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► **To cite this version:**

Sylvie Boldo, Jean-Michel Muller. Some Functions Computable with a Fused-mac. 17th IEEE Symposium on Computer Arithmetic, Jun 2005, Cape Cod, Massachusetts, USA, pp.52-58. inria-00000895

**HAL Id: inria-00000895**

**<https://inria.hal.science/inria-00000895v1>**

Submitted on 1 Dec 2005

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# Some Functions Computable with a Fused-mac

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## Abstract

*The fused multiply accumulate instruction (fused-mac) that is available on some current processors such as the Power PC or the Itanium eases some calculations. We give examples of some floating-point functions (such as  $\text{ulp}(x)$  or  $\text{Nextafter}(x, y)$ ), or some useful tests, that are easily computable using a fused-mac. Then, we show that, with rounding to the nearest, the error of a fused-mac instruction is exactly representable as the sum of two floating-point numbers. We give an algorithm that computes that error.*

## 1 Introduction

The fused multiply accumulate instruction (fused-mac) is available on some current processors such as the IBM Power PC or the Intel/HP Itanium. That instruction computes an expression  $\pm ax \pm b$  with one final rounding error only. This allows one to perform correctly rounded division using Newton-Raphson division [17, 7, 16] (the main idea behind that is that if  $q$  approximates  $x/y$  with enough accuracy, then the remainder  $x - yq$  will be exactly computed with a fused-mac, allowing to correct the quotient estimation). Also, this makes evaluation of scalar products and polynomials faster and, generally, more accurate than with conventional (addition and multiplication) floating-point operations. This is important, since scalar products appear everywhere in linear algebra, and since polynomials are very often used for approximating functions.

It is well known [9, 2, 3] that (assuming rounding to nearest) the error of a floating point addition or the remainder of a square root is exactly representable as a floating-point number of the same format. This is also true (for any rounding mode) for the error of a multiplication or the remainder of a division. A natural question arises: is there a similar property for the fused-mac operation?

Also, expert floating-point programming sometimes requires the evaluation of functions such as  $\text{Nextafter}(x, y)$ ,

or the successor of a given floating-point number, or (for error estimation),  $\text{ulp}(x)$ . We may also, for some calculations, need to know if the last bit of the significand of a number is a zero [4]. These various functions can always be computed at a low level, using masks and integer arithmetic: this results in software that is not portable, and sometimes quite slow, since the corresponding calculations are not performed in the floating-point pipeline. With conventional arithmetic, designing portable software for these functions is feasible [5] but might be costly. We aim at showing that the availability of a fused-mac instruction facilitates portable yet efficient implementation of such functions.

## 2 Definitions and notations

Define  $\mathbb{M}_n$  as the set of exponent-unbounded,  $n$ -bit significand, binary floating-point (FP) numbers (with  $n \geq 1$ ), that is:  $\mathbb{M}_n = \{M \times 2^E, 2^{n-1} \leq M \leq 2^n - 1, M, E \in \mathbb{Z}\} \cup \{0\}$ . It is an “ideal” system, with no overflows or underflows. We will show results in  $\mathbb{M}_n$ . These results will remain true in actual systems that implement the IEEE-754 standard [6, 1], provided that no overflows or underflows do occur. The **mantissa** or **significand** of a nonzero element  $M \times 2^E$  of  $\mathbb{M}_n$  is the number  $m(x) = M/2^{n-1}$ , its **integral significand**, noted  $M_x$  is  $M$  and its corresponding **exponent**, noted  $e_x$  is  $E$ . We assume that the reader is familiar with the basic notions of floating-point arithmetic: rounding modes, ulps, ... See [10] for definitions. In the following  $\circ(t)$  means  $t$  rounded to the nearest even.

## 3 Previous results and preliminary properties

We will use the 2sum and Fast2Sum algorithm, presented below. They do not require the availability of a fused-mac, and make it possible to compute the error of a

floating-point addition exactly, represented by an FP number. The first one [14, 18] only assumes  $a$  and  $b$  are normalized FP numbers (i.e., elements of  $\mathbb{M}_n$ ).

**Property 1 (2Sum Algorithm)** *Let  $a, b \in \mathbb{M}_n$ . Define  $x$  and  $y$  as*

$$\begin{aligned} x &= \circ(a + b) \\ b' &= \circ(x - a) \\ a' &= \circ(x - b') \\ \epsilon_b &= \circ(b - b') \\ \epsilon_a &= \circ(a - a') \\ y &= \circ(\epsilon_a + \epsilon_b) \end{aligned}$$

We have:

- $x + y = a + b$  exactly;
- $|y| \leq \frac{1}{2} \text{ulp}(x)$ . □

If we know in advance that  $|a| \geq |b|$  (as a matter of fact, it suffices to have  $e_a \geq e_b$ ), a much faster algorithm can be used [9, 14]:

**Property 2 (Fast2Sum Algorithm)** *Let  $a, b \in \mathbb{M}_n$ , with  $|a| \geq |b|$ . Define  $x$  and  $y$  as*

$$\begin{aligned} x &= \circ(a + b) \\ b' &= \circ(x - a) \\ y &= \circ(b - b') \end{aligned}$$

We have:

- $x + y = a + b$  exactly;
- $|y| \leq \frac{1}{2} \text{ulp}(x)$ . □

Although we have presented these properties assuming a radix-2 number system, it is worth being noticed that the 2Sum algorithm (property 1) works in any radix  $\geq 2$ , and that the Fast2Sum algorithm (property 2) works in radices 2 and 3. And yet, rounding to nearest is mandatory: with “directed” roundings it is possible [14] to exhibit cases where the difference between the computed value of  $a + b$  and the exact value cannot be exactly expressed as an FP number.

The 2Sum algorithm satisfies the following property, that will be needed in Section 5.

**Property 3** *If  $(x, y) = 2\text{Sum}(a, b)$  then  $|y| \leq |b|$ .* □

**Proof.**  $x$  is the FP number that is closest to  $(a + b)$ . This implies that  $x$  is closer to  $(a + b)$  than  $a$ . Hence,  $|(a + b) - x| = |y|$  is smaller than  $|(a + b) - a| = |b|$ . □

A well known and useful property of the fused-mac instruction, noticed by Karp and Markstein [13], is that it allows to very quickly compute the product of two FP numbers  $x$  and  $y$  exactly, expressed as the sum of two FP numbers  $u$  and  $v$ . More precisely,

**Property 4 (Fast2Mult Algorithm)** *Let  $a, b \in \mathbb{M}_n$ . Define  $x$  and  $y$  as*

$$\begin{aligned} x &= \circ(ab) \\ y &= \circ(ab - x) \end{aligned}$$

we have:

- $x + y = ab$  exactly;
- $|y| \leq \frac{1}{2} \text{ulp}(x)$ . □

Without a fused-mac, computing  $x$  and  $y$  is possible, but requires much more computation [9] (the significands of  $x$  and  $y$  are splitted, then partial products are computed and summed up).

## 4 Basic functions computable with a fused-mac

### 4.1 Checking if the last bit of the significand of some number is a zero

Brisebarre, Muller and Raina [4] have suggested an algorithm for division by a constant that works when the last bit of the divisor significand is a zero. Checking that condition is easily done with a fused-mac.

**Property 5 (Algorithm IsEven)** *The following algorithm on  $x$  checks if the last significand bit of  $x$  is a zero.*

$$\begin{aligned} \alpha &= \circ(3x) \\ \beta &= \circ(\alpha - 2x) \\ \text{IsEven} &= (\beta = x) \end{aligned}$$

□

One may notice that the same algorithm also works with the usual (addition and multiplication) floating-point instructions. The availability of a fused-mac, here, only saves one operation.

## 4.2 Checking if a number is a power of 2.

The following algorithm requires storage of the constant

$$C = 2^n - 1.$$

Of course,  $C \in \mathbb{M}_n$ : it is exactly representable as a floating-point number.

**Property 6 (Algorithm IsAPowerOf2)** *The following algorithm on  $x$  returns “true” if  $x$  is a power of 2.*

$$\begin{aligned} y_h &= \circ(xC) \\ y_\ell &= \circ(xC - y_h) \\ \text{IsAPowerOf2} &= (y_\ell = 0). \end{aligned}$$

□

**Proof** if  $x$  is not a power of 2 then  $M_x$  has at least a prime factor different from 2, thus  $M_x C$  is of the form  $P2^\alpha$ , where  $P$  is odd and larger than  $2^n$ . Hence  $P$  cannot be exactly representable with  $n$  bits, hence  $y_h \neq xC$ , hence  $y_\ell \neq 0$ . □

**Important remark** The above given algorithm works in the “ideal” set  $\mathbb{M}_n$ , which means that with “real world” floating-point arithmetic it will work provided that no overflow or underflow occur. To minimize the risk of overflow/underflow, one should choose

$$C = (2^n - 1)/(2^n),$$

instead of the previously given constant. The proof will be the same, overflow will never occur, and underflow will occur only where  $x$  is a subnormal FP number.

## 4.3 Floating-point successors

There are several notions of “floating-point successor” that can be defined. The IEEE-754 standard for FP arithmetic<sup>1</sup> [1] *recommends* (but does not *require*) the availability of function `Nextafter`. `Nextafter( $x, y$ )` returns the next representable neighbor of  $x$  in the direction toward  $y$ . If  $x = y$ , then the result is  $x$  without any exception being signaled. If either  $x$  or  $y$  is a NaN, then the result is a NaN. Overflow is signaled when  $x$  is finite but `Nextafter( $x, y$ )` is infinite; underflow is signaled when the result is subnormal or zero. Cody and Coonen [5] provide a portable C version of that function.

Let us show how such a function can be implemented using fused-mac instructions. First, define the following four functions.

<sup>1</sup>See <http://754r.ucbtest.org/standards/754.txt>

**Definition 1** *The successor of an FP number  $x$ , denoted  $x^+$  is the smallest FP number larger than  $x$ . The predecessor  $x^-$  of  $x$  is the largest FP number less than  $x$ . The symmetrical successor of  $x$ , denoted  $\text{succ}(x)$  is  $x^-$  if  $x < 0$ , and  $x^+$  if  $x > 0$ . The symmetrical predecessor  $\text{pred}(x)$  of  $x$  is  $x^+$  if  $x < 0$  and  $x^-$  if  $x > 0$ .*

The following algorithm will use the constant

$$s = 2^{-n} + 2^{-2n+1}.$$

Notice that  $s \in \mathbb{M}_n$ . Even on “real life” floating-point systems,  $s$  will be representable: on all floating-point systems of current use, the number of significand bits is less than the absolute value of the smallest exponent. This is required by the IEEE-854 Standard for Floating-Point arithmetic [12], that says that  $(E_{max} - E_{min})/n$  shall exceed 5 and should exceed 10, and that  $b^{E_{max}+E_{min}+1}$  should be the smallest integral power of  $b$ , that is greater than or equal to 4, where  $b$  is the radix.

**Property 7** *Computation of  $\text{succ}(x)$  If  $n \geq 2$  and  $x \neq 0$ , then*

$$\text{succ}(x) = \circ(x + sx)$$

□

**Proof** Assume  $x > 0$  and  $2^e \leq x < 2^{e+1}$  (i.e., the exponent of  $x$  is  $e$ ). Since, in that case,  $\text{succ}(x) = x + 2^{e-n+1}$  and  $\text{ulp}(x) = 2^{e-n+1}$ , to show that  $\circ(x + sx)$  is equal to  $\text{succ}(x)$  it suffices to show that

$$x + 2^{e-n} < x + sx < x + 3 \times 2^{e-n}$$

(i.e., that  $x + sx$  is within  $1/2\text{ulp}$  from  $\text{succ}(x)$ ).

Thus, it suffices to show that

$$2^{e-n} < sx < 3 \times 2^{e-n}. \quad (1)$$

Since  $x \geq 2^e$ ,  $sx > 2^{e-n}$ . Since  $x < 2^{e+1}$ ,  $sx < 2(1 + 2^{-n+1})2^{e-n}$ , which is less than  $3 \times 2^{e-n}$  as soon as  $n \geq 2$ . □

Property 7 shows that  $\text{succ}(x)$  can be computed with one fused-mac only.

Function  $\text{pred}(x)$  is also computable with one fused-mac only. The proof is very similar to that of Property 7.

**Property 8** *Computation of  $\text{pred}(x)$  If  $n \geq 2$  and  $x \neq 0$ , then*

$$\text{pred}(x) = \circ(x - sx)$$

□

Now, from functions `succ` and `pred`, one can very easily compute functions `Nextafter`,  $x^+$  and  $x^-$ :

**Property 9** *if  $x \neq 0$  then*

$$\begin{aligned} x^+ &= \circ(x + s|x|) \\ x^- &= \circ(x - s|x|) \\ \text{Nextafter}(x, y) &= \begin{cases} x^+ & \text{if } y > x \\ x & \text{if } y = x \\ x^- & \text{if } y < x \end{cases} \end{aligned}$$

□

**Important remark:** although we have proven these algorithms assuming an ideal FP arithmetic with unbounded exponents, they work well with “real life” arithmetic. From the definition of `succ`( $x$ ), underflow is impossible. Also, if  $|x|$  is equal to the largest representable FP number, then on a machine compliant with the IEEE 754 standard,  $\pm\infty$  (depending on the sign of  $x$ ) will be returned<sup>2</sup>, which is the right answer. If  $x$  is a NaN, then the fused-mac operation will return a NaN. Hence, our algorithm for `succ`( $x$ ) is always correct, unless  $x$  is a subnormal number. Function `pred`( $x$ ) cannot generate an overflow, correctly propagates NaNs, and correctly signal underflows, however, it does not work correctly if  $x$  is a subnormal number: that (rare) case should be handled separately.

If we use rounding to nearest, then a fused-mac instruction looks mandatory for designing such algorithms (at least, they cannot be implemented using one addition or multiplication). For example:

**Property 10** *Apart from the “toy case”  $n \leq 2$ , there is no constant  $C \in \mathbb{M}_n$  such that  $\circ(xC)$  always equals `succ`( $x$ ).*

□

**Proof:** Suppose that there exists  $C \in \mathbb{M}_n$  such that  $\circ(xC)$  always equals `succ`( $x$ ). Assume  $1 \leq x < 2$  (the other cases are easily deduced from this one). This implies

$$x + 2^{-n} \leq Cx \leq x + 3 \times 2^{-n}.$$

Hence,

$$2^{-n} \leq (C - 1)x \leq 3 \times 2^{-n}$$

for any  $x \in \mathbb{M}_n$ ,  $1 \leq x < 2$ . For  $x = 1$ , this implies  $C \geq 1 + 2^{-n}$ . Since the smallest element of  $\mathbb{M}_n$  larger than or equal to  $1 + 2^{-n}$  is  $1 + 2^{-n+1}$ , we then have  $C \geq 1 + 2^{-n+1}$ . And yet, for  $x$  equal to the largest element

<sup>2</sup>This is due to the definition of rounding to the nearest: the standard specifies that *An infinitely precise result with magnitude at least  $2^{Emax}(2 - 2^{-n})$  shall round to  $\infty$  with no change in sign.*

of  $\mathbb{M}_n$  less than 2 (i.e.,  $2 - 2^{-n+1}$ ),  $C \geq 1 + 2^{-n+1}$  implies  $(C - 1)x \geq 2^{-n+1}(2 - 2^{-n+1}) = 4 \times 2^{-n} - 2^{-2n+2}$ . Therefore, in that case,  $(C - 1)x > 3 \times 2^{-n}$ , unless  $n \leq 2$ . □

This may be different with other rounding modes. For instance, if rounding towards zero  $\mathcal{Z}(x)$  is used, then  $\mathcal{Z}(x\sigma)$  returns `pred`( $x$ ) for any nonzero  $x \in \mathbb{M}_n$ , with  $\sigma = 1 - 2^{-n}$ . One can also implement  $x^+$  as  $(x + \epsilon)$  rounded towards  $+\infty$ , where  $\epsilon$  is the smallest positive nonzero subnormal number. And yet, in practice, changing the rounding mode may be quite time consuming: this is why an algorithm that works in the default mode (i.e., round-to-nearest) is preferable.

#### 4.4 Function `ulp`( $x$ )

Function `ulp` (unit in the last place) is very frequently used for expressing the accuracy of a floating-point result. Several definitions have been given (see [11] for a discussion on that topic), they differ near the powers of 2. If we use as a definition, when  $x$  is an FP number:

$$\text{ulp}(x) = |x|^+ - |x|$$

then (if  $x \in \mathbb{M}_n$  is nonzero) one can compute function `ulp` through the following sequence

$$\begin{aligned} y &= \circ(x + sx) \\ \text{ulp} &= |y - x| \end{aligned}$$

where  $s$  is the same constant as in Section 4.3. If we define `ulp`( $x$ ) as

$$\text{ulp}(x) = |x| - |x|^-$$

then function `ulp` is computed through

$$\begin{aligned} y &= \circ(x - sx) \\ \text{ulp} &= |y - x| \end{aligned}$$

The two functions differ only when  $x$  is a power of 2. The first one is compatible with Goldberg’s definition [10] (which is given for *real* numbers, not only for floating-point ones), the second is compatible with Kahan’s one<sup>3</sup> and Harrison’s one [11] (they differ for real numbers but coincide on FP numbers).

## 5 Computing the error term of a fused-mac

We require here that  $n \geq 3$ . The correcting term cannot be a single FP number, even in rounding to the nearest. We will therefore compute two FP numbers such that their sum is the exact correcting term of the fused-mac.

<sup>3</sup>Kahan’s definition is: `ulp`( $x$ ) is the gap between the two finite floating-point numbers nearest  $x$ , even if  $x$  is one of them (But `ulp`(NaN) is NaN.)

## 5.1 The algorithm ErrFmac

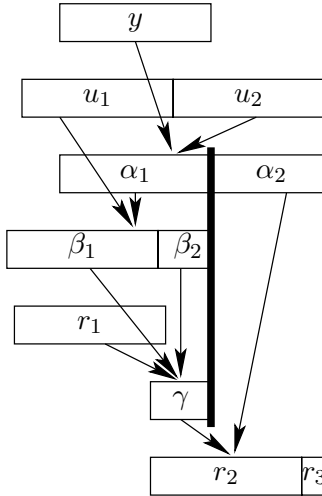
**Property 11 (Algorithm ErrFmac)** Let  $a, x, y \in \mathbb{M}_n$ . Define  $r_1, r_2$  and  $r_3$  as

$$\begin{aligned} r_1 &= \circ(ax + y) \\ (u_1, u_2) &= \text{Fast2Mult}(a, x) \\ (\alpha_1, \alpha_2) &= \text{2Sum}(y, u_2) \\ (\beta_1, \beta_2) &= \text{2Sum}(u_1, \alpha_1) \\ \gamma &= \circ(\circ(\beta_1 - r_1) + \beta_2) \\ (r_2, r_3) &= \text{Fast2Sum}(\gamma, \alpha_2) \end{aligned}$$

we have:

- $ax + y = r_1 + r_2 + r_3$  exactly;
- $|r_2 + r_3| \leq \frac{1}{2}\text{ulp}(r_1)$ ;
- $|r_3| \leq \frac{1}{2}\text{ulp}(r_2)$ . □

Figure 1 gives the idea behind the algorithm: we want to exactly add the 3 FP numbers  $y, u_1$  and  $u_2$ . This is usually difficult, but as we know the correct answer ( $r_1$ ) thanks to the fused-mac computation, we just have to get the two error terms. We first compute the “small” error, namely  $\alpha_2$ . Then the other terms  $u_1$  and  $\alpha_1$  are bigger than this value and can be combined with  $r_1$  into a single value  $\gamma$ .



**Figure 1. Intermediate values of the ErrFmac algorithm.**

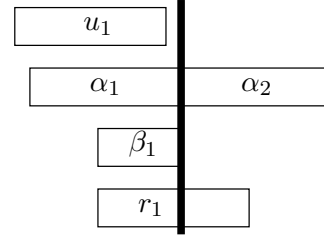
## 5.2 Proof of the correctness of the ErrFmac algorithm

If  $\gamma = \circ(\circ(\beta_1 - r_1) + \beta_2)$  is equal to  $(\beta_1 - r_1) + \beta_2$ , then  $r_1 + r_2 + r_3 = r_1 + \gamma + \alpha_2 = r_1 + \beta_1 - r_1 + \beta_2 + \alpha_2 =$

$u_1 + \alpha_1 + \alpha_2 = u_1 + u_2 + y = y + ax$ . If this equality holds, we easily also have that  $|r_2 + r_3| \leq \frac{1}{2}\text{ulp}(r_1)$  and  $|r_3| \leq \frac{1}{2}\text{ulp}(r_2)$ , from previous properties.

There is left to prove that  $\beta_1 - r_1$  and  $(\beta_1 - r_1) + \beta_2$  are in  $\mathbb{M}_n$ . If they are, then they are exactly computed and the algorithm is correct. To guarantee that a value  $v$  is in  $\mathbb{M}_n$ , we just have to find an exponent  $e$  such that  $v2^{-e}$  is an integer and  $|v2^{-e}| < 2^n$ . There may exist more than one suitable  $e$ , but the existence of one is enough. We split the proof into two subcases.

**If we have  $\beta_2 = 0$ ,**



**Figure 2. Intermediate values when  $\beta_2 = 0$ .**

Figure 2 reminds the compared positions of the FP numbers involved. As  $\beta_2 = 0$ , we have left to prove that  $\beta_1 - r_1$  is in  $\mathbb{M}_n$ . If  $\beta_1 = 0$ , then this is correct. Let us assume that  $\beta_1 \neq 0$ . We then know that  $r_1 = \circ(\beta_1 + \alpha_2)$  as  $\beta_2 = 0$ .

But we also have that  $|\alpha_2| \leq \frac{1}{2}\text{ulp}(\alpha_1)$  from Property 1 and that  $|\alpha_2| \leq |u_2| \leq \frac{1}{2}\text{ulp}(u_1)$  from Property 3 and by definition  $\beta_1 = \circ(u_1 + \alpha_1)$ . This means that  $|\alpha_2| \ll |\beta_1|$ . More precisely, we either have:

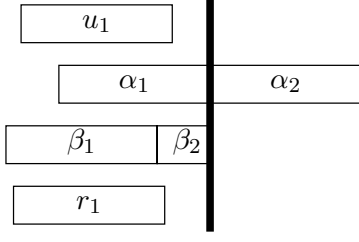
- the general case:  $|\beta_1| \geq 4|\alpha_2|$ ;
- the special case where  $\beta_1$  is a result of a near-total cancellation:  $\beta_1 = 2^{\min(e_{u_1}, e_{\alpha_1})}$  and  $|\beta_1| \geq 2|\alpha_2|$ .

In the general case, we are in the conditions of Sterbenz’s theorem [19]:  $r_1$  and  $\beta_1$  share the same sign and

$$\begin{aligned} |r_1| &\leq \frac{|\beta_1 + \alpha_2|}{1 - 2^{-n}} \leq \frac{5}{4} \frac{1}{1 - 2^{-n}} |\beta_1| \leq 2|\beta_1| \\ |r_1| &\geq \frac{|\beta_1 + \alpha_2|}{1 + 2^{-n}} \geq \frac{3}{4} \frac{1}{1 + 2^{-n}} |\beta_1| \geq \frac{1}{2} |\beta_1| \end{aligned}$$

In the special case, we have  $4|\alpha_2| > |\beta_1| \geq 2|\alpha_2|$ . As  $\beta_1$  is a power of 2, we know that  $e_{\beta_1} - 1 \leq e_{r_1} \leq e_{\beta_1}$ , so  $e_{r_1}$  is a suitable exponent for  $\beta_1 - r_1$  and

$$\begin{aligned} |\beta_1 - r_1|2^{-e_{r_1}} &= |\beta_1 - \circ(\beta_1 + \alpha_2)|2^{-e_{r_1}} \\ &\leq \left( \frac{1}{2}\text{ulp}(r_1) + |\alpha_2| \right) 2^{-e_{r_1}} \\ &\leq \frac{1}{2} + |\beta_1|2^{-e_{r_1}-1} \\ &\leq \frac{1}{2} + (2^n - 1)2^{e_{r_1}+1-e_{r_1}-1} < 2^n. \end{aligned}$$



**Figure 3. Intermediate values when  $\beta_2 \neq 0$ .**

If we have  $\beta_2 \neq 0$ ,

Figure 3 reminds the compared positions of the FP numbers involved. In the general case, we have here that  $\beta_1 = r_1$ , then of course  $\beta_1 - r_1 = 0$  and  $(\beta_1 - r_1) + \beta_2 = \beta_2$  are in  $\mathbb{M}_n$ . If not, as  $\beta_2 \neq 0$ , the only possibility for  $\beta_1 = \circ(\beta_1 + \beta_2)$  not to be equal to  $\circ(\beta_1 + \beta_2 + \alpha_2) = r_1$  is that either  $|\beta_2| = \frac{1}{2}\text{ulp}(\beta_1)$  or  $\beta_2 = -\frac{1}{4}\text{ulp}(\beta_1)$  if  $\beta_1$  is a power of 2.

We also deduce that the exponent of  $r_1$  and of  $\beta_1$  differ from at most 1. Lastly, we know that  $|\alpha_2| \leq |\beta_2| \leq 2^{e_{\beta_1}-1}$ . The value  $\min(e_{r_1}, e_{\beta_1})$  is a suitable exponent for  $\beta_1 - r_1$  and

$$\begin{aligned} & |\beta_1 - r_1|2^{-\min(e_{r_1}, e_{\beta_1})} \\ &= |\beta_1 - \circ(\beta_1 + \beta_2 + \alpha_2)|2^{-\min(e_{r_1}, e_{\beta_1})} \\ &\leq \left(\frac{1}{2}\text{ulp}(r_1) + |\beta_2| + |\alpha_2|\right)2^{-\min(e_{r_1}, e_{\beta_1})} \\ &\leq (2^{e_{r_1}-1} + 2^{e_{\beta_1}-1} + 2^{e_{\beta_1}-1})2^{-\min(e_{r_1}, e_{\beta_1})} \leq 4 \end{aligned}$$

So  $\beta_1 - r_1 \in \mathbb{M}_n$  as  $n \geq 3$ . There is left to prove that  $(\beta_1 - r_1) + \beta_2 = u_1 + \alpha_1 - r_1$  is in  $\mathbb{M}_n$ . We know that  $e_{\beta_1} + 1 \geq e_{r_1} \geq e_{\beta_1} - 1$  and that  $\beta_2$  is either  $2^{e_{\beta_1}-1}$  or  $2^{e_{\beta_1}-2}$ , so  $e_{\beta_1} - 2$  is a suitable exponent for  $(\beta_1 - r_1) + \beta_2$  and

$$\begin{aligned} & |(\beta_1 - r_1) + \beta_2|2^{-e_{\beta_1}+2} \\ &= |u_1 + \alpha_1 - \circ(u_1 + \alpha_1 + \alpha_2)|2^{-e_{\beta_1}+2} \\ &\leq \left(\frac{1}{2}\text{ulp}(r_1) + |\alpha_2|\right)2^{-e_{\beta_1}+2} \\ &\leq (2^{e_{r_1}-1} + 2^{e_{\beta_1}-1})2^{-e_{\beta_1}+2} \leq 6 \end{aligned}$$

So  $(\beta_1 - r_1) + \beta_2 \in \mathbb{M}_n$  as  $n \geq 3$ .  $\square$

### 5.3 With other rounding modes

Such correcting terms for the fused-mac are only representable when the rounding is to the nearest. For example, when rounding up, if  $a = x = 2^n - 1$  and  $y = 2^{4n}$  then  $ax + y = 2^{4n} + 2^{2n} - 2^{n+1} + 1$  and therefore  $r_1$  must be strictly greater than  $2^{4n}$  so  $r_1 = \Delta(ax + y) = 2^{4n} + 2^{3n+1}$ .

So  $r_2 + r_3$  should be exactly equal to  $-2^{3n+1} + 2^{2n} - 2^{n+1} + 1$  that cannot be represented as the sum of two FP numbers in  $\mathbb{M}_n$ .

### 5.4 Cost of the algorithm

The basic cost of the algorithm is 20 fused-mac delays (FMD), but this can be tremendously reduced.

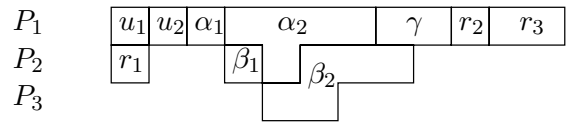
The first enhancement is when we know that  $|y| \geq |ax|$  or that  $|y| \geq |u_1|$ . Then, the first 2Sum is useless as  $\alpha_1 = y$  and  $\alpha_2 = u_2$ . This is typically the case in range reduction [8, 15].

The second enhancement is to get rid of the final Fast2Sum: this means that the result will not be compressed. It means that we only have:

- $ax + y = r_1 + r_2 + r_3$  exactly;
- $|r_2 + r_3| \leq \frac{1}{2}\text{ulp}(r_1)$ ;
- $r_2 = 0$  or  $|r_2| > |r_3|$ .

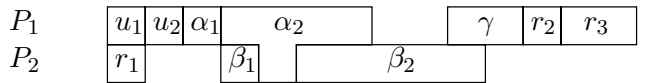
The last enhancement is if the processor can use several floating-point units (FPUs) in parallel. There are indeed several computations that can be done either at the same time or at consecutive steps in a pipe-line, as there is no dependence between them. For example, the computations of  $a'$  and  $\epsilon_b$  in the 2Sum algorithm (Property 1) can be performed in parallel.

If 3 FPUs are available, the algorithm only costs 12 FMDs. The tasks given to each processor are given in Figure 4. More FPUs are useless to speed up the algorithm.



**Figure 4. Task repartition when 3 FPUs are available.**

If only 2 FPUs are available, the algorithm costs 14 FMDs. The tasks given to each processor are shown in Figure 5.



**Figure 5. Task repartition when 2 FPUs are available.**

The following table gives the cost of the ErrFmac algorithm depending on the conditions (number of FPUs, final

compression and knowledge that the inequality  $|y| \geq |ax|$  holds):

Cost (in FMDs)	1 FPU	2 FPUs	3 FPUs
Given algorithm	20	14	12
Without the final compression	17	11	9
When $ y  \geq  ax $	14	10	10
When $ y  \geq  ax $ and without the final compression	11	7	7

## 6 Conclusion

We have shown that the fused-mac instruction makes it possible to implement efficiently and in a portable way many functions that are useful for expert floating-point programming. We also have shown that, assuming rounding to nearest, the error of a fused-mac operation in a given format is exactly representable as a sum of two floating-point numbers of the same format. We have given a fast and portable algorithm that returns that error. We can take advantage of this algorithm for implementing a very accurate range reduction.

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