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# Note on the Veltkamp/Dekker Algorithms with Directed Roundings

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## Abstract

The Veltkamp/Dekker algorithms are very useful for double-double arithmetic, when no fused-multiply-add is available in hardware. Their analysis is well-known for rounding to nearest-even [6, 1, 5]. We study how they behave with directed roundings in radix 2.

## 1 Veltkamp’s Splitting

We recall Veltkamp’s algorithm in radix 2, using the description from [6, Algorithm 4.9] and [1], where the floating-point constant  $C = 2^s + 1$  is precomputed, and  $\circ(\cdot)$  denotes a rounding:

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**Algorithm 1** Split( $x, s$ )

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**Input:** a  $p$ -bit floating-point number  $x$ , an integer  $s$ ,  $2 \leq s \leq p - 2$ **Output:** floating-point numbers  $x_h, x_\ell$  such that  $x = x_h + x_\ell$ 

- 1:  $C = 2^s + 1$
  - 2:  $\gamma = \circ(Cx)$
  - 3:  $\delta = \circ(x - \gamma)$
  - 4:  $x_h = \circ(\gamma + \delta)$
  - 5:  $x_\ell = \circ(x - x_h)$
- 

In [6], it is said that Dekker [2] proves this algorithm in radix 2 (for rounding to nearest-even), with the implicit assumption that neither overflow nor underflow occurs, and that Boldo [1] shows that for any radix and any precision  $p$ , provided that  $Cx$  does not overflow, the algorithm works (again for rounding to nearest-even). We consider in this note the case where  $\circ(\cdot)$  is a directed rounding from IEEE 754: towards  $-\infty$ , towards  $+\infty$ , or towards zero. We further assume the *same rounding* is used in all steps of Veltkamp’s algorithm.

In [3, Algorithm 4], the authors consider another variant where Step 3 is replaced by  $\delta = \circ(\gamma - x)$  and Step 4 is replaced by  $x_h = \circ(\gamma - \delta)$ . They “prove” Theorem 1 for rounding towards  $\pm\infty$ , but as pointed out by Stef Graillat, there is a hole in their proof: for example in binary64 with rounding towards  $+\infty$  and  $s = 27$ ,  $x = 2 - 2^{-52}$ , we get  $\gamma = 2^{28} + 2$ ,  $\delta = 2^{28} + 2^{-24}$ ,  $x_h = 2 - 2^{-24}$ ,  $x_\ell = 2^{-24} - 2^{-52}$ , and  $x_\ell$  does not fit on  $s$  bits. In [4], the authors say that Veltkamp’s splitting is no more valid in case of directed roundings, and therefore they design alternate split functions for rounding towards  $\pm\infty$ . These alternate algorithms use one more operation, require more time (up to 17% and 34% on an AMD Opteron), moreover they require that the rounding mode is known

to call the correct variant, whereas Theorem 1 shows that the “right” variant from [6] is valid both in rounding to nearest and in the IEEE 754 directed rounding modes.

**Theorem 1** *When  $\circ(\cdot)$  is a directed rounding from IEEE 754 (towards zero, towards  $+\infty$  or towards  $-\infty$ ), assuming no overflow occurs, the outputs  $x_h, x_\ell$  from Veltkamp’s algorithm satisfy  $x = x_h + x_\ell$ ,  $x_h$  fits in  $p - s$  bits, and  $x_\ell$  fits in  $s$  bits.*

Since Veltkamp’s algorithm is proven in [6] for rounding “RN”, where RN denotes either to-nearest-even or to-nearest-away, with Theorem 1 it is proven for all IEEE 754 rounding modes (in binary).

**Proof:** Let us first assume that  $\text{ulp}(x)$  is greater or equal to the smallest positive normal number, say  $y$ . Since  $|\gamma| \geq |x|$ , it is easy to see that all numbers  $\gamma, \delta, x_h, x_\ell$  are integer multiples of  $y$ , thus there is no underflow in the algorithm. We mimic the proof of [6]. The rounding towards zero satisfies  $\circ(-e) = -\circ(e)$  for any expression  $e$ , thus it is easy to see that the values of  $\gamma, \delta, x_h, x_\ell$  obtained by Veltkamp’s algorithm for  $x$  and  $-x$  are opposite. Similarly, if we run Veltkamp’s algorithm once for  $x$  and rounding towards  $+\infty$ , and then for  $-x$  and rounding towards  $-\infty$ , the values obtained for  $x$  and  $x'$  are opposite. We can thus restrict to  $x > 0$  (the case  $x = 0$  being trivial). If we multiply  $x$  by  $2^k$  for an integer  $k$ , all variables in the algorithm are scaled by  $2^k$ , thus we can assume  $1 \leq x < 2$ . If  $x = 1$ , then  $\gamma = C = 2^s + 1$ ,  $\delta = -2^s$ ,  $x_h = 1$ ,  $x_\ell = 0$ , thus the statement of the theorem holds. We can thus assume  $1 < x < 2$ , which yields  $1 + 2^{-p+1} \leq x \leq 2 - 2^{-p+1}$  since  $x$  is a  $p$ -bit floating-point number.

**Computation of  $\gamma$ .**  $Cx = (2^s + 1)x$  implies  $2^s + 1 < Cx < 2^{s+2}$ , therefore  $2^{s-p+1} \leq \text{ulp}(Cx) \leq 2^{s-p+2}$ . This gives  $\gamma = (2^s + 1)x + \varepsilon_1$  with  $|\varepsilon_1| < 2^{s-p+2}$ , and  $\gamma$  is a multiple of  $2^{s-p+1}$ .

**Computation of  $\delta$ .** We have  $\delta = x - \gamma + \varepsilon_2$  with  $\varepsilon_2$  the rounding error in the computation of  $\delta$ . This yields  $\delta = -2^s x - \varepsilon_1 + \varepsilon_2$ .

We have  $x - \gamma = -2^s x - \varepsilon_1$ , thus  $|x - \gamma| < 2^s(2 - 2^{-p+1}) + 2^{s-p+2} = 2^{s+1} + 2^{s-p+1}$ . But for  $|x - \gamma|$  to exceed  $2^{s+1}$ , we need  $\varepsilon_1 > 0$ , thus rounding towards  $+\infty$ ; in that case since  $-2^{s+1} - 2^{s-p+1} < x - \gamma < -2^{s+1}$ ,  $x - \gamma$  is rounded to  $-2^{s+1}$ , thus in all cases  $|\delta| \leq 2^{s+1}$ . We now prove that  $|\delta| \geq 2^s$  by distinguishing three cases: (i) for  $x \neq 1 + 2^{-p+1}$ , then  $x \geq 1 + 2^{-p+2}$  thus  $|x - \gamma| > 2^s(1 + 2^{-p+2}) - 2^{s-p+2} = 2^s$ ; (ii) if  $x = 1 + 2^{-p+1}$  and we round towards  $+\infty$ , then  $\varepsilon_1 \geq 0$ , and  $|x - \gamma| \geq 2^s x > 2^s$ ; (iii) finally if  $x = 1 + 2^{-p+1}$  and we round towards zero or  $-\infty$ , then it is easy to see that  $\gamma = 2^s + 1 + 2^{s-p+1}$ , which is exactly representable in precision  $p$ , thus  $\varepsilon_1 = -2^{-p+1}$ , and  $|x - \gamma| = 2^s + 2^{s-p+1} - 2^{-p+1} > 2^s$ . Therefore, in all cases,  $\delta$  lies in the binade  $[2^s, 2^{s+1}]$ . This implies that  $\delta$  is a multiple of  $2^{s-p+1}$ . From Sterbenz’s lemma, since  $-\gamma$  and  $\delta$  are within a factor of 2 from each other,  $\gamma + \delta$  is computed exactly:  $x_h = \gamma + \delta = x + \varepsilon_2$ . Since both  $\gamma$  and  $\delta$  are multiples of  $2^{s-p+1}$ , so is  $x_h$ . We also have  $|\varepsilon_2| < 2^{s-p+1}$ : either  $|x - \gamma|$  lies in the binade  $[2^s, 2^{s+1}]$ , and it follows from the fact that  $\varepsilon_2 < \text{ulp}(x - \gamma)$ , otherwise we have seen that  $-2^{s+1} - 2^{s-p+1} < x - \gamma < -2^{s+1}$ , and that  $x - \gamma$  is rounded to  $-2^{s+1}$ , thus  $|\varepsilon_2| < 2^{s-p+1}$  too. From  $x < 2$  and  $x_h = x - \varepsilon_2$ , one deduces  $x_h < 2 + 2^{s-p+1}$ , but the only multiple of  $2^{s-p+1}$  between 2 and  $2 + 2^{s-p+1}$  is 2, so  $x_h \leq 2$ .

**Computation of  $x_\ell$ .** By Sterbenz’s lemma,  $x - x_h$  is computed exactly, thus  $x_\ell = x - x_h = \varepsilon_2$ . As a consequence,  $|x_\ell| = |\varepsilon_2|$  is less than  $2^{s-p+1}$ . Moreover  $x_\ell$  is a multiple of  $2^{-p+1}$ , since  $x$  and  $x_h$  are multiples of  $2^{-p+1}$ .

In summary, we have  $x = x_h + x_\ell$  with:

- $x_h \leq 2$  and  $x_h$  multiple of  $2^{s-p+1}$  imply that  $x_h$  fits in  $p-s$  bits;
- $|x_\ell| < 2^{s-p+1}$  and  $x_\ell$  multiple of  $2^{-p+1}$  imply that  $x_\ell$  fits in  $s$  bits.

Now assume that  $\text{ulp}(x)$  is smaller than the smallest positive normal number  $y$ . Letting  $2^e \leq |x| < 2^{e+1}$ , then by scaling the bounds of the above proof we find the bounds  $|\varepsilon_1| < 2^{e+s-p+2}$  and  $|\varepsilon_2| < 2^{e+s-p+1}$ . If  $2^{e+s-p+1}$  is greater or equal to the smallest positive subnormal number  $\alpha$ , these bounds hold, and the theorem too. Now assume  $2^{e+s-p+2} \leq \alpha$ : this means that  $|x| < 2^{e+1} \leq 2^{p-s-1}\alpha$ . It follows  $(2^s + 1)|x| < (2^s + 1)2^{p-s-1}\alpha < 2^p\alpha$ , thus  $\gamma$  is computed exactly,  $\delta = -2^s x$ ,  $x_h = x$ ,  $x_\ell = 0$ , and the statement holds. ■

Note that the statement from Theorem 1 does not hold when different roundings are used for the computation of  $\gamma$  and  $\delta$ . For example with  $p = 4$ ,  $s = 2$ ,  $x = 15/8$ , if  $\gamma$  is rounded upwards we get  $\gamma = 10$ , then if  $\delta$  is rounded downwards we get  $\delta = -9$ , which yields  $x_h = 1$  and  $x_\ell = 7/8$ , which does not fit in 2 bits. It does not hold either when rounding “away from zero” (which is not in IEEE 754): in binary64 with  $s = 27$  and  $x = 2 - 2^{-52}$ , we would get  $\gamma = 2^{28} + 2$ ,  $\delta = -2^{28} - 2^{-24}$ ,  $x_h = 2 - 2^{-24}$ ,  $x_\ell = 2^{-24} - 2^{-52}$ , and  $x_\ell$  does not fit on  $s$  bits (this is the same example that fails with the variant from [3]).

If overflow occurs in the computation of  $\gamma = \circ(Cx)$ , the error bound  $\gamma = (2^s + 1)x + \varepsilon_1$  with  $|\varepsilon_1| < 2^{e+s-p+2}$  no longer holds. Thus the statement Theorem 1 is in some sense the best possible.

## 2 Dekker’s Product

Again, we use the algorithm from [6, Algorithm 4.10]:

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### Algorithm 2 Dekker’s product

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**Input:**  $x, y$  two  $p$ -bit floating-point numbers

**Output:** floating-point numbers  $r_1, r_2$  such that  $x \cdot y = r_1 + r_2$

- 1:  $s = \lceil p/2 \rceil$
  - 2:  $(x_h, x_\ell) = \text{Split}(x, s)$
  - 3:  $(y_h, y_\ell) = \text{Split}(y, s)$
  - 4:  $r_1 = \circ(xy)$
  - 5:  $t_1 = \circ(-r_1 + \circ(x_h y_h))$
  - 6:  $t_2 = \circ(t_1 + \circ(x_h y_\ell))$
  - 7:  $t_3 = \circ(t_2 + \circ(x_\ell y_h))$
  - 8:  $r_2 = \circ(t_3 + \circ(x_\ell y_\ell))$
- 

**Theorem 2** *When  $\circ(\cdot)$  is a directed rounding from IEEE 754 (towards zero, towards  $+\infty$  or towards  $-\infty$ ) and  $p$  is even, assuming no overflow nor underflow occurs, the outputs  $r_1, r_2$  from Dekker’s algorithm satisfy  $xy = r_1 + r_2$ .*

Since Dekker’s algorithm is proven in [6] for rounding “RN”, where RN denotes either to-nearest-even or to-nearest-away, with Theorem 2 it is proven for all IEEE 754 rounding modes (in binary).

**Proof:** Again, we mimic the proof from [6]. From Theorem 1, we know that  $x_h, y_h$  fit in  $p-s$  bits, that  $x_\ell, y_\ell$  fit in  $s$  bits,  $x = x_h + x_\ell$ , and  $y = y_h + y_\ell$ . Since  $p$  is even,  $s = p - s = p/2$ , and all four products  $x_h y_h, x_h y_\ell, x_\ell y_h, x_\ell y_\ell$  fit in  $p$  bits, thus  $t_1 = \circ(-r_1 + x_h y_h)$ ,  $t_2 = \circ(t_1 + x_h y_\ell)$ ,

$t_3 = \circ(t_2 + x_\ell y_h)$ ,  $r_2 = \circ(t_3 + x_\ell y_\ell)$ . Without loss of generality, we can assume  $1 \leq x, y < 2$ . We have seen in the proof of Theorem 1 that  $x_h, y_h < 2$ , and that  $|x - x_h|, |y - y_h| \leq 2^{s-p+1}$ . From

$$xy - x_h y_h = (x - x_h)y + (y - y_h)x_h,$$

we deduce

$$|xy - x_h y_h| \leq 2^{s-p+3}.$$

Since we also have

$$|xy - r_1| \leq \text{ulp}(xy) \leq 2^{-p+2},$$

we get

$$|r_1 - x_h y_h| \leq 2^{-p+2} + 2^{s-p+3}.$$

This shows that  $r_1$  and  $\circ(x_h y_h) = x_h y_h$  are very close, so Sterbenz's lemma can be applied to their subtraction. As a consequence,

$$t_1 = -r_1 + x_h y_h.$$

Now,  $xy - x_h y_h = x_h y_\ell + x_\ell y_h + x_\ell y_\ell$ , so

$$\begin{aligned} |t_1 + x_h y_\ell| &= |-r_1 + x_h y_h + x_h y_\ell| \\ &= |-r_1 + xy + (x_h y_h + x_h y_\ell - xy)| \\ &\leq |-r_1 + xy| + |x_\ell(y_h + y_\ell)| \\ &< 2^{-p+2} + (2^{s-p+1} - 2^{-p+1}) \cdot 2 \leq 2^{s-p+2}, \end{aligned}$$

where we have used the fact that  $|x_\ell| < 2^{s-p+1}$  and  $x_\ell$  is a multiple of  $2^{-p+1}$ , thus  $|x_\ell| \leq 2^{s-p+1} - 2^{-p+1}$ , and  $|y_h + y_\ell| = |y| < 2$ . Since  $x_h$  is a multiple of  $2^{s-p+1}$  and  $y_\ell$  is a multiple of  $2^{-p+1}$ ,  $x_h y_\ell$  is a multiple of  $2^{s-2p+2}$ . This implies that  $t_1 +_h y_\ell$  is a multiple of  $2^{s-2p+2}$ . This and  $|t_1 +_h y_\ell| < 2^{s-p+2}$  imply that  $t_1 +_h y_\ell$  is exactly representable. Therefore,

$$t_2 = -r_1 + x_h y_h + x_h y_\ell.$$

Now,  $t_2 + x_\ell y_h = (-r_1 + xy) - x_\ell y_\ell$ ; therefore,

$$\begin{aligned} |t_2 + x_\ell y_h| &\leq |-r_1 + xy| + |x_\ell y_\ell| \\ &\leq 2^{-p+2} + 2^{2s-2p}. \end{aligned}$$

Since  $2s \leq p + 1$ ,

$$|t_2 + x_\ell y_h| \leq 2^{-p+2} + 2^{-p+1} \leq 2^{-p+3}.$$

This and the fact that  $t_2 + x_\ell y_h$  is a multiple of  $u := 2^{s-2p+2}$  imply that  $|t_2 + x_\ell y_h| \leq 2^{p+1-s} \cdot u$ , thus  $t_2 + x_\ell y_h$  is exactly representable; therefore,

$$t_3 = t_2 + x_\ell y_h = -r_1 + x_h y_h + x_h y_\ell + x_\ell y_h.$$

Finally,  $|t_3 + x_\ell y_\ell| = |-r_1 + xy| \leq 2^{-p+2}$ , and it is a multiple of  $2^{-2p+2}$ , thus  $t_3 + x_\ell y_\ell$  is exactly computed. Therefore,

$$r_2 = xy - r_1.$$

■

Overflow can occur in the computation of  $r_1 = \circ(xy)$ , in which case the error bounds no longer hold; in particular if  $|xy|$  exceeds twice the largest representable number  $\Omega$ , there is no way we can write  $xy = r_1 + r_2$  with  $|r_1|, |r_2| \leq \Omega$ . Underflow can occur in the computation of  $\circ(x_\ell y_\ell)$ , in which case the error bounds no longer hold; in particular if the exact product  $xy$  is not an integer multiple of the smallest positive subnormal  $\alpha$ , there is no way we can write  $xy = r_1 + r_2$  with both  $r_1, r_2$  multiples of  $\alpha$ . (In case of underflow, it is given in [1] a bound of  $7/2 \cdot \alpha$ , and it is said it might be improved to  $3\alpha$  in binary.) The bound of Theorem 2 is thus the best possible.

For  $p$  odd, the statement of Theorem 2 does not hold (see Lemma 1 below in that case). Take for example  $p = 3$ ,  $x = y = 5/4$ , and rounding towards zero. We get  $x_h = y_h = 2$ ,  $x_\ell = y_\ell = -3/4$ . It follows  $r_1 = 3/2$ ,  $t_1 = 5/2$ ,  $t_2 = 1$ ,  $t_3 = -1/2$ ,  $r_2 = 0$ . The issue is that since  $x_\ell$  and  $y_\ell$  both fit in 2 bits, their product fits in 4 bits and not 3, thus  $x_\ell y_\ell$  is not exact.

Note that in Algorithm 2, a different rounding mode can be used to split  $x$  in Step 2, since we only used the facts that  $x = x_h + x_\ell$ ,  $x_h$  fits in  $p - s$  bits, and  $x_\ell$  fits in  $s$  bits (and similarly for  $y$ ).

**Lemma 1** *When  $\circ(\cdot)$  is a directed rounding from IEEE 754 (towards zero, towards  $+\infty$  or towards  $-\infty$ ) and  $p$  is odd, assuming no overflow nor underflow occurs, the outputs  $r_1, r_2$  from Dekker's algorithm satisfy  $xy = r_1 + r_2 + \varepsilon$  with  $|\varepsilon| \leq 2^{e_x + e_y - 2p}$ , where  $2^{e_x - 1} \leq |x| < 2^{e_x}$  and  $2^{e_y - 1} \leq |y| < 2^{e_y}$  (assuming neither  $x$  nor  $y$  is zero).*

**Proof:** All the reasoning of the proof of Theorem 2 holds, except that  $x_\ell y_\ell$  is not exactly representable. Let  $t_4 = \circ(x_\ell y_\ell)$ . Since  $|x_\ell y_\ell| < 2^{2s - 2p + 2}$ ,  $t_4 = x_\ell y_\ell + \varepsilon$  with  $|\varepsilon| < \text{ulp}(x_\ell y_\ell) \leq 2^{2s - 3p + 2}$ . Moreover, since  $p$  is odd,  $2s = p + 1$ , thus  $|\varepsilon| < 2^{-2p + 3}$ . We thus have  $r_2 = \circ(-r_1 + xy + \varepsilon)$ . In fact, since  $x_\ell y_\ell$  is a multiple of  $2^{-2p + 2}$ , so is  $\varepsilon$ , and  $\varepsilon$  can only take three values: 0 and  $\pm 2^{-2p + 2}$ . We thus have  $|-r_1 + xy + \varepsilon| \leq |-r_1 + xy| + 2^{-2p + 2}$ . Since  $|-r_1 + xy| < \text{ulp}(xy) \leq 2^{-p + 2}$  and  $|-r_1 + xy|$  is a multiple of  $2^{-2p + 2}$ , we have  $|-r_1 + xy| \leq 2^{-p + 2} - 2^{-2p + 2}$ . It follows  $|-r_1 + xy + \varepsilon| \leq 2^{-p + 2}$ , and as in the case  $p$  even,  $-r_1 + xy + \varepsilon$  is exactly representable, thus  $r_2 = -r_1 + xy + \varepsilon$ . ■

The example above with  $p = 3$  has  $|\varepsilon| = 2^{-2p + 2}$ , which shows the bound from Lemma 1 is optimal.

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