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## Minimizing spurious diapycnal mixing induced by horizontal tracer advection

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James C. McWilliams

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# Minimizing spurious diapycnal mixing induced by horizontal tracer advection

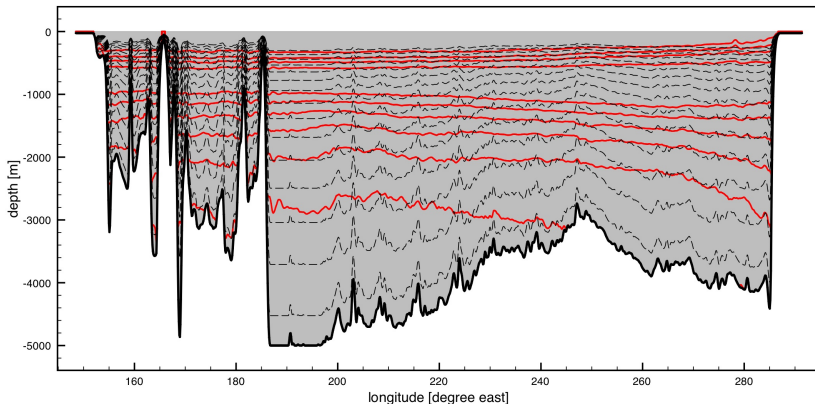
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and J.C. McWilliams<sup>1</sup>

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## Spurious diapycnal mixing, why ?

- Strong horizontal mixing along iso-sigma surfaces can lead to spurious diapycnal fluxes of heat and salt (and hence density) when it is applied in regions with sloping isopycnals.



*Vertical section in the southern Pacific*

# Outline

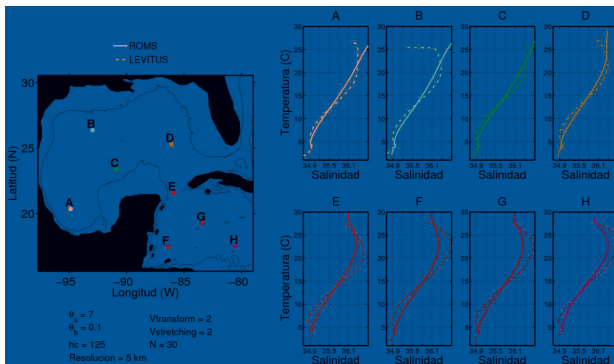
- 1 Introduction and motivations
- 2 Neutral diffusion in ocean models
- 3 Advection and rotated diffusion coupling
- 4 Preliminary numerical test
- 5 Conclusion and perspectives

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## Identification of the problem

- The implicit diffusion associated to *odd-order* advection schemes may be large enough to produce excessive diapycnal mixing.
- South West Pacific Region (Marchesiello *et. al*, 2009 ; Couvelard *et. al*, 2008)
- Gulf of Mexico



Courtesy of Lorena Guerrero and Julio Sheinbaum

## Why do we like the third order upstream advection scheme (U3H / UP3)?

- *odd-ordered* scheme = next higher ordered scheme plus a dissipation term

$$\tilde{q}_{i+\frac{1}{2}} = \underbrace{\frac{7(q_i + q_{i+1}) - (q_{i-1} + q_{i+2})}{12}}_{4^{\text{th}} \text{ order advection}} + \text{Sign}(u_{i+\frac{1}{2}}) \underbrace{\frac{3(q_{i+1} - q_i) - (q_{i+2} - q_{i-1}))}{12}}_{\text{implicit diffusion}}$$

- Vertical advection not considered in this talk (small inherent diffusion)

### 3rd upstream-biased, parabolic interpolation

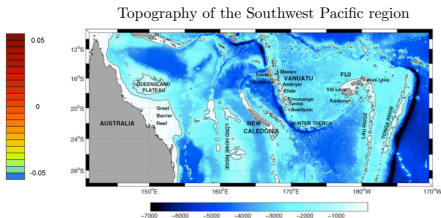
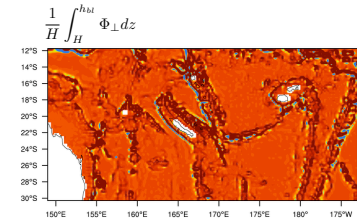
- Good compromise between accuracy, stability, simplicity and computational efficiency
- Provides a resolution dependent and scale-selective parameterization of dissipation

⇒ **However the adiabatic property of advection is broken**

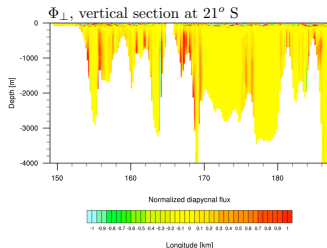
# Diapycnal component of the implicit diffusion

→ Online computation :

$$\Phi_{\perp} = \left( (\vec{F}_{C4} - \vec{F}_{UP3}) \cdot \hat{\rho} \right) \rho_3 \text{ with } \hat{\rho} = (\rho_1, \rho_2, \rho_3)^t = \frac{\vec{\nabla} \rho^{(ad)}}{\|\vec{\nabla} \rho^{(ad)}\|}$$

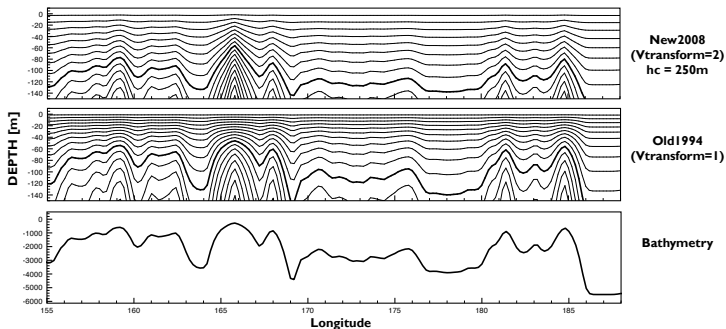


(Marchesio *et al.*, 2009)



## A first remedy : tuning of the vertical grid (and bathymetry smoothing)

- Take advantage of the absence of restriction  $h_c < h_{\min}$  with the new ("z -  $\sigma$ ") coordinates  $\Rightarrow$  flattening of the near surface  $\sigma$ -levels

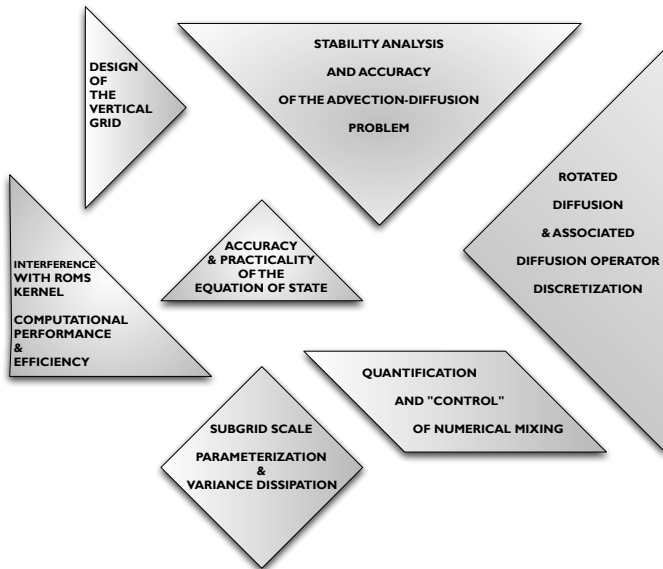


Shchepetkin's law :  $h_c \approx 1.5 \times H_{\text{cline}}$

- Bathymetry smoothing : see scaling of diapycnal mixing in Marchesiello *et al*, 2009

## Putting the Pieces of the Puzzle Together

→ The aim is not to suppress diffusion but to keep it "under control"



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## Neutral diffusion in ocean models

→ Isopycnal mixing via **Coordinate Rotation** (Redi, 1982)

$$\mathcal{D} = \nabla \cdot (K_h \mathbf{K} \cdot \nabla q)$$

$\mathbf{K}$  = diffusion (Redi) tensor

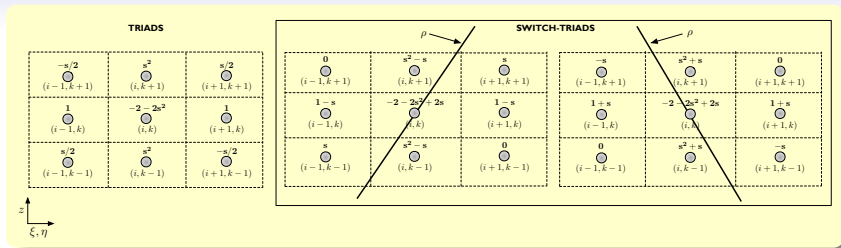
→ Example in 2D ( $\xi - z$ )

$$\mathcal{D} = K_h \partial_\xi^2 q \quad \rightarrow \quad \frac{K_h}{1 + \left[ \frac{\partial_\xi \rho}{\partial_z \rho} \right]^2} \left( \partial_\xi - \left[ \frac{\partial_\xi \rho}{\partial_z \rho} \right] \partial_z \right)^2 q$$

In the following  $\vartheta = - \left[ \frac{\partial_\xi \rho}{\partial_z \rho} \right] = \text{const}$

→ Local computation of **neutral directions** (cf Sasha's talk)

$$\mathcal{D} = \frac{1}{\xi} \left[ F_{i+\frac{1}{2},k}^{(\xi)} - F_{i-\frac{1}{2},k}^{(\xi)} \right]$$



- **TRIADS Scheme** ( Griffies *et al.*, 1998)

⇒ Self-adjoint property

$$F_{i+\frac{1}{2},k}^{(\xi)} = K_h \left[ \frac{q_{i+1,k} - q_{i,k}}{\Delta \xi} - \frac{1}{4} \frac{\rho_{i+1,k} - \rho_{i,k}}{\Delta \xi} \right] \left\{ \frac{q_{i,k} - q_{i,k-1}}{\rho_{i,k} - \rho_{i,k-1}} + \frac{q_{i+1,k+1} - q_{i+1,k}}{\rho_{i+1,k+1} - \rho_{i+1,k}} + \frac{q_{i,k+1} - q_{i,k}}{\rho_{i,k+1} - \rho_{i,k}} + \frac{q_{i+1,k} - q_{i+1,k-1}}{\rho_{i+1,k} - \rho_{i+1,k-1}} \right\}$$

- **SWITCH Scheme** (Beckers *et al.*, 2000, Shchepetkin, 1998 (unpublished) )

$$F_{i+\frac{1}{2},k}^{(\xi)} = K_h \left[ \frac{q_{i+1,k} - q_{i,k}}{\Delta \xi} - \frac{1}{2} \frac{\max(\rho_{i+1,k} - \rho_{i,k}, 0)}{\Delta \xi} \left\{ \frac{q_{i,k} - q_{i,k-1}}{\rho_{i,k} - \rho_{i,k-1}} + \frac{q_{i+1,k+1} - q_{i+1,k}}{\rho_{i+1,k+1} - \rho_{i+1,k}} \right\} - \frac{1}{2} \frac{\min(\rho_{i+1,k} - \rho_{i,k}, 0)}{\Delta \xi} \left\{ \frac{q_{i,k+1} - q_{i,k}}{\rho_{i,k+1} - \rho_{i,k}} + \frac{q_{i+1,k} - q_{i+1,k-1}}{\rho_{i+1,k} - \rho_{i+1,k-1}} \right\} \right]$$

# Stability and accuracy of the isopycnal-mixing equation

- **Truncation error**

$$\varepsilon = \mathcal{N}(\mathcal{D}) - \mathcal{D} = \frac{\Delta\xi^2}{12} \partial_\xi^4 q + \vartheta^2 \frac{\Delta z_{k+\frac{1}{2}}^3 + \Delta z_{k-\frac{1}{2}}^3}{24\Delta z_k} \partial_z^4 q + \vartheta \frac{(\Delta z_{k+\frac{1}{2}} + \Delta z_{k-\frac{1}{2}})\Delta\xi}{4} \partial_\xi^2 \partial_z^2 q$$
$$+ 2\vartheta \left[ \frac{\Delta z_{k+\frac{1}{2}}^2 + \Delta z_{k-\frac{1}{2}}^2}{12} \partial_\xi \partial_z^3 q + \frac{\Delta\xi^2}{6} \partial_\xi^3 \partial_z q \right] + \mathcal{O}(\Delta z^3, \Delta\xi^3, \Delta\xi^2 \Delta z, \Delta\xi \Delta z^2)$$

⇒ the rotation adds terms of dispersive nature (SWITH-TRIADS ⇒ green diffusive term)

- **Stability**

with Euler time stepping :

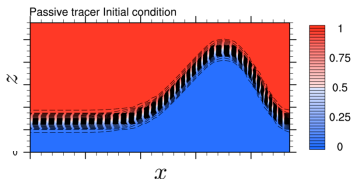
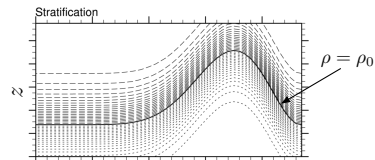
$$q^{n+1} - q^n = \Delta t K_h \left( \partial_\xi^2 q^n + 2\vartheta \partial_\xi \partial_z q^n + \vartheta^2 [(1-\alpha) \partial_z^2 q^n + \alpha \partial_z^2 q^{n+1}] \right)$$

Stability constraint :

$$\alpha = 0 \quad \rightarrow \quad \frac{K_h \Delta t}{\Delta\xi^2} \left( 1 + \frac{\Delta z^2}{\Delta\xi^2} \vartheta^2 \right) \leq \frac{1}{2} \quad ; \quad \alpha = 1 \quad \rightarrow \quad \frac{K_h \Delta t}{\Delta\xi^2} \leq \frac{1}{2}$$

For the full tensor and  $\alpha = 0$   $\rightarrow \frac{K_h \Delta t}{\Delta\xi^2} \leq \frac{1}{2}$ .

- Isopycnal diffusion should not affect a tracer homogeneous along isopycnals



- Diffusion equation discretized in  $(x,z)$
- The diffusion is rotated along isopycnals

Two stratifications :

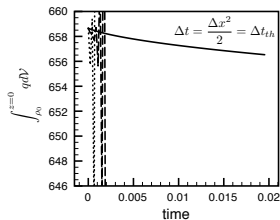
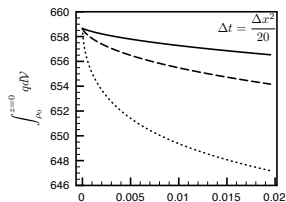
- 1 Large slope  $s_{\max} = 2$  ( $\approx 48^\circ$ )
- 2 Moderate slope  $s_{\max} = 0.5$  ( $\approx 25^\circ$ )

Two discretizations of the tensor :

- 1 Small slope approximation
- 2 Full tensor

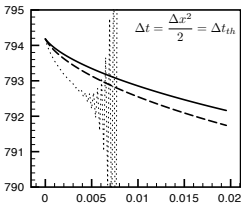
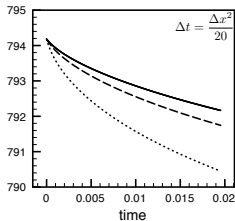
# Numerical stability, overshots and cross-isopycnal flux

Case 1 :  $s_{\max} = 2$

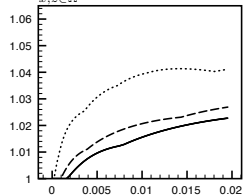


- Full Tensor
- - - Small slope [SWITCH]
- ..... Small slope [TRIADS]

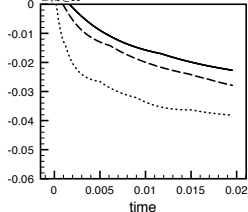
Case 2 :  $s_{\max} = 0.5$



$\max(q)$   
 $x, z \in \Omega$



$\min(q)$   
 $x, z \in \Omega$



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### RSUP3

$$\left\{ \begin{array}{l} q^{n+1,*} = q^{n-1} - 2\Delta t [u\partial_x q^n]_{C4} \\ q^{n+1/2} = \frac{5}{12}q^{n+1,*} + \frac{2}{3}q^n - \frac{1}{12}q^{n-1} \\ q^{n+1} = q^n - \Delta t [u\partial_x q^{n+1/2}]_{C4} + \Delta t \mu(\Delta x, u) \Delta^2 (\alpha_1 q^{n+1/2} + \alpha_2 q^n) \\ \text{with } \Delta = (\partial_x + \vartheta \partial_z)^2 \end{array} \right.$$

However let's look if we can find

- either a simpler alternative
- or an alternative with a wider stability range

- **Idea** : take advantage of the form of the truncation error of the first-order upstream scheme (i.e. an anisotropic Laplacian in 2D)

### Scheme 1

$$\left\{ \begin{array}{l}
 q^{n+1,\star} = q^{n-1} - 2\Delta t [u\partial_x q^n]_{C4} \\
 q^{n+1/2} = \frac{5}{12}q^{n+1,\star} + \frac{2}{3}q^n - \frac{1}{12}q^{n-1} \\
 q^{n+1} = q^n - \Delta t [u\partial_x q^{n+1/2}]_{C4} \\
 q^\star = q^n - \Delta t [u\partial_x q^n]_{UP1} \\
 q^{n+1} = q^{n+1} - \mu(\Delta x)\Delta(q^\star - q^{n+1}) \qquad \Delta = \partial_x^2
 \end{array} \right.$$

## Scheme 1 rotated anisotropic "implicit" laplacian

$$\left\{ \begin{array}{l} q^{n+1,\star} = q^{n-1} - 2\Delta t [u\partial_x q^n]_{C4} \\ q^{n+1/2} = \frac{5}{12}q^{n+1,\star} + \frac{2}{3}q^n - \frac{1}{12}q^{n-1} \\ q^{n+1} = q^n - \Delta t [u\partial_x q^{n+1/2}]_{C4} \\ q^\star = q^n - \Delta t [u\partial_x q^n]_{UP1} \\ q^{n+1} = q^{n+1} - \mu(\Delta x)\Delta(q^\star - q^{n+1}) \end{array} \right. \quad \Delta = (\partial_x + \vartheta\partial_z)^2$$

- extremely simple to implement and computationally efficient
- does not correspond exactly to a rotated biharmonic

Rotated biharmonic = Rotation of a non-rotated Laplacian +  $\mathcal{O}(\vartheta^3, \vartheta^4)$

Perfectly valid for small slopes and thus for rotation along geopotentials

- aims at reducing spurious mixing to an acceptable level but not at suppressing it

## Scheme 2

$$\left\{ \begin{array}{l}
 q^{n+1,*} = q^{n-1} - 2\Delta t [u\partial_x q^n]_{C4} \\
 q^{n+1,**} = q^{n+1,*} + 2\Delta t \Delta(q^n) \quad \Delta = \partial_x^2 \\
 q^{n+1/2} = \frac{5}{12}q^{n+1,*} + \frac{2}{3}q^n - \frac{1}{12}q^{n-1} \\
 q^{n+1} = q^n - \Delta t [u\partial_x q^{n+1/2}]_{C4} \\
 q^{n+1} = q^{n+1} - \mu(\Delta x, u)\Delta(q^{n+1,**} - q^{n+1}) \quad \Delta = \partial_x^2
 \end{array} \right.$$

- We split the biharmonic in a first laplacian during the predictor and a second in the corrector

## Scheme 2 rotated

$$\left\{ \begin{array}{l}
 q^{n+1,*} = q^{n-1} - 2\Delta t [u\partial_x q^n]_{C4} \\
 q^{n+1,**} = q^{n+1,*} + 2\Delta t \Delta(q^n) \quad \Delta = (\partial_x + \vartheta\partial_z)^2 \\
 q^{n+1/2} = \frac{5}{12}q^{n+1,*} + \frac{2}{3}q^n - \frac{1}{12}q^{n-1} \\
 q^{n+1} = q^n - \Delta t [u\partial_x q^{n+1/2}]_{C4} \\
 q^{n+1} = q^{n+1} - \mu(\Delta x, u)\Delta(q^{n+1,**} - q^{n+1}) \quad \Delta = (\partial_x + \vartheta\partial_z)^2
 \end{array} \right.$$

- Two rotated laplacians are simpler than a rotated biharmonic
- makes it possible to derive an unconditionally stable scheme

## Scheme 2\* (for the vertical component of the tensor)

$$\left\{ \begin{array}{l} q^{n+1,*} = q^{n-1} - 2\Delta t [w\partial_z q^n] \\ q^{n+1,**} = q^{n+1,*} + 2\Delta t \Delta (q^{n+1,**}) \\ q^{n+1/2} = \frac{5}{12} q^{n+1,*} + \frac{2}{3} q^n - \frac{1}{12} q^{n-1} \\ q^{n+1} = q^n - \Delta t [w\partial_z q^{n+1/2}] + \mu \Delta (q^{n+1} - q^{n+1,**}) \end{array} \right. \quad \Delta = \partial_z^2$$

- resolution of one (possibly two) **tridiagonal systems** at each time steps
- this results into an **implicit-in-time biharmonic** in the vertical direction
- however no need to decrease the time step
- but increase of the **complexity of the code**

- **Scheme 1**

$$\mathcal{R}(q) = \partial_{\xi}^4 q + \vartheta^2 \partial_z^2 \partial_{\xi}^2 q + 2\vartheta \partial_{\xi}^3 \partial_z q$$

⇒ Should not require any additional limiters for stability.

- **Scheme 2**

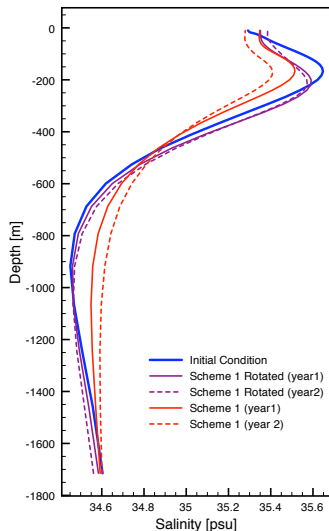
$$\mathcal{R}(q) = \partial_{\xi}^4 q + 6\vartheta^2 \partial_z^2 \partial_{\xi}^2 q + 4\vartheta \partial_{\xi}^3 \partial_z q + 4\vartheta^3 \partial_{\xi} \partial_z^3 q + \vartheta^4 \partial_z^4 q$$

⇒ Need to incorporate *ad-hoc* prescriptions in order to keep stability for steep slopes.

- **Scheme 2\*** : unconditionally stable

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ROMS-UCLA : Southwest Pacific Region

$\Delta x = 20\text{km}$

- 1 Scheme 1 (no rotation)
- 2 Scheme 1 rotated along geopotentials

For this particular testcase no need to decrease the time step

$$\Delta t = \Delta t_{\text{rotated}} = 2160s$$

(N=40 vertical levels)

⇒ reduction of the error by 60%

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## en route to a physical parameterization of diapycnal mixing for the ocean interior

By restoring (part of) the adiabaticity in the ocean interior we open the door to a closure scheme for diapycnal mixing

$$\mathbf{K}^{\text{small}} = K_h \begin{pmatrix} 1 & 0 & \vartheta_x \\ 0 & 1 & \vartheta_y \\ \vartheta_x & \vartheta_y & \varepsilon + \vartheta_y^2 + \vartheta_x^2 \end{pmatrix}$$

$\varepsilon = \frac{K_D}{K_h}$

- The formalism of the tensor allows for the definition of a diapycnal diffusivity  $K_D$
- e.g. connection between internal waves and diapycnal mixing

# Putting the Pieces of the Puzzle Together : the missing piece !

