implication fails ($\mathrm{B} \cap \mathrm{R} \neq \emptyset$). For a more illustrative case, consider when $V = \{a, a', b, b'\}$, and $\mathrm{B} = \{aa', bb'\}$. If the premise of the implication does not fail, we have $\mathrm{B} \cap \mathrm{R} = \emptyset$; if there is an æ-cycle, it means that $ab, a'b' \in \mathrm{R}$ (or $ab', a'b \in \mathrm{R}$, but it is a symmetrical case). If the æ-cycle contains a chord, this chord is ab' (or ba', but it is a symmetrical case). But, being SP, R is P_4-free (cf. proposition 5). Thus, at least one of the three edges aa', bb', $a'b$ must be in R. The first two are prohibited — the premise of our implication would fail; therefore $a'b \in \mathrm{R}$, and aa', bb' is a bow-tie of $\mathcal{C}$.

Assume the result holds for all $\mathsf{G}' = (V'; \mathrm{B}', \mathrm{R}')$ and $\mathcal{C}'$ with $|V'| < |V|$ or $|V'| = |V|$ and $\lg(\mathcal{C}') < \lg(\mathcal{C})$, and let us show that it holds for G and $\mathcal{C}$.

We may assume that $\lg(\mathcal{C}) \geq 4$ — otherwise the premise of the implication would fail.

1 Assume that $\mathcal{C}$ is not Hamiltonian, i.e. does not use a vertex of G — and therefore, being an æ-cycle does not use some B-edge xy of G. Then we have $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C})$ from the induction hypothesis. Indeed, let us consider the restriction G' of G to $V \backslash \{x, y\}$. Because of proposition 13, we have $\forall \mathcal{C}' \in$ æ-c(G) $\mathsf{Chord}(\mathsf{G}', \mathcal{C})$

and therefore, by induction hypothesis, we have $\mathsf{Bowtie}(\mathsf{G}', \mathcal{C})$. But then, by proposition 13 we have $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C})$.

2 So we may assume that $\mathcal{C}$ is Hamiltonian. By hypothesis, $\mathcal{C}$ contains a chord, say $x_i x_j$, (thus $x_i \neq x_j$) and each of x_i and x_j has an R-neighbor along $\mathcal{C}$, say $x_{i\pm1}$ and $x_{j\pm1}$. As $\mathcal{C}$ is alternate and elementary, no two R-edges of $\mathcal{C}$ are adjacent, so $x_{i\pm1} \neq x_{j\pm1}$. So its makes four distinct vertices in the same connected component of R — unless $x_i x_{i\pm1} = x_j x_{j\pm1} = x_i x_j$ but, then, $x_i x_j$ would not be a chord of $\mathcal{C}$. So we can consider one of the connected component of R, say $R \subset V$ which has at least four vertices. Since R is connected component of R with more than one vertex, we know from proposition 5 that $R = S \uplus T$ with $\mathrm{R} \supset S * T$.

2.1 Assume there exists a part of $\mathcal{C}$ such that its first vertex is in S, its last vertex is in T (or the other way round) and each intermediate vertex is outside R. Let $x_i, x_{i+1}, \ldots, x_{i+p}, x_{i+p+1}$ be such a part of $\mathcal{C}$, with $s \in S$, $t \in T$ and $x_{i+q} \notin R$ for all $q \in [1, p]$. Because $\mathrm{R} \supset S * T$, there is an R-edge x_i, x_{i+p+1} in G, and this part defines an æ-cycle $x_i, x_{i+1}, \ldots, x_{i+p}, x_{i+p+1}, x_i$. Indeed, because R is a connected component of R, $x_i \in S \subset R$, and $x_{i+1} \notin R$ we have $x_i x_{i+1} \in \mathrm{B}$ and, symmetrically, $x_{i+p} x_{i+p+1} \in \mathrm{B}$ — thus p is necessarily even. Because R contains at least four vertices and $\mathcal{C}$ is Hamiltonian, $\mathcal{C}$ can not be reduced to this æ-cycle. So this æ-cycle $\mathcal{C}' = s, x_1, \ldots, x_p, t$ is strictly shorter, and by induction hypothesis, one has $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C}')$. But none of the two R-edges of the bow-tie may be $x_i x_{i+p+1}$ since there may not be any R-edge $x_i x_{i+q}$ with $q \in [1, p]$ — since R is a connected component of R while $x_i \in R$ and $x_{i+q} \notin R$. Thus the two R-edges of the bow-tie lie on $\mathcal{C}$, and thus $\mathcal{C}$ contains a bow-tie, i.e. $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C})$.

2.2 Thus we may assume (*): for each part of $\mathcal{C}$ such that its first vertex is in R, its last vertex is in R, and all intermediate vertices are outside R, either the first and last vertices both belong to S, or both belong to T.

Consider now two vertices x_i, x_{i+p} of $\mathcal{C}$ such that $x_i \in S$ and $x_{i+1+p} \in T$ and p is minimal. Let x_{i+q} be an intermediate vertex, i.e. $q \in [1, p]$. Then x_{i+q} may not belong to R (otherwise x_{i+q} would be in S or in T, and this would conflict with p being minimal) thus all the x_{i+q} are not in R. But this conflicts with (*). Therefore $p = 0$, i.e. there is no intermediate vertex, and thus $\mathcal{C}$ contains an R-edge $x_i x_{i+1}$ with $x_i \in S$ and $x_{i+1} \in T$.

Consider the first vertex x_k after x_i along $\mathcal{C}$ which belongs to S too. Consider the part $(x_i \in S), (x_{i+1} \in T), x_{i+2}, \ldots, x_{k-1}, x_k \in S$ of $\mathcal{C}$ — we know that

$x_{i+2}, \ldots, x_{k-1} \notin S$. Let x_j be the last point in T along this part of $\mathcal{C}$: in between x_j and x_k there may neither be any vertex of S, nor of T; thus because of (*) there may not be any vertex in between x_j and x_k. Therefore the vertex x_{k-1} previous to x_k along $\mathcal{C}$ belongs to T.

But because G contains no chordless æ-cycle, while there are all possible R-edges between S and T, there is no B-edge between S and T. Thus $x_{k-1}x_k$ is an R-edge of $\mathcal{C}$.

2.2.1 If there is no R-edge $x_{i+1}x_{k-1}$ then $x_ix_{i+1}, x_{k-1}x_k$ is a bow-tie of $\mathcal{C}$.

2.2.2 So we may assume that there is an R-edge $x_{i+1}x_{k-1}$**.** This R-edge defines a smaller æ-cycle $\mathcal{C}'$, which, by induction hypothesis, contains a bow-tie.

2.2.2.1 If $x_{i+1}x_{k-1}$ **does not belong to this bow-tie**, it is a bow-tie of $\mathcal{C}$ and $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C})$.

2.2.2.2 If $x_{i+1}x_{k-1}$ **belong to this bow-tie** let $x_{i+1}x_{k-1}, x_l, x_{l+1}$ be this bow-tie of $\mathcal{C}'$ and assume that $x_{i+1}x_l \notin$ R — the other case $x_ix_{l+1} \notin R$ being symmetrical. Firstly, observe that since R is a connected component of R, and $x_{i+1}x_{k-1} \in T \subset R$, one has $x_l, x_{l+1} \in R$, and as there is no vertex of S in between x_i and x_k, we have $x_l, x_{l+1} \in T$. We already know that $x_{l+1}x_{i+1} \in$ R and as $x_i \in S$ and $x_l \in T$, $x_ix_l \in$ R while $x_{i+1}x_l \notin$ R. Therefore x_ix_{i+1}, x_lx_{l+1} is a bow-tie of $\mathcal{C}$, and thus $\mathsf{Bowtie}(\mathsf{G}, \mathcal{C})$. □

6 Lemmata on R&B-graphs

Actually this section is the proof of sequentialisation of SP-R&B-proof-nets to be stated in the forthcoming section. The key notion is a pair of more or less inverse transformations on R&B-graphs (non necessarily SP). One is the replacement of a B-edge and its adjacent R-edges into a complete "tripartite" R-graph — i.e. three complete bipartite graphs $(V_1 * V_2) \uplus (V_2 * V_3) \uplus (V_3 * V_1)$ — the V_i being pairwise disjoint. In particular for some special class of SP-graphs to be introduced, these transformations behave especially well w.r.t. the notions we are interested in: æ-cycle free, bow-tied (or chorded), bridges, R being SP. That is the reason why it works.

6.1 Particular SP-graphs, bow-ties in SP-graphs

An SP-term is said to be of **depth nought** whenever it is a graph with no edge. The SP-terms of **depth one** are the closure under parallel composition of the terms of depth nought and the binary series composition of terms of depth nought. In other words, in a term of depth one written as a binary tree, there is never a $*$ below a $*$, i.e. a term of depth one may be written as

$$\mathsf{t} = (\mathsf{t}_1 * \mathsf{t}'_1) + \cdots (\mathsf{t}_p * \mathsf{t}'_p) + x_1 + \cdots + x_k \text{ with } \mathsf{t}_i \text{ of depth nought}$$

Given an SP-term t, we defines is differential to be the term of depth one obtained from it by turning all the $*$ operation below a $*$ operation by $+$. If ones wants to be really formal, one may define ∂t as follows:

$$\begin{array}{rclcrcl} \partial x & = & x & & \flat x & = & x \\ \partial(\mathsf{t}' + \mathsf{t}'') & = & \partial(\mathsf{t}') + \partial(\mathsf{t}'') & & \flat(\mathsf{t}' + \mathsf{t}'') & = & \flat(\mathsf{t}') + \flat(\mathsf{t}'') \\ \partial(\mathsf{t}' * \mathsf{t}'') & = & \flat(\mathsf{t}') * \flat(\mathsf{t}'') & & \flat(\mathsf{t}' * \mathsf{t}'') & = & \flat(\mathsf{t}') + \flat(\mathsf{t}'') \end{array}$$

Beware that this operation is not defined on SP-graphs, i.e. the resulting SP-graph really depends on the way we wrote the original SP-graph as a binary SP-term: it makes a difference w.r.t. the chosen bracketing for the nested $*$.

Proposition 15 *Let* t *be a term describing an* SP*-relation* R*, and let* R_1 *be the* SP*-relation associated with* $\partial\mathsf{t}$*. Assume that* $ab, cd, ac, bd \in \mathrm{R}$ *and that* $ad \notin \mathrm{R}$*. If* $ab, cd \in \mathrm{R}_1$ *then* $ab, cd, ac, bd \in \mathrm{R}_1$*, and still* $ad \notin \mathrm{R}_1$*.*

Proof: We proceed by induction on t.

Assume $\mathsf{t} = \mathsf{t}'(x'_1, \ldots, x'_n) * \mathsf{t}''(x''_1, \ldots, x''_p)$. Then $\partial\mathsf{t} = \flat(\mathsf{t}'(x'_1, \ldots, x'_n)) * \flat(\mathsf{t}''(x''_1, \ldots, x''_p))$. On one hand we know that both $\mathsf{t}'(x'_1, \ldots, x'_n)$ and $\mathsf{t}''(x''_1, \ldots, x''_p)$ of depth nought i.e. correspond to SP-graphs with no edge. On the other hand we know that $ab, cd \in \mathrm{R}_1 \subset \mathrm{R}$. Therefore,

either $a, c \in \{x'_i, i \in [1, n]\}$ and $b, d \in \{x''_i, i \in [1, p]\}$
or $a, d \in \{x'_i, i \in [1, n]\}$ and $b, c \in \{x''_i, i \in [1, p]\}$.

But the first case is excluded: we would have $ad \in \mathrm{R}_1$, and a fortiori $ad \in \mathrm{R}$. In the second case, one has $\mathrm{R}_1\big|_{a,b,c,d} = \{ab, cd, ac, bd\}$.

Assume $t = t'(x'_1, \ldots, x'_n) + t''(x''_1, \ldots, x''_p)$, the corresponding SP-graphs being denoted by R' and R''. Then either $\{a, b, c, d\} \subset \{x'_i, i \in [1, n]\}$ or $\{a, b, c, d\} \subset$

$\{x''_i, i \in [1,p]\}$. Indeed they all belong to the same component of R. Assume it is the first case, the other one being symmetrical. Clearly, $\mathrm{R}\big|_{a,b,c,d} = \mathrm{R}'\big|_{a,b,c,d}$, thus ab, cd is a bow-tie of R′. Let R'_1 be the SP-graph associated with $\partial\mathsf{t}'$: as $\partial(\mathsf{t}' + \mathsf{t}'') = \partial\mathsf{t}' + \partial\mathsf{t}''$ we have $\mathrm{R}'_1\big|_{a,b,c,d} = \mathrm{R}_1\big|_{a,b,c,d}$. Thus the hypothesis of the proposition also holds when restricting the SP-graphs to the vertices to $\{x'_i, i \in [1,n]\}$. Thus $ab, cd, ac, bd \in \mathrm{R}'_1$ while $ad \notin \mathrm{R}'_1$, and the same holds in R_1. □

Let us define a subclass of the SP-graphs, the class of **unary** SP-graph. This subclass of the SP-graphs is the smallest class of graphs including ONE, closed by disjoint union and the following restricted form of series composition: if $\mathsf{G} = (V, \mathrm{R})$ is unary, and $x \notin V$ then $\mathsf{G} * x$ is unary. Then one has:

Proposition 16 *If an* SP*-graph is unary then none of its induced subgraphs is isomorphic to* $\mathtt{K}_{2,2}$.

Proof: By induction on $|V|$, noticing that x is adjacent to every other vertex in $x * G$, and thus may not be a vertex of an induced $\mathtt{K}_{2,2}$. □

Proposition 17 *Let* t *be an* SP*-term describing an* SP*-graph* R*, and let* R_1 *be the* SP*-graph corresponding to* $\partial\mathsf{t}$. *If an* SP*-graph* $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$ *is chorded, so is* $\mathsf{G}_1 = (V; \mathrm{B}, \mathrm{R}_1)$.

Proof: Let $\mathcal{C}$ be an æ-cycle of G_1. Then $\mathcal{C}$ is an æ-cycle of G as well. Thus it contains a bow-tie in G, that is to say four vertices $x_i, x_{i+1}, x_j, x_{j+1}$ such that: $x_ix_{i+1}, x_jx_{j+1}, x_ix_j, x_{i+1}x_{j+1} \in \mathrm{R}$ while $x_ix_{j+1} \notin \mathrm{R}$ or $x_{i+1}x_j \notin \mathrm{R}$. Applying proposition 15 we see that x_ix_{i+1}, x_jx_{j+1} is a bow-tie of $\mathcal{C}$ in G_1 too. Thus, G_1 is bow-tied too. □

Proposition 18 *A chorded* SP-R&B*-graph* $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$ *with* R *being unary is æ-cycle free.*

Proof: Assume we have an æ-cycle. Consider an æ-cycle $\mathcal{C}$ the length of which is minimal. As G is chorded, $\mathcal{C}$ must contain a bow-tie x_ix_{i+1}, x_jx_{j+1}. Notice that $x_ix_{j+1} \notin \mathrm{R}$: this would yield a shorter æ-cycle. Symmetrically, $x_jx_{i+1} \notin \mathrm{R}$. Thus this bow-tie is isomorphic to $\mathtt{K}_{2,2}$. But not induced subgraph of a unary SP-graph is isomorphic with $\mathtt{K}_{2,2}$, as we have seen in proposition 16. □

6.2 Folding, unfolding

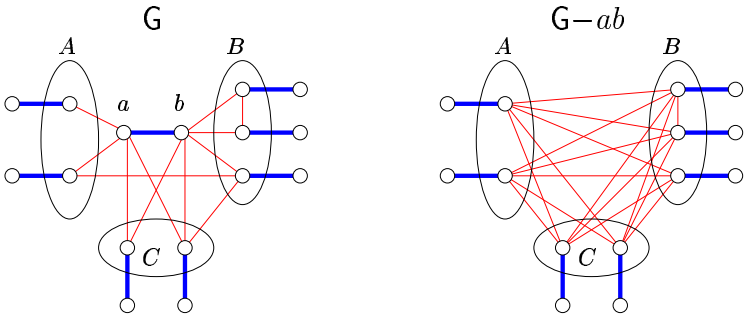
We define here two transformations on R&B-graphs, which are, up to some inessential details the inverse one of the other. Let us insists that we define these transformations for all R&B-graphs, not just for SP-R&B-graphs.

6.2.1 Unfolding

Let $ab \in (\mathrm{B} \setminus \mathrm{R})$ be a B-edge of G. Let $A = (\mathrm{R}(a) \setminus \mathrm{R}(b))$, $B = (\mathrm{R}(b) \setminus \mathrm{R}(a))$, and $C = (\mathrm{R}(a) \cap \mathrm{R}(\mathrm{B}))$. The **unfolding of** G **at** ab is the following R&B-graph:

$$\mathsf{G} - ab = \begin{pmatrix} V' = V \setminus \{a, b\}; \\ \mathrm{B} \setminus \{ab\}, \\ \mathrm{R}\big|_{V'} \cup \Big(A * B * C\Big) \end{pmatrix}$$

A little picture shows better how the unfolding of $ab \in \mathrm{B}$ acts upon an R&B-graph:



In general, the unfolding does not preserve the absence of æ-cycle, but:

Proposition 19 *If* $|\mathrm{R}(a)| \leq 1$ *or* $|\mathrm{R}(b)| \leq (1)$*, and if* G *is æ-cycle free, then* $\mathsf{G}{-}ab$ *is æ-cycle free too.*

Proof: The vertices a and b play symmetric rôles. If $\mathrm{R}(a) = \emptyset$ $\mathsf{G}{-}ab$ is the restriction of G to $V - \{a, b\}$, which is æ-cycle free if G is.

If $\mathrm{R}(a) = \{u\}$ then $u \neq b$, as G is æ-cycle free. All R-edges of $\mathsf{G}{-}ab$ which are not in G are in $u * B$. If an æ-cycle of $\mathsf{G}{-}ab$ does not use any of them then it is itself an æ-cycle in G. But an æ-cycle may only use one of them, say ub'. Replacing ub' with $ua \in \mathrm{R}$, $ab \in \mathrm{B}$ $bb' \in \mathrm{R}$ we would obtain an æ-cycle in G. □

Proposition 20 *If* $|\mathrm{R}(a)| \leq 1$ *and* $|\mathrm{R}(b)| \leq 1$*, and if a* B*-edge* $xy \neq ab$ *is bridge in* $\mathsf{G}-ab$ *then it is a bridge in* G*.*

Proof: We prove the contrapositive. Assume that xy lies on a cycle in $\underline{\mathsf{G}}$, which we may assume to be elementary. If this cycle does not use any edge incident to a or b, it is itself a cycle in $\underline{\mathsf{G}-ab}$. Otherwise, since it is an elementary cycle, is uses $a'a, ab, bb'$ and replacing this sequence with $a'b'$ we obtain in $\underline{\mathsf{G}-ab}$ a cycle on which xy lie on — unlike in the general case, the cycle in $\underline{\mathsf{G}}$ may not use edges $a'a, aa''$ with $a'a''$ in $\mathrm{R}(a)$, and vanish because $|\mathrm{R}(b)| = 0$. □

In general, the unfolding does not turn an SP-R&B-graph into an SP-R&B-graph. Nevertheless, in some case it may be viewed as the substitution of a disjoint SP-graph for a vertex.

Proposition 21 *If* R *is series parallel, and if* R *may be written as*

$$\mathrm{R} = \Big(a * \mathsf{t}'(x_1, \ldots, x_n)\Big) + \mathsf{t}''(b, y_1, \ldots, y_p)$$

then $\mathrm{R} - ab$ *is series parallel and may be written as:*

$$\mathrm{R}{-}ab = \mathsf{t}''\Big(\mathsf{t}'\left(x_1, \ldots, x_n\right), y_1, \ldots, y_p\Big)$$

As a particular case, if $\mathrm{R} = \Big(a * \mathsf{t}'(x_1, \ldots, x_n)\Big) + \Big(b * \mathsf{t}''(y_1, \ldots, y_p)\Big)$ *then* $\mathrm{R}{-}ab = \mathsf{t}'(x_1, \ldots, x_n) * \mathsf{t}''(y_1, \ldots, y_p)$.

Proof: Let $\mathrm{R}, \mathrm{R}', \mathrm{R}'', \mathrm{R}_0$ be the SP-graphs respectively associated with $\mathsf{t}, \mathsf{t}', \mathsf{t}'', \mathsf{t}''[\mathsf{t}'/b]$. Observe that $\mathrm{R} = \mathrm{R}_0$ whenever:

- $\forall x, y \in \mathrm{dom}(\mathsf{t}') \qquad xy \in \mathrm{R} \Leftrightarrow xy \in \mathrm{R}_0$
- $\forall x \in \mathrm{dom}(\mathsf{t}')\ \forall y \in \mathrm{dom}(\mathsf{t}'') \setminus \{b\} \qquad xb \in \mathrm{R}'' \Leftrightarrow xy \in \mathrm{R}_0$

The result follows from this observation. □

Next comes the main reason for introducing the unfolding:

Proposition 22 *If both* G *and* $\mathsf{G}-ab$ *are* SP-R&B*-graphs — e.g. like in the previous proposition — and if* G *is chorded, so is* $\mathsf{G}-ab$*.*

Proof: Firstly, an observation: as R is SP, if $C \neq \emptyset$, than $\mathrm{R} \subset (A * C) \uplus (B * C)$. Indeed, for all $a' \in A$ and $c \in C$, $a'a, ac, cb \in \mathrm{R}$ and since R is P_4 free, one must either have $a'b \in \mathrm{R}$ (but this yields $a' \in C$ while $C \cap A = \emptyset$) or $ab \in \mathrm{R}$ (but this conflicts with G being chorded) or $a'c \in \mathrm{R}$. Thus for all $c \in C$ we have that for all $a' \in A$, $a'c \in \mathrm{R}$, and, symmetrically, for all $b' \in B$, $cb' \in \mathrm{R}$.

Let $\mathcal{C}$ be an æ-cycle in $\mathsf{G}-ab$.

1 If does not use any R-edge of $A * B * C$ than it is an itself an æ-cycle in G.

2 If it uses exactly one R-edge in $A * B * C$ this edge is either $a'b'$, $a'c'$ or $b'c'$, with $a' \in A$, $b' \in B$, $c' \in C$, the latter two cases being symmetrical. If this edge is $a'b'$, replacing it with the three edges $a'a, ab, bb'$ we obtain an æ-cycle in G, which contains a chord.

If this chord is neither incident to a nor b, it is itself a chord on the æ-cycle of $\mathsf{G}-ab$ unless it is $a'b'$ ($a'b'$ may be an R-edge of G). In this particular case, the æ-cycle of $\mathsf{G}-ab$ is itself an æ-cycle of G, and its chord is a chord in $\mathsf{G}-ab$.

If this chord is ax (resp. bx) then $b'x$ (resp. $a'x$) is a chord of $\mathcal{C}$ in $\mathsf{G}-ab$.

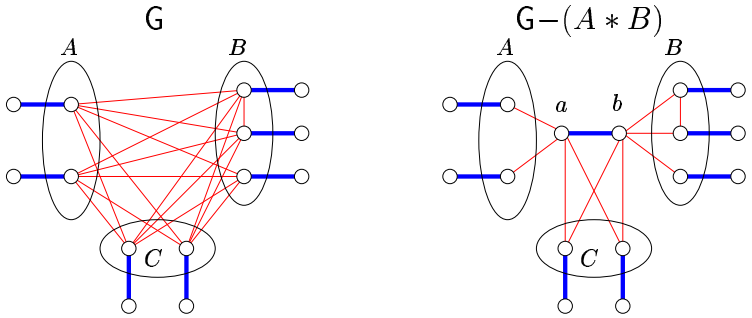
3 If $\mathcal{C}$ use at least two R-edges $xy, uv \in A * B * C$, then at least two of these four vertices are both in A or both in B or both in C, while we know that x and y (resp. u and v) are not in the same part. Assume for instance that x and u are in the same part A, B or C. Then xv is a chord of the æ-cycle in G. □

6.2.2 Folding

Let A and B be two disjoint subsets of V such that the complete bipartite graph $A * B$ is included in R. Let $C = \{x | \mathrm{R}(x) \supset (A \cup B)\}$. The **folding of** G **at** $A * B$ is the following R&B-graph, where $a, b \notin V$:

$$\mathsf{G}-(A * B) = \left(\begin{array}{l} V \uplus \{a, b\}; \\ \mathrm{B} \uplus \{ab\}, \\ \mathrm{R}^* = (\mathrm{R} \setminus A * B * C) \\ \quad \uplus (a * A) \uplus (B * b) \quad \uplus (a * C) \uplus (b * C) \end{array} \right)$$

A little picture shows better how the folding of $A * B$ acts upon an R&B-graph:



Proposition 23 *The set of* R*-edges* $A * B$ *is a bipartite cut in* G *if and only if the* B*-edge* ab *is a bridge in* $\mathsf{G}-(A * B)$.

Proof: This is quite straightforward once you notice that if ab is a bridge, then $\mathrm{R}(a) \cap \mathrm{R}(b) = \emptyset$, and conversely, if $A * B$ is a bipartite cut then $C = \emptyset$. □

In the cases we are interested in, the folding turns an SP-R&B-graph into another SP-R&B-graph:

Proposition 24 *If* R *may be written as*

$$\mathsf{t}_0 + \Big((\mathsf{t} + \mathsf{t}'') * t' \Big)$$

then the complete bipartite graph dom(t)∗*dom*(t') *is a subgraph of* R *and the folding of* G *at dom*(t') ∗ *dom*(t') *yields an* SP*-graph* R^* *which may be described as:*

$$r^* = \mathsf{t}_0 + (\mathsf{t} * a) + \big((b + \mathsf{t}'') * \mathsf{t}' \big)$$

Proof: Easy computation. □

Proposition 25 *If* G *is chorded, so is* $\mathsf{G}-(A * B)$.

Proof: Let $\mathcal{C}$ be an æ-cycle of $\mathsf{G}-(A * B)$. If $\mathcal{C}$ does not uses the B-edge ab, then it is itself an æ-cycle in G, which is chorded. The chord, being neither incident to a nor to b is a chord of $\mathsf{G}-(A * B)$.

So assume that $\mathcal{C}$ uses ab, and thus contains a sequence u, a, b, v.

- If $u \in A$ and $v \in C$, (resp. $u \in C$ and $v \in B$) av (resp. vb) is a chord of $\mathcal{C}$ in $\mathsf{G}-(A * B)$.
- If $u, v \in C$, then av is a chord of $\mathcal{C}$ in $\mathsf{G}-(A * B)$.
- If $u \in A$ and $v \in B$, replacing this sequence with $uv \in \mathrm{R}$ yields an æ-cycle $\mathcal{C}'$ in G which contains a chord. If this chord is not in $A * B * C$, it is a chord of $\mathcal{C}$ in $\mathsf{G}-(A * B)$. Otherwise, $\mathcal{C}'$ contains a vertex u in $A \uplus B \uplus C$ and au or bu is a chord of $\mathcal{C}$ in $\mathsf{G}-(A * B)$.

□

Proposition 26 *Let* $\mathsf{G} = (V; \mathrm{B}, \mathrm{R})$ *be an* SP-R&B*-graph such that* R *is the disjoint union of complete bipartite graphs* $A_i * A'_i$*,* $i \in [1, p]$ *and isolated vertices — in other words any* SP*-term describing* R *is of depth one. Consider the folding* $\mathsf{G}-(A_i * A'_i)_{i\in[1,p]} = (V \uplus \{a_i, a'_i\}; \mathrm{B} \uplus \{a_i a'_i\}, \mathrm{R}^*)$ *of all the* B_i *— which clearly commutes in this particular case. Then* R^* *is unary and* $\forall x \in V\ |\mathrm{R}^*(x)| \leq 1$.

Proof: Assume that $\mathrm{R} = \biguplus_{i=1}^{p} (\{x_i^1, \ldots, x_i^{p_i}\} * \{y_i^1, \ldots, y_i^{q_i}\}) \uplus \{z_1, \ldots, z_l\}$. Thus the term describing R, up to the associativity and commutativity of $+$ and to the commutativity of $*$ is: $\mathsf{t} = (\mathsf{t}_1 * \mathsf{t}'_1) + \cdots + (\mathsf{t}_p * \mathsf{t}'_p) + z_1 + \cdots + z_l$
with $\mathsf{t}_i = x_i^1 + \cdots + x_i^{p_i}$ and $\mathsf{t}'_i = y_i^1 + \cdots + y_i^{q_i}$.

Then, as observed in proposition 24 a term t^* describing R^* up to the associativity and commutativity of $+$ and to the commutativity of $*$ is:

$\mathsf{t}^* = \Big((\mathsf{t}_1 * a_1) + (a'_1 * \mathsf{t}'_1)\Big) + \cdots + \Big((\mathsf{t}_p * a_p) + (a'_p * \mathsf{t}'_p)\Big) + z_1 + \cdots + z_l$

which may be written as:

$\mathsf{t}^* = (\mathsf{t}_1 * a_1) + \cdots + (\mathsf{t}_p * a_p) + (a'_1 * \mathsf{t}'_1) + \cdots + (a'_p * \mathsf{t}'_p) + z_1 + \cdots + z_l$

As the t_i and the t'_i are empty SP-graphs they are unary, and thus, so are the $(\mathsf{t}_i * a_i)$ and the $(a'_i * \mathsf{t}'_i)$. So t^* is a sum of unary SP-graphs, and therefore is itself unary. Finally, observe that $|\mathrm{R}(z_s)| = |\emptyset| = 0$ while $|\mathrm{R}^*(x_i^k)| = |\{a_i\}| = 1$ and $|\mathrm{R}^*(y_i^l)| = |\{a'_i\}| = 1$. □

7 Sequentialisation of SP-R&B-proof-nets

Let SP-R&B-OK be the class of SP-R&B-graph generated as follows:

- A single B-edge is in SP-R&B-OK.
- SP-R&B-OK is closed under disjoint union.
- SP-R&B-OK is closed under the following operation:
  If $(V; \mathrm{B}, \mathrm{R}_1 + \mathrm{R}_2), (V'; \mathrm{B}, \mathrm{R}'_1 + \mathrm{R}'_2) \in$ SP-R&B-OK, with $V \cup V' = \emptyset$
  then $(V \uplus V'; \mathrm{B} \uplus \mathrm{B}', (\mathrm{R}_1 * \mathrm{R}'_1) + \mathrm{R}_2 + \mathrm{R}'_2) \in$ SP-R&B-OK

The class strict SP-R&B-OKis defined similarly except that we never use the disjoint union operation.

Remark.— These are exactly the translations of the rules of the MLL+MIX sequent calculus given in section 3: a single B-edge is an axiom, disjoint union corresponds exactly to MIX, and the other one exactly corresponds to *tensor*. Remember that the *par* rule does not affect the SP-R&B-graph.

Theorem 27 *An* SP-R&B*-graph is in* SP-R&B-OK *if and only if it is chorded.*

Furthermore, an SP-R&B*-graph is critically chorded if and only if it is in strict* SP-R&B-OK.

The reconstruction of the inductive definition of a chorded SP-R&B*-graph, — i.e. the sequentialisation – may be done in $O(v^2)$: indeed the recognition and reconstruction of an* SP*-graph is linear in the number of arcs plus the number of edges, i.e. less than $O(v^2)$, see [6].*

The proof of this theorem is essentially nothing more than the following lemma, which the previous little propositions of the previous section leads to.

Lemma 28 *Let* G *be a chorded* SP-R&B*-graph. Let* t *a binary tree describing* R*. Then one of the main $*$ operation of* t *defines a complete bipartite graph of* G *which is a bipartite cut of* $\underline{\mathsf{G}}$*.*

Proof: Of course we will rely on the equivalence proved in lemma 14, and used every other line in the little propositions of the previous section.

Let R_1 be the SP-relation corresponding to $\partial\mathsf{t}$, and consider the SP-R&B-graph $\mathsf{G}_1 = (V; \mathrm{B}, \mathrm{R}_1)$. By proposition 17 it is chorded too.

Because R_1 is the parallel composition of complete bipartite graphs and isolated vertices, if we fold all the complete bipartite graphs, as in proposition 26, we obtain an SP-R&B-graph

$$\mathsf{G}_{\mathrm{fan}} = (V \uplus \{a_i, a'_i\}; \mathrm{B} \uplus \{a_i a'_i\}, \mathrm{R}_{\mathrm{fan}})$$

such that

- G_{fan} is chorded, by proposition 25
- R_{fan} is unary, by proposition 26
- the R-degree of each vertex in V is either 0 or 1 by proposition 26

As G_{fan} is an SP-R&B-graph with R_{fan} being unary we know by proposition 18 that G_{fan} is æ-cycle free.

The original set of B-edges $\mathrm{B} \subset \mathrm{B}_{\text{fan}}$ is of course a perfect matching in G_{fan} of the vertices in V, and we know that for every vertex in V, $|\mathrm{R}_{\text{fan}}(x)| \leq 1$. Thus if we unfold all B-edges $xy \in \mathrm{B}$ in G_{fan} we obtain an R&B-graph $G_{\otimes}$ which is still æ-cycle free by proposition 19.

Therefore, by lemma 12 one of the B-edges of G_{fan}, i.e. one $a_i a'_i$ is a B-bridge of $G_{\otimes}$.

Because of proposition 20 $a_i a'_i$ is a B-bridge in G_{fan}. Hence, because of proposition 23 this means that $\mathrm{dom}(\mathsf{t}_i) * \mathrm{dom}(\mathsf{t}'_i)$ is a bipartite cut of $\underline{\mathsf{G}_1}$.

If $\mathrm{dom}(\mathsf{t}_i) * \mathrm{dom}(\mathsf{t}'_i)$ were not a bipartite cut of $\underline{\mathsf{G}}$ there would exists a path $\mathcal{P}$ in $\underline{\mathsf{G}'}$ where $\mathsf{G}' = (V; \mathrm{B}, \mathrm{R} \setminus (\mathrm{dom}(\mathsf{t}_i) * \mathrm{dom}(\mathsf{t}'_i))$ from some $x_i \in \mathrm{dom}(\mathsf{t}_i)$ to some $x'_i \in \mathrm{dom}(\mathsf{t}'_i)$. We can assume that all the vertices of $\mathcal{P}$ but x_i and x'_i are not in $\mathrm{dom}(\mathsf{t}_i) \uplus \mathrm{dom}(\mathsf{t}'_i)$, by shortening $\mathcal{P}$ otherwise.

If $\mathcal{P}$ does not use any R-edge $a_j b_j$ with $j \neq i$ and $a_j, b_j \in \mathrm{dom}(\mathsf{t}_j)$ or $a_j b_j \in \mathrm{dom}(\mathsf{t}'_j)$ then $\mathcal{P}$ is a path in $\underline{\mathsf{G}_1}$ from x_i to x'_i not using any edge of $\mathrm{dom}(\mathsf{t}_i) * \mathrm{dom}(\mathsf{t}'_i)$, contradiction.

Otherwise, any such R-edge may be replaced with two R-edges $a_j c_j$ and $c_j b_j$ where $c_j \in \mathrm{dom}(\mathsf{t}'_j)$ if $a_j b_j \in \mathrm{dom}(\mathsf{t}_j)$ and $c_j \in \mathrm{dom}(\mathsf{t}_j)$ if $a_j b_j \in \mathrm{dom}(\mathsf{t}'_j)$. We thus obtain a path of $\underline{\mathsf{G}_1}$ from x_i to x'_i not using any edge of $\mathrm{dom}(\mathsf{t}_i) * \mathrm{dom}(\mathsf{t}'_i)$, contradiction.

□

Proof: [of Theorem 27]

Any SP-R&B-graph in SP-R&B-OK is chorded This has already been proved in proposition 7.

G is critically chorded if and only if its inductive definition does not use disjoint union. This has already been proved in proposition 8.

Each chorded SP-R&B**-graph is in** SP-R&B-OK If G contains no R-edge, then G is obtained by a sequence of disjoint union (MIXrules) from B-edges (axioms).

Otherwise, let us arbitrarily chose an SP-term denoting R:

$$\mathrm{R}' = (\mathsf{t}_1 * \mathsf{t}'_1) + (\mathsf{t}_2 * \mathsf{t}'_2) + (\mathsf{t}_3 * \mathsf{t}'_3) + \cdots + (\mathsf{t}_p * \mathsf{t}'_p) + z_1 + z_s$$

By lemma 28, we know that one of the main $*$ operations, say $\mathsf{t}_1 * \mathsf{t}'_1$, defines a complete bipartite graph which is a bipartite cut. Let

$$\mathrm{R}' = (\mathsf{t}_1 + \mathsf{t}'_1) + (\mathsf{t}_2 * \mathsf{t}'_2) + (\mathsf{t}_3 * \mathsf{t}'_3) + \cdots + (\mathsf{t}_p * \mathsf{t}'_p) + z_1 + z_s$$

Now the connected components of $\underline{\mathsf{G}'}$ where $\mathsf{G}' = (V; \mathrm{B}, \mathrm{R}')$ defines a partition of the vertices of V, and we may regroup the vertices of the connected components of $\underline{\mathsf{G}'}$ in $\mathrm{dom}(\mathsf{t}_1) \uplus A$ and $\mathrm{dom}(\mathsf{t}'_1) \uplus A' \uplus C$ where:

- A is the set of the vertices $x \in V \setminus \mathrm{dom}(\mathsf{t}_1)$ whose connected component of $\underline{\mathsf{G}'}$ contains some $y \in \mathrm{dom}(\mathsf{t}_1)$.
- A' is the set of the vertices $x \in V \setminus \mathsf{t}'_1$ whose connected component of $\underline{\mathsf{G}'}$ contains some $y \in \mathrm{dom}(\mathsf{t}'_1)$.
- C is the set of the vertices $x \in V$ whose connected component of $\underline{\mathsf{G}'}$ does not contain any $y \in \mathrm{dom}(\mathsf{t}_1) \uplus \mathrm{dom}(\mathsf{t}'_1)$.

Because $\mathsf{t}_1 * \mathsf{t}'_1$ defines a bipartite cut of G, $(A \uplus \mathrm{dom}(\mathsf{t}_1)) \cap (A' \uplus \mathrm{dom}(\mathsf{t}'_1)) = \emptyset$ and thus $V = (\mathrm{dom}(\mathsf{t}_1) \uplus A) \uplus (\mathrm{dom}(\mathsf{t}'_1) \uplus A' \uplus C)$.

Now observe that whenever $xy \in \mathrm{B}$ then either $x, y \in \mathrm{dom}(\mathsf{t}_1) \uplus A$ or $x, y \in \mathrm{dom}(\mathsf{t}'_1) \uplus A'$ or $x, y \in C$.

Thus $\mathrm{B} = \mathrm{B}\big|_{\mathrm{dom}(\mathsf{t}_1)\uplus A} \uplus \mathrm{B}\big|_{\mathrm{dom}(\mathsf{t}'_1)\uplus A'\uplus C}$ and $\mathrm{R}' = \mathsf{t}_1 \uplus \mathsf{t}'_1 \uplus \mathrm{R}\big|_A \uplus \mathrm{R}\big|_{A'\uplus C}$ and therefore $\mathrm{R} = (\mathsf{t}_1 * \mathsf{t}'_1) \uplus \mathrm{R}\big|_A \uplus \mathrm{R}\big|_{A'\uplus C}$.

Consequently G is obtained be *tensor* composition of

$$\mathsf{G}_0 = (V;\ \mathrm{B}\big|_{\mathrm{dom}(\mathsf{t}_1)\uplus A},\ \mathrm{R}\big|_{\mathrm{dom}(\mathsf{t}_1)\uplus A} = \mathsf{t}_1 + \mathrm{R}\big|_A)$$

$$\mathsf{G}'_0 = (V;\ \mathrm{B}\big|_{\mathrm{dom}(\mathsf{t}'_1)\uplus A'\uplus C},\ \mathrm{R}\big|_{\mathrm{dom}(\mathsf{t}'_1)\uplus A'\uplus C} = \mathsf{t}'_1 + \mathrm{R}\big|_{A'})$$

As observed in proposition 13, each of the two SP-R&B-graphs is chorded. The result follows by iterating this argument.

Observe, by the way, that disjoint union (or MIX) is only use between family of isolated B-edges (axioms).

Complexity of reconstructing the inductive definition of an SP-R&B-OK **graph.** As the connected components of a graph may be computed in $O(v^2)$, reconstructing the inductive definition of an SP-graph maybe done in $O(v^3)$. (Maybe there are better algorithm, cf. the previous foot note.)

Once we have an SP-term describing R, computing G_{fan} may be done in linear time. In G_{fan}, the number of edges is proportional to the number of vertices. Computing the bridges is thus performed in $O(v)$. We then find a bridge which is a B-edge $a_i a_i'$ and proceed with the two components. This gives an algorithm in $O(v^2)$.

□

8 Intuitions: relation between R&B and SP-R&B-proof-nets, intermediates structures

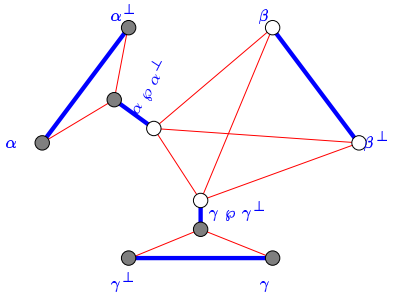
We did mot have a sudden apparition of the abstract SP-R&B-proof-nets. They rather are an outcome of some work on pomset logic [16, 17]. In this calculus there is an SP-order on the conclusions of the formulae, *par* correponding to parallel composition, and before ($<$) corresponding to directed series composition. But the *tensor* does not fit well with this orders, so we added to these the symmetrical series composition, which corresponds to *tensor* — this yields a class of relations containing both series-parallel orders and series-parallel graphs, see [4].

We already knew that it was possible to transform a proof-net with a conclusion $A \parr B$ (resp. $A < B$) by two conclusions which are twins w.r.t. the order on conclusions, in such a way that the proof-net before this expansion is correct if and only if the one after expansion is correct. So what we did here is to treat *tensor* similarly.

Let us define a **generalized** R&B**-proof structure** as a R&B-proof structure plus an SP-graph R_c between the conclusions of the R&B-proof structure. To distinguish the conclusions, which are not anymore pendant, we will denote them with white circles. Let us say that these generalized R&B-proof structure are generalized **MIX** R&B**-proof nets** (resp. generalized R&B-proof-nets) whenever they are chorded (resp. critically chorded).

Observe that the SP-graph R of a generalized R&B-proof structure may be written as: $\mathrm{R} = \mathrm{R}_c \uplus \mathrm{R}_1 \uplus \cdots \uplus \mathrm{R}_k$ where R_c is the SP-graph between conclusions and each R_i is isomorphic to $\mathtt{K}_3$ or $\mathtt{K}_{1,2}$ — which are SP-graphs — i.e. each R_i is a link, respectively *tensor* or *par*.

Here is an example of a generalized R&B-proof structure/net:



A few observations:

- When R_c is empty a generalized R&B-proof structure is an R&B-proof structure. Observe that the (MIX) R&B-proof-nets exactly are (MIX)-R&B-proof-nets. Indeed, because of the link structure an æ-path or cycle in an R&B-proof structure may not contain any chord: the connected components of R in an R&B-proof-net have exactly three vertices, while a chord with the two adjacent R-edges of the æ-cycle or path makes four vertices within the same connected component of R. Hence to say that each æ-cycle contains a chord or to say there is no æ-cycle is the same thing.

- When $k = 0$, i.e. where is no link, the generalized R&B-proof structure is an SP-R&B-proof structure, and the two criteria for being a (MIX) proof net coincide.

Let $\circ$ be either $*$ or $+$, and $\bullet$ be either $\otimes$ or $\wp$ accordingly. Let us denote the other end vertex of the B-edge of a compound subformula C by $\overline{C}$. Thus the R-edges of a link $A \circ B$ are $(A \bullet B) * \overline{A \circ B}$.

Now let us see how we can gradually turn an R&B-proof structure into the corresponding SP-R&B-proof structure, and vice versa, in such a way that the (MIX)-R&Bproof-net correspond to the (MIX) SP-R&B-proof nets. For instance the example given above of a generalized R&B-proof net is an intermediate step between the example of an R&B-proof net given in section 2 and the example given in section 3.

8.1 From R&B-proof-nets to SP-R&B-proof-nets

Let $A \circ B$ be any compound conclusion of a generalized R&B-proof-net G, the other conclusions being $C_1, \ldots, C_n$, and the SP-graph between conclusions being R_c, possibly described by the SP-term t_c.

Consider the following transformation of G. Suppress the final $\circ$ link $A \circ B$, thus obtaining an R&B-proof-net with conclusions $A, B, C_1, \ldots C_n$, and replace the SP-graph $\mathsf{t}_c(A \circ B, C_2, \ldots, C_n)$ with

$$\mathsf{t}_c((A \bullet B), C_1, \ldots, C_2, \ldots, C_n)$$

This is a special instance of the unfolding described in subsection 6.2. It is the unfolding at the B-edge $(A \circ B)\overline{(A \circ B)}$ and proposition 21 applies in this case. So we know by proposition 22 that the result of this transformation is chorded too — and critically chorded if the original was, this is not difficult to see.

If we iterate this transformation starting with an R&B-proof-net we obtain the SP-R&B-graph corresponding to the R&B-proof-net we started with, since the only conclusions will be the atoms of the sequent and the SP-graph the sequent written as an SP-graph. Observe that the obtained SP-R&B-proof structure is an SP-R&B-proof net when the (MIX) R&B-proof-net with started with is a (MIX) R&B-proof net.

8.2 From SP-R&B-proof-nets to R&B-proof-nets

Now consider a generalized R&B-proof-net, with relation R_c between conclusions. Because of proposition 5, if it has more than a single conclusion, than it contains two twins conclusions w.r.t. the SP-graph R_c or R since $\mathrm{R} = \mathrm{R}_c + \mathrm{R}_1 + \mathrm{R}_p$.

Thus there exists an SP-term describing R with the following pattern:

$$\mathsf{t}\big((C \bullet C'), C_1, \ldots, C_n\big) + \mathsf{t}_1 + \cdots + \mathsf{t}_p$$

where the t_i correspond to links.

As C and C' are twin conclusions, $\mathsf{t}\big((C \bullet C'), C_1, \ldots, C_n\big)$ contains the complete bipartite graph $\{C, C'\} * (\mathrm{R}(C) \setminus \{C'\})$ — since $\mathrm{R}(C) \setminus \{C'\} = \mathrm{R}(C') \setminus \{C\}$ for twin vertices. So, let us fold this complete bipartite graph as in subsection 6.2, and let us call $\overline{C \circ C'}$ and $C \circ C'$ the two fresh vertices linked by the new B-edge. As we have seen in proposition 24 the resulting SP-term is:

$$\mathsf{t}\big((C \circ C'), C_1, \ldots, C_n\big) + \big((C \bullet C') * \overline{C \circ C'}\big) + \mathsf{t}_1 + \cdots + \mathsf{t}_p$$

That is to say we have added a $\circ$ link the premises of which are C and C', while the new conclusion is $C \circ C'$ replaces C (or C', they were twins) in the SP-graph on conclusions. Furthermore, because of the proposition 25, the generalized R&B-proof-net thus obtained is still chorded, and clearly an SP-R&B-graph.

If we iterate this transformation starting with an SP-R&B-proof-net until there are no more twin vertices in R, i.e. until the generalized proof-net has a single conclusion, we obtain an R&B-proof-structure which is chorded, i.e. an R&B-proof-net, i.e. æ-cycle free.

Notice that as expected, unlike the converse transformation, this one is not canonical but relies on the choices of twin vertices, i.e. in the choices of an SP-term for describing the sequent — i.e. associativity of $\otimes, \wp$ and the presence or absence of the final *par*'s.

8.2.1 Another proof of sequentialisation

These transformations provides another way to prove the sequentialisation of SP-R&B-proof-nets by reducing it to the sequentialisation of R&B-proof-nets. But to work it out with full details is unfortunately not simpler than the direct proof we gave.

9 A graph-theoretical consequence

In [4] we have shown that:

Theorem 29 *The following rewrite system, up to commutativity and associativity, is a complete rewrite system for the inclusion of series-parallel graphs.*

$$\begin{array}{lrcl}
\xrightarrow[(4)]{} : & (t+t')*(u+u') & \xrightarrow[(4)]{} & (t+u)*(t'+u') \\
\xrightarrow[(3)]{} : & t*(u+u') & \xrightarrow[(3)]{} & (t*u)+u' \\
\xrightarrow[(2)]{} : & t*u & \xrightarrow[(2)]{} & t+u
\end{array}$$

This means that an SP-graph $G = (V; \mathrm{R})$ is included in an SP-graph $G = (V; \mathrm{R}')$ whenever the SP-term t describing R (which is unique up to commutativity and associativity) rewrites onto t' using associativity and commutativity of $*$ and $+$ and the rewriting rules $\xrightarrow[(4)]{}, \xrightarrow[(3)]{}$ *and* $\xrightarrow[(2)]{}$.

Proposition 30 *Regarding the correctness of* SP-R&B*-proof-nets and* MIX-SP-R&B*proof-nets, rewrite rules for* SP*graphs behave as follows. Given* $\Pi = (V; \mathrm{B}, \mathrm{R})$*, and* $\Pi' = (V; \mathrm{B}, \mathrm{R}')$ *let us write* $\Pi \xrightarrow[(2)]{} \Pi'$ *whenever* $\mathrm{R} \xrightarrow[(2)]{} \mathrm{R}'$*,* $\Pi \xrightarrow[(3)]{} \Pi'$ *whenever* $\mathrm{R} \xrightarrow[(3)]{} \mathrm{R}'$*etc.*

If $\Pi \xrightarrow[(3)]{} \Pi'$ **then** $(\Pi\ chorded) \Rightarrow (\Pi'\ chorded)$
$(\Pi\ critically\ chorded) \Rightarrow (\Pi'\ critically\ chorded)$

If $\Pi \xrightarrow[(2)]{} \Pi'$ **then** $(\Pi\ chorded) \Rightarrow (\Pi'\ chorded)$

None of the aforementioned implications is an equivalence, and every unmentioned preservation property fails — in particular $\xrightarrow[(4)]{}$ *does not preserve the chordedness.*

Proof: There are two ways to prove this, the first one being a direct proof on SP-R&B-graphs.

They also can be proven logically: an SP rewrite rule corresponds to a deduction rule on MLL formulae. When the deduction rule is valid, the fact that proof maps onto correct SP-R&B-graphs ensure that the SPrewrite rule preserves the correctness. When the deduction rule is not valid, finding a counter example yields a counter example on the SP-R&B-graphs because of the sequentialisation theorem. □

It is clear that any *chorded* SP-R&B-graph G with vertices $V = \{p_i, p'_i, 1 \leq i \leq n\}$ and B-edges $\mathrm{B} = \{(p_i, p'_i), 1 \leq i \leq n\}$ is included into the SP-R&B-graph with the same vertices and B-edges and the following R-edges:

$$\mathrm{R} = \bar{\mathrm{B}} = \{xy | x \in V,\ y \in V\ xy \notin \mathrm{B}\} = \mathbin{*}_{1 \leq i \leq n}(p_i + p'_i)$$

Indeed, G being chorded, there cannot be any R-edge $p_i p'_i$. Let us call these SP-R&B-graphs the **complete** SP-R&B-**graphs**, since their underlying graph is a complete graph. Complete SP-R&B-graph obviously are proof-nets, i.e. are critically chorded. The sequent calculus proof simply consists in axioms $\vdash p_i, p_i^{\perp}$, followed by $\wp$ rules yielding $\vdash p_i \wp p_i^{\perp}$, and then by $\otimes$ rules yielding $\vdash \bigotimes_{1 \leq i \leq n}(p_i \wp p_i^{\perp})$. [5]

Because of the previous theorem and proposition, we know that:

- All the SP-R&B-graphs with $2n$ vertices which are derived from the complete SP-R&B-graphs with $2n$ vertices by $\xrightarrow[(3)]{}$ (resp. $\xrightarrow[(2)]{}$ and $\xrightarrow[(3)]{}$) are chorded (resp. critically chorded).

[5] For simplicity I assume the p_i to be distinct. Otherwise, they should be distinguished as vertices of the graph, and then identifi ed as propositional variables.

- All SP-R&B-graphs (chorded or not) are derived from the corresponding complete SP-R&B-graph by $\overrightarrow{(4)}$, $\overrightarrow{(3)}$, $\overrightarrow{(2)}$.

So a natural question is whether, if we restrain the rewriting system to the rules preserving correction ($\overrightarrow{(3)}$ for MLL, $\overrightarrow{(3)}$ and $\overrightarrow{(2)}$ for MLL+mix) do we obtain *exactly all* the correct proof-nets (critically chorded for MLL, chorded for MLL+mix) by rewriting from the complete SP-R&B-graphs?

The answer is yes:

Theorem 31 *Let $\Pi = (V; \mathrm{B}, \mathrm{R})$ a chorded* SP-R&B*-graph; then* $\bar{\mathrm{B}}$ *rewrites to* R *using only $\overrightarrow{(3)}$ and $\overrightarrow{(2)}$ but not $\overrightarrow{(4)}$ — thus any* SP-R&B*-graph appearing in the sequence is itself chorded. Furthermore if Π is critically chorded the rewriting rule $\overrightarrow{(2)}$ is not used — thus any* SP-R&B*-graph appearing in the sequence is itself critically chorded.*

Presently we only have a logical proof of this result — thus, in some sense, this is an application of linear logic to graph theory. The proof is an immediate combination of sequentialisation of SP-R&B-proof-nets, theorem 6, proposition 30 and of the following well-known result, which is easily established by induction on sequent calculus proofs:

Proposition 32 *Let*

$$AX = \left\{ \bigotimes_{1\leq i\leq n} (p_i \parr p_i{}^{\perp}) \,\middle|\, \{p_i, 1 \leq i \leq n\}\text{: multiset of propositional variables} \right\}$$

Let $\overrightarrow{(3-LL)}$ and $\overrightarrow{(2-LL)}$ be the following rewrite rules, modulo commutativity and associativity:

$\overrightarrow{(3-LL)}$ $F \otimes (G \parr H) \to (F \otimes G) \parr H$

$\overrightarrow{(2-LL)}$ $F \otimes G \to F \parr G$

*Then the closure of AX under $\overrightarrow{(3-LL)}$ (resp. $\overrightarrow{(3-LL)}$ and $\overrightarrow{(2-LL)}$) exactly is the set of the theorems of MLL (resp. MLL+*MIX*).*

10 Conclusion

We hope that we have convinced the reader that the combinatorial structure of proof-nets is a deep one which may be presented in a neat and standard way.

Of course this approach suggests various questions and further developments.

For instance we did not say a word on cut elimination. In case all atoms are atomic, a cut consists in a *tensor* between two dual formulae, that is to say a complete bipartite graph between two SP-graph which are the complement one of the other. Clearly cut-elimination is to replace this complete bipartite graph with a set of pairwise non-adjacent R-edges, linking each atom with the corresponding dual atom. To eliminate the cuts, we need to compute vertices on both side gets linked by æ-paths, when we pass back and forth inside the two parts corresponding to the cuts formulae. This is quite close to what Girard once called turbo cut elimination. Although the preservation of the correctness obviously holds, it is not an immediate property, but may be deduced from our work. The reduction of cuts in case there are not only atomic axioms involves some computing on the SP-graph: first check that the single point on one side of the *tensor* rule is labeled by the dual of the R-relation expressed in the other parts. Then it is observed that the elimination of this cut, i.e. of the axiom is nothing less than a contraction of this edge, which will produce, as expected, a replacement of the cut-formulae by its SP-graph.

Another question is to adapt SP-R&B-proof-nets for non-commutative linear logic. The representation and simple criterion obtained by Abrusci and Maringelli [14, 3] with R&B-proof nets, gives hope for this. If such a presentation was found, i do believe it would bring some lights on the open question of the complexity of decidability of the calculus of Lambek, Abrusci and Yetter, since there is a wide range of well-known algorithms for such structures.

An easier question is to find a similar description of the proof-nets for Pomset logic [16, 17] in the setting of SP-R&B-graphs, since we actually started with this question. This is already done in [18]. But a more difficult result is an inductive definition for the resulting SP-R&B-graphs, that is to say a complete sequent calculus for Pomset logic.

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