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Asymptotic estimations of a perturbed symmetric eigenproblem

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Abstract

We study ill-conditioned positive definite matrices that are disturbed by the sum of m rank-one matrices of a specific form. We provide estimates for the eigenvalues and eigenvectors. When the condition number of the initial matrix tends to infinity, we bound the values of the coordinates of the eigenvectors of the perturbed matrix. Equivalently, in the coordinate system where the initial matrix is diagonal, we bound the rate of convergence of coordinates that tend to zero.

Keywords: Perturbation of symmetric matrices, estimation of eigenvectors

2020 MSC: 15A42, 15B57

1. Introduction

Given a $d \times d$ symmetric matrix with known eigenvectors and eigenvalues denoted B , and a rank-one matrix vv^T where $v \in \mathbb{R}^d$, eigenvalues and eigenvectors of matrices of the form

$$A = B + vv^T \quad (P_1)$$

have been widely studied, notably in the context of perturbation theory. For instance, the eigenvalues of (P_1) can be estimated and a formula for the eigenvectors is known [1, 2, 3]. Specifically, if B is a diagonal matrix $\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_d)$ with distinct eigenvalues and v has only nonzero entries, then the component j of the unit eigenvector associated to eigenvalue ν_i of the updated matrix A satisfies the so-called Bunch-Nielsen-Sorensen formula

$$C_i \times \frac{[v]_j}{\lambda_j - \nu_i} \quad \text{for } i, j = 1, \dots, d \quad (1)$$

where C_i is a nonzero normalization constant and $[.]_j$ denotes the j -th coordinate, a notation we will continue to use in the sequel. Several results have been established for additive perturbations of rank 1 [4, 5] and of higher rank [6, 7]. Symmetric and nonsymmetric perturbation eigenvalue problems have been studied [8] as well as perturbation results for invariant subspaces [9]. In this paper, we provide relative perturbation bounds for the eigenvectors of positive definite matrices. In contrast to previous relative perturbation results for eigenvalues [10] and invariant subspaces [11, 12], the bounds in our result depend on the eigenvalues of the initial matrix B rather than the norm of the perturbation, see Eq. (2) below.

Specifically, we consider the perturbation with a sum of m rank-one matrices of the form

$$A^{(m)} = B + \sqrt{B} \sum_{i=1}^m [v^{(i)}][v^{(i)}]^T \sqrt{B} \quad (P_m)$$

with $B = P \text{diag}(\lambda_1, \dots, \lambda_d) P^T$ where P is an orthogonal matrix, $\lambda_1 \geq \dots \geq \lambda_d > 0$ are the eigenvalues of B , and $v^{(1)}, \dots, v^{(m)} \in \mathbb{R}^d$. The square root $\sqrt{B} := P \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}) P^T$ is defined as the unique symmetric positive

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definite matrix such that $\sqrt{B} \times \sqrt{B} = B$, see e.g. [13, Theorem 7.2.6]. Matrices of the form (P_m) are used in various applications in different domains. For instance, low rank updates of covariance matrices are used in stochastic optimization [14, 15], system identification [16, p. 369], and adaptive Markov Chain Monte Carlo methods [17]. Our motivation is to study the eigenvectors of A in (P_m) , denoted as $e_i^{(m)}$ in the sequel, *when the matrix B is highly ill-conditioned*. When $d = 2$ and $m = 1$ we can compute the eigenvectors of $A^{(1)}$ explicitly. As an example, consider $B = \text{diag}(\lambda_1, 1)$ where $\lambda_1 > 1$ and $v^{(1)} = [1, 1]^T$. Then, the unit eigenvector associated to the largest eigenvalue of $A^{(1)}$ obeys $\sqrt{1+s^2} \times e_1^{(1)} = [1, s]^T$ with $s = \lambda_1^{1/2} \times (1 - \lambda_1^{-1} - \sqrt{1 - \lambda_1^{-1} + \lambda_1^{-2}}) = -\lambda_1^{-1/2}/2 + \mathcal{O}(\lambda_1^{-3/2})$ and hence $[e_1^{(1)}]_2 = \lambda_1^{-1/2} + o(\lambda_1^{-1/2})$ when $\lambda_1 \rightarrow \infty$. Hence, the (second) coordinate of the (first) unit eigenvector of $A^{(1)}$ vanishes like $1/\sqrt{\lambda_1}$ when $\lambda_1 \rightarrow \infty$. In this paper, we generalize this result to the case where $d \geq 2$ and $m \geq 1$, as summarized in the following theorem which directly follows from Theorem 4 below.

Theorem 1. *If $e_i^{(m)}$ is a unit eigenvector corresponding to the i -th largest eigenvalue (counted with multiplicity) of $A^{(m)}$ in (P_m) and $e_j^{(0)}$ is a unit eigenvector corresponding to the j -th largest eigenvalue λ_j of B , then*

$$\left| \langle e_i^{(m)}, e_j^{(0)} \rangle \right| \leq C_m \times \sqrt{\frac{\min\{\lambda_i, \lambda_j\}}{\max\{\lambda_i, \lambda_j\}}} \quad (2)$$

where $C_m > 0$ is a constant which depends polynomially on d and $\max_{k=1, \dots, m} \|v^{(k)}\|$.

When B is diagonal, $e_j^{(0)}$ is the j -th canonical unit vector. Hence $|\langle e_i^{(m)}, e_j^{(0)} \rangle| = [e_i^{(m)}]_j$ and the theorem implies in particular that the j -th coordinate of $e_i^{(m)}$ converges to zero at least as fast as $\sqrt{\min\{\lambda_i, \lambda_j\}/\max\{\lambda_i, \lambda_j\}}$ when the latter tends to 0 (which is tight in the above example when $d = 2$ and $m = 1$), thereby limiting the change of the angle between these eigenvectors. Considerations on the angle between eigenspaces have been made previously [18], however matrices on the form of (P_m) have not been studied in this context. In the remainder, we always choose w.l.o.g. the coordinate system where the matrix B of (P_m) is diagonal and has decreasingly ordered diagonal values.

This inequality is crucial to study the stability of a Markov chain underlying the CMA-ES algorithm [19, 15]. Proofs of linear convergence for Evolutionary Strategies (ES) rely on a drift condition [20, Theorem 17.0.1] to prove the ergodicity of an underlying Markov chain, see e.g. [21, 22]. To apply this approach to CMA-ES, a potential function is defined on the state-space of this Markov chain and its expected decrease is proven outside a compact set. The state space includes a covariance matrix, updated as

$$C_{t+1} = (1 - c)C_t + c \sqrt{C_t} \sum_{i=1}^m w_i U_i U_i^T \sqrt{C_t}, \quad (3)$$

where $c \in [0, 1]$, w_1, \dots, w_m are positive weights that sum to 1, and the vectors U_i , $i = 1, \dots, m$, are Gaussian vectors ranked according to a fitness function [15, Eq. (11)]. Hence, (P_m) encompasses the update of this covariance matrix. Eq. (2) is needed to bound the expected condition number of the updated covariance matrix, since it controls the influence of small eigenvalues on the growth of the largest eigenvalues.

This paper is organized as follows. In Section 2, we study the eigenvalues of (P_m) . In Section 3, we provide bounds for the coordinates of the eigenvectors using Eq. (1), and provide an empirical result suggesting that these bounds are tight.

2. Bounds on the eigenvalues of (P_m)

The Bunch-Nielsen-Sorensen formula (1) which we will use in Section 3 requires the eigenvalues of the *updated* matrix. Thus, we first derive bounds on the (decreasingly ordered) eigenvalues

$$\lambda_i(A^{(m)}) = \max_{V \subset \mathbb{R}^d, \dim V = i} \min_{v \in V, v \neq 0} \frac{v^T A^{(m)} v}{v^T v} = \min_{V \subset \mathbb{R}^d, \dim V = d - i + 1} \max_{v \in V, v \neq 0} \frac{v^T A^{(m)} v}{v^T v} \quad \text{for } i = 1, \dots, d \quad (4)$$

where the equalities ensue from the min-max principle and from Gersgorin's circle theorem [13, Theorems 4.2.6 and 6.1.1].

Theorem 2.7 in [10] and Theorem 2.1 in [23, p. 175] provide an estimation for the eigenvalues of (P_m) :

$$\lambda_i \leq \nu_i \leq \lambda_i \times \left(1 + md \times \max_{k=1, \dots, m} \|v^{(k)}\|_\infty^2\right) \quad \text{for } i \in \{1, \dots, d\}. \quad (5)$$

The next lemma provides a slightly tighter upper bound on these eigenvalues after a single rank-one perturbation ($m = 1$) and is used in Proposition 3.

Lemma 2. *Let $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ be a diagonal matrix with $\lambda_1 > \dots > \lambda_d > 0$. Let $v \in \mathbb{R}_{\neq 0}^d$ be a vector with only nonzero entries. Let $A = D + \sqrt{D}vv^T\sqrt{D}$ and $\nu_1 \geq \nu_2 \geq \dots \geq \nu_d$ denote the eigenvalues of A . Then,*

$$\nu_i \leq \lambda_i \times \left(1 + (d - i + 1)\|v\|_\infty|v_{j_i}|\right) \quad \text{for all } i \in \{1, \dots, d\} \quad (6)$$

where $j_i \in \text{Arg max}_{j=i, \dots, d} \left\{|v_j| \mid \lambda_j \geq \lambda_i \times \left(1 - \sqrt{\frac{\lambda_j}{\lambda_i}}(d - i + 1)\|v\|_\infty|v_{j_i}|\right)\right\}$.

Proof. Fix $i \in \{1, \dots, d\}$ and remark that, by Eq. (4), we have $\nu_i \leq \max_{v \in \bar{V}_i, \|v\|=1} v^T A v = \lambda_1([A]_{i:d, i:d})$, where $\bar{V}_i = \text{Vect}(e_i, \dots, e_d)$ with e_i being the i^{th} vector of the standard basis of \mathbb{R}^d , and with $[A]_{i:d, i:d}$ denoting the submatrix of A from rows and columns with indices between i and d included. But, by [13, Theorem 6.1.1], we also have that

$$\lambda_1([A]_{i:d, i:d}) \leq \max_{j=i, \dots, d} \left(\sum_{k=i}^d |[A]_{j,k}| \right) =: \max_{j \geq i} B_j.$$

Since $A = D + \sqrt{D}vv^T\sqrt{D}$, then $|[A]_{j,k}| \leq \sqrt{\lambda_j \lambda_k}(\mathbb{1}\{j=k\} + \|v\|_\infty|v_j|)$. If $j \geq i$ is such that $\lambda_j \geq \lambda_i \times (1 - \sqrt{\lambda_j/\lambda_i}(d - i + 1)\|v\|_\infty|v_{j_i}|)$, then by definition of j_i we have then $|v_j| \leq |v_{j_i}|$, yielding to $B_j \leq \lambda_i \times (1 + (d - i + 1)\|v\|_\infty|v_{j_i}|)$. Any other $j \geq i$ satisfies $\lambda_j < \lambda_i - \sqrt{\lambda_j \lambda_i}(d - i + 1)\|v\|_\infty|v_{j_i}|$, hence by sum $B_j \leq \lambda_i$. All in all, $\max_{j \geq i} B_j \leq \lambda_i \times (1 + (d - i + 1)\|v\|_\infty|v_{j_i}|)$, proving Eq. (6). \square

3. Estimating the eigenvectors of (P_m)

We use the bounds from Lemma 2 and Eq. (5) to estimate the eigenvectors of (P_m) by applying Eq. (1), for $m = 1$ in the next section and $m \geq 1$ in Section 3.2.

3.1. Rank-one perturbation

In Proposition 3, we obtain bounds on the coordinates of the eigenvectors of (P_m) (B is assumed to be diagonal) when $m = 1$, which comes as a consequence of Eq. (1).

Proposition 3. *Let $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ be a diagonal matrix with $\lambda_1 \geq \dots \geq \lambda_d > 0$. Let $v \in \mathbb{R}^d$ and $V := \max\{d^{-1/2}, \|v\|_\infty\}$. Consider the matrix $A = D + \sqrt{D}vv^T\sqrt{D}$ and $\nu_1 \geq \dots \geq \nu_d$ its eigenvalues and $(e_1^{(1)}, \dots, e_d^{(1)})$ a corresponding orthonormal basis of eigenvectors. Then,*

$$|[e_i^{(1)}]_j| \leq 5d^2 V^4 \sqrt{\frac{\min\{\lambda_i, \lambda_j\}}{\max\{\lambda_i, \lambda_j\}}} \quad \text{for all } i, j \in \{1, \dots, d\} \quad (7)$$

Proof. We prove first that, if $\max\{\lambda_i, \lambda_j\} > (1 + dV^2) \times \min\{\lambda_i, \lambda_j\}$, then

$$|[e_i^{(1)}]_j| \leq (d - i + 1)V^2 \times \frac{\inf_{\rho \in (0,1)} \psi(\rho, (d - i + 1)V^2)}{1 - (1 + (d - i + 1)V^2) \frac{\min\{\lambda_i, \lambda_j\}}{\max\{\lambda_i, \lambda_j\}}} \sqrt{\frac{\min\{\lambda_i, \lambda_j\}}{\max\{\lambda_i, \lambda_j\}}} \quad (8)$$

with $\psi(\rho, W) = \max\{2(1 - \rho)^{-1/2}, 2\rho^{-1}W\}$, from which we deduce Eq. (7).

First suppose that the eigenvalues λ_i of D are distinct, and that all entries of v are nonzero. Then, by Eq. (1), we have for $i, j \in \{1, \dots, d\}$ that $[e_i^{(1)}]_j = C_i \frac{[\sqrt{D}v]_j}{\lambda_j - \nu_i} = C_i \frac{\sqrt{\lambda_i}[v]_j}{\lambda_j - \nu_i}$, where $C_i \in \mathbb{R}$ is chosen such that $\|e_i^{(1)}\| = 1$, hence

$$|C_i| = \left\| \left(\frac{\sqrt{\lambda_j}[v]_j}{\lambda_j - \nu_i} \right)_{j=1, \dots, d} \right\|^{-1} = \left(\sum_{j=1}^d \left| \frac{\sqrt{\lambda_j}[v]_j}{\lambda_j - \nu_i} \right|^2 \right)^{-1/2} \leq \min_{1 \leq j \leq d} \frac{|\lambda_j - \nu_i|}{\sqrt{\lambda_j}[v]_j}. \quad (9)$$

Combining [23, Theorem 2.1, p. 175] with Eq. (6), we have $\lambda_i < \nu_i \leq \lambda_i \times (1 + (d - i + 1)V|[v]_{j_i}|)$, where j_i is defined in Lemma 2. By definition of j_i we have $0 \leq \lambda_i - \lambda_{j_i} \leq \lambda_i(d - i + 1)V|[v]_{j_i}|$. By sum, we obtain $0 < \nu_i - \lambda_{j_i} \leq 2\lambda_i(d - i + 1)V|[v]_{j_i}|$. We apply this to Eq. (9) to get

$$|C_i| \leq \frac{\nu_i - \lambda_{j_i}}{\sqrt{\lambda_{j_i}}|[v]_{j_i}|} \leq \frac{2\lambda_i(d - i + 1)V|[v]_{j_i}|}{\sqrt{\lambda_{j_i}}|[v]_{j_i}|} = \frac{\lambda_i}{\sqrt{\lambda_{j_i}}} 2(d - i + 1)V. \quad (10)$$

Let $\rho \in (0, 1)$. If $(d - i + 1)V^2 \sqrt{\lambda_{j_i}} \leq \rho \sqrt{\lambda_i}$, then by definition of j_i , $\lambda_{j_i} \geq \lambda_i \times (1 - \rho)$, and by Eq. (10), $|C_i| \leq (1 - \rho)^{-1/2} \times 2(d - i + 1)V \sqrt{\lambda_i}$. Otherwise, $|C_i| \leq \rho^{-1} \times 2(d - i + 1)^2 V^3 \sqrt{\lambda_i}$. All in all, for $\rho \in (0, 1)$,

$$|C_i| \leq (d - i + 1)V \sqrt{\lambda_i} \times \max \{ 2(1 - \rho)^{-1/2}, 2\rho^{-1}(d - i + 1)V^2 \} =: C_\rho \sqrt{\lambda_i}. \quad (11)$$

Then, $[e_i^{(1)}]_j = |C_i| \sqrt{\lambda_j}[v]_j / |\lambda_j - \nu_i| \leq \sqrt{\lambda_i \lambda_j} / |\lambda_j - \nu_i| \times \inf_{\rho \in (0, 1)} C_\rho$. By Eq. (6), when $\lambda_j < \lambda_i$, $[e_i^{(1)}]_j \leq \min_{\rho \in (0, 1)} C_\rho \times (1 - \lambda_j / \lambda_i)^{-1} \sqrt{\lambda_j / \lambda_i}$. By Eq. (5), when $\lambda_j > (1 + dV^2)\lambda_i$, $[e_i^{(1)}]_j \leq \min_{\rho \in (0, 1)} C_\rho \times (1 - (1 + dV^2)\lambda_i / \lambda_j)^{-1} \sqrt{\lambda_i / \lambda_j}$.

If the eigenvalues of D are not distinct or not all entries of v are nonzero, we consider a sequence of diagonal matrices $\{D_k = \text{diag}(\lambda_1^k, \dots, \lambda_d^k)\}_{k \in \mathbb{N}}$ such that the diagonal elements $\lambda_1^k > \lambda_2^k > \dots > \lambda_d^k > 0$ are distinct and $D_k \rightarrow D$ when $k \rightarrow \infty$, and a sequence of vectors $\{v_k \in \mathbb{R}_{\neq 0}^d\}_{k \in \mathbb{N}}$ with only nonzero entries where $\|v_k\|_\infty \leq N$ and $v_k \rightarrow v$ when $k \rightarrow \infty$. Denote then $A_k = D_k + \sqrt{D_k} v_k v_k^T \sqrt{D_k}$ and $\nu_1^k \geq \dots \geq \nu_d^k$ its eigenvalues. Note that $A_k \rightarrow A$ when $k \rightarrow \infty$, so by continuity of the eigenvalues, $\nu_i^k \rightarrow \nu_i$ when $k \rightarrow \infty$. Furthermore, we just proved that if e_1^k, \dots, e_d^k are unit eigenvectors of A_k corresponding respectively to the eigenvalues ν_1^k, \dots, ν_d^k , then e_1^k, \dots, e_d^k and $\lambda_1^k, \dots, \lambda_d^k$ satisfy Eq. (8). Moreover, the vectors e_i^k all belong to the unit sphere of \mathbb{R}^d , so up to considering a subsequence of $\{A_k\}_{k \in \mathbb{N}}$, we can assume w.l.o.g. that each e_i^k tends to a vector $e_i^{(1)} \in \mathbb{R}^d$ when $k \rightarrow \infty$. As (e_1^k, \dots, e_d^k) is an orthonormal system of \mathbb{R}^d , so is its limit $(e_1^{(1)}, \dots, e_d^{(1)})$ and $e_i^{(1)}$ is an eigenvector of A corresponding to the eigenvalue ν_i . Therefore, Eq. (8) holds by taking the limit $k \rightarrow \infty$ in the equation satisfied by e_1^k, \dots, e_d^k and $\lambda_1^k, \dots, \lambda_d^k$.

To obtain Eq. (7), note that when $\rho = 1/2$, we have $2(1 - \rho)^{-1/2} \leq 4 \leq 4dV^2$, and $2\rho^{-1}(d - i + 1)V^2 \leq 4dV^2$, and when $\max\{\lambda_i, \lambda_j\} / \min\{\lambda_i, \lambda_j\} > 1 + 4dV^2$, by Eq. (8), then,

$$(1 - (1 + dV^2) \min\{\lambda_i, \lambda_j\} / \max\{\lambda_i, \lambda_j\})^{-1} \leq (1 + 4dV^2) / (4dV^2) \leq 5/4,$$

as $V \geq d^{-1/2}$, and thus Eq. (7) holds. If otherwise $\max\{\lambda_i, \lambda_j\} \leq (1 + 4dV^2) \min\{\lambda_i, \lambda_j\}$, as $\max\{\lambda_i, \lambda_j\} / \min\{\lambda_i, \lambda_j\} \geq 1$, we find $[e_i]_j \leq 1 \leq (1 + 4dV^2) \sqrt{\max\{\lambda_i, \lambda_j\} / \min\{\lambda_i, \lambda_j\}}$. Since $1 \leq dV^2$, then $(1 + 4dV^2) \leq 5dV^2$ and Eq. (7) holds. \square

3.2. Sum of m rank-one matrices perturbation

Our final Theorem 4 generalizes Proposition 3 to any value $m \geq 1$ and is obtained by induction using Eq. (8). Theorem 4 implies in particular Theorem 1 via the spectral theorem.

Theorem 4. *Let $D = \text{diag}(\lambda_1, \dots, \lambda_d)$ be a diagonal matrix with $\lambda_1 \geq \dots \geq \lambda_d > 0$. Let $V \geq 1/\sqrt{d}$ and consider a sequence of vectors $v^{(i)} \in \mathbb{R}^d$ such that $\|v^{(i)}\|_\infty \leq V$ for all $i \in \mathbb{N}$. For $m \in \mathbb{N}$, let $A^{(m)} = D + \sqrt{D} \sum_{i=1}^m [v^{(i)}][v^{(i)}]^T \sqrt{D}$ and $\nu_1^{(m)} \geq \dots \geq \nu_d^{(m)}$ the eigenvalues of $A^{(m)}$ and $(e_1^{(m)}, \dots, e_d^{(m)})$ a corresponding orthonormal system of eigenvectors. Then*

$$|[e_i^{(m)}]_j| \leq C_m \sqrt{\frac{\min\{\lambda_i, \lambda_j\}}{\max\{\lambda_i, \lambda_j\}}} \quad \text{for all } i, j \in \{1, \dots, d\} \text{ and } m \in \mathbb{N} \quad (12)$$

with $C_0 = 1$ and $C_{m+1} = 5d^7 V^4 C_m^5 \sqrt{1 + dmV^2}$.

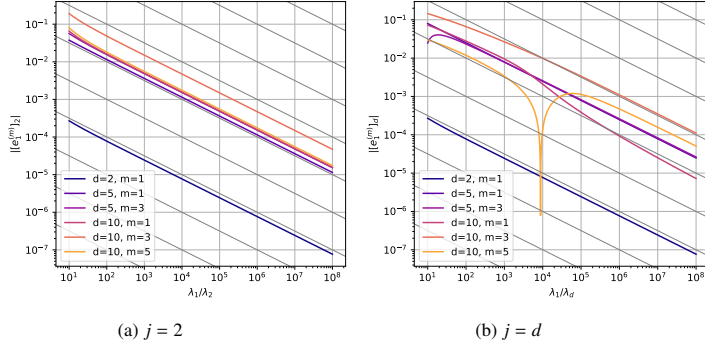


Figure 1: Value of $|[e_1^{(m)}]_j|$ as a function of λ_1/λ_j where $e_1^{(m)}$ is an eigenvector associated to the largest eigenvalue of $A^{(m)}$ from (P_m) for different dimensions and values of m as given in the legend. The $v^{(l)}$ are independent standard Gaussian vectors (with the same realization for all values of λ_1) and the eigenvalues of the diagonal matrix B are chosen uniformly on a log scale between $\lambda_d = 1$ and λ_1 . The value $|[e_1^{(m)}]_j|$ behaves consistent with $\Theta(\sqrt{\lambda_j/\lambda_1})$.

Proof. For $a, b > 0$, denote $\alpha(a, b) = \sqrt{\min\{a, b\}/\max\{a, b\}}$. Let $m \in \mathbb{N}$ and assume that Eq. (12) holds which is true if $m = 0$ since $C_0 = 1$. Observe now that $A^{(m+1)} = A^{(m)} + \sqrt{D}[v^{(m+1)}][v^{(m+1)}]^T \sqrt{D}$. In the system of coordinates $\mathcal{B}^{(m)} := (e_1^{(m)}, \dots, e_d^{(m)})$, $A^{(m)}$ writes as $D^{(m)} := \text{diag}(v_1^{(m)}, \dots, v_d^{(m)})$. Since $\lambda_1, \dots, \lambda_d > 0$, and as $A^{(m)} \geq D$, then $v_i^{(m)} \geq \lambda_i > 0$ for $i \in \{1, \dots, d\}$, and

$$\langle \sqrt{D}v^{(m+1)}, e_i^{(m)} \rangle = \sum_{j=1}^d [e_i^{(m)}]_j \sqrt{\lambda_j} [v^{(m+1)}]_j = \sqrt{D_{ii}^{(m)}} \times \sum_{j=1}^d \sqrt{\lambda_j/v_i^{(m)}} [e_i^{(m)}]_j [v^{(m+1)}]_j =: [\sqrt{D^{(m)}} w^{(m+1)}]_i.$$

Hence $[A^{(m+1)}]_{\mathcal{B}^{(m)}} = D^{(m)} + \sqrt{D^{(m)}} [w^{(m+1)}] [w^{(m+1)}]^T \sqrt{D^{(m)}}$ with

$$\|w^{(m+1)}\|_\infty \leq \sum_{j=1}^d \sqrt{\lambda_j/v_i^{(m)}} |[e_i^{(m)}]_j| \times V \leq \sum_{j=1}^d \sqrt{\lambda_j/\lambda_i} |[e_i^{(m)}]_j| \times V \leq dV \times C_m.$$

We apply Proposition 3 to $[A^{(m+1)}]_{\mathcal{B}^{(m)}}$ so that, for $i, k \in \{1, \dots, d\}$,

$$|\langle e_i^{(m+1)}, e_k^{(m)} \rangle| = |[e_i^{(m+1)}]_{\mathcal{B}^{(m)}}]_k| \leq 5d^6 V^4 C_m^4 \alpha(v_i^{(m)}, v_k^{(m)}).$$

By Eq. (5), $|\langle e_i^{(m+1)}, e_k^{(m)} \rangle| \leq (5d^6 V^4 C_m^4 (1 + dmV^2)^{1/2} \alpha(\lambda_i, \lambda_k))$. Since $|[e_k^{(m)}]_j| \leq C_m \alpha(\lambda_i, \lambda_k)$, then,

$$|[e_i^{(m+1)}]_j| \leq \sum_{k=1}^d |[e_k^{(m)}]_j| \times |\langle e_i^{(m+1)}, e_k^{(m)} \rangle| \leq dC_m \times 5d^6 V^4 C_m^4 (1 + dmV^2)^{1/2} \alpha(\lambda_i, \lambda_j).$$

This proves by induction that Eq. (12) holds for all $m \in \mathbb{N}$. □

3.3. Tightness

Figure 1 shows numerical computations of coordinates of the first eigenvector of $A^{(m)}$ in dimension 2, 5, 10. The coordinates seem to obey $\Theta(\min\{\lambda_i, \lambda_j\}/\max\{\lambda_i, \lambda_j\})$ in all cases which suggests that this rate in our upper bounds is tight. However we do not expect the constant C_m given in Theorem 4 to be tight.

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