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Control of an NV center as a two-qubit system [★]

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Abstract: Nitrogen-vacancy (NV) centers are promising experimental platforms for quantum information processing. In this paper, we investigate their controllability and fundamental quantum speed limit for two-qubit gates. Such a quantum system consists of two coupled spins, an electronic and a nuclear spin, where only the former can be controlled directly via microwave pulses. We discuss the various physical approximations that lead to the system model before studying its controllability. We use this control issue as an example to demonstrate how standard geometric control tools can be applied to spin networks. We complete this analysis with a computation of the quantum speed limit using known analytical techniques on Lie groups and their algebras. We finally demonstrate, thanks to preliminary optimal control numerical experiments, that this limit can be approached while keeping a reasonable energy of the control field.

Keywords: spin systems, NV centers, quantum control

1. INTRODUCTION

The development of quantum technologies for computing or sensing requires an extremely high level of control due to the fragility of quantum systems, which are highly dependent on the interaction with their environment Acín et al. (2018). This fragility delays the transfer of results obtained in the laboratory in a well-controlled environment to devices manufactured on a larger scale in an industrial setting. Therefore, the emergence of these new technologies requires the significant development of control techniques that consider the limitations and uncertainties of the device under study Koch et al. (2022); Brif et al. (2010). Optimal quantum control is a rapidly developing field that provides a toolbox of mathematical and numerical techniques for manipulating quantum systems using external electromagnetic fields Borziet al. (2017); Boscain et al. (2021); Ansel et al. (2024). Optimal control performs basic operations while minimizing costs such as control time, and satisfying the experimental constraints imposed by the setup. This approach is currently being applied systematically in many platforms Koch et al. (2022), ranging from cold atoms to superconducting qubits and spin systems in color centers Rembold et al. (2020). In this

study, our focus is on the latter, specifically a benchmark control problem: the optimal control of an electronic spin coupled to a nuclear spin, in which only the former can be controlled. There has been extensive research in the literature on the control of spin systems, with general analyses of the controllability of spin networks Albertini and D'Alessandro (2018, 2021) and geometric optimal control based on Lie algebra structures Khaneja et al. (2001) or the Pontryagin maximum principle Bonnard et al. (2012); Bonnard and Sugny (2012), as well as numerical optimal control Schulte-Herbrüggen et al. (2005); Machnes et al. (2011). We propose to use these approaches to demonstrate how control theory tools can be applied to study the controllability of an Nitrogen-Vacancy (NV) center system Doherty et al. (2013), and to estimate the associated quantum speed limit (QSL) for state-to-state transfer via numerical optimal control.

The paper is organized as follows. First, we describe the model system and the various physical approximations that are used to derive it. A general controllability analysis is then performed. Finally, we demonstrate how a lower bound of the QSL can be derived from a Cartan decomposition of the Hamiltonian Lie algebra and how a gradient-based algorithm allows us to approach this fundamental limit while keeping a bounded control amplitude. Conclusions and perspectives are presented in the final section.

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2. MODEL

In quantum mechanics, the evolution of a system is defined through the Schrödinger equation (with $\hbar = 1$) as:

$$\dot{U}(t) = -iH(t)U(t), \quad U(0) = \mathbb{1}, \quad (1)$$

where $H(t) = H_0(t) + \sum_{j=1}^m u_j(t)H_j(t)$ is the system Hamiltonian and where $U(t)$ is the state evolution operator, also called propagator. In this work we consider the system composed of two spin-1/2 particles coupled through dipole-dipole hyperfine coupling, so the associated Hamiltonian is:

$$H_{\text{LAB}} = \omega_S S_z \otimes \mathbb{1} + \omega_I \mathbb{1} \otimes I_z + A_{\parallel} S_z \otimes I_z + A_{\perp} S_z \otimes I_x + u S_x \otimes \mathbb{1} + v \mathbb{1} \otimes I_x \quad (2)$$

where ω_S and ω_I are the zero-field splitting of each individual spin, A_{\parallel}, A_{\perp} are the parallel and perpendicular hyperfine coupling strength, respectively, and u and v are the controls. Up to multiplication by the complex unit i , the matrices used to define H_{LAB} belong to the Lie algebra $\mathfrak{su}(4)$ (skew-Hermitian traceless matrices) of the special unitary group $SU(4)$ on which the state $U(t)$ lives. These matrices are obtained from tensor products of the standard Pauli Matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

More precisely, the spin matrices are defined as $S_{\alpha} = I_{\alpha} = \sigma_{\alpha}/2$ for $\alpha \in \{x, y, z\}$, and the S_{α} matrices act on the electron spin while the I_{α} ones act on the nuclear spin. To simplify the notation, we adopt the standard conventions: $S_{\alpha} = S_{\alpha} \otimes \mathbb{1}$, $I_{\alpha} = \mathbb{1} \otimes I_{\alpha}$ and $S_{\alpha} I_{\beta} = S_{\alpha} \otimes I_{\beta}$.

2.1 Rotating-wave approximation

The goal of the rotating-wave approximation (RWA) is to understand which terms oscillate faster than the system response time in the Schrödinger equation so that they can be time-averaged (Burgarth et al. (2022)). This process simplifies the system model and its analytical study. The system dynamic time scale of driven quantum systems such as (2) is given by the strengths of the control fields, here u and v . For standard NV center experimental studies, they are in the MHz and KHz regimes, respectively, while the zero-field splitting of the electron spin ω_S is in the GHz regime. Using the reference Hamiltonian $H_R = \omega_S S_z$, the rotating frame Hamiltonian H_{RF} is defined as $U(t)^{\dagger}(H - H_R)U(t)$ with $U(t) = e^{-itH_R}$. In other words:

$$H_{\text{RF}} = \omega_I I_z + A_{\parallel} S_z I_z + A_{\perp} S_z I_x + u \cos(\omega_S t) S_x - u \sin(\omega_S t) S_y + v I_x.$$

The control pulse driving the electron spin is modulated around the central frequency ω_S and can therefore be expressed as $u(t) = u_0(t) \cos(\omega_S t + \phi_S(t))$; $u_0(t)$ is linked to the Rabi frequency, the frequency of the spin population oscillation, $\Omega_R(t) = |u_0(t)|/2$ and $\phi_S(t)$ is the azimuthal angle defining the rotation axis in the Bloch sphere representation. In our study, we assume that the control field strength is bounded in amplitude, $|u_0(t)| \leq \Omega$. The Hamiltonian can thus be rewritten according to

$$H_{\text{RF}} = \omega_I I_z + A_{\parallel} S_z I_z + A_{\perp} S_z I_x + \frac{1}{2} u_0(t) [\cos(2\omega_S t + \phi_S(t)) + \cos(\phi_S(t))] S_x - \frac{1}{2} u_0(t) [\sin(2\omega_S t + \phi_S(t)) + \sin(\phi_S(t))] S_y + v I_x. \quad (3)$$

Now, define the time averaged Hamiltonian

$$H_{\text{RWA}} := \overline{H}_{\text{RF}} = \omega_I I_z + A_{\parallel} S_z I_z + A_{\perp} S_z I_x + \frac{1}{2} u_0 \cos(\phi_S) S_x - \frac{1}{2} u_0 \sin(\phi_S) S_y + v I_x. \quad (4)$$

We follow Corollary 1 in (Burgarth et al. (2022)) and consider a family of integrable time-dependent Hamiltonians $H_{\kappa}(t)$, self-adjoint and bounded for all $t \in [0, T]$, indexed by $\kappa \in R \subset \mathbb{R}$, a subset with a limit point κ_0 . If

$$\|S_{\kappa}\|_{\infty, T} (1 + \|H_{\kappa}\|_{1, T}) \rightarrow 0 \text{ as } \kappa \rightarrow \kappa_0,$$

then $U_{\kappa}(t) \rightarrow \overline{U}(t)$ uniformly for $t \in [0, T]$ as $\kappa \rightarrow \kappa_0$, and the difference between the exact propagator $U_{\kappa}(t)$ and the propagator \overline{U} associated with the time averaged Hamiltonian \overline{H} satisfies the estimate

$$\|U_{\kappa}(t) - \overline{U}(t)\| \leq \|S_{\kappa}\|_{\infty, T} (1 + \|H_{\kappa}\|_{1, T} + \|\overline{H}\|_{1, T}).$$

In the previous expressions, we have used the notation

$$S_{\kappa}(t) := \int_0^t (H_{\kappa}(\tau) - \overline{H}(\tau)) d\tau.$$

with the L^{∞} and L^1 norms of an operator-valued function $A(t)$ as

$$\|A\|_{\infty, T} = \text{ess sup}_{\tau \in [0, t]} \|A\|, \quad \|A\|_{1, T} = \int_0^t \|A\| d\tau.$$

respectively, with $t > 0$ and $\|A\|$ the usual spectral norm. In our case, $H_{\kappa}(t)$ is the rotating-frame Hamiltonian H_{RF} and the parameter κ is the frequency ω_S . One has

$$\|S_{\text{RF}}\|_{\infty, T} \leq \frac{\Omega}{4\omega_S}$$

and

$$\|H_{\text{RF}}\|_{1, T} \leq T(\omega_I + A_{\parallel} + A_{\perp}) + \frac{\Omega}{2\omega_S} (2\omega_S + 1).$$

As a result, $\|S_{\text{RF}}\|_{\infty, T} (1 + \|H_{\text{RF}}\|_{1, T}) \rightarrow 0$ when $\Omega/\omega_S \rightarrow 0$. This approximation is relevant for our NV center system as $\Omega/\omega_S \lesssim 10^{-4}$. Moreover, since $\|H_{\text{RWA}}\|_{1, T} \leq T(\omega_I + A_{\parallel} + A_{\perp})$, the convergence error for a time $T > 0$ is bounded by

$$\|U_{\text{RF}} - U_{\text{RWA}}\| \leq \frac{\Omega}{4\omega_S} \left(1 + 2Tc + \frac{\Omega}{2\omega_S} (2\omega_S + 1) \right).$$

with $c = \omega_I + A_{\parallel} + A_{\perp}$.

3. CONTROLLABILITY

In this section, we will study the controllability of (1) with Hamiltonian (4) whose propagators live in the Lie group $SU(4)$. There are different notions of controllability for the Schrödinger equation, and we will focus on the standard notion of state controllability for finite dimensional control systems. The approach consists in evaluating the reachable set of system (1) when controls span some admissible set \mathcal{U} (typically a subset of measurable essentially bounded functions). Because of the particular structure of the system, one has the following result (see, e.g., D'Alessandro (2007)):

Theorem 1. (Lie algebra rank condition). Consider the system (1), the set of reachable states is the connected Lie group whose Lie algebra \mathcal{L} is generated by the family $\{-iH(u), u \in \mathcal{U}\}$.

In our framework, consider the Hamiltonian (4). In order to apply Theorem 1, we consider

$$H_{\text{RWA}} = (H_0 + u_1 H_1 + u_2 H_2 + v H_3)$$

with

$$\begin{aligned} H_0 &= \omega_I I_z + A_{\parallel} S_z I_z + A_{\perp} S_z I_x, \\ H_1 &= S_x, \\ H_2 &= S_y, \\ H_3 &= I_x, \end{aligned}$$

under the constraint $u_1^2 + u_2^2 \leq 1$ (this convexification does not change the reachable set). So the set \mathcal{U} of admissible controls is comprised of measurable controls u_1, u_2 and v , with (u_1, u_2) valued in the disc, and v also assumed to be bounded, valued in $[-1, 1]$. We can actually consider the family of available Hamiltonians to be $\{iH_0, iH_1, iH_2, iH_3\}$, noting that the drift vector field defined by iH_0 is recurrent and can be included. We will use the notation

$$iH_{jk} := [iH_j, iH_k], \quad iH_{jkl} := [iH_j, [iH_k, iH_l]],$$

etc.

Proposition 2. The system associated with Hamiltonian (4) with A_{\parallel} and A_{\perp} both positive is controllable: the propagator can be an arbitrary element of $SU(4)$.

Proof. It is enough to prove that the Lie span

$$\text{Lie}\{iH_0, iH_1, iH_2, iH_3\}$$

is equal to $\mathfrak{su}(4)$, which has a natural basis for $\alpha, \beta \in \{x, y, z, 0\}$,

$$\mathfrak{su}(4) = \text{span}\{iS_{\alpha}I_{\beta} : (\alpha, \beta) \neq (0, 0)\}.$$

We have

$$\begin{aligned} iH_{10} &= -i(A_{\parallel} S_y I_z + A_{\perp} S_y I_x) \\ iH_{110} &= i(A_{\parallel} S_z I_z + A_{\perp} S_z I_x) \\ iH_{33110} &= -iA_{\parallel} S_z I_z \end{aligned}$$

and

$$\begin{aligned} iH_{110} + iH_{33110} &= iA_{\perp} S_z I_x = iH_4 \\ iH_{433110} &= -iA_{\perp} A_{\parallel} I_y. \end{aligned}$$

Computing the commutator between iS_x, iS_y gives iS_z . Therefore, the Lie span contains all iS_{α} with $\alpha \neq 0$. Similarly, it contains all iI_{β} with iH_3, iH_4 and iH_{433110} . Doing the same with the elements iS_{α}, iI_{β} with iH_{33110} , we obtain all possible pairs $iS_{\alpha}I_{\beta}$ with non-zero α, β as

$$\begin{aligned} [iS_{\alpha}, -iA_{\parallel} S_z I_z] &= \pm iA_{\parallel} S_{[\alpha, z]} I_z \\ [iI_{\beta}, -iA_{\parallel} S_z I_z] &= \pm iA_{\parallel} S_z I_{[\beta, z]}. \end{aligned}$$

Summarizing, $\text{Lie}\{iH_0, iH_1, iH_2, iH_3\}$ spans

$$\text{Lie}\{iS_{\alpha}I_{\beta} : (\alpha, \beta) \neq (0, 0)\}$$

which is exactly $\mathfrak{su}(4)$, proving the result.

Proposition 3. If $v = 0$, the system associated with Hamiltonian (4) with A_{\parallel} and A_{\perp} both positive remains controllable.

Proof. According to the previous result, it suffices to prove that

$$\text{Lie}\{iH_0, iH_1, iH_2\} = \text{Lie}\{iH_0, iH_1, iH_2, iH_3\}. \quad (5)$$

One has

$$iH_0 - iH_{110} = i\omega_I I_z,$$

and

$$\begin{aligned} [i\omega_I I_z, [i\omega_I I_z, iH_0]] &= i\omega_I^2 A_{\perp} S_z I_x, \\ iH_{110} - iA_{\perp} S_z I_x &= iA_{\parallel} S_z I_z, \end{aligned}$$

So

$$[iA_{\perp} S_z I_x, iA_{\parallel} S_z I_z] = iA_{\perp} A_{\parallel} I_y,$$

which generates $i\omega_I A_{\perp} A_{\parallel} H_3$ if we bracket with $i\omega_I I_z$. This gives the equality (5) and proves the result.

Proposition 4. If $v = 0$ and $A_{\perp} = 0$, the system associated with Hamiltonian (4) with A_{\parallel} positive is not controllable.

Proof. We have

$$\begin{aligned} iH_{10} &= -iA_{\parallel} S_y I_z, \\ iH_{110} &= iA_{\parallel} S_z I_z, \end{aligned}$$

and

$$iH_0 - iH_{110} = i\omega_I I_z = iH_3,$$

So we only generate the following Lie algebra:

$$\text{Lie}\{iS_{\alpha}, iS_{\beta}I_z \mid \alpha \in \{x, y, z\}, \beta \in \{x, y, z, 0\}\}. \quad (6)$$

It is a strict subalgebra of $\mathfrak{su}(4)$, so the system is not completely controllable on $SU(4)$.

Accordingly, the system with the following Hamiltonian is not completely controllable,

$$\omega_I I_z + A_{\parallel} S_z I_z + \frac{u_0(t)}{2} (\cos(\phi_S) S_x - \sin(\phi_S) S_y), \quad (7)$$

as it is impossible to generate new terms on the second qubit since the only available operator acting on it is I_z . It might still be sufficient to perform standard single and two-qubit gate. In particular, for quantum information processing, an important gate to consider is the CNOT gate, which entangles two qubits together. The CNOT gate corresponds to flipping the electron spin population if the nuclear spin is in the state $|1\rangle_n$, a matrix representation of this operator is given as:

$$\text{CNOT} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

Although CNOT does not belong to the reachable set, it can be shown that the following identity is true,

$$e^{-i\frac{\pi}{4}} \text{CNOT} = e^{i\frac{\pi}{4} (S_x I_z - I_z - S_x)} \quad (9)$$

ensuring that, up to a global shift in the phase (here, $\pi/4$), the gate is attainable.

4. QUANTUM SPEED LIMIT

In this section we study system (1) with the Hamiltonian

$$H = A_{\parallel} S_z I_z + \frac{u_0}{2} (\cos(\phi_S) S_x - \sin(\phi_S) S_y). \quad (10)$$

and investigate the minimal time required to steer the system between two distinguishable states. To study the quantum speed limit (QSL) of the system we assume that the control amplitude is unbounded; thanks to this strong relaxation, we can generate arbitrarily fast rotations around the control Hamiltonians H_j . We recall the following result:

Theorem 5. ((Khaneja et al. (2001))). Let G be a compact semi-simple Lie Group with a bi-invariant metric $\langle X, Y \rangle_G$, and K be a closed subgroup. Let \mathfrak{g} and \mathfrak{k} be their Lie algebras, respectively, such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is a Cartan decomposition and the dimension of the maximal Abelian subalgebra of \mathfrak{p} has rank one. Consider the system (1) where $H_0 \in \mathfrak{p}$ and $Lie\{iH_j\} = \mathfrak{k}$. Then, the minimal time T is the smallest value $t > 0$ such that we can solve $U_f = Q_1 \exp(-itH_0)Q_2$ with $Q_1, Q_2 \in K$.

In our case, let be G the group with the associated Lie algebra $\mathfrak{g} = Lie\{iS_z I_z, iS_x, iS_y\}$ and K be a subgroup with the associated Lie algebra $\mathfrak{k} = Lie\{iS_x, iS_y\}$. Since $G \cong SU(2) \times SU(2)$ and $K \cong SU(2)$, the quotient $G/K \cong SU(2)$ is a homogeneous space, which has the direction of the flow given by the control Hamiltonian. Furthermore, consider the orthogonal subalgebra such that $\mathfrak{p} = \mathfrak{k}^\perp$ under the bi-invariant metric $\langle X, Y \rangle_G = \text{Tr}(XY)$. Then, $\mathfrak{p} = Lie\{iS_x I_z, iS_y I_z, iS_z I_z\}$ and \mathfrak{g} can be decomposed along Cartan decomposition, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Note that $\{iS_z I_z\} \subset \mathfrak{p}$ is a maximal Abelian subalgebra of rank one. Using the Theorem 5, the minimal time to steer the system from $\mathbb{1}$ to U_f is the small time $t > 0$ such that $U_f = Q_1 \exp(\pm tA_{\parallel} S_z I_z)Q_2$ with $Q_1, Q_2 \in K$. Since the drift is 2π periodic, the worst case scenario is when the system needs to make a π rotation, which gives the minimal time equal to π/A_{\parallel} .

5. OPTIMAL CONTROL FOR TWO-QUBIT GATE

While the Cartan decomposition gives a theoretical lower bound on the time required to implement a given quantum gate, it relies on the assumption of unbounded control amplitudes which is not physically realisable. In practice, the implementation of control pulses on quantum systems is constrained by inherent physical limitations, e.g. finite control power and bandwidth limitations, such that the theoretical QSL is not experimentally reachable. Our goal is therefore to numerically determine a control field that performs the target operation in a time as close to the QSL as possible while taking into account the limited available power of the control field.

To this end, we utilise Quantum Optimal Control (QOC), a set of analytical and numerical tools that formulates the design of control fields as an optimisation problem-Borziet al. (2017); Boscain et al. (2021). In particular, we employ the Gradient Ascent Pulse Engineering (GRAPE) algorithm Khaneja et al. (2005); Ansel et al. (2024), which iteratively adjusts the control pulses, considered as a piecewise constant functions, to maximize a given figure of merits (often the operation fidelity).

To take into account the limited control power, we perform a series of numerical optimizations for a given fixed Rabi frequency, denoted as Ω_R such that $u_0(t) = 2\Omega_R$. In this setting, the only optimized control parameter is the phase of the control field $\phi_S(t)$. Thus, the numerically studied Hamiltonian is:

$$H = A_{\parallel} S_z I_z + \Omega_R [\cos(\phi_S(t)) S_x - \sin(\phi_S(t)) S_y]. \quad (11)$$

The aim of our work is to provide preliminary results on the Rabi frequency strength required to approach the QSL for implementing a CNOT gate. To this end, the optimization problem is formulated as a conditional spin

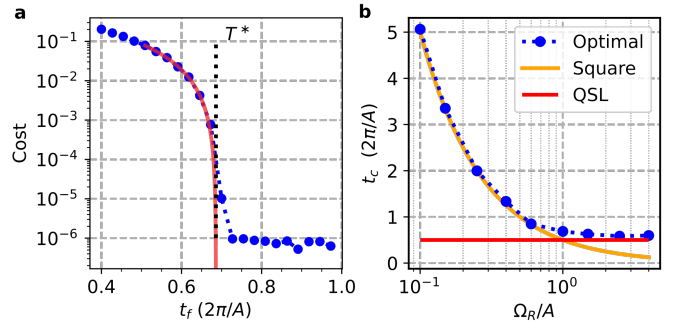


Fig. 1. **a** The final costs obtained from the GRAPE iterative optimisation process for varying final time t_f in the case of $\Omega_R = A$ are shown in blue. The red curve corresponds to the fit of the exponential decay part of the curve, the fit is only shown in the time interval used to fit the numerical data point. The extracted values of the optimal control time T^* is indicating with the dashed vertical line. **b** Shown in blue are the numerical optimal times T^* for varying Ω_R , the dashed line being a visual aid. The orange curve corresponds to the time of a standard π -pulse using a square envelope pulse. The red horizontal lines corresponds to the theoretical QSL.

flip: the electron spin population is flipped if the nuclear spin is in a given state, without imposing any constraint on the phase of the final state. Formally, if the target CNOT operator is denoted by U_t , the figure of merit to minimize is defined as:

$$C = 1 - F = 1 - \frac{1}{4} \sum_{i,j \in \{0,1\}} |\langle i, j | U_t^\dagger U(t_f) | i, j \rangle|^2, \quad (12)$$

where $U(t_f)$ is the evolution operator generated by the the dynamics under a given control field over the total time t_f , and $\{|i, j\rangle\}_{i,j \in \{0,1\}}$ is the canonic computational basis of the two-spin Hilbert space with $|i, j\rangle = |i\rangle_e \otimes |j\rangle_n$.

To determine the minimal time required to perform the CNOT operation in a state-to-state manner for a given Rabi frequency Ω_R , we carry out a series of optimizations for various control times t_f . Each optimization consists of iteratively modifying the control field based on the gradient of the cost function with respect to the control parameters, here the set of finites values taken by the phase ϕ_S considered as a piecewise constant function, until a local minimum of the cost function is reached. To increase the likelihood of finding the global minimum, multiple initial guesses for the control field are tested.

In Fig. 1a, we illustrate how, for $\Omega_R = A$, the cost function evolves as a function of t_f . We observe a sharp exponential decrease followed by a plateau, whose value ($C = 10^{-6}$) is the numerical threshold we impose during the optimization process. The optimal time T^* is defined as the smallest control time for which the fidelity F reaches unity, *i.e.*, $C = 0$. We therefore model the fidelity near the optimal time as:

$$F(t_f) = \exp(\alpha(t_f - T^*)^2), \quad (13)$$

where α is a constant. Consequently, the cost function can be expressed as:

$$C(t_f) = 1 - \exp(\alpha(t_f - T^*)^2). \quad (14)$$

To extract the optimal time for a given Rabi frequency Ω_R , we fit the cost function curve before the plateau (highlighted in red in the figure), allowing us to estimate T^* . Repeating this procedure for various values of Ω_R enables us to determine the minimal time required to implement the CNOT gate for a given control field strength. The numerical results are presented in blue in Fig. 1b. For comparison, we also display in orange the time required for a standard π -pulse: $T_\pi = \pi/\Omega_R$, control pulses for which the fidelity is equal to 0.5 when Ω_R is the same order of magnitude than A_{\parallel} .

As expected, the optimal time decreases as the control field power increases and asymptotically approaches the constant value π/A_{\parallel} , which corresponds to the quantum speed limit. Interestingly, for low value of Ω_R , the optimal times T^* are close to the standard π -pulse time T_π . Furthermore, for $\Omega_R = A_{\parallel}$, the optimal time is in the order of magnitude of the QSL (up to a factor 1.35), suggesting that this control strength is already sufficient from an experimental point of view. Any further increase in control field amplitude would yield only marginal improvements in gate duration, while significantly increasing the complexity of the pulse shape and potentially introducing additional experimental challenges due to hardware limitations. This behavior is reflected in the convexity of the curve, which shows steep gains at low power followed by diminishing returns at higher power, making $\Omega_R = A_{\parallel}$ the practical compromise between speed and practicality.

In other words, when designing complex quantum algorithms based on physical qubits, a useful guideline would thus be that the time required to perform a CNOT operation between two qubits is approximately the inverse of the coupling strength, reachable for a Rabi frequency comparable to this coupling. This estimate is highly relevant to design scalable quantum architectures for quantum information processing.

6. CONCLUSION

In this paper, we apply tools of control theory to study the optimal control of two coupled spins in NV centers. For a specific configuration, we investigate both the system controllability and the estimation of the QSL for state-to-state transfer. This paper opens the way to a series of interesting issues in the control of spin systems. A first step is to study the controllability of complex spin networks, with spins potentially larger than $1/2$. Another problem is the minimum control time in such systems, and its connection to the network complexity. A final objective concerns the experimental implementation of the derived optimal control protocols.

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