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THE GAUSS-BONNET-CHERN THEOREM FOR α -GRUSHIN MANIFOLDS

IVAN BESCHASTNYI, PAULA CEREJEIRAS & SIMÃO LUCAS

Abstract

In this paper we provide a Gauss-Bonnet-Chern formula for α -Grushin manifolds. We establish a result that yields a geometric invariant linking asymptotically the geometry and topology of manifolds that exhibit a not necessarily connected submanifold of singularities, that we shall call singular set. This invariant is obtained by means of the classical Gauss-Bonnet-Chern formula for manifolds with boundary and asymptotic contraction of the tubular neighbourhood of the singular set. As a consequence, a Gauss-Bonnet formula for 2D almost-Riemannian structures can be recovered.

1. Introduction

The goal of this article is to establish asymptotic Gauss-Bonnet-Chern formulas for a large class of singular manifolds, known as α -Grushin manifolds. In order to motivate their definition, consider \mathbb{R}^2 with coordinates (x, y) and an operator introduced by Grushin himself [15]:

$$(1) \quad \Delta_G = \frac{1}{2} (X_0^2 + X_1^2),$$

where $X_0 = \frac{\partial}{\partial x}$ and $X_1 = x \frac{\partial}{\partial y}$. It is known that the properties of this operator are intimately connected to the properties of a metric g on \mathbb{R}^2 for which X_0 and X_1 are orthonormal [9, 16]. This metric is given by the following matrix with respect to the standard Euclidean basis

$$g(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix}.$$

The pair (\mathbb{R}^2, g) is called the standard Grushin plane, and represents the most simple example of what we will later call a Grushin manifold.

The standard Grushin plane has many interesting properties. For example, even though there is a singularity at the set $x = 0$, the whole plane \mathbb{R}^2 is a

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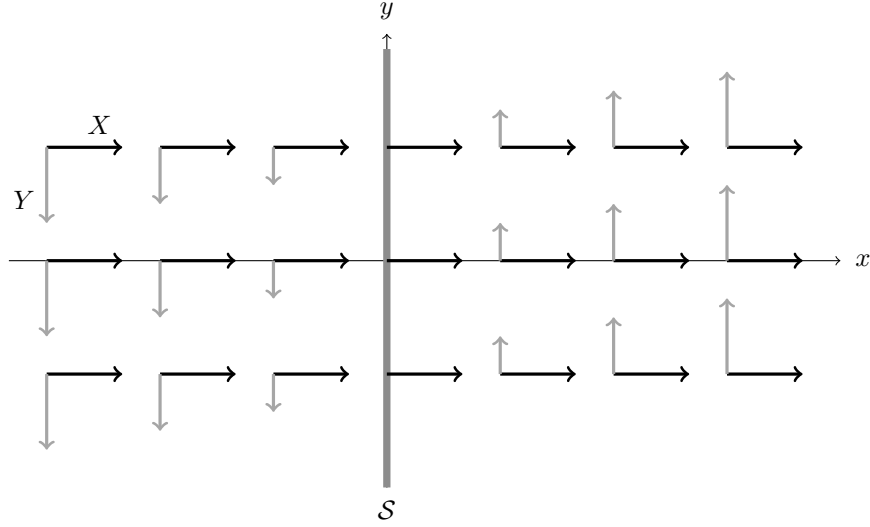


Figure 1. The Grushin Plane

well-defined metric space and geodesics can cross the singularity while remaining smooth [1]. Interestingly enough, the second order part of the associated Laplace-Beltrami operator coincides with the Grushin operator (1). In addition, it is essentially self-adjoint on the half-plane $x > 0$. By Stone's theorem, this implies that a quantum particle modelled by such structure would be confined by the singularity. This is known as the quantum confinement, which is a phenomenon previously explored in various works [4, 5, 7, 8, 14, 17, 23].

In this work, we study the relation between metric and topological properties on the following generalization of this basic example.

Definition 1.1 ([6]). Let $\alpha \in \mathbb{R}$. An α -*Grushin manifold* is a triple (M, \mathcal{S}, g) consisting of a smooth manifold M of dimension $2m = n + 1$, a co-orientable embedded submanifold $\mathcal{S} \subset M$ of co-dimension one, which we call the *singular set*, and a metric g which in a tubular neighbourhood of \mathcal{S} takes the form

$$(2) \quad g = dx^2 + \frac{1}{|x|^{2\alpha}} g_{x, \mathcal{S}},$$

where $g_{x, \mathcal{S}}$ is a smooth family of Riemannian metrics on the level sets $x = \text{const}$ and $|x|$ is the distance to \mathcal{S} .

There are several motivations for this definition. Firstly, they include several important particular cases. For $\alpha = -2$ they give rise to cusp singularities, for $\alpha = -1$ cone singularities, for $\alpha = 0$ they are essentially a subclass of manifolds with boundaries and for $\alpha = 1$ and $n = 1$ they give rise to generic

almost-Riemannian manifolds in dimension 2 without tangency points [2]. More generally, for $\alpha \in \mathbb{N}$, Grushin manifolds in the sense of Definition 1.1 form a natural subclass of almost-Riemannian manifolds (see [1, Chapter 3] for precise definition and discussion). Grushin manifolds constitute a class of singular manifolds general enough to include many interesting examples, and at the same time particular enough to have at our disposal a variety of tools that might be not available for other types of singularities.

In this article, we study the limit Gauss-Bonnet-Chern formulas for α -Grushin manifolds. The classical Gauss-Bonnet formula for a closed 2D Riemannian manifold (M, g) is

$$\chi(M) = \frac{1}{2\pi} \int_M K dA$$

where K is the Gaussian curvature, dA is the Riemannian area element and $\chi(M)$ is the Euler characteristic. The Gauss-Bonnet formula connects very nicely metric invariants (the Gaussian curvature) with topological invariants (Euler characteristic) of manifolds M and is one of the most important results in differential geometry. This formula was generalized by Chern via a completely intrinsic argument to general even-dimensional closed orientable Riemannian manifolds and reads as

$$(3) \quad \chi(M) = \frac{1}{(2\pi)^{\frac{\dim M}{2}}} \int_M \text{Pf}(\Omega).$$

where $\text{Pf}(\Omega)$ is the Pfaffian that will be defined in Section 2.

It is known that this formula does not hold for manifolds with singularities [3]. Instead, one is interested in computing the so-called Gauss-Bonnet defect, which is the difference between the left and the right-hand side of (3). Probably the simplest singular manifolds are manifolds with boundary, and in this case the Gauss-Bonnet defect is well known and can be found in the classical reference [24] (see also Theorem 2.3). In the presence of other singularities, the defect is usually computed by cutting out the singularity using a small ε -tubular neighbourhood. This gives a manifold with boundary M_ε . Then we can compute the limit of

$$(4) \quad \chi(M_\varepsilon) - \frac{1}{(2\pi)^{\frac{\dim M}{2}}} \int_{M_\varepsilon} \text{Pf}(\Omega)$$

as $\varepsilon \rightarrow 0^+$ using the Gauss-Bonnet-Chern formula for manifolds with boundary. As we will see, in the case of α -Grushin manifolds this quantity does not converge for $\alpha > -1$ due to singularities in the curvature. This excludes many interesting cases. For this reason, we look instead at the asymptotic

expansion of (4) as $\varepsilon \rightarrow 0^+$ and compute the principal term of the corresponding asymptotic expansion. In particular, we prove the following asymptotic Gauss-Bonnet-Chern formulas.

Theorem 1.2. *Let M be a compact oriented α -Grushin manifold of dimension $2m = n + 1$. If $\alpha > -1$ and $\alpha \neq 0$, then*

$$(5) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{n(\alpha+1)} \left(\chi(M) - \frac{1}{(2\pi)^m} \int_{|x| \geq \varepsilon} \text{Pf}(\Omega) \right) = \frac{2\alpha^n (n-1)!!}{(2\pi)^m} \text{vol } \mathcal{S}.$$

If $\alpha < -1$, then

$$(6) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{\alpha+1} \left(\chi(M) - \frac{1}{(2\pi)^m} \int_{|x| \geq \varepsilon} \text{Pf}(\Omega) \right) = \frac{2\alpha}{(2\pi)^m} \int_{\mathcal{S}} \kappa_{n-1}(\mathcal{S}),$$

where $\text{vol } \mathcal{S}$ is the Riemannian volume and $\kappa_{n-1}(\mathcal{S})$ is a Lipschitz-Killing curvature of the singular set \mathcal{S} with respect to $g_{0,\mathcal{S}}$ defined in Definition 1.1.

The two missing cases were already known in the literature: for $\alpha = 0$ this is just the Gauss-Bonnet-Chern formula for manifolds with boundaries [24], and for $\alpha = -1$ this reduces to the formula for manifolds with conic singularities [21]. Versions of the Gauss-Bonnet-Chern theorems and formulas for Gauss-Bonnet defects for other types of singularities were studied by many groups. This includes 4D manifolds with certain conformally flat ends and singular points of finite area [10, 11], manifolds with conical singularities [21, 22], closed surfaces in the Heisenberg group seen as a sub-Riemannian manifold [13, 25]. A variant of the Gauss-Bonnet formula for 2D Grushin manifolds ($n = 1$, $\alpha = 1$ in our notations) was obtained in [2]. However, it is significantly different from our result in two regards. First, instead of taking KdA as the Pfaffian of the curvature form would give, the authors take $KdA_{\mathcal{S}}$ as integrand, where $dA_{\mathcal{S}}$ is the signed area form (see precise definition in Section 3.3). This allows them to prove that the Gauss-Bonnet defect is zero, while in our case it is divergent. Nevertheless, one does not obtain a zero defect for other values of α in dimension 2 with this approach. In addition, they make an assumption that the manifold M is separated by the singular set into two disjoint parts, which greatly limits the use of this formula. Instead, our formula is valid for any 2D Grushin manifold with any α in the considered range.

The article has the following structure. In section 2, we introduce the concepts needed from differential geometry, namely, the Pfaffian $\text{Pf}(\Omega)$ and the Lipschitz-Killing curvatures $\kappa_{2k}(M)$. Also, we state the Gauss-Bonnet-Chern theorem for manifolds with boundary with slight modifications in relation to the literature. Namely, we set the dimension of the manifold as $2m = n + 1$ and consider the unit normal vector to the boundary as the first one of the frame

X_0 , while in the literature it is usually considered the last one X_n . In section 3, we derive explicit formulas for the Christoffel symbols associated with the metric (2) to be used later to obtain explicit expressions for the curvature forms $\Omega_{\sigma(i)}^{\sigma(j)}$ and present the proof of Theorem 1.2. Finally, in section 3.3, we discussed a particular case of Theorem 1.2, namely, its version in two dimensions, and point out the main differences between our result and an already existent one for two-dimensional almost-Riemannian manifolds.

2. Gauss-Bonnet-Chern Theorem Revisited

In this section, we recall the classical Gauss-Bonnet-Chern theorem for manifolds with boundary, as well as the concept of Lipschitz-Killing curvatures. We refer to [21] for further details.

We begin with some definitions, and use the Einstein summation convention throughout the rest of the paper. Consider a general Riemannian manifold of dimension n , a local orthonormal moving frame X_1, \dots, X_n and its dual frame $\theta^1, \dots, \theta^n$. Given a Levi-Civita connection ∇ , we can define the Christoffel symbols Γ_{jk}^i as

$$\nabla_{X_i} X_j := \Gamma_{ij}^k X_k.$$

A connection can also be defined in terms of a family of one forms θ_i^j as

$$\nabla_X X_i := \theta_i^j(X) X_j,$$

where X is any linear combination of elements of the frame. Given the values of the Γ_{ki}^j , we can define one forms θ_i^j by

$$\theta_i^j := \Gamma_{ki}^j \theta^k.$$

Using the orthonormality of the frame, the metric compatibility condition can be written in terms of Christoffel symbols as

$$\Gamma_{ki}^j + \Gamma_{kj}^i = 0,$$

and in terms of forms as

$$\theta_i^j + \theta_j^i = 0.$$

In particular, we have $\theta_i^i = 0$ and $\Gamma_{ji}^i = 0$.

The Cartan method of moving frames condenses into two main equations, whose solutions give us the connection and curvature forms.

Theorem 2.1 (Structure equations). *The connection and curvature forms of a Levi-Civita connection are the unique solutions of the equations*

1. (first structure equation) $d\theta^j = \theta^i \wedge \theta_i^j$;
2. (second structure equation) $\Omega_i^j = d\theta_i^j - \theta_i^k \wedge \theta_k^j$,

for all $i = 1, \dots, n$.

An important step into the generalisation of the Gauss-Bonnet theorem to higher dimensions is the introduction of a curvature invariant called Pfaffian. Consider now a Riemannian manifold of dimension $2m$. The Pfaffian, which is also the Gauss-Bonnet-Chern integrant, is given by

$$(7) \quad \text{Pf}(\Omega) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \Omega_{\sigma(2)}^{\sigma(1)} \wedge \dots \wedge \Omega_{\sigma(2m)}^{\sigma(2m-1)},$$

where S_{2m} is the set of all permutations of order $2m$.

It is important to emphasize that once an orientation is fixed, the Pfaffian is frame invariant [24, Vol. 5]. Indeed, the Levi-Cevita connection can be thought of as a connection on the frame bundle, which is a principle $SO(n)$ -bundle in the oriented case. We will denote this connection by $\tilde{\theta}$ and its curvature by $\tilde{\Omega}$. The one-forms θ_j^i and two-forms Ω_j^i are components of the pull-backs $s^*\tilde{\theta}$ and $s^*\tilde{\Omega}$, where s is the section of the orthonormal frame bundle corresponding to the orthonormal frame X_1, \dots, X_n . Since $\tilde{\Omega}$ is a curvature of a connection on a principle $SO(n)$ -bundle, it carries an $SO(n)$ -action. The Pfaffian is an invariant of this action that is recovered from the classical invariant theory. Strictly speaking, for this reason, we should write $\text{Pf}(\tilde{\Omega})$ instead of $\text{Pf}(\Omega)$. However, as mentioned, $\text{Pf}(\tilde{\Omega})$ is $SO(n)$ -invariant and hence descends to a form on a manifold M .

The form $\text{Pf}(\Omega)$ is a particular example of a family of curvature invariants known as the Lipschitz-Killing curvatures.

Definition 2.2. Let (M, g) be a Riemannian manifold of dimension n and let θ^i be an orthonormal coframe in a neighbourhood of $p \in M$. The Lipschitz-Killing curvature of order $0 \leq 2k \leq n$ is given by

$$\kappa_{2k}(M) = \frac{1}{2^k k! (n-2k)!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Omega_{\sigma(2)}^{\sigma(1)} \wedge \dots \wedge \Omega_{\sigma(2k)}^{\sigma(2k-1)} \wedge \theta^{\sigma(2k+1)} \wedge \dots \wedge \theta^{\sigma(n)}.$$

The Lipschitz-Killing curvatures of odd orders are zero.

Similarly to the Pfaffian all the Lipschitz-Killing curvatures are actually $SO(n)$ -invariant forms on the principle bundle. Some particular cases include

$$\begin{aligned} \kappa_0(M) &= d\text{vol}, \\ \kappa_2(M) &= \frac{S}{2} d\text{vol} \end{aligned}$$

where S is the scalar curvature. When $\dim M = 2m$, then

$$\kappa_{2m} = \text{Pf}(\Omega).$$

They can be defined in a more invariant way using the language of double forms and their contractions. See, for example, in [19] or [21]. They appear, among other, in the Weyl tube formula [26].

Finally, we recall the Gauss-Bonnet-Chern theorem for manifolds with boundary.

Theorem 2.3. *Let M be a compact oriented Riemannian manifold with boundary of dimension $2m = n+1$, and ∇ a metric compatible connection on its tangent bundle TM with curvature matrix $\Omega = [\Omega_i^j]$ relative to an orthonormal frame X_0, \dots, X_n . Let X_0 be the inward-pointing unit normal vector to ∂M . Then there exists a differential form $\Pi \in \Omega^n(SM)$ on the spherical bundle SM such that*

$$(8) \quad \chi(M) = \frac{1}{(2\pi)^m} \int_M \text{Pf}(\Omega) + \int_{\partial M} X_0^* \Pi$$

In a local trivialization, this form has the following expression

$$X_0^* \Pi = \sum_{k=0}^{m-1} c_{k,2m} i^* \Theta_k,$$

where $i : \partial M \rightarrow M$ is the inclusion map and

$$(9) \quad \Theta_k = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Omega_{\sigma(2)}^{\sigma(1)} \wedge \dots \wedge \Omega_{\sigma(2k)}^{\sigma(2k-1)} \wedge \theta_0^{\sigma(2k+1)} \wedge \dots \wedge \theta_0^{\sigma(n)}.$$

and

$$c_{k,2m} = \frac{(-1)^{k+1}}{2^{k+m} \pi^m k! (2m - 2k - 1)!}.$$

Remark 2.4. If X_0 is an outward-pointing unit normal, then one has to take

$$\bar{c}_{k,2m} = \frac{1}{2^{k+m} \pi^m k! (2m - 2k - 1)!}.$$

See [21, p. 78] and note that we only consider even-dimensional manifolds.

Lemma 2.5. *Let M be a Riemannian manifold. Suppose that θ^j is the dual frame to an orthonormal frame X_j . Then, the curvature forms are given by*

$$(10) \quad \Omega_i^j = \left(X_l \left[\Gamma_{mi}^j \right] + \Gamma_{ki}^j \Gamma_{ml}^k + \Gamma_{lk}^j \Gamma_{mi}^k \right) \theta^l \wedge \theta^m,$$

where we use the notation $\Omega_{ilm}^j := X_l \left[\Gamma_{mi}^j \right] + \Gamma_{ki}^j \Gamma_{ml}^k + \Gamma_{lk}^j \Gamma_{mi}^k$.

Proof. Consider the Levi-Civita connection, and connection forms defining it. Then, from the second structure equation, we have

$$\Omega_i^j = d \left(\Gamma_{ki}^j \theta^k \right) + \Gamma_{kl}^j \Gamma_{mi}^k \theta^l \wedge \theta^m = X_l \left[\Gamma_{ki}^j \right] \theta^l \wedge \theta^k + \Gamma_{ki}^j d\theta^k + \Gamma_{lk}^j \Gamma_{mi}^k \theta^l \wedge \theta^m.$$

Moreover, from the first structure equation, we have

$$\begin{aligned}\Omega_i^j &= X_l \left[\Gamma_{ki}^j \right] \theta^l \wedge \theta^k + \Gamma_{ki}^j \Gamma_{ml}^k \theta^l \wedge \theta^m + \Gamma_{lk}^j \Gamma_{mi}^k \theta^l \wedge \theta^m \\ &= \left(X_l \left[\Gamma_{mi}^j \right] + \Gamma_{ki}^j \Gamma_{ml}^k + \Gamma_{lk}^j \Gamma_{mi}^k \right) \theta^l \wedge \theta^m.\end{aligned}$$

q.e.d.

3. Asymptotic expansion of the Gauss-Bonnet defect

3.1. Non-regular Christoffel Symbols Formulas. Let (M, g) be an α -Grushin manifold. Let us consider an orthonormal moving frame with respect to g given by

$$(11) \quad X_0 = \partial_x, \quad X_i = |x|^\alpha Y_i,$$

where the family $X_0 = Y_0, Y_1, \dots, Y_n$ represents an orthonormal moving frame with respect to the regularized metric

$$(12) \quad \tilde{g} = dx^2 + g_{x,S}$$

in a tubular neighbourhood of g . The dual frame to the family X_0, \dots, X_n is then given by

$$(13) \quad \theta^0 = dx = \omega^0, \quad \theta^i = \frac{1}{|x|^\alpha} \omega^i,$$

where $\omega^1, \dots, \omega^n$ is the dual frame to Y_1, \dots, Y_n .

A consequence of the Koszul formula is the following theorem, that allows us to compute the respective Christoffel symbols for the frame Y_0, \dots, Y_n .

Theorem 3.1 ([1], Theorem 16.21). *Let M be a Riemannian manifold of dimension $n + 1$. Given an orthonormal frame Y_0, \dots, Y_n , its Christoffel symbols are given by*

$$\Gamma_{IJ}^K = \frac{1}{2} (c_{IJ}^K - c_{JK}^I + c_{KI}^J), \quad I, J, K = 0, \dots, n,$$

where c_{IJ}^K are structure functions defined by the identities $[Y_I, Y_J] = \sum_{K=0}^n c_{IJ}^K Y_K$, for $I, J = 0, \dots, n$.

We now deduce a way to compute the Christoffel symbols of the Levi-Cevita connection corresponding to metric g by means of those corresponding to metric \tilde{g} .

Proposition 3.2. *Let Γ and $\bar{\Gamma}$ be the Christoffel symbols of the Levi-Cevita connection, corresponding to the metrics \tilde{g} and g , respectively. Then Christoffel*

symbols $\bar{\Gamma}$ are given by

$$(14) \quad \begin{aligned} \bar{\Gamma}_{ij}^0 &= \frac{\alpha}{x} \delta_{ij} + \Gamma_{ij}^0 \\ \bar{\Gamma}_{0j}^i &= \Gamma_{0j}^i \\ \bar{\Gamma}_{ij}^k &= |x|^\alpha \Gamma_{ij}^k, \end{aligned}$$

where $i, j, k = 1, \dots, n$.

Proof. Let the capital letters I, J, K, \dots denote the indices in the range $0, \dots, n$. Since Y_i are tangent to the level sets $x = \text{const}$, by Theorem 3.1, computing the Lie brackets for Y_i shows that the structure constants c_{IJ}^0 and c_{0I}^0 must be zero. Moreover, as a consequence of Theorem 3.1 Christoffel symbols $\Gamma_{IJ}^0, \Gamma_{0I}^0$ are zero.

Computing the Lie bracket between the elements of this orthonormal frame, we have

$$\begin{aligned} [X_0, X_i] &= \frac{\alpha \operatorname{sgn}(x)}{|x|} X_i + c_{0i}^j X_j \\ [X_i, X_j] &= |x|^\alpha c_{ij}^k X_k. \end{aligned}$$

From these, the structure constants \bar{c}_{JK}^I of the frame X_0, \dots, X_n are found to be

$$\bar{c}_{0i}^j = \frac{\alpha}{x} \delta_{ij} + c_{0i}^j, \quad \bar{c}_{ij}^k = |x|^\alpha c_{ij}^k.$$

The result then follows by Theorem 3.1.

q.e.d.

3.2. Proof of Theorem 1.2. Let us start by describing what aspect the curvature forms acquire when we consider the frame (11) and its dual (13). We aim to use formula (9) where the indices $\sigma(j)$ and $\sigma(i)$ in the terms $\Omega_{\sigma(i)}^{\sigma(j)}$ are permutations of the set $\{1, \dots, n\}$. Due to the heaviness of the indices on the formulas of $\Omega_{\sigma(i)}^{\sigma(j)}$, we simplify by writing $\sigma(j) := j$ and $\sigma(i) := i$. Therefore, we have only the following options for the coefficients Ω_{ilm}^j in (10),

$$(15) \quad \Omega_{ilm}^j = \left(X_l \left[\bar{\Gamma}_{mi}^j \right] + \bar{\Gamma}_{ki}^j \bar{\Gamma}_{ml}^k + \bar{\Gamma}_{lk}^j \bar{\Gamma}_{mi}^k \right) = |x|^{2\alpha} \left(Y_l \left[\Gamma_{mi}^j \right] + \Gamma_{ki}^j \Gamma_{ml}^k + \Gamma_{lk}^j \Gamma_{mi}^k \right)$$

$$(16) \quad \begin{aligned} \Omega_{i0m}^j &= \left(X_0 \left[\bar{\Gamma}_{mi}^j \right] + \bar{\Gamma}_{ki}^j \bar{\Gamma}_{m0}^k + \bar{\Gamma}_{0k}^j \bar{\Gamma}_{mi}^k \right) \\ &= |x|^\alpha \left(\frac{\alpha}{x} \Gamma_{mi}^j + Y_0 \left[\Gamma_{mi}^j \right] + \Gamma_{ij}^k \left(-\frac{\alpha}{x} \delta_{mk} - \Gamma_{mk}^0 \right) + \Gamma_{0k}^j \Gamma_{mi}^k \right) \end{aligned}$$

$$(17) \quad \Omega_{i0}^j = \left(X_l \left[\bar{\Gamma}_{0i}^j \right] + \bar{\Gamma}_{ki}^j \bar{\Gamma}_{0l}^k + \bar{\Gamma}_{lk}^j \bar{\Gamma}_{0i}^k \right) = |x|^\alpha \left(Y_l \left[\Gamma_{0i}^j \right] + \Gamma_{ki}^j \Gamma_{0l}^k + \Gamma_{lk}^j \Gamma_{0i}^k \right)$$

Since the θ^i , $i = 0, \dots, n$, are precisely given by (13), the Ω_i^j in the basis $\theta^l \wedge \theta^m$ (see Lemma 2.5) become

$$(18) \quad \begin{aligned} \Omega_i^j &= \left(\frac{\alpha}{x} \Gamma_{mi}^j + Y_0 \left[\Gamma_{mi}^j \right] + \Gamma_{ij}^k \left(-\frac{\alpha}{x} \delta_{mk} - \Gamma_{mk}^0 \right) + \Gamma_{0k}^j \Gamma_{mi}^k \right. \\ &\quad \left. - Y_l \left[\Gamma_{0i}^j \right] - \Gamma_{ki}^j \Gamma_{0l}^k - \Gamma_{lk}^j \Gamma_{0i}^k \right) dx \wedge \omega^m + \left(Y_l \left[\Gamma_{mi}^j \right] + \Gamma_{ki}^j \Gamma_{ml}^k + \Gamma_{lk}^j \Gamma_{mi}^k \right) \omega^l \wedge \omega^m. \end{aligned}$$

Since the curvature forms Ω_i^j (18) are pullbacks on ∂M , the term $dx \wedge \omega^m$ vanishes. Therefore, the only remaining terms are $\omega^l \wedge \omega^m$. As pullbacks on ∂M , the forms θ_0^j are given by:

$$(19) \quad \theta_0^j = \bar{\Gamma}_{A0}^j \theta^A = \bar{\Gamma}_{00}^j \theta^0 + \bar{\Gamma}_{k0}^j \theta^k = -\bar{\Gamma}_{kj}^0 \theta^k = -\left(\frac{\alpha}{x} \delta_{kj} + \Gamma_{kj}^0 \right) \frac{1}{|x|^\alpha} \omega^k,$$

using the fact from the proof of Proposition 3.2 that the $\bar{\Gamma}_{00}^j$ are zero. Asymptotically the θ_0^j become

$$(20) \quad \theta_0^j = -\frac{\alpha}{x|x|^\alpha} \omega^j + O(|x|^{-\alpha}).$$

Since pullbacks of Ω_i^j are regular smooth, we can just focus on the part with θ_0^j in the formula of Θ_k , see (9). Plugging the asymptotic (20), we obtain the following,

$$\begin{aligned} \theta_0^{\sigma(2k+1)} \wedge \dots \wedge \theta_0^{\sigma(n)} &= \left(-\frac{\alpha}{x|x|^\alpha} \omega^{\sigma(2k+1)} + O(|x|^{-\alpha}) \right) \\ &\quad \wedge \dots \wedge \left(-\frac{\alpha}{x|x|^\alpha} \omega^{\sigma(n)} + O(|x|^{-\alpha}) \right) \\ &= -\frac{\alpha^{n-2k}}{x^{n-2k} |x|^{\alpha(n-2k)}} \omega^{\sigma(2k+1)} \wedge \dots \wedge \omega^{\sigma(n)} \\ &\quad + O\left(|x|^{-(\alpha(n-2k)+n-2k-1)} \right). \end{aligned}$$

Note that the minus remains due to n being odd from the definition of M . Plugging all this into Θ_k , these become

$$(21) \quad \begin{aligned} \Theta_k &= - \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Omega_{\sigma(2)}^{\sigma(1)} \wedge \dots \wedge \Omega_{\sigma(2k)}^{\sigma(2k-1)} \\ &\quad \wedge \frac{\alpha^{n-2k}}{x^{n-2k} |x|^{\alpha(n-2k)}} \omega^{\sigma(2k+1)} \wedge \dots \wedge \omega^{\sigma(n)} + O\left(|x|^{-(\alpha(n-2k)+n-2k-1)} \right). \end{aligned}$$

Let us now consider the formula

$$(22) \quad \int_{\partial M} X_0^* \Pi = \int_{\partial M} \sum_{k=0}^{m-1} c_{k,2m} \Theta_k = \int_{\partial M} c_{0,2m} \Theta_0 + \int_{\partial M} c_{1,2m} \Theta_1 + \int_{\partial M} c_{2,2m} \Theta_2 + \dots$$

The idea with this formula, and Theorem 2.3, is that we shall consider a tubular neighbourhood of radius ε of the singular set \mathcal{S} of M , and the manifold $M_\varepsilon \simeq M \setminus (\mathcal{S} \times [-\varepsilon, \varepsilon])$ with boundary at $x = \varepsilon$ and $x = -\varepsilon$. It is important to note that the boundary is homeomorphic to the singular set. Note that X_0 in (11) is outward pointing at $x = -\varepsilon$ and inward pointing at $x = \varepsilon$. Following this, we shall compute

$$(23) \quad \sum_{k=0}^{m-1} \left[c_{k,2m} \int_{x=\varepsilon} \Theta_k + \bar{c}_{k,2m} \int_{x=-\varepsilon} \Theta_k \right],$$

where the constants $c_{k,2m}$ and $\bar{c}_{k,2m}$ are taken accordingly with Theorem 2.3 and Remark 2.4. Then we look for what happens when we let ε go to zero.

For $\alpha > -1$ and $\alpha \neq 0$ the singularity is concentrated in the terms θ_0^j , while the pullbacks of Ω_j^i are smooth. For this reason, the principal term in the asymptotics of (23) must be the integral of Θ_0 . That is, by (23) the principal term is given by

$$\begin{aligned} c_{0,2m} \int_{x=\varepsilon} \Theta_0 + \bar{c}_{0,2m} \int_{x=-\varepsilon} \Theta_0 &= c_{0,2m} \int_{x=\varepsilon} - \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\alpha^n}{\varepsilon^n |\varepsilon|^{n\alpha}} \omega^{\sigma(1)} \\ &\quad \wedge \cdots \wedge \omega^{\sigma(n)} + O\left(|\varepsilon|^{-(n\alpha+(n-1))}\right) \\ &\quad + \bar{c}_{0,2m} \int_{x=-\varepsilon} - \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\alpha^n}{(-\varepsilon)^n |\varepsilon|^{n\alpha}} \omega^{\sigma(1)} \\ &\quad \wedge \cdots \wedge \omega^{\sigma(n)} + O\left(|\varepsilon|^{-(n\alpha+(n-1))}\right). \end{aligned}$$

Since $c_{0,2m} = -\bar{c}_{0,2m}$, we, therefore, have

$$\begin{aligned} c_{0,2m} \int_{x=\varepsilon} \Theta_0 + \bar{c}_{0,2m} \int_{x=-\varepsilon} \Theta_0 &= 2\bar{c}_{0,2m} \int_{\mathcal{S}} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\alpha^n}{\varepsilon^{n(\alpha+1)}} \omega^{\sigma(1)} \wedge \cdots \wedge \omega^{\sigma(n)} \\ &\quad + O\left(\varepsilon^{-(n\alpha+(n-1))}\right). \end{aligned}$$

This is due to (21) and the fact that n is odd from the definition of M . Hence, we can compute the integral of the right-hand side, to obtain

$$\sum_{k=0}^{m-1} \left[c_{k,2m} \int_{x=\varepsilon} \Theta_k + \bar{c}_{k,2m} \int_{x=-\varepsilon} \Theta_k \right] = \frac{2\alpha^n \bar{c}_{0,2m} n!}{\varepsilon^{n(\alpha+1)}} (\operatorname{vol} \mathcal{S} + O(\varepsilon))$$

We now consider the formula (8) of Theorem 2.3, taking into account M_ε , and plug in (23) to arrive at the equality

$$\varepsilon^{n(\alpha+1)} \left(\chi(M_\varepsilon) - \frac{1}{(2\pi)^m} \int_{|x| \geq \varepsilon} \operatorname{Pf}(\Omega) \right) = 2\alpha^n \bar{c}_{0,2m} n! \operatorname{vol} \mathcal{S} + O(\varepsilon).$$

By taking the limit on both sides and computing the constant $\bar{c}_{0,2m}$, we get

$$(24) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{n(\alpha+1)} \left(\chi(M_\varepsilon) - \frac{1}{(2\pi)^m} \int_{|x| \geq \varepsilon} \text{Pf}(\Omega) \right) = \frac{2\alpha^n (n-1)!!}{(2\pi)^m} \text{vol } \mathcal{S}.$$

A distinction when $\alpha < -1$ has to be made. This is because θ_0^j are now vanishing instead of being singular, see (20). Due to this fact the principal term of the asymptotic expansion of $X_0^* \Pi$ appears in the term with the maximum number of curvature forms $\Omega_{\sigma(i)}^{\sigma(j)}$, i.e. when $k = m - 1$ in (23).

We take on from (21) and a straightforward computation, considering now that $\alpha < -1$, gives the following asymptotic,

$$(25) \quad \sum_{k=0}^{m-1} \left[c_{k,2m} \int_{x=\varepsilon} \Theta_k + \bar{c}_{k,2m} \int_{x=-\varepsilon} \Theta_k \right] = \frac{2\alpha}{\varepsilon^{(\alpha+1)}} \left(\bar{c}_{m-1,2m} \int_{\mathcal{S}} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Omega_{\sigma(2)}^{\sigma(1)} \right. \\ (26) \quad \left. \wedge \cdots \wedge \Omega_{\sigma(n-1)}^{\sigma(n-2)} \wedge \omega^{\sigma(n)} + O(\varepsilon) \right).$$

We arrive at this equality by following a similar process as in the case $k = 0$, and by noticing that $k = m - 1$ is even (m is odd), which by Theorem 2.3 and Remark 2.4 implies that $c_{m-1,2m} = -\bar{c}_{m-1,2m}$.

An interpretation can be given to the sum

$$(27) \quad \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Omega_{\sigma(2)}^{\sigma(1)} \wedge \cdots \wedge \Omega_{\sigma(n-1)}^{\sigma(n-2)} \wedge \omega^{\sigma(n)}.$$

This sum is, in fact, a particular Lipschitz-Killing curvature up to a multiplicative constant. It can be obtained by replacing k by $m - 1$ in Definition 2.2. That is,

$$(28) \quad 2^{m-1} (m-1)! \kappa_{n-1}(\mathcal{S}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Omega_{\sigma(2)}^{\sigma(1)} \wedge \cdots \wedge \Omega_{\sigma(n-1)}^{\sigma(n-2)} \wedge \omega^{\sigma(n)}.$$

Note that we have the Lipschitz-Killing curvature of \mathcal{S} , not of M , since the sum is taken over permutations of order $n = 2m - 1$ and the forms are restricted to the boundary. Also, all restriction on K are satisfied and, in fact, $m - 1$ is the biggest integer k can take.

In the light of this conclusion, formula (25) becomes

$$\sum_{k=0}^{m-1} \left[c_{k,2m} \int_{x=\varepsilon} \Theta_k + \bar{c}_{k,2m} \int_{x=-\varepsilon} \Theta_k \right] = \frac{2\alpha}{\varepsilon^{\alpha+1}} \left(\bar{c}_{m-1,2m} \int_{\mathcal{S}} 2^{m-1} (m-1)! \kappa_{n-1}(\mathcal{S}) + O(\varepsilon) \right).$$

Therefore, when $\alpha < -1$, formula (24) becomes

$$(29) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{\alpha+1} \left(\chi(M_\varepsilon) - \frac{1}{(2\pi)^m} \int_{|x| \geq \varepsilon} \text{Pf}(\Omega) \right) = \frac{2\alpha}{(2\pi)^m} \int_{\mathcal{S}} \kappa_{n-1}(\mathcal{S}),$$

after taking the limit on both sides and computing the constant $\bar{c}_{m-1,2m}$.

In order to obtain our final formulas, we must compute $\chi(M_\varepsilon)$. The sets M_ε and $\mathcal{S} \times (-\varepsilon', \varepsilon')$, where $\varepsilon' > \varepsilon$, form an open cover of M such that $M \simeq M_\varepsilon \cup (\mathcal{S} \times (-\varepsilon', \varepsilon'))$. Moreover, the Euler characteristic is homotopy invariant, and the singular set \mathcal{S} is homotopic to the cylinder $\mathcal{S} \times (-\varepsilon', \varepsilon')$, which contains the tubular neighbourhood considered before. The Euler characteristic satisfies the Inclusion-Exclusion principle. Thus, we have

$$\chi(M) = \chi(M_\varepsilon) + \chi(\mathcal{S} \times (-\varepsilon', \varepsilon')) - \chi(M_\varepsilon \cap (\mathcal{S} \times (-\varepsilon', \varepsilon'))) = \chi(M_\varepsilon)$$

where $\chi(\mathcal{S} \times (-\varepsilon', \varepsilon')) = \chi(\mathcal{S}) = 0$ and $\chi(M_\varepsilon \cap (\mathcal{S} \times (-\varepsilon', \varepsilon'))) = 0$ due to the dimension of \mathcal{S} being odd, and $M_\varepsilon \cap (\mathcal{S} \times (-\varepsilon', \varepsilon'))$ becoming a disjoint union of two copies of \mathcal{S} . Combining with formulas (24) and (29) proves the theorem.

3.3. Particular Case. An interesting special case of Theorem 1.2 is the one in two dimensions. The reasons are that the Pfaffian of the curvature forms matrix $\text{Pf}(\Omega)$ becomes the Gaussian curvature K times the Riemannian area element dA and formulas (5) and (6) condense into one.

The next theorem is of particular importance to conclude the previous fact. It tells how the Gaussian curvature is related to the curvature forms.

Theorem 3.3 ([18], Proposition 2.6). *If M is a two-dimensional Riemannian manifold, then for an orthonormal frame we have $\Omega_0^1 = -K\theta^0 \wedge \theta^1$, where K is the Gaussian curvature of M .*

Corollary 3.4 (of Theorem 1.2). *Let M be a compact oriented two-dimensional α -Grushin manifold, with $\alpha \in \mathbb{R} \setminus \{-1, 0\}$. Then*

$$(30) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{\alpha+1} \left(\chi(M) - \frac{1}{2\pi} \int_{|x| \geq \varepsilon} K dA \right) = \frac{\alpha}{\pi} \text{vol } \mathcal{S}.$$

Proof. We start by considering Theorem 1.2 in dimension two. This implies that $m, n = 1$. Then, formulas (5) and (6) become,

$$(31) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{\alpha+1} \left(\chi(M) - \frac{1}{2\pi} \int_{|x| \geq \varepsilon} \text{Pf}(\Omega) \right) = \frac{\alpha}{\pi} \text{vol } \mathcal{S},$$

$$(32) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{\alpha+1} \left(\chi(M) - \frac{1}{2\pi} \int_{|x| \geq \varepsilon} \text{Pf}(\Omega) \right) = \frac{\alpha}{\pi} \int_{\mathcal{S}} \kappa_0(\mathcal{S}),$$

where we recall that the Lipschitz-Killing curvature $\kappa_0 = d\text{vol}$, which implies that the integral on the right-hand side becomes $\text{vol } \mathcal{S}$.

Let us now consider a two-dimensional frame of the type (11), and its dual frame. Then, by Theorem 3.3 and formula (7), we have

$$\text{Pf}(\Omega) = K\theta^0 \wedge \theta^1,$$

where $\theta^0 \wedge \theta^1$ is precisely the Riemannian area element dA of M . q.e.d.

4. Conclusion and comparison with previous work

We conclude by pointing out the differences between our result and the already known one for two-dimensional almost-Riemannian manifolds, which we state in the following.

Theorem 4.1 ([2], Theorem 21). *Let M be a compact oriented two-dimensional α -Grushin manifold, with $\alpha = 1$. Then*

$$\int K dA_{\mathcal{S}} = 2\pi(\chi(M^+) - \chi(M^-)),$$

where $dA_{\mathcal{S}}$ is an area element given by $\pm dA$, where dA is the Riemannian area element on M .

We state the theorem with respect to our definition of α -Grushin manifolds. This is possible due to the fact that, when $\alpha = 1$ and $m, n = 1$, it precisely coincides with the assumptions made on the two-dimensional almost-Riemannian manifolds. One can check [2], where this fact is discussed systematically.

Even though we are still in two dimensions, we obtained a more general result than Theorem 4.1. Firstly, with $\alpha \in \mathbb{R} \setminus \{-1, 0\}$, Theorem 3.4 holds for several different interesting cases, including the case when $\alpha = 1$, that is, of generic almost-Riemannian manifolds, as already mentioned. Secondly, although in both cases the premiss is the same, i.e., dA diverges nearing \mathcal{S} , the approach followed in [2] is different. In that work, the idea was to consider a Riemannian area element $dA_{\mathcal{S}}$ that changes sign when crossing the singular set. This implies a splitting of M into two manifolds with boundary with opposite orientations, defined by $dA_{\mathcal{S}}$. Denoted by M_{ε}^- if orientation is opposite and M_{ε}^+ if it coincides with the one of M . This is not always possible and excludes a large class of manifolds. The ε in the notation represents the points that are at a distance greater or equal than ε from the singular set, that is, $|x| \geq \varepsilon$. The choice of the specific Riemannian area element $dA_{\mathcal{S}}$ is also the reason why the authors were able to obtain a Gauss-Bonnet formula (4.1) where the Gauss-Bonnet defect is zero, while in our case (Theorem 3.4) the Riemannian area element keeps the same sign on both sides of the singular set, which does not make it possible to get the Gauss-Bonnet defect as zero. Thirdly, their proof relies on a normal form for the orthonormal frame near

the singular set. Namely, $X(x, y) = (1, 0)$ and $Y(x, y) = (0, e^{\phi(x, y)})$, where $\phi \in C^\infty(M)$ such that $\phi(0, y) = 0$. Although it would be possible to adapt this specific pair for two-dimensional α -Grushin manifolds, we freed ourselves from this restriction and work with the general case that depends only on the natural induced curvature form on the singular set by the frame on M .

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