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# Bidomain Model Coupled To A Realistic Multi-Electrode Device In Cardiac Electrophysiology

SimCardioTest European Project

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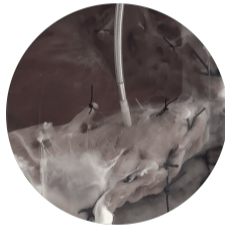
  
Institut de  
Mathématiques de  
Bordeaux

## Context & Motivation

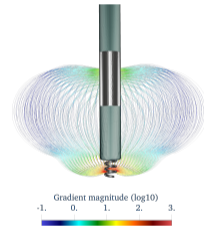
WP2 of SimCardioTest focuses on Pacing Leads and Catheters, especially on **energy required to trigger a cardiac beat** with a pacemaker.



(a) BOREA™ pacemaker



(b) Lead implanted in the septum

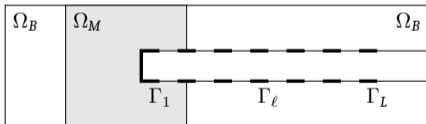


(c) Simulation of electrical field

## Problematic

Electrophysiological propagation models already exist, **but we need to describe the stimulation**

- Three discontinuous parts by stimulation  $\Rightarrow$  **Device circuit must be modeled**  
Structural analysis of electric circuits and consequences for MNA,  
D. Estévez Schwarz, C. Tischendorf, 2000
- Presence of a reactive layer at electrode boundaries  $\Rightarrow$  **Layer must be modeled too**  
Existence and Uniqueness for Electrode Models for Electric Current Computed Tomography,  
E. Somersalo, M. Cheney and D. Isaacson, 1992



(a) Geometric setup

+ Contacts +



(b) BOREA™



## Principle & Notations - II

- Only contains **capacitors, resistors and voltage sources**

- Split** graph and matrix accordingly:

$$B = B_c \cup B_g \cup B_s \text{ and } A = [A_c, A_g, A_s]$$

- $AJ = 0 \Rightarrow \boxed{A_c J_c + A_g J_g + A_s J_s = 0}$

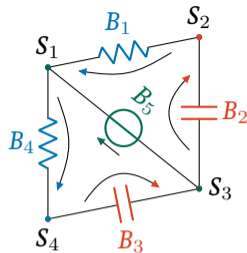
- $V = A^T U \Rightarrow V_c = A_c^T U, V_g = A_g^T U \text{ and } V_s(t) = A_s^T U$

- Dipole laws:**  $J_c = C \frac{dV_c}{dt} = C \frac{dA_c^T U}{dt}$  and  $J_g = G V_g = G A_g^T U$

- Differential algebraic equations (DAE)**

$(U, J_s)$  solution of

$$\begin{cases} A_c C \frac{dA_c^T U}{dt} + A_g G A_g^T U + A_s J_s = 0 \\ A_s^T U = V_s \end{cases}$$



$$U = (U_{S_1}, U_{S_2}, U_{S_3}, U_{S_4})$$

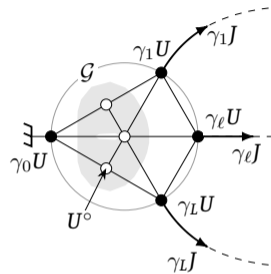
$$J = (J_{B_1}, J_{B_2}, J_{B_3}, J_{B_4}, J_{B_5})$$

$$\begin{matrix} & B_1 & B_2 & B_3 & B_4 & B_5 \\ \begin{matrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{matrix}$$

# Generic Models

## Multi-Electrode Device Model

- Graph  $(S, B)$  oriented with  $L + 1$  **specialized nodes**
- **Potentials** split as  $U = (U^\circ, \gamma U)$  of size  $N_n$ , internal potentials  $U^\circ$
- **Dirichlet boundary condition**,  $\gamma_0 U = 0$  (ground)
- **Coupling boundary potentials**  $\gamma_1 U, \dots, \gamma_L U$
- **Coupling output currents**  $\gamma J = (\gamma_1 J, \dots, \gamma_L J)$



$(U, J_s)$  solution of DAE

$$\begin{cases} A_c C \frac{dA_c^T U}{dt} + A_g G A_g^T U + A_s J_s + \gamma A \gamma J = 0 \\ A_s^T U = V_s \end{cases}$$

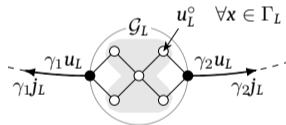
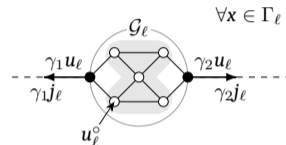
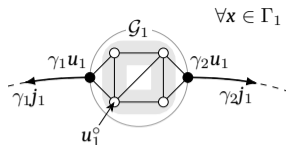
with  $\gamma A := [0, Id_L]^T$ .

# Generic Models

Equivalent Models for each electrode

On each electrode  $\ell = 1, \dots, L$  (without voltage sources)

- **Defined spatially:**  $\forall x \in \Gamma_\ell$
- **Potentials** split as  $u_\ell = (u_\ell^\circ, \gamma u_\ell)$  of size  $N_\ell$
- **Coupling boundary potentials**  $\gamma_1 u_\ell, \gamma_2 u_\ell$
- **Coupling output currents**  $\gamma j_\ell = (\gamma_{1j_\ell}, \gamma_{2j_\ell})$



$u_\ell$  solution of x-ODE

$$A_{\ell,c} C_\ell \frac{\partial A_{\ell,c}^\top u_\ell}{\partial t} + A_{\ell,g} G_\ell A_{\ell,g}^\top u_\ell + \gamma A_\ell \gamma j_\ell = 0$$

on  $\Gamma_\ell$

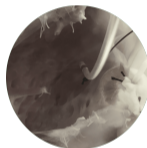
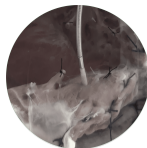
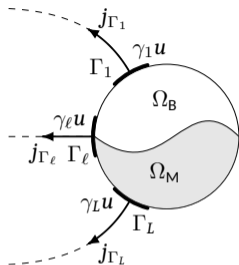
with  $\gamma A_\ell := [0, \text{Id}_2]^\top$ .



# Generic Models

## Bidomain Model (PDE)

- **Tissue**  $\Omega_M$  - intracellular and extracellular
- **Bath/Blood**  $\Omega_B$  - passive media
- **Transmembrane voltage**  $[u] = u_i - u$
- **Conductivity tensors**  $\sigma$  and  $\sigma_i$
- **$L$  electrodes** at boundary  $\bigcup_{\ell=1}^L \Gamma_\ell \subset \partial(\overline{\Omega_M} \cup \overline{\Omega_B})$
- $\gamma_\ell u = u|_{\Gamma_\ell}$ , and  $-\sigma \nabla u \cdot \mathbf{n} = j_{\Gamma_\ell}$  on  $\Gamma_\ell$

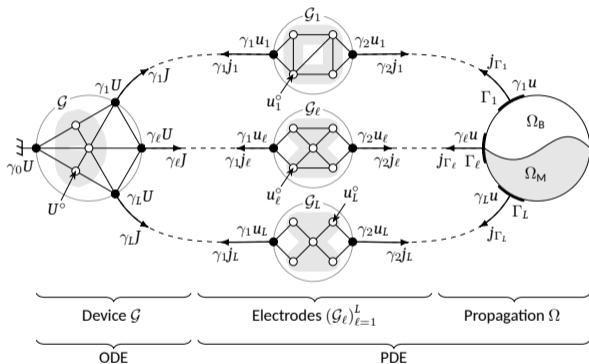


$$(u, u_i) \text{ solution of PDE } \begin{cases} -\operatorname{div}(\sigma_i \nabla u_i) = -j_m & \text{in } \Omega_M \\ -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega_B \end{cases}$$

with membrane current density  $j_m = \chi (c_m \partial_t [u] + I_{\text{ion}}([u], h, t))$ , and Ionic Model on  $h$ .

# Generic Models

## Transmission conditions



### – Potentials

$$\gamma_\ell U = \gamma_1 u_\ell(x), \text{ for } x \in \Gamma_\ell$$

$$\gamma_2 u_\ell(x) = \gamma_\ell u(x), \text{ for } x \in \Gamma_\ell$$

### – Currents

$$\gamma_\ell J + \int_{\Gamma_\ell} \gamma_1 j_\ell(x) = 0$$

$$\gamma_2 j_\ell(x) + j_{\Gamma_\ell}(x) = 0, \text{ for } x \in \Gamma_\ell$$

## Sketch of the proof - I

### Global variational formulation

For all  $\phi = (\phi, \phi_i, \phi_1, \dots, \phi_L, \Phi) \in C_c^\infty$  of  $[0, T] \times \{\overline{\Omega}, \overline{\Omega_M}, (\overline{\Gamma_\ell})^{N_\ell}, \mathbb{R}^{N_n}\}$ ,

find  $\mathbf{u} := (u, \mathbf{u}_i, u_1, \dots, u_L, U)$  such that  $\gamma_0 U = 0$  + transmission conditions on potentials, and  $J_s$ , verifying

$$\left\{ \begin{array}{l} \int_0^T \int_{\Omega_M} \sigma_i \nabla \mathbf{u}_i \cdot \nabla \phi_i + \int_0^T \int_{\Omega} \sigma \nabla u \cdot \nabla \phi + \int_0^T \int_{\Omega_M} \left( c_m \frac{\partial [u]}{\partial t} [\phi] + f([u], h, t) [\phi] \right) \\ + \int_0^T \left( c \frac{dA_c^\top U}{dt} \cdot A_c^\top \Phi + GA_g^\top U \cdot A_g^\top \Phi + A_s J_s \cdot \Phi \right) \\ + \sum_{\ell=1}^L \int_0^T \int_{\Gamma_\ell} \left( c_\ell \frac{\partial A_{\ell,c}^\top u_\ell}{\partial t} \cdot A_{\ell,c}^\top \phi_\ell + G_\ell A_{\ell,g}^\top u_\ell \cdot A_{\ell,g}^\top \phi_\ell \right) = 0, \\ A_s^\top U = V_s \end{array} \right.$$

+ IPP  $\Rightarrow$  weak solution

## Sketch of the proof - II

Global variational formulation semi-discretized in time

Find  $\mathbf{u}^n := (u^n, u_i^n, u_1^n, \dots, u_L^n, U^n) \in \mathcal{H}_0$ , and  $J_s^n \in \mathbb{R}^{N_s}$  with

$$\mathcal{H} := H^1(\Omega_M) \times H^1(\Omega) \times \prod_{\ell=1}^L L^2(\Gamma_\ell)^{N_\ell} \times \mathbb{R}^{N_n},$$

$$\mathcal{H}_0 := \{\mathbf{u} \in \mathcal{H} \text{ s.t. } \boxed{\gamma_0 U = 0, \gamma_\ell U = \gamma_1 u_\ell, \gamma \mathbf{u} = \gamma_2 u_\ell}, \ell = 1, \dots, L\},$$

verifying, for all  $\phi := (\phi, \phi_i, \phi_1, \dots, \phi_L, \Phi) \in \mathcal{H}_0$ ,

$$a\left(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}, \phi\right) + b(\mathbf{u}^n, \phi) + (f([\mathbf{u}^n]), [\phi])_{\Omega_M} + \mathbf{A}_s J_s^n \cdot \Phi = 0,$$

$$\mathbf{A}_s^\top U^n = V_s^n$$

$$a(\delta \mathbf{u}, \phi) = (c_m[\delta \mathbf{u}], [\phi])_{\Omega_M} + \sum_{\ell=1}^L \left( \mathbf{C}_\ell \mathbf{A}_{\ell,c}^\top \delta u_\ell, \mathbf{A}_{\ell,c}^\top \phi_\ell \right)_{\Gamma_\ell} + \mathbf{C} \mathbf{A}_c^\top \delta U \cdot \mathbf{A}_c^\top \Phi,$$

$$b(\mathbf{u}, \phi) = (\sigma_i \nabla u_i, \nabla \phi_i)_{\Omega_M} + (\sigma \nabla \mathbf{u}, \nabla \phi)_\Omega + \sum_{\ell=1}^L \left( \mathbf{G}_\ell \mathbf{A}_{\ell,g}^\top u_\ell, \mathbf{A}_{\ell,g}^\top \phi_\ell \right)_{\Gamma_\ell} + \mathbf{G} \mathbf{A}_g^\top U \cdot \mathbf{A}_g^\top \Phi.$$

## Sketch of the proof - III

### Existence

#### — Hypothesis

$A_s$  has full column rank

#### — Consequence

$\ker(A_s) = \{0\}$  and  $\text{Im}(A_s^T) = \ker(A_s)^\perp = \mathbb{R}^{N_s}$ ,  $\Rightarrow \forall V_s^n \in \mathbb{R}^{N_s}$  there exists  $U_0^n \in \mathbb{R}^{N_n}$  such as  $A_s^T U_0^n = V_s^n$  and  $\|u_0^n\|_{\mathcal{H}} \leq cst \|V_s^n\|_{\mathbb{R}^{N_n}}$ .

#### — Variational formulation in the constraint space

Find  $u_\star^n \in \mathcal{H}_0^{\ker}$ , with  $u^n = u_\star^n + u_0^n$  and  $\mathcal{H}_0^{\ker} := \{u \in \mathcal{H}_0 \mid U \in \ker(A_s^T)\}$ , verifying for all  $\phi_\star \in \mathcal{H}_0^{\ker}$ ,

$$a\left(\frac{u_\star^n + u_0^n - u^{n-1}}{\Delta t}, \phi_\star\right) - b(u_\star^n + u_0^n, \phi_\star) + (f([u^n]), [\phi])_{\Omega_M} = 0$$

Lax-Milgram and Banach fixed-point theorems: existence of  $u_\star^n$  in  $\mathcal{H}_0^{\ker}$ .

#### — Resolution of $J_s^n$ equation in $\mathbb{R}^{N_s}$

- $\forall \{e_k\}$  basis of  $\mathbb{R}^{N_s}$ ,  $\exists \{\phi_k\} \subset \mathcal{H}_0$  s.t.  $e_k = A_s^T \phi_k$ , and  $J_s^n$  solves the linear systems  $J_s^n \cdot e_k = w_k$ ,
- $w_k$  are independent of the choice of  $\{\phi_k\}$ .

## Sketch of the proof – IV

### Compactness & passing to the limit

We have a sequence  $(\mathbf{u}_0^n)$  bounded by data, and  $(\mathbf{u}_*^n)$  in  $\mathcal{H}_0^{\text{ker}}$  with *a priori* estimates.

— *A priori* estimates on

$$a(\mathbf{u}^n, \mathbf{u}^n) = \sum_n \Delta t b(\mathbf{u}^n, \mathbf{u}^n) = \sum_n \Delta t a\left(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}, \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}\right)$$

for compactness (as  $f$  is nonlinear)

— **Compactness:** discrete Aubin-Simon (Gallouët-Latché theorem)

⇒ up to subsequence  $\mathbf{u}^n \rightarrow \mathbf{u}$  (strong convergence)

— **Passing to the limit**

⇒  $\mathbf{u}$  solution of the continuous problem.



SIM  
CARDIO  
TEST

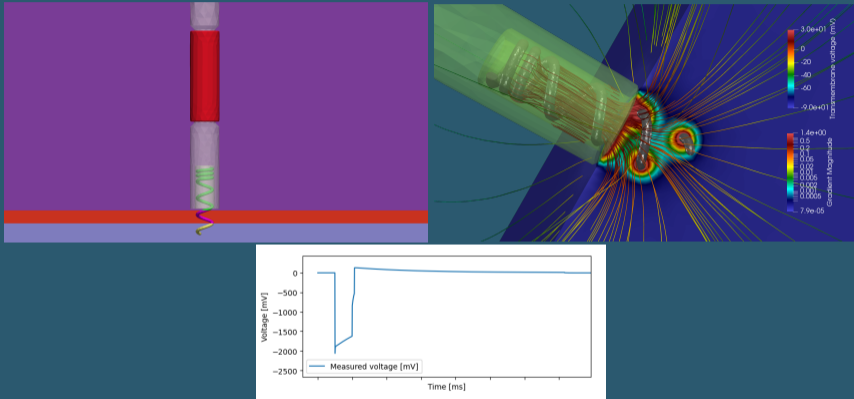


Figure: Pacemaker + 2 R//C contacts + geometry inspired of a standard VEGA lead