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Full nonlinearity in weakly dispersive Boussinesq models: luxury or necessity ?

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Abstract

Boussinesq type (BT) models play a major role in coastal engineering applications. Due to their asymptotic nature, these models have lent themselves to constant study, and improvement based on the inclusion/removal of asymptotically small (or not) terms. In this work we go back to the fundamental question of whether full nonlinear in weakly dispersive BT models is a necessity. We re-consider the tests by [67] to address this issue, as well as a number of more demanding ones. We also consider different families of weakly nonlinear BT models, with different shoaling characteristics, especially when nonlinear waves are involved. Our study allows to point out that for many cases it is quite hard to conclude in favor of one class of model or another. There are a few discriminant cases, which are unfortunately not those mostly used in the literature proposing new models or new numerical methods.

1 Introduction

No one can dispute the importance of the Boussinesq type (BT) models in coastal engineering. In the last 50 years significant developments have been achieved both at the level of modeling, as well as in the development of related numerical methods. Boussinesq type models are depth-integrated, phase resolving models for surface wave propagation. Two main parameters characterize these type of models. The first one is the dispersion parameter μ , which is the ratio of the reference water depth d over the characteristic wavelength L . The second is nonentity parameter ϵ , defined as the ratio of the characteristic wave amplitude α_0 over d . Very long waves are characterized by negligible values of μ^2 . In this case, the very well known system of shallow water, or Saint-Venant, equations can be recovered. In near shore applications, involving wave transformations and their impact in coastal structures, the most widely used models are the so-called weakly-dispersive ones, which are obtained in the hypothesis that $\mu^2 \ll 1$, and neglect all effects of order $O(\mu^4)$. This paper focuses on this class of Boussinesq models. Note that all these PDE systems involve weak dispersive perturbations of the shallow water equations, and usually involve the inversion of an elliptic operator in the time advancement of the depth averaged velocity.

Within weakly dispersive BT equations, we can further distinguish two important sub-families. Weakly nonlinear Boussinesq models are obtained assuming that $\epsilon \approx O(\mu^2)$, which allows to neglect all effects of order $O(\mu^4, \epsilon\mu^2)$. This ansatz allows to obtain PDE systems which can be written as the nonlinear shallow water equations with a linear dispersive perturbation. The linearity of the dispersive operator translates, in the time independent nature of the elliptic equation. A large number of the most well known models in the coastal engineering community belong to this class: from the simple model by Boussinesq [7] to Peregrine's equations [50], to the dispersion enhanced models by Nwogu [45], Madsen and Sørensen [38] or Beji and Nadaoka [4]. The linear character of the dispersive operator, translates in its time independent nature which allows efficient implementation strategies for its inversion [54]. There exist a number of known codes implementing these models, e.g. BOUSS-2D

45 by the US Army [44], DHI’s commercial platform MIKE21 [12], the code BOSZ initially developed
46 at U. of Hawaii [57], or the more recent open source platform CELERIS [61, 62].

47 The BT models obtained assuming that $\epsilon = O(1)$ are referred to as fully nonlinear. In this case
48 the depth averaged equations involve a strongly nonlinear dispersive operator, making the associated
49 numerical resolution inherently more expensive. Despite of this fact, this choice is without doubt
50 physically more relevant, and provides the well known models of Serre [58], Green Naghdi [21], and
51 enhanced variants discussed for example in [67], [32, 10], or [11]. Model of this type is implemented
52 in many operational codes among which the most well known ones are probably FUNWAVE [59, 14],
53 and Coulwave [37, 15], but similar models have been implemented in other platforms as e.g. the open
54 source projects Basilisk [16], and other research platforms less visible than the above.

55
56 Note that, other parameters may be introduced to characterize depth averaged equations in terms
57 of mild (or not) slope, presence (or not) of surface tension effects etc. For an extensive review on the
58 BT models we refer to [34, 31, 8] and references therein. In this work, we will limit our study to the
59 impact of full/weak nonlinearity.

60
61 As already mentioned, fully nonlinear models are more complex than weakly nonlinear ones, and
62 require the inversion of a nonlinear, and thus time dependent, elliptic operator at every time step. The
63 natural question arising in this case is whether this additional complexity and computational cost is
64 necessary. Many think that this issue is fully answered in the landmark paper by Wei and collaborators
65 [67] which concludes that full nonlinearity is absolutely necessary to correctly predict water heights.
66 Yet, the literature is still filled with simulations using weakly nonlinear models, showing accurate
67 predictions of wave heights and phase, for now almost standard cases, in both small amplitude and
68 configurations close to/or in incipient breaking.

69 So in this paper we go back to the question of whether full nonlinearity in weakly dispersive BT
70 models is a necessity. We re-consider the tests by [67], as well as a number of more demanding
71 ones. We also consider different families of weakly nonlinear BT models, with different shoaling
72 characteristics. As we will demonstrate, for many cases it is quite hard to conclude in favor of one
73 class of model or another. There are however, a few discriminant cases in which the height predictions
74 are very negatively affected by the lack of full nonlinearity. These cases are however seldom used in
75 the literature proposing new numerical methods or, more importantly, dispersive PDE models.

76
77 The structure of the paper is the following. We first discuss the BT models considered, their main
78 characteristics, and briefly recall their numerical treatment. Then we review the results on benchmarks
79 of increasing complexity in one and two space dimensions. We end the paper with a conclusive section
80 that tries to provide a few recommendation concerning the choice and benchmarking of the models.

81 **2 Weakly dispersive Boussinesq PDE models**

82 In this section we briefly describe the Boussinesq systems used in this work, and recall some of their
83 significant properties. As mentioned in the introduction the Boussinesq-type models derived since the
84 90’s are numerous. In this work we use some of the most well known among these.

85 2.1 Weakly nonlinear BT equations

86 Among the weakly nonlinear models the ones of Madsen and Sørensen [40, 39] and Nwogu's [45]
 87 are two typical and widely used models. These two models have quite comparable linear dispersion
 88 properties, but different shoaling characteristics. For this reason, we also consider the model Nwogu-
 89 Abbot introduced in [18], which has the exact same linear properties of the Nwogu equations but is
 90 only equal to the latter within an $\mathcal{O}(\epsilon\mu^2, \mu^4)$ correction allowing to write all the dispersive terms using
 91 derivatives of the volume flux rather than of the velocity. As shown in [18], this leads to a system
 92 which behaves like the Nwogu one for small amplitude waves, but much closer to the one of Madsen
 93 and Sørensen in more nonlinear regimes.

94

95 All these models can be written in a pseudo balanced law form:

$$\begin{aligned} h_t + \nabla \cdot \mathbf{q} &= \mathcal{D}_h \\ \mathbf{U}_t + \nabla \cdot (\mathbf{u} \otimes \mathbf{q}) + gh\nabla\eta + c_f(h, \mathbf{q})\mathbf{q} &= \mathcal{D}_\mathbf{q} \end{aligned} \quad \text{on } \Omega \times [0, t] \subset \mathbb{R}^2 \times \mathbb{R}^+ \quad (1)$$

96 where h is the total water depth and \mathbf{q} the volume flux $h\mathbf{u}$, with \mathbf{u} the depth averaged velocity. The
 97 free surface is denoted here by $\eta = h + b$ with b the bathymetry. The friction term $c_f\mathbf{q}$ in this work
 98 is approximated with the usual Manning law $c_f = gn_M^2\|\mathbf{u}\|/h^{4/3}$ with n_M the Manning coefficient.
 99 Finally, all the dispersive terms are on the right hand sides of (1). The definition of \mathcal{D}_h , of $\mathcal{D}_\mathbf{q}$, and of
 100 the pseud-momentum variables \mathbf{U} is what allows to distinguish between different models.

101

102 **The equations of Madsen and Sørensen** This system can be seen as a dispersion-enhanced variant
 103 of the Abbot equations, which are an equivalent of Peregrine's [50] within an $\mathcal{O}(\epsilon\mu^2, \mu^4)$ correction
 104 allowing to write all the dispersive terms using derivatives of the volume flux rather than of the velocity
 105 [18]. In two space dimensions, the equations are obtained from (1) setting $\mathcal{D}_h = 0$, and

$$\mathcal{D}_\mathbf{q} = Bgd^2\nabla\nabla \cdot (d\nabla\eta) \quad (2)$$

106 where $d = d(x, y) = h_0 - b(x, y)$ is the still water depth. The momentum variable is defined by

$$\mathbf{U} = \mathbf{q} - \left(B + \frac{1}{3}\right) d^2\nabla(\nabla \cdot \mathbf{q}) - d \left[\frac{1}{3} (\nabla d \otimes \nabla \mathbf{q}) + \frac{1}{6} (\nabla d \otimes (\nabla \mathbf{q})^\perp) - \frac{1}{6} ((\nabla d)^\perp \otimes \nabla \mathbf{q}) \right] \quad (3)$$

107 The value of $B = 1/15$ is the one for which the phase relation of the system matches a second-order
 108 Padé approximation of the exact phase of the Euler equations [40].

109 **The equations of Nwogu** Introduced in [45] the BT model due to Nwogu can be obtained from
 110 Peregrine's by replacing the depth averaged velocity with a velocity at an arbitrary distance from the
 111 still water level z_θ . The value $z_\theta = -0.531d$ is the one that optimizes the dispersion relation of the
 112 system wrt the one of the Euler equations. The model is defined as in (1) with

$$\mathcal{D}_h = -\nabla \cdot \left[\left(\frac{z_\theta^2}{2} - \frac{d^2}{6} \right) d\nabla(\nabla \cdot \mathbf{u}) + \left(z_\theta + \frac{d}{2} \right) d\nabla[\nabla \cdot (d\mathbf{u})] \right], \quad (4)$$

$$\mathcal{D}_\mathbf{q} = -\mathbf{u}\mathcal{D}_h + h_t \frac{z_\theta^2}{2} \nabla(\nabla \cdot \mathbf{u}) + h_t z_\theta \nabla(\nabla \cdot d\mathbf{u}) \quad \text{and} \quad (5)$$

$$\mathbf{U} = \mathbf{q} + h \left[\frac{z_\theta^2}{2} \nabla(\nabla \cdot \mathbf{u}) + z_\theta \nabla(\nabla \cdot d\mathbf{u}) \right]. \quad (6)$$

113 **The equations of Nwogu-Abbott** These equations can be derived starting from the ones of Nwogu
 114 and in the aim of find an asymptotically equivalent system in an amplitude-volume flux form which
 115 degenerates to the same linearized equations of [45]. This derivation, detailed in [18], boils down to
 116 add order $\mathcal{O}(\epsilon\mu^2, \mu^4)$ correction terms to pass from derivatives of du to derivatives of \mathbf{q} . In this case,
 117 the equations are written as in (1) but now we have $\mathcal{D}_{\mathbf{q}} = 0$ and:

$$\mathcal{D}_h = -\nabla \cdot \left[\left(\frac{z_\theta^2}{2} - \frac{d^2}{6} \right) d\nabla\nabla \cdot \left(\frac{\mathbf{q}}{d} \right) + \left(z_\theta + \frac{d}{2} \right) d\nabla\nabla \cdot \mathbf{q} \right], \quad (7)$$

$$\mathbf{U} = \mathbf{q} + \frac{dz_\theta^2}{2} \nabla \left(\nabla \cdot \left(\frac{\mathbf{q}}{d} \right) \right) + dz_\theta \nabla (\nabla \cdot \mathbf{q}) \quad (8)$$

118 2.2 Fully nonlinear BT modeling: the enhanced Green-Naghdi equations

Maybe one of the most well know fully nonlinear Boussineq models are the equations of Green-Naghdi [21]. The study of improved formulations of these equations and their numerical approximation has been the object of intense efforts in the last 15 years, one can refer for example to [6, 20, 42, 46, 35, 52, 11, 13, 36, 47, 34, 51, 63] (and many others, we are certainly forgetting many) and references therein. Here we consider the enhanced equations in the form used in [19, 28] which reads as in (1) with $\mathcal{D}_h = 0$, $\mathbf{U} = [h, \mathbf{q}]^T$ and

$$\mathcal{D}_{\mathbf{q}} = \frac{gh\nabla\eta}{\alpha} + \Psi$$

119 where Ψ is the solution of the elliptic problem

$$(I + \alpha\mathcal{T})\Psi = \mathcal{W} - \mathcal{R}. \quad (9)$$

We define

$$\mathcal{W} = -\frac{gh}{\alpha} \nabla\eta \quad \text{and} \quad \mathcal{R} = h\mathcal{Q} \left(\frac{\mathbf{q}}{h} \right)$$

120 where the operators \mathcal{T} , \mathcal{Q} and \mathcal{R} are

$$T(\cdot) = -\frac{1}{3} \nabla \left(h^3 \nabla \cdot \left(\frac{\cdot}{h} \right) \right) - \frac{h^2}{2} \left(\nabla \cdot \left(\frac{\cdot}{h} \right) \right) \nabla b + \frac{1}{2} \nabla \left(h^2 \nabla b \cdot \left(\frac{\cdot}{h} \right) \right) + h \left(\nabla b \cdot \left(\frac{\cdot}{h} \right) \right) \nabla b, \quad (10)$$

$$\begin{aligned} Q(\cdot) &= \frac{2}{3h} \nabla \left(h^3 \left(\nabla(\cdot)_1 \cdot \nabla^\perp(\cdot)_2 + (\nabla \cdot (\cdot))^2 \right) \right) + h^2 \left(\nabla(\cdot)_1 \cdot \nabla^\perp(\cdot)_2 + (\nabla \cdot (\cdot))^2 \right) \nabla b + \\ &+ \frac{1}{2h} \nabla \left(h^2 \left((\cdot) \cdot \left((\cdot) \cdot \nabla \right) \nabla b \right) \right) + \left((\cdot) \cdot \left((\cdot) \cdot \nabla \right) \nabla b \right) \nabla b. \end{aligned} \quad (11)$$

121 α is a parameter used to improve the dispersion properties of the model w.r.t those of the full Euler
 122 equations [32]. By setting $\alpha = 1$ we recover the classical Green-Naghdi model. We can write the
 123 operator \mathcal{T} in such a way as to guarantee its inversion properties and the coersivity of the operator
 124 $(I + \alpha\mathcal{T})$, see [18],[30] and references therein for further details.

125 2.3 A short summary on the properties of the models

126 We provide here a small recap of the main properties of the models considered. In terms of dispersive
 127 propagation, the linear dispersion relation for the nominal values of the models' constants proposed in
 128 literature is reported on the leftmost picture on figure 1. In the classical near-shore window of values
 129 $kd \in [0, \pi]$, all models provide errors below 2% wrt the Airy theory [60]. We highlight that Nwogu

130 and Nwogu-Abbot models have the same linearized form so the same phase relation. The usual choice
 131 of parameters provides a monotone relation only for the MS and GN model. The Nwogu and eGN
 132 equations pass from phase lag to phase advance when going from long to short waves, the crossing
 133 taking place within the frequency window considered for the Nwogu model.

134 An other relevant parameter to consider is the shoaling coefficient (S) which relates the rate of
 135 change in wave amplitude to the rate of change in water depth. The procedure to produce explicit
 136 expressions of the coefficient can be found in [25, 39] among others. The middle picture in figure
 137 1 presents graphically the value of S for each model. The standard values for the MS equations are
 138 optimized in terms of linear shoaling, and this is clearly visible in the picture. In the linear case, the
 139 eGN shows an over shoaling behavior for short waves, while the Nwogu and GN models under-predict
 140 the linear increase in height.

141 The situation is actually much different in the nonlinear case, as the tests in this paper show.
 142 To advance some of the results, we also report in the rightmost picture amplification of second order
 143 Stoke-Waves obtained with a second order analysis accounting for $\mathcal{O}(\epsilon)$ effects (see e.g. [18] and
 144 references therein). The plot shows that amplitudes are systematically under-predicted for the models
 145 in amplitude-flux form, and in particular for the MS and NA models considered here. When some
 146 nonlinearity is accounted for the Nwogu model, an over-prediction in the amplitudes, up to values
 147 $kd \approx 1.7$, is observed. The same holds for all the well known amplitude-velocity models as shown in
 148 [18] and in contrast with what the linear shoaling shows. Clearly this behavior is not universal for BT
 149 models, which is also a part of the motivation of this work.

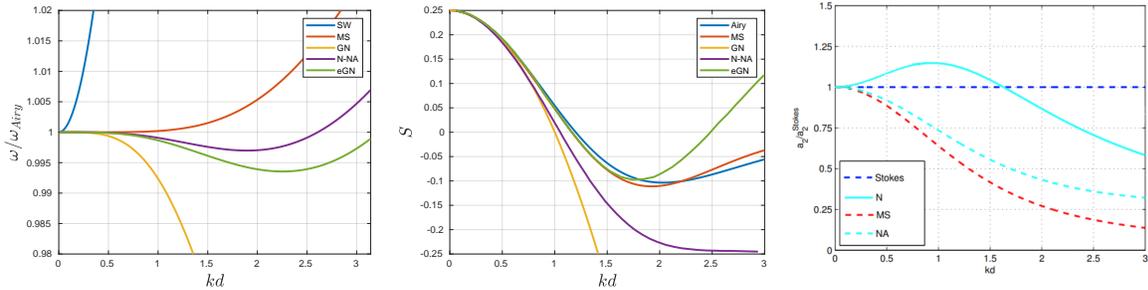


Figure 1: Linear dispersion with respect to the one of the Airy theory and linear shoaling properties for the models considered

150 2.4 Numerical approximation, wet/dry treatment, and breaking closure

151 **Numerical framework** The numerical discretization of system (1) is performed on unstructured
 152 mesh discretizations of the physical domain, and brings together work from the present authors in the
 153 last 10 years. It relies on two main building blocks: a high order node centered finite volume solver
 154 on dual cells to evolve in time vector ($h \mathbf{U}$) at all mesh nodes; a nodal elliptic solver used to recover
 155 the velocities/fluxes at all mesh nodes starting from the values of \mathbf{U} , or evaluate the dispersive forcing
 156 solving (9) for the GN equations. The nodal finite volume solver, described in detail in [26, 19, 28] uses
 157 classical Roe fluxes, with appropriate modifications to deal with wet/dry fronts and to guarantee the
 158 well balanced approximation of the bathymetric source. Third order polynomial reconstructions are
 159 used to evaluate the fluxes, based either on the MUSCL approach, or on more recent compact iterative
 160 Green-Gauss polynomial recovery [28]. Where necessary, the high order polynomial reconstructions

161 also allow to evaluate high order derivatives in the dispersive terms on the right hand side of (1). As
 162 sections §2.1 and §2.2 show these terms are considerably more cumbersome for the fully nonlinear
 163 equations than for the weakly nonlinear models, representing one of the computational overheads
 164 of accounting for full nonlinearity. Where necessary, as e.g. for some wave breaking closures,
 165 discontinuity capturing is guaranteed by decreasing the degree of the polynomial approximation using
 166 either the van Albada limiter, of the high order limiter proposed in [41]. Time integration is achieved
 167 through a third order explicit Strong Stability Preserving Runge-Kutta method.

168 The elliptic systems are discretized with linear finite elements. The coupling with the FV method
 169 to express the necessary quantities at nodes is performed via the cell-wise high order polynomial (see
 170 [28] for details). Note that for fully nonlinear models these systems involve the depth h and are thus
 171 time dependent. They require a new assembly step and inversion at each time step, which is another
 172 computational overhead of accounting for full nonlinearity. The appropriate formulation of boundary
 173 condition for this step is an open research area, especially on non-straight boundaries. Usually ad-hoc
 174 techniques involving sponge layers, and internal wave generators are used in combination with simple
 175 treatments of the elliptic component. The interested reader can refer to [33, 24, 43] and references
 176 therein for more on this aspect. Note also that we have not considered here less classical variants of
 177 the GN model as those proposed in [35] which involve two elliptic solves at each time step, albeit of
 178 constant (in time) elliptic operators. The general boundary treatment for the extra auxiliary variable is
 179 even less clear. We refer however to the reference for details on this promising approach.

180 As a final note that we would like to stress that *unless otherwise specified, all results presented in*
 181 *this paper are grid independent.*

182 **Wet/dry transitions** In wet/dry fronts special care needs to be taken in to account in order to keep
 183 the well balancing of the scheme, and avoid the blow up of the velocities. This involves modifications
 184 to the states used to evaluate the numerical fluxes, as well as the bathymetry, which are somewhat
 185 classical in the approximation of the shallow water equations. We refer the interested reader to [55,
 186 26, 27, 53, 1] and references therein for details.

187 For dispersive models, an important issue is related to the so-called “dispersive run-up/rundown”.
 188 Weakly nonlinear models include in their dispersion operator the still water level $d(x, y)$, which is a
 189 somewhat problem dependent variable. Above the level $d = 0$, usually corresponding to the initial
 190 position of the shorelines, these terms do not make sense, as d itself changes sign. For this reason, in
 191 all weakly nonlinear models dispersion is (smoothly) turned off in initially dry areas. For some cases
 192 involving moving waves interacting with shorelines this transition can be seen to affect the results. In
 193 contrast, the fully nonlinear GN model involves the actual depth h in the dispersive terms. These terms
 194 are in practice numerically limited only close to the current location of the wet/dry front, thus also
 195 computed in initially dry regions.

Wave breaking Last but not least, an appropriate closure is required to account for the dissipation in
 breaking waves. There are many approaches to obtain such closure. The interested reader is referred
 to [27, 30] and references therein for an overview. A wave breaking closure consists of two parts: a
 detection step; a dissipation mechanism. In this work we use the simple, local, and robust detection
 criteria proposed in [27, 19]. A cell is flagged as breaking when:

$$|\eta_t| \geq \gamma \sqrt{gH} \text{ surface variation criterion}$$

or/and

$$\|\nabla\eta\| \geq \tan(\phi_c) \text{ local slope criterion}$$

196 Usually, the flagging is extended according to some physical and numerical criteria discussed e.g. in
 197 [27, 19, 2]. The parameters γ and ϕ_c depend on the type of breaking regime. Some values are provided
 198 in the last references. In this work we use $\gamma = 0.6$ and $\phi_c = 30^\circ$ unless otherwise stated.

199 Concerning the dissipation mechanism the results presented in this paper have been obtained with
 200 two approaches discussed in [30]: the hybrid closure in which dispersion is turned off in breakers
 201 which eventually become shallow water (hydrostatic) bores; an eddy viscosity closure involving the
 202 solution of a balance equation for the turbulent kinetic energy (TKE). As shown in [30] both approaches
 203 allow to describe correctly wave transformation and breaking in large scales. But when we want to
 204 obtain grid converged solutions, as in most of the 1d results presented here, only the TKE closure can
 205 be used, the hybrid closure introducing unbounded physical oscillations at the transition region when
 206 the mesh is refined (see also [2]).

207 3 One dimensional benchmarks

208 3.1 Shoaling of a solitary wave

209 This is a benchmark used in two landmark papers [22, 67]. In the first the authors discuss comparisons
 210 between experimental data and a fully nonlinear/dispersive potential model. The second paper uses this
 211 benchmark to compare fully and weakly nonlinear variants of a Boussinesq model, based on Nwogu's
 212 equations [45], with the experimental and potential results. The benchmark setup is quite simple. The
 213 computational domain consists of a ramp of constant slope, whose toe is located at $x = 0m$. The depth
 214 on the left of the ramp is of $d = 0.44m$. A solitary wave of amplitude $A = \epsilon d$ propagates and shoals
 215 on the ramp. Here, differently from [67], we compare weakly Boussinesq models with different linear
 216 and non-linear shoaling characteristics, as well as the fully nonlinear Green-Naghdi one. For all the
 217 models, the usual analytical sech^2 solitary wave shape is used to initialize the computation.

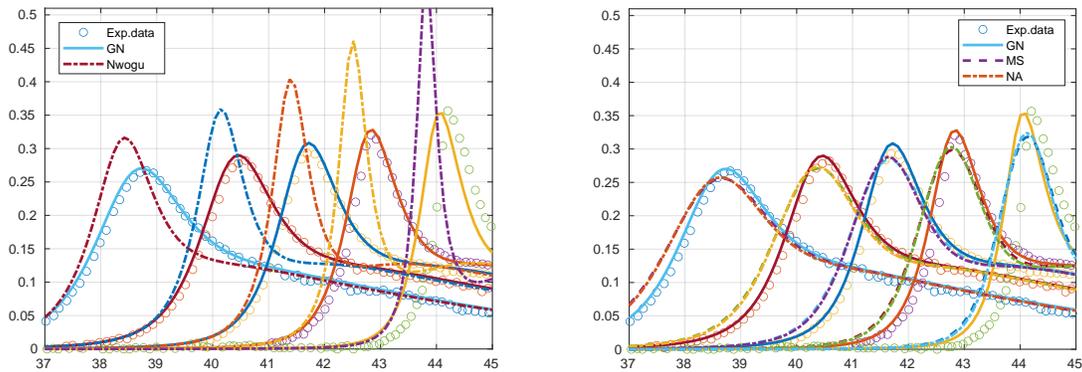


Figure 2: Free surface elevation on the wave gauges 1, 3, 5, 7, 9 for GN and Nwogu (left), and for MS, NA, and GN (right).

218 We start from the case $\epsilon = 0.2$, with a slope $s = 1 : 35$, and report the wave height measured at
 219 different wave gauges along the slope in figure 2. On the left we compare the experimental data to the
 220 fully nonlinear Green-Naghdi results as well as to those obtained with the weakly nonlinear model of
 221 Nwogu. On the right we compare the data with the results of the fully nonlinear equations and with

222 those obtained with the equations of Madsen and Sørensen, and Nwogu-Abbot. The result on the left
 223 is essentially the one discussed in [67]. We can clearly see that the weakly nonlinear model provides
 224 a large overestimation of the wave heights. The right figure however is much different, the weakly
 225 nonlinear MS and NA models providing a small underestimation of the elevations. Similar results
 226 have been obtained in [18] using as initial data for each model its own semi-analytical soliton shape.
 227

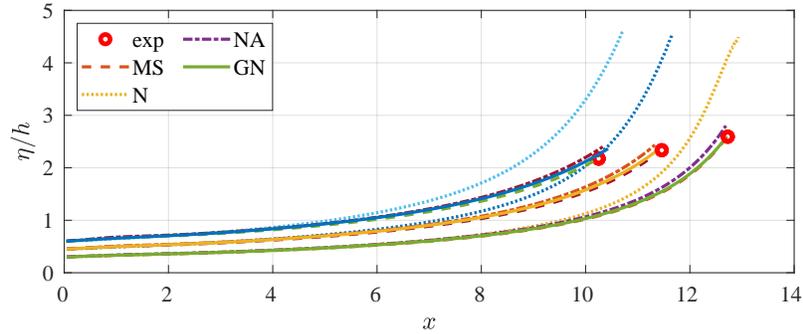


Figure 3: Relative wave height computed with all the models on a 1:15 slope.

228 Next, following the test performed in [67], we let different solitary waves of nonlinearity $\epsilon =$
 229 0.3, 0.45, 0.6 shoal on a slope $s = 1 : 15$, and plot the envelopes of the wave peaks along the shoal.
 230 The results are plotted on figure 3 (soliton nonlinearity increasing from left to right curves). The red
 231 circles represent the experimental breaking point of each wave. Our results confirm only partly the
 232 ones by [67]. Indeed, for the equations of Nwogu, the breaking point is “missed” by the model, which
 233 predict a substantial over-shoaling of the waves. The more nonlinear the wave, the stronger the over
 234 prediction. However, the weakly nonlinear equations written in amplitude flux form provide results
 235 much closer to the fully nonlinear model, and a reasonable approximation of the wave peak at the
 236 breaking point. This is somewhat in contrast with the dogmatic view that all the weakly nonlinear
 237 models over predict wave heights, and that provide wrong values of the free surface elevation. To
 238 complete the picture, in figure 4 we report the solitary shape for the case $\epsilon = 0.3$ at breaking point.
 239 We can see that only the Nwogu model provides the nonphysical peak also obtained in [67], the other
 240 models giving a wave with similar topology.

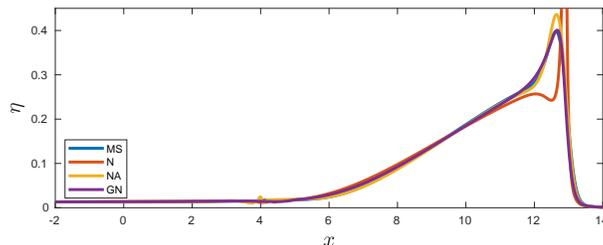


Figure 4: Free surface for the different models for $\epsilon = 0.3$, $s = 1 : 15$ at breaking point $t = 11.32$.

241 As a final remark, note that in [18] it has thoroughly shown that weakly nonlinear models written

242 starting from the Peregrine equations, thus with linear dispersive effects expressed as space-time
 243 derivatives of the velocity, have very close nonlinear behavior. Indeed, results similar to those of
 244 Nwogu’s model can be obtained with other BT systems, e.g. Peregrine’s [50], Beji-Nadaoka [4], etc.
 245 (see [18] and references therein).

246

247 To conclude, despite showing important differences in the nonlinear behavior of the models,
 248 this benchmark does not settle the question on whether full nonlinearity is necessary, the models in
 249 amplitude-flux form following relatively closely the fully nonlinear model up to the breaking point as
 250 shown clearly from Figure 3.

251 3.2 Undular bore propagation

252 The propagation of undular bores is also a classical problem, studied for example in [49, 67], and for
 253 which well known experiments also exist as e.g. [17, 65]. The initial setup is essentially the same
 254 as in [67]. We consider a computational domain of 300 m with wall conditions at both domain ends.
 255 A transition between uniform flow and still water is initially centered at $x = 0$ m, with velocity and
 256 surface elevation given by

$$u = \frac{1}{2}u_0 \left(1 - \tanh\left(\frac{x}{a}\right)\right), \quad \eta = u + \frac{1}{4}u^2$$

257 where u_0 is the velocity of uniform flow from the left boundary and a set to 5 m as in [67]. We consider
 258 two initial states with jumps in the initial surface elevation $\eta_0 = 0.1$ and 0.3 , which also correspond
 259 to nonlinearity values $\epsilon = 0.1$ and $\epsilon = 0.3$, and to Froude numbers $Fr = 1.07$ and 1.22 respectively.
 260 Both waves evolve in undulating bores propagating through the channel. The amplitude as well as the
 261 wavelength of the oscillations depend on the initial data.

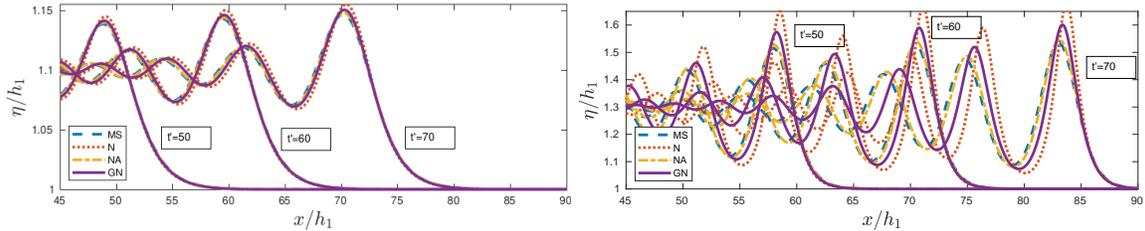


Figure 5: Evolution of an undular bore. Up Left $\epsilon = 0.1$ Up right $\epsilon = 0.3$.

262 As in [67] we plot the bore profiles at different (dimensionless) times $t' = 50, 60,$ and 70 . The
 263 results are shown on figure 5. As one could expect, for the least nonlinear case the models provide
 264 nearly identical results, with only slight over amplification of the oscillations by the Nwogu model,
 265 and even less visible under amplifications for the MS and NA models. As the nonlinearity grows these
 266 differences become more visible. The wave heights predicted by Nwogu’s model are substantially
 267 larger compared to the GN solution. Note that the latter has been shown to provide results almost
 268 identical to a fully nonlinear potential model [9]. On the contrary, both the MS and NA models
 269 underestimate the amplitude of the oscillation. Note that this difference in behavior is somewhat
 270 in line with the second order analysis done in [18] showing that amplitudes of Stokes-Waves are
 271 systematically under-predicted by BT models in amplitude-flux, while models in amplitude-velocity

272 underestimate very short waves, and over estimate longer ones (see also figure 1).
 273

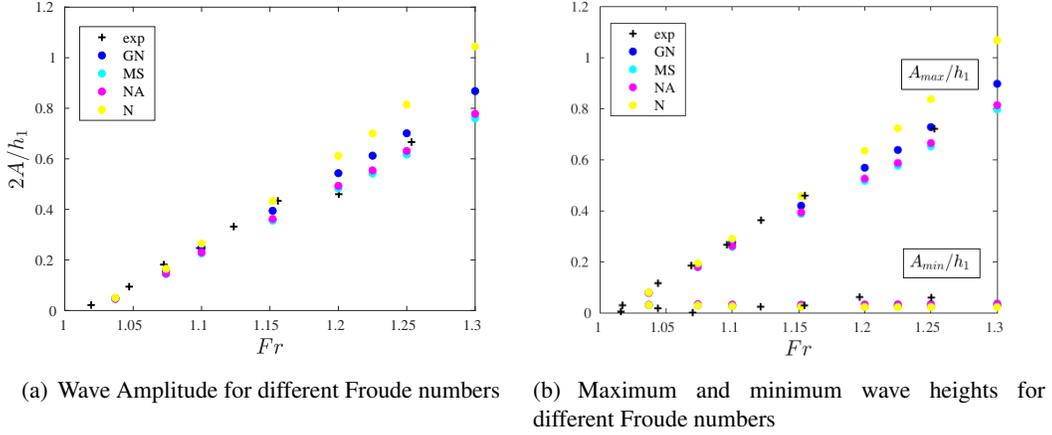


Figure 6: Undular bore propagation. +: Data of Treske and Favre, o: Numerical simulations

274 To have a more quantitative view of these differences, we look at the dependence of the amplitudes
 275 of the secondary waves on the Froude number. Some experimental data are also available for this from
 276 Favre [17] and Treske [65]. So simulations are run by considering different values of u_0 corresponding
 277 to bore Froude numbers in between 1 and 1.3. Amplitudes are computed as the difference between
 278 the heights of the first peak and the first trough. On figure 6 we report the plots of both the amplitude,
 279 and the peak/trough elevations A_{\max} and A_{\min} . The amplitude plot shows a growth with Froude
 280 which looks perfectly linear. For a Froude number below roughly 1.15 all the models provide similar
 281 predictions, all close to the experimental values. For more nonlinear situations, we see three family of
 282 models: the weakly nonlinear model of Nwogu over-predicting the amplitude; the models of MS and
 283 NA under-predicting the amplitudes; the fully nonlinear GN model in between the previous two, and
 284 somewhat closer to the experiments. It is worth mentioning that as the Froude increases the difference
 285 of the result for Nwogu’s equations wrt the data (and to the GN results) is at least twice as much as
 286 that of the other weakly nonlinear models which remain in a range of amplitudes reasonably close to
 287 the data. From the right figure, we can see that the elevation of the first trough does not change much
 288 with the Froude number, so the behavior observed is really associated to the dynamics of the first peak
 289 which grows much faster with models in amplitude-velocity form, as Nwogu’s, as remarked at the end
 290 of the previous section (see also [18]).

291
 292 As the previous one, this benchmark does show the impact of the nonlinear behavior of the models.
 293 However, looking at the amplitudes on figure 6, the improvement of using the fully nonlinear GN
 294 model is not striking, compared to the MS and NA ones.

295 3.3 Monochromatic waves on a submerged bar

296 We consider not the laboratory experiments performed by Beji and Battjes [3] on monochromatic
 297 waves over a submerged bar (cf. sketch in figure 7). According to [3] the experimental set up was
 298 designed to investigate both frequency dispersion characteristics of monochromatic waves, and non-
 299 linear interactions. This is possible the most popular set benchmarks in the literature on numerical

300 models for dispersive free surface waves, widely used for the validation of new BT models as well
 301 as of their discretization. We refer to the original reference for the setup, or to one of the numerous
 302 papers using this test, see e.g. [26, 54, 27, 18, 2] by the present authors and references therein.

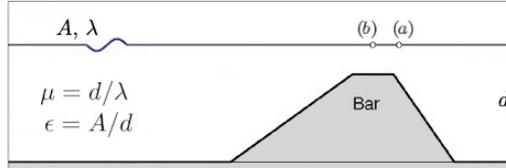


Figure 7: Monochromatic waves on a submerged bar, benchmark sketch

303 We consider here two cases. The first is a mildly dispersive case with incoming wave is $A =$
 304 $0.01m$, and the nonlinearity and dispersion parameters $\epsilon = 0.025$ and $\mu = 0.11$. The second, is a
 305 mildly nonlinear case with incoming waves of amplitude $A = 0.054m$ and nonlinearity and dispersion
 306 equal to $\epsilon = 0.125$ and $\mu = 0.083$ respectively. In the second case, the breaking point is reached and
 307 waves breaking is observed above the plateau of the submerged bar. Wave breaking is modeled for this
 308 case with the hybrid approach discussed in section §2.4 for all the BT models considered.

309
 310 The first case involving incoming waves with very little nonlinearity, we expect all models to
 311 behave similarly. This is absolutely the case, as can be seen in the pictures of the left column of figure
 312 8. The plots report a few periods of the water elevation time series in the gauge (a) right after the
 313 plateau (cf figure 7). The results of the GN, MS and Nwogu model are compared to the experimental
 314 data. This gauge allows to include effects of both the increased nonlinearity acquired at the end of the
 315 shoaling phase, and some dispersive effects during the propagation along the plateau. The numerical
 316 results obtained with the enhanced Green-Naghdi model (taken from [19]), and those obtained with
 317 the MS and Nwogu models (taken from [18]) are extremely close to one another.

318 For the mildly nonlinear case, with wave breaking, one might expect to see a substantial difference
 319 between different weakly nonlinear models, and wrt the fully nonlinear one. The right column on
 320 figure 8 reports the results for this case in terms of water elevation time series in gauge (b) of figure
 321 8, in correspondence of which incipient breaking is observed. The results obtained with the enhanced
 322 Green-Naghdi model (taken from [19]), those obtained with the MS model (taken from [2]), and those
 323 provided by Nwogu's equations (taken from [27]) using the exact same wave breaking detection, and
 324 hybrid breaking closure, and the same mesh size of 4 cm are essentially the same.

325
 326 Despite being one of the most commonly benchmarks, and while possibly relevant to asses other
 327 aspects, this test seems to be little sensitive to the amount of linearity accounted for in the BT model,
 328 unless of course the breaking closure is removed to reach nonphysical regimes as done in [18].

329 3.4 Propagation, breaking, and over-topping of a reef

330 We consider now a more complex case, involving highly nonlinear waves. It consists of the propagation,
 331 breaking and reef over-topping of a soliton, investigated numerically and experimentally in [56], and
 332 numerically in several other more recent works (see [64, 57, 30, 29] and references therein). The
 333 geometry of the problem is sketched in the left picture on figure 9: it consists of a reef with a constant
 334 fore slope, hiding on the lee side a shallow flat with a wall on its right end. Water height distributions
 335 at time series in 14 wave gauges have been measured in the flume experiments of [56].

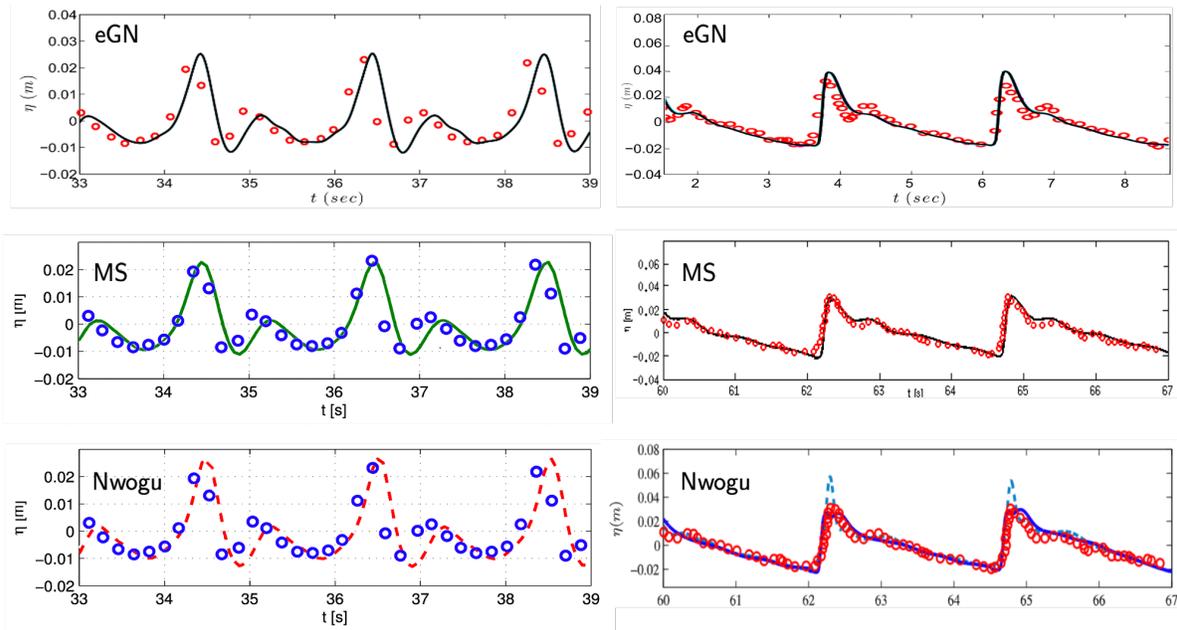


Figure 8: Left column: Free surface elevation for GN, MS, and Nwogu’s models for the case where initial wave’s amplitude is $A=0.01$ m (non breaking case). Right column: Free surface elevation for GN MS and Nwogu’s models when $A=0.054$ m (breaking case)

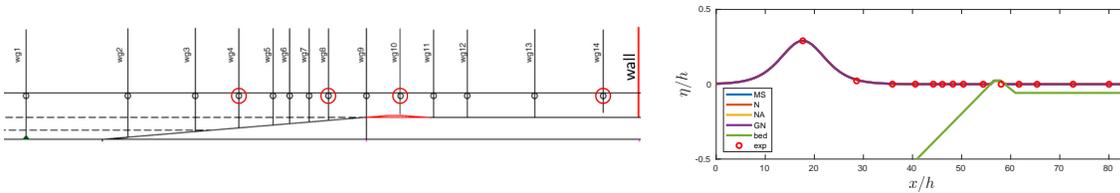


Figure 9: Solitary wave over-topping a reef: geometry and gauges (left); initial propagation (right).

336 A solitary wave of offshore nonlinearity $\epsilon = 0.3$ propagates on the left of the reef, as shown on
 337 the right on figure 9. Simulations with all the BT models of sections §2.1 and §2.2 are discussed.
 338 Breaking is modeled using the Turbulent Kinetic Energy (TKE) closure (see section §2.4 and [30]).
 339 We visualize the results via a few snapshots of the free surface distribution and water elevation series
 340 in 4 gauges denoted by red circles in the left picture on figure 9.

341

342 During propagation there is hardly any difference in the results as seen in the right picture on
 343 figure 9. Minor differences are observed at the end of the shoaling phase (top left on figure 10), and
 344 during over-topping (top right on figure 10) with the MS and NA models giving almost identical height
 345 distributions, and Nwogu’s model giving the most strongly damped wave, possibly as a result of the
 346 slight over-shoaling observed in the top left picture triggering wave breaking a little earlier. After the
 347 over-topping, we see the generation of a very nonlinear bore propagating in the shallow area, mid-
 348 left picture on figure 10. Here again we can see some differences in the nature of the bore (some
 349 dispersion still appearing in the MS and GN results), or in its speed and magnitude (faster and stronger

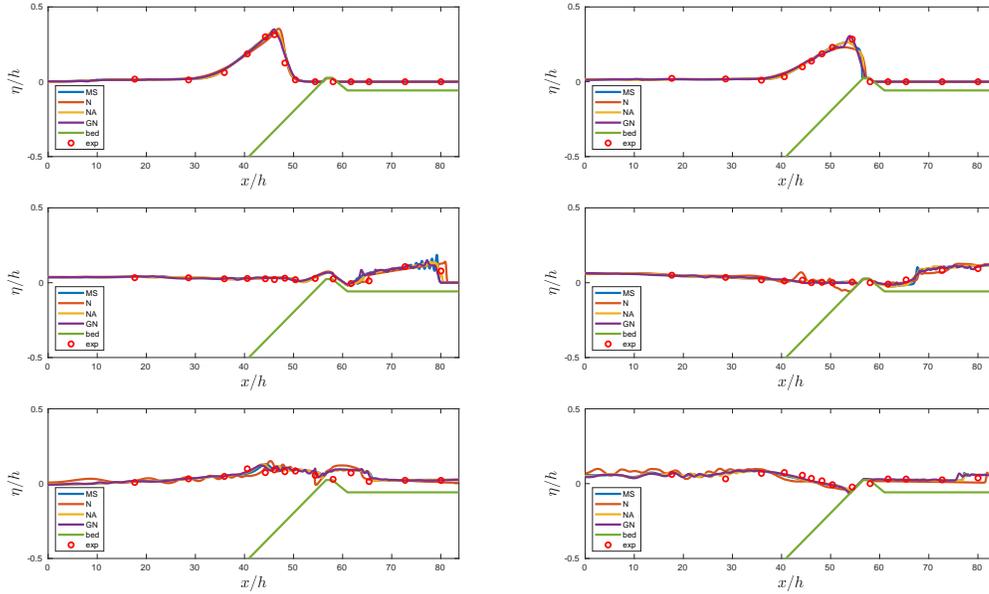


Figure 10: Solitary wave overtopping a reef: free surface snapshots.

350 for Nwogu’s model). In the next snapshot (mid-right picture on figure 10) we see the propagation of
 351 the wall reflected bore for which all models are again in phase. Perhaps a main feature to remark is the
 352 fact that the first backwash on the front side is stronger with Nwogu’s model. After the reflected bore
 353 overtops the reef (bottom-left picture on figure 10) we can see a weak dispersive jump on the front
 354 side of the reef, and a secondary reflected more in the shallow area. On the shallow side, the results of
 355 Nwogu’s model now lag phase compared to the others. This is even more clear in the wall reflection
 356 of the bore in the bottom-right picture on figure 10. The same figure shows the second backwash
 357 on the front side of the reef, for which all models are very close to one another.

358 Going to the wave gauge signals, we can see that only minor differences between the GN, MS,
 359 and NA models. These are mostly small details of the undular bore generated by the second over-
 360 topping, and the first bore in the shallow area which shows a lot more dispersion with the MS model.
 361 Compared to the data (in red) one can however hardly judge one better than another. Nwogu’s results
 362 are not far off but show high amplitude oscillations in the reflected wave from the first backwash,
 363 visible at dimensionless times $\sim 45 - 50$ in the first two wave gauges, and from the first overtopping,
 364 dimensionless times $\sim 55 - 60$ in the first two wave gauges. The phase lag of the secondary bores in
 365 the shallow area is also visible in the lagging arrival of these waves in the third and fourth gauge, at
 366 dimensionless times ~ 55 and ~ 68 approximately.

367 Despite its nonlinearity, and relevance in mimicking the inundation process, we cannot claim this
 368 test to allows to evaluate between the benefits of using fully or weakly nonlinear BT models. The
 369 differences observed, when not minor, are mostly originating in very shallow areas. This suggests that
 370 they may be more closely related to the parametrisation of wave breaking closure and of the wet/dry
 371 treatment, than to the choice of the propagation model. The interested reader can refer to e.g. [23]
 372 for a study on the sensitivity of breaking closures, or to [29] for a thorough study of this case using
 373 the hybrid closure. So far, we do seem to see some sort of trend, suggesting that weakly-nonlinear
 374 amplitude-flux models such as the MS and NA provide consistently results close to the fully nonlinear

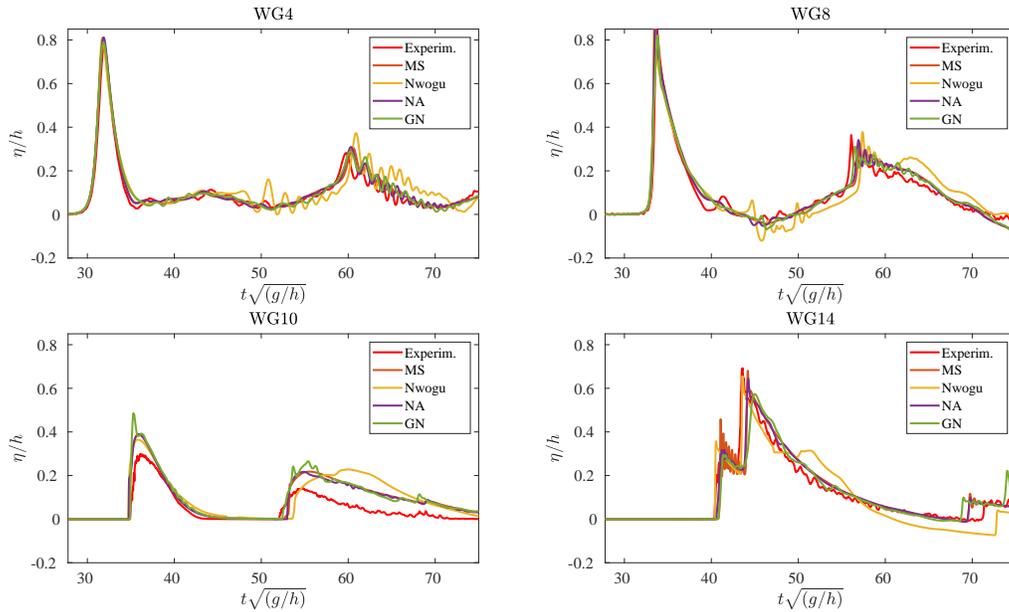


Figure 11: Surface elevation on the wave gauges WG4-WG8-WG10-WG14

375 GN model up to the breaking point. This will be contradicted in the following benchmarks.

376 3.5 Extreme run-up on a vertical barrier

377 Introduced in [66], this benchmark is relevant for wave structure applications, as e.g. the design of
 378 coastal structures like sea walls and breakwaters. The core of the problem is to correctly predict the
 379 wall run-up which can exceed six times their initial wave amplitude due to the interaction of steepening
 380 and dispersion processes. The setup involves a wave tank with flat bottom of depth $d = 10$ m and
 381 length $2L$. The actual physical domain is from 0 to L while from L to $2L$ a mirror image of the
 382 initial wave is placed (cf figure 12). This symmetric arrangement allows to free the analysis of the
 383 dependence of numerical wall boundary fluxes at the point $x = L$. In [66] the problem is studied using
 384 the free surface Euler equations, and the classical (non enhanced) Green-Naghdi equations. Here we
 385 perform the study with all the BT models discussed in section §2. Both the effects of changing the
 386 nonlinearity and dispersion parameters are investigated.

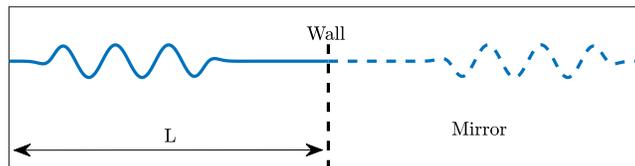


Figure 12: Initial conditions and set up of the physical and computational domains

387 The initial conditions are given by a linear wave multiplied by an envelope function i.e

$$\eta(x) = W(x)\alpha_0 \sin k_0(x - x_0), \quad (12)$$

$$u(x) = W(x)\alpha_0 \frac{\omega_0}{dk_0} \sin k_0(x - x_0) \quad (13)$$

388 where

$$W(x) = \frac{1}{2} \tanh\left(\frac{x - x_0}{\delta}\right) - \frac{1}{2} \tanh\left(\frac{x - x_0 - N_w \lambda_0}{\delta}\right). \quad (14)$$

389 We have denoted by $\lambda_0 = \frac{2\pi}{k}$ is the initial wavelength, and with x_0 and α_0 the initial position and
 390 amplitude of the wave, respectively. The main parameters that characterize the initial conditions are
 391 the frequency ω_0 , which depends on the dispersion relation of the propagation model, the amplitude α_0
 392 and the initial wave's location x_0 or in other words, the wave's distance from the wall. The remaining
 393 parameters N_w and δ denote respectively the number of waves in the physical domain, set to 3 like
 394 in [66], and a thickness parameter affecting the wave dynamics set to $0.2\lambda_0$ as in the original reference.

395

396 We first we examine the effect of the wavelength on the run-up. For a fixed initial wave amplitude
 397 $\alpha_0 = 0.05d$ we span the range $\lambda_0/d \in [30, 625]$, roughly corresponding to $\mu \in [0.0016, 0.033]$.
 398 In all simulations we set $L = 6\lambda_0$. The maximum run-up as a function of the initial wavelength is
 399 plotted on figure 13 for all the BT models considered. The reference solution of the Euler model is
 400 taken from [66]. For this case the situation is clearly reversed compared to the previous ones. The fully
 401 nonlinear models are quite close to the reference as expected, the enhanced model providing the best
 402 results. In this case, the amplitude-flux models, namely the MS and NA ones, provide a considerable
 403 underestimation of the run-up (up to 15%) for a wide range of wavelengths. The classical Nwogu
 404 equations instead perform quite well, as well as the enhanced Green-Naghdi model, for the longest
 405 waves, reversing the trend set up from the previous benchmarks. For short waves, the Nwogu model
 406 still shows a noticeable over-prediction of the run-up, compared to the Euler equations. Only the
 407 fully nonlinear model provides a correct prediction of these waves, the weakly nonlinear equations in
 408 amplitude-flux form still underestimating the run-up.

409 Next, as in [66], we examine the effect of the normalized initial amplitude, for a fixed wavelength
 410 set to $\lambda_0 = 125d$, roughly corresponding to a value $\mu \approx 8 \times 10^{-3}$. We consider nonlinearities in
 411 the range $\epsilon = \alpha_0/d \in [0.01, 0.15]$. The resulting maximum run-up is plotted for all the models in
 412 figure 14 (reference solution of the Euler model taken from [66]). For small nonlinearity $\epsilon \leq 0.02$ all
 413 the models provide almost superimposed results, with a small underestimation of the run-up height,
 414 compared to the Euler equations. As in [66] we observe a sharp transition taking place in the range
 415 $0.02 < \epsilon < 0.05$. This is the region in which the run up regime changes from linear to fully nonlinear.
 416 Above $\epsilon = 0.05$ some sort of equilibrium is attained between all the processes, with the Euler equations
 417 providing run-up values all close to about $R_{max} = 6d$. Both the slopes in the second phase, as well
 418 as the run-up values and slopes in the third one show significant differences between the models. For
 419 both the MS and NA models the run-up increase in the second phase is slower, and reaches a value
 420 roughly 10% lower than all other models. In the last phase, we see a decrease in run-up, leading for
 421 the most nonlinear waves to an underestimation of the run-up of the order of 25% compared to the
 422 Euler equations. All the remaining models, including the weakly nonlinear Nwogu equations, provide
 423 a very close transition in the second phase, all reaching values of R_{max}/d in between 5.5 and 5.75 for
 424 $\epsilon = 0.05$, and with values of for higher nonlinearities considerably closer to the Euler model than the
 425 MS and NA equations.

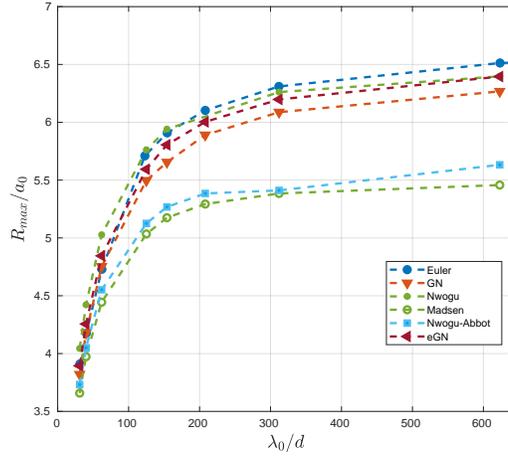


Figure 13: Normalized maximum run-up vs non-dimensional initial wavelength for different BT models and fixed $\alpha_0 = 0.05d$.

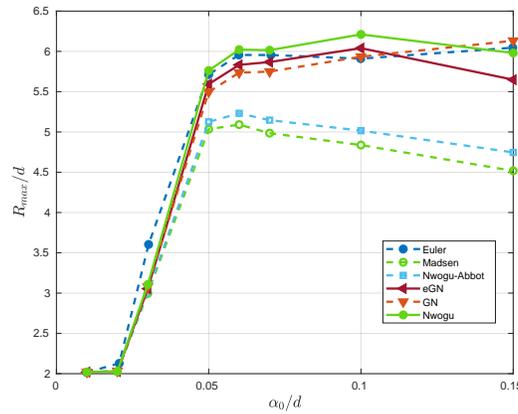


Figure 14: Normalized maximum run-up vs non-dimensional initial amplitude for different BT models and fixed $\lambda_0 = 125d$.

426 Note that, for the range of kd used in this test, the difference between the weakly nonlinear models
 427 is somewhat in line with the second order analysis recalled in section §2.3 (cf. rightmost picture in
 428 figure 1), and already invoked to interpret the results observed for the propagation of undular bores.
 429 Also it is important to stress that for all the waves considered the hypothesis $\epsilon = \mathcal{O}(\mu^2)$ underlying
 430 the weakly nonlinear models is violated systematically, the nonlinearity exceeding this limit by at least
 431 one order of magnitude. Despite of this, we are far from breaking conditions (we experimentally
 432 confirmed that the breaking sensor is far from breaking onset). So all the weakly nonlinear propagation
 433 models are used well outside their initial ansatz.

434 It is interesting to see in figure 15 the comparison of the time series of the run-up at the wall for
 435 the different BT models. We present the time series for two initial wave amplitudes, $\alpha_0 = 0.1d$ and

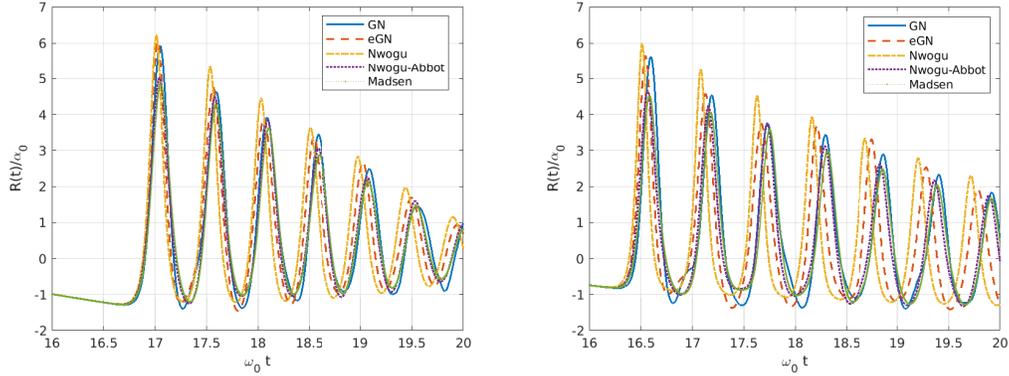


Figure 15: Run-up at the wall for two initial wave amplitudes $\alpha_0 = 0.1d$ (left) and $\alpha_0 = 0.15d$ (right) and fixed $\lambda_0 = 125d$.

436 $\alpha_0 = 0.15d$. We focus on the first dispersive wave, where $16 < \omega_0 t < 20$ in order to best appreciate the
 437 differences. All the weakly nonlinear models written in an amplitude-flux form (MS and NA) produce
 438 results with a slight phase lag and, as seen before, an under-prediction of the amplitudes of the trailing
 439 waves with respect to the ones of the eGN equations. The later are really close to the Euler equations,
 440 see [66], while Nwogu's model shows a phase advance which grows as the nonlinearity of the initial
 441 wave grows.

442 3.6 Long distance propagation, shoaling and run-up of a rectangular wave

443 This test case was first proposed as a benchmark for tsunami-type surface wave simulation codes in [5]
 444 as a part of the French TANDEM project. The main idea was to compare the codes based on different
 445 mathematical models, from Euler to BT and shallow water equations, on the dynamics of wave train
 446 propagation and run up on the vertical walls of the domain. It involves the formation of an undular
 447 bore which propagates over a large distance, shoals on a near coastal bathymetry and finally runs up
 448 on a vertical wall. The computational domain consists of a numerical wave tank of 30km length and
 449 50m depth. The sloping bed of 1:125 is placed at the right part of the domain where $x \in [25, 29\text{km}]$
 450 followed by flat shelf where the water depth is constant and equals 18m. The initial conditions of the
 451 free surface elevation are:

$$452 \eta(x) = \begin{cases} 5\text{km}, & \text{if } 0 \leq x < 1900\text{m} \\ 2.5(1 - \tanh(229.756 \frac{x}{2000} - 1)), & \text{if } 1900\text{m} \leq x \leq 2100\text{m} \\ 0, & \text{if } 2100\text{m} < x \leq 30000\text{m} \end{cases} \quad (15)$$

452 Wall boundary conditions are imposed at both ends of the domain. Figures 16-17 show the first hour
 453 of life of the wave, including its propagation, shoaling, and reflection on the right and left wall. In the
 454 large scale the numerical solution provided by all models looks almost identical, at least up to the first
 455 reflection. Some differences in phase and secondary wave structure can be seen after the first reflection
 456 (bottom-right picture on figure 16. These differences are somewhat compensated during propagation
 457 toward the left wall, and the position and structure of the wave up to the 5th or 6th peak is quite similar
 458 for all models before and after the second reflection (middle- and bottom- right pictures on figure 17),

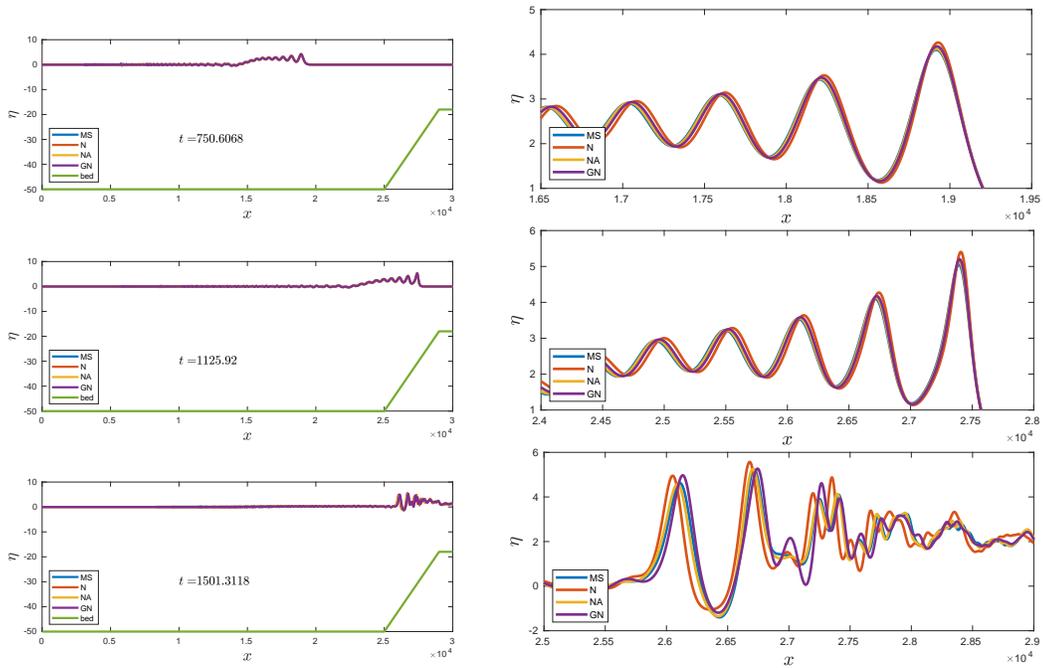


Figure 16: Propagation, shoaling, and wall reflection of the undular bore. A zoom on the wave trains is presented in the right column.

459 aside perhaps from a larger phase advance and amplitude over-prediction for the Nwogu model. A
 460 closer look of the run-up of the waves on the vertical right wall is provided in figure 18. This plot
 461 allows to better visualize the differences between all models. We see again a clear under-estimation
 462 of the maximum run-up from the amplitude-flux weakly nonlinear models by 15%-20% compared to
 463 the GN equations. Note that the reference value of the maximum run-up [5] is of about 24.5 m, which
 464 means that the GN model already provides an underestimation of roughly 18.4%, and the MS and NA
 465 models have errors of 30%-35% wrt this value. These are much worse numbers than for the previous
 466 case. We suspect that the implementation of the wall boundary conditions may affect these values and
 467 we are investigating this point. The message is however that, as for the previous case, weakly nonlinear
 468 BT models which seem to perform well in other quite nonlinear cases, under estimate substantially the
 469 run-up on vertical walls. The tendency to over-predict amplitudes somehow leads to better results for
 470 the Nwogu model for this case as for the previous.

471 4 Two dimensional benchmarks

472 4.1 Bore propagation in a trapezoidal channel

473 We consider now a well known two dimensional variant of the bore propagation problem of section
 474 §3.2. The problem is similar, but involves topographic variations across the section which has a
 475 trapezoidal shape. As illustrated by the experimental work Treske [65], and by the numerical and
 476 theoretical study of [9], the bore dynamics are much more complex in this case. More precisely
 477 Treske already showed that the secondary wave field has strong variations in crest height across the

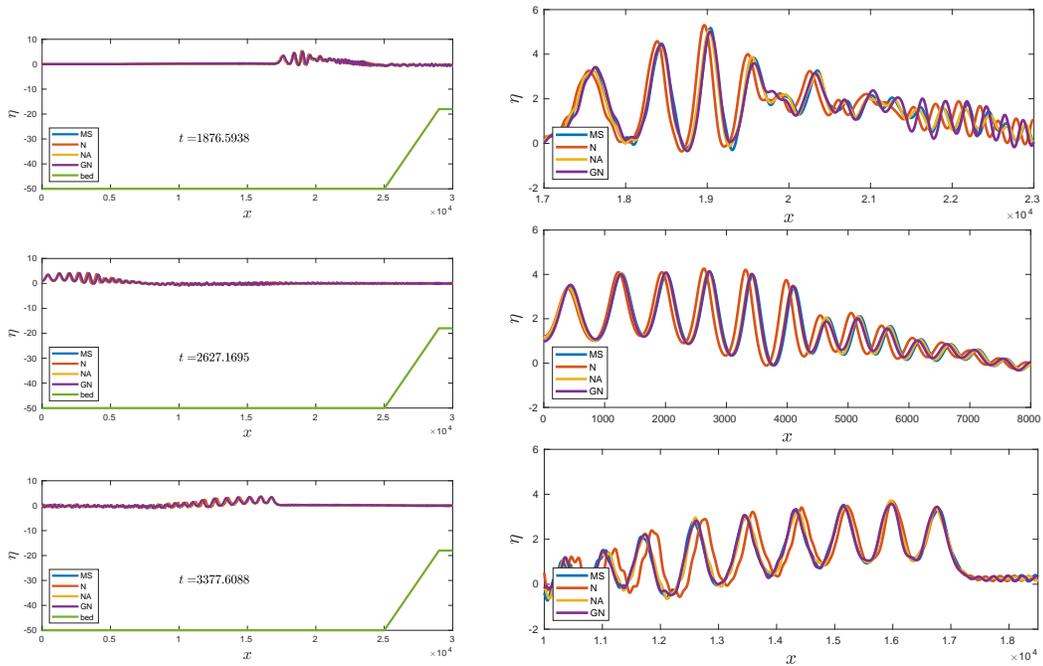


Figure 17: Reflected wave propagation, and reflection on the left hand wall. A zoom on the wave trains is presented in the right column.

478 channel section, and may even have variations in frequency content, depending on the Froude number,
 479 a transition taking place around a critical value $Fr_t = 1.15$. According to [9] for Froude above Fr_t
 480 the flow is truly multi-dimensional. The waves along the centerline of the channel are similar to the
 481 Favre waves and have higher amplitudes, while much longer and less steep waves are observed on
 482 the banks. Below Fr_t the flow is quasi-one dimensional with longer waves of higher amplitudes
 483 on the banks of the channel. In this regime the same wavelength is observed all across the section. This
 484 benchmark, featured quite nonlinear waves, with Froude numbers approaching, and locally exceeding,
 485 the breaking limit.

486 We performed simulations for a range of Froude numbers from 1.05 to 1.25 using the GN, Nwogu
 487 and Nwogu-Abbot models. This range of numbers contains the transition, and remains below the
 488 transition to a fully breaking bore along the whole channel section. The computational domain of
 489 27566 nodes, consist of a channel with $(x, y) \in [0, 25] \times [0, 1.5]$ in which the half of the channel is
 490 placed. More on the set up of the problem can be found in [9]. Free surface visualizations for Froude
 491 1.05, 1.17 and 1.25 are reported in figures 19, 20 and 21 respectively. We can visually see the the
 492 appearance of more oscillations along the same length of channel shown, and the presence of stronger
 493 cross-sectional waves for higher Froude. To confirm these observations, the distribution of the free
 494 surface elevation along the banks and centerline are reported in figure 22 for the three models.

495 Following [9] we post processed the results in order to provide a more quantitative assessment. In
 496 particular, we have estimated the height of the first peak and trough, and the wavelength. The results
 497 are compared in figure 23 with the experimental data of Treske [65]. The main difference observed
 498 among the models is related to the wave amplitudes after the transition, i.e for Froude numbers grater
 499 that 1.15. This is roughly the limit after which wave breaking starts taking place close to the banks.

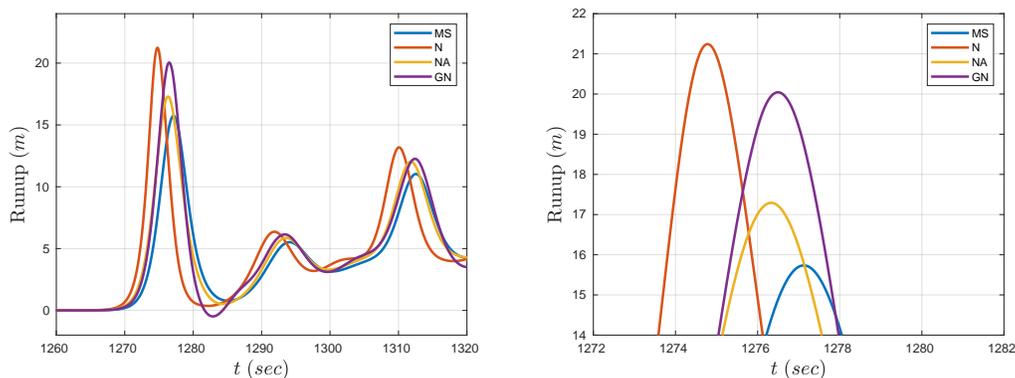


Figure 18: Run up of the wave on the right end. A zoom of the peak is presented in the right column.

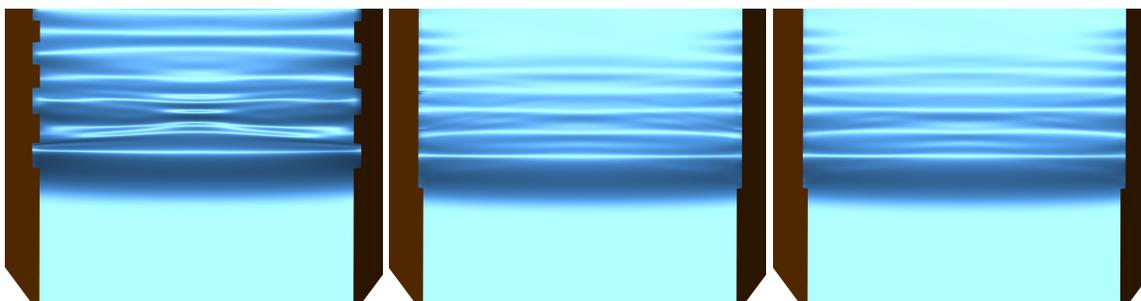


Figure 19: 2 dimensional free surface elevation for GN (left) Nwogu (centered) Nwogu-Abott (right) at $t \approx 13.5$ sec. The Froude number is 1.05

500 The computed wavelengths before and after the transition are almost identical for all the models. The
 501 only difference observed is for the model of Nwogu in which the transition seems to take place for
 502 slightly lower Froude. This may be related to the over shoal of the wave on the banks, leading to
 503 earlier breaking and possibly earlier transition.

504 Despite the strong nonlinearities involved, and Froude numbers close to (or higher than) the
 505 breaking point, the amount of nonlinearity embedded in the BT models doesn't seem to affect their
 506 capability to correctly predict both amplitudes, and wavelengths.

507 4.2 The seaside test case

508 The last test case presented is the laboratory benchmark test case for tsunami inundation through urban
 509 waterfront. The physical model was a 1:50 scale idealization of the town Seaside, Oregon, designed
 510 to observe the complex tsunami flow around the macro-roughness such as buildings idealized as
 511 impermeable, rectangular blocks. The laboratory benchmark took place in the Oregon State University.
 512 It has been served as a blind benchmark test case for the NTHMP Mapping and Modeling Benchmarking
 513 Workshop: Tsunami Currents. All the experimental data are available¹. Numerical predictions using
 514 fully nonlinear using Boussinesq type equations can be found in [48, 28]. The numerical set up of

¹http://coastal.usc.edu/currents_workshop/index.html

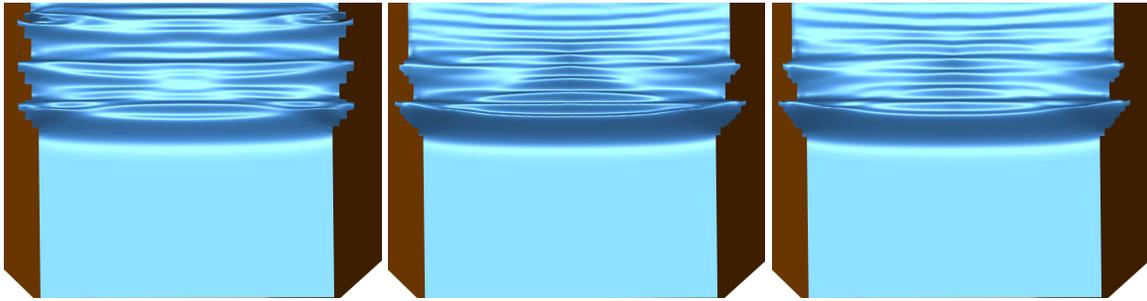


Figure 20: 2 dimensional free surface elevation for GN (left) Nwogu (centered) Nwogu-Abott (right) at $t \approx 13.5$ sec. The Froude number is 1.17

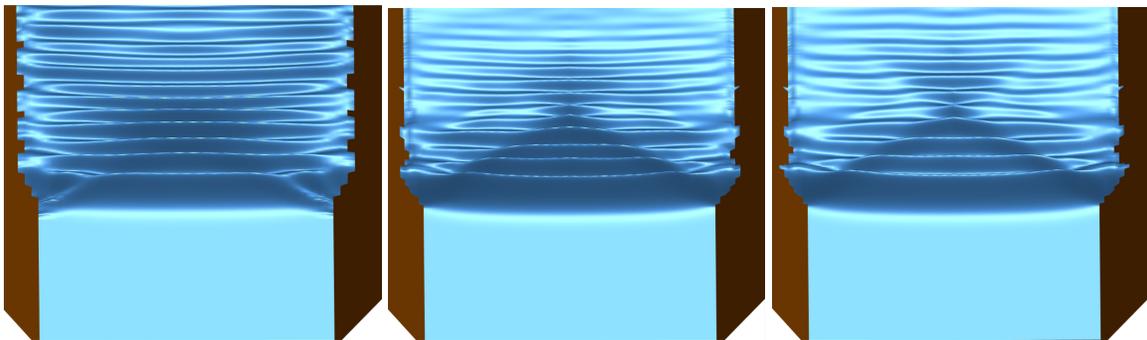


Figure 21: 2 dimensional free surface elevation for GN (left) Nwogu (centered) Nwogu-Abott (right) at $t \approx 13.5$ sec. The Froude number is 1.25

515 the test case is as follows: the rectangular domain of 42m in x and 14.5m in y is discretized using
 516 an unstructured triangulation with 172854 nodes, refined in the region where the buildings are placed.
 517 The maxim side of the triangles is $h_{ref} \approx 0.7$ m and the minimum one $h_{ref} \approx 0.03$ m. The individual
 518 structures and buildings are approximated as steep-sided topography and a manning coefficient of 0.01
 519 is used. The available experimental data involve the topography, the forcing signal and the time series
 520 of the waves measured at the wave gauges. The forcing signal is entering the computational domain
 521 through the left boundary and wall boundary conditions are placed elsewhere. The computational
 522 domain is shown in 24. The 31 wave gauges, measure the free surface and the velocity and are
 523 depicted in figure 24. The incoming solitary wave shoals and breaks close to the seawall and then
 524 floods an urban area which contains buildings of different height. A detailed description of behavior
 525 of the free surface elevation can be found in [48, 28]. Figure 25 shows snapshots of the free surface
 526 elevation, for each model, at $t \approx 30$ sec during the flooding of the urban area.

527 Figures 26-29 show the free surface elevation for the 31 wave gauges. The experimental data are
 528 compared with the numerical results of the equations of Nwogu, Nwogu-Abbot and Green-Naghdi.
 529 We observe that in almost all the wave gauges the numerical results are very close to the experimental
 530 data and that, especially for the first wave are almost indistinguishable. A difference in the arrival
 531 time of the secondary wave at roughly 60sec, is observed in the wave gauges located in front of the
 532 buildings.

533 Overall, when comparing the free surface visualizations, and the time histories at gauges, for

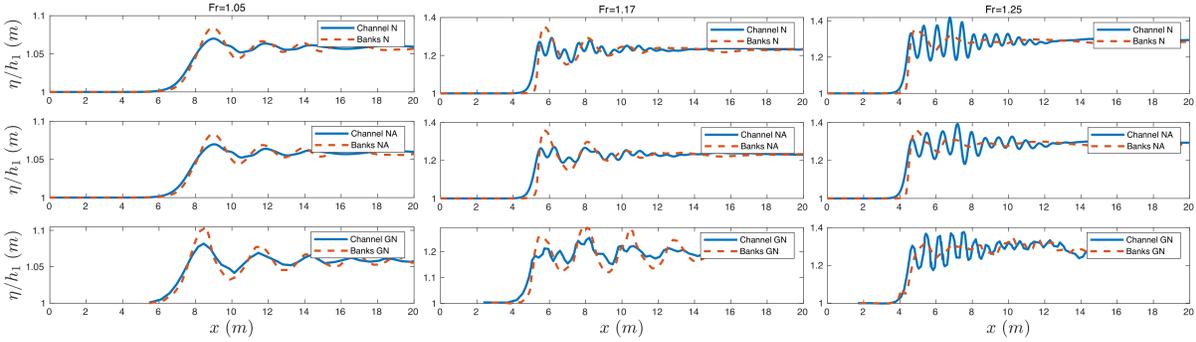


Figure 22: Free surface profile for different Froude numbers. Continuous line: axis. Dashed line: banks.

534 this challenging case, we cannot see extreme differences in the main features of the flow. One major
 535 difference we see, which also affects some of the previous tests, is in very shallow areas, and especially
 536 in initially dry flooding areas, where all weakly nonlinear models never include dispersion. This
 537 explains in our opinion the details of the wave gauge histories for this tests which somewhat favor
 538 the fully nonlinear model, especially maybe for Locations A and B. But the differences are in small
 539 details, and also some of these results maybe could be improved by “tuning” the roughness.

540 5 Discussion and conclusions

541 In this paper we have reviewed a certain number of benchmarks for dispersive wave propagation
 542 models. It is not simple to draw clear conclusions. One disappointing fact is perhaps that, when
 543 considering both amplitude-flux and amplitude-velocity weakly-nonlinear models, many very well
 544 known and common test cases are not adequate to reveal differences between fully nonlinear and
 545 weakly nonlinear equations. This is the case even for quite nonlinear problems in which the breaking
 546 point is reached and exceeded.

547 One exception is perhaps Favre waves. Indeed bore propagation on flat topography shows some
 548 advantage of having full nonlinearity in a short window of high Froude numbers close to the breaking
 549 limit. The relatively more recent benchmarks proposed in [66] and [5] allow to highlight more clearly
 550 the need of full nonlinearity at least when considering wave run-up on vertical walls, already for mild
 551 offshore nonlinearity, and as long as breaking conditions are not reached.

552 These are however relatively academic tests. In complex physical benchmarks with bathymetric
 553 variations, wave breaking, and wet/dry transitions, both in one and two space dimensions it seems
 554 much harder to draw clear conclusions. For models which are well known to over-estimate wave
 555 amplitudes (as Nwogu’s model) we do observe some early breaking affecting the overall wave dynamics.
 556 This is however not the case with all weakly-nonlinear models. Some details of the secondary waves
 557 appearing in complex problems as e.g. the Seaside case, also show some impact of not including
 558 full-nonlinearity, which is certainly related to the inability of weakly nonlinear models to represent
 559 some weak dispersive propagation as soon as it takes place above the initial shoreline. However the
 560 differences are again not striking. Some genuinely 2D extensions of the 1D cases which reveal most
 561 clearly the need of nonlinearity, while being of physical and engineering relevance, with possibly
 562 openly available reference solutions or experimental data, seem still missing in literature.

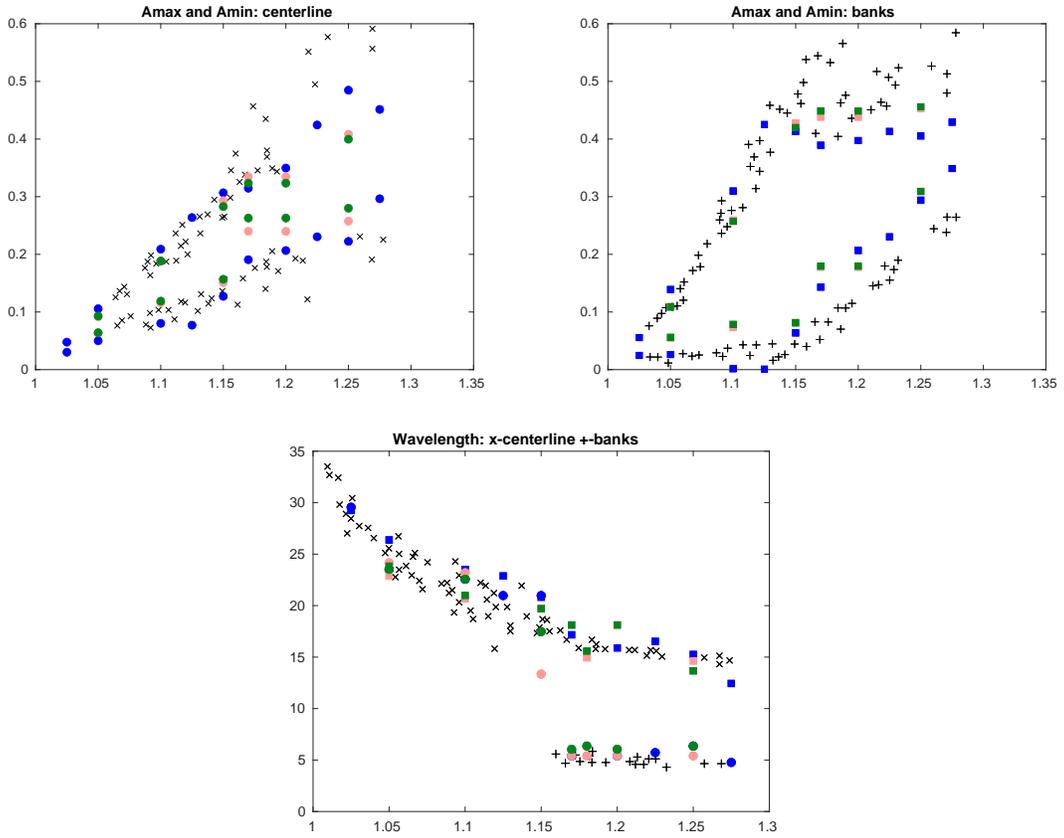


Figure 23: Wavelength and water level of undular bores in a channel with trapezoidal section. up-left: Water elevation (axis), up right: water elevation (banks), down wavelength. x: Data of Treske (1994) axis, +: data of Treske (1994) banks, circle: simulated values on the channel axis, squares: simulated values on the banks. Blue color: GN, Pink color: Nwogu, Green color: Nowgu-Abbot

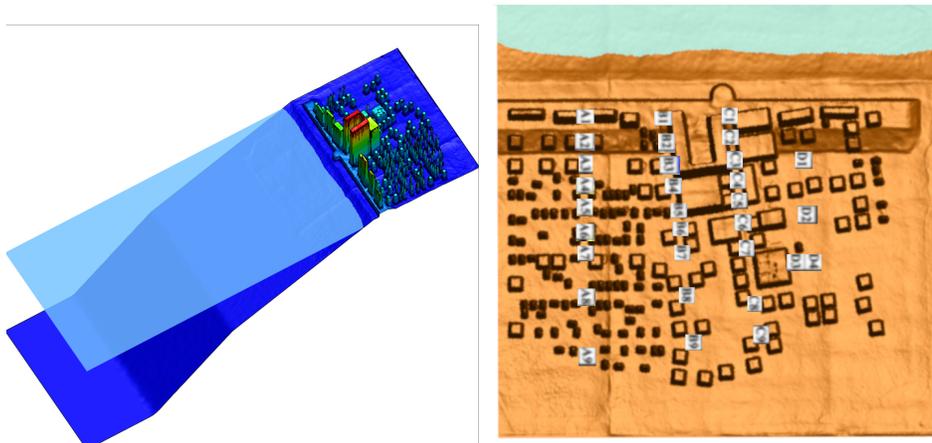


Figure 24: Set up and location of the wave gauges for the seaside test case.

563

564 We can thus draw two main conclusions. The first is that benchmarking habits need to be slightly
 565 revised. Authors, editors, and reviewers must be more careful in considering/suggesting appropriately
 566 chosen tests, so that when new models/schemes are proposed their capability for highly nonlinear
 567 propagation are clearly highlighted together with all other aspects (phase prediction, breaking location
 568 and strength, and so on). Also it should be always underlined when a benchmark, although involving
 569 highly nonlinear waves, is not sensitive to the fully/weakly nonlinear nature of the model. This work
 570 by no means aims at performing this task, which is possibly to be performed with a larger set of
 571 models/codes. The second conclusion we draw from our tests is that of course fully nonlinear models
 572 allow to safely approach problems of different nature and involving highly nonlinear waves. However,
 573 as a rule of thumb weakly nonlinear models can still be safely used in many problems of practical
 574 interest not involving planar non-breaking wave fronts reflecting on steep coastal structures in deep
 575 waters. This is true even for quite complex urban run-up of nonlinear waves, as seen in the Seaside
 576 example.

577 To answer the question in the title, the humble opinion of the authors is that full nonlinearity is
 578 neither luxury nor necessity. There is a large class of problems for which both appropriately chosen
 579 weakly and fully nonlinear models provide reasonable answers. So it is up to the user to be aware of
 580 the differences existing, and choose which of these models is most suited for his application.

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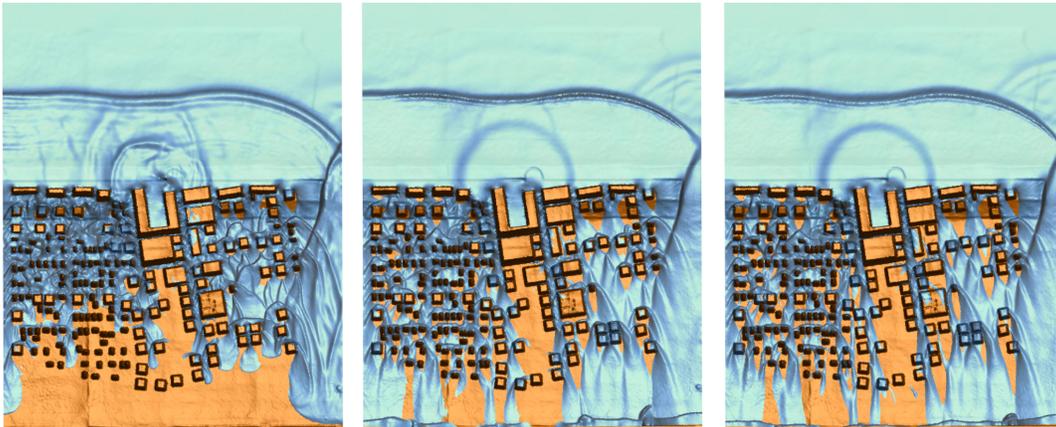


Figure 25: 2 dimensional free surface elevation for GN (left) Nwogu (centered) Nwogu-Abott (right) at $t \approx 30$ sec.

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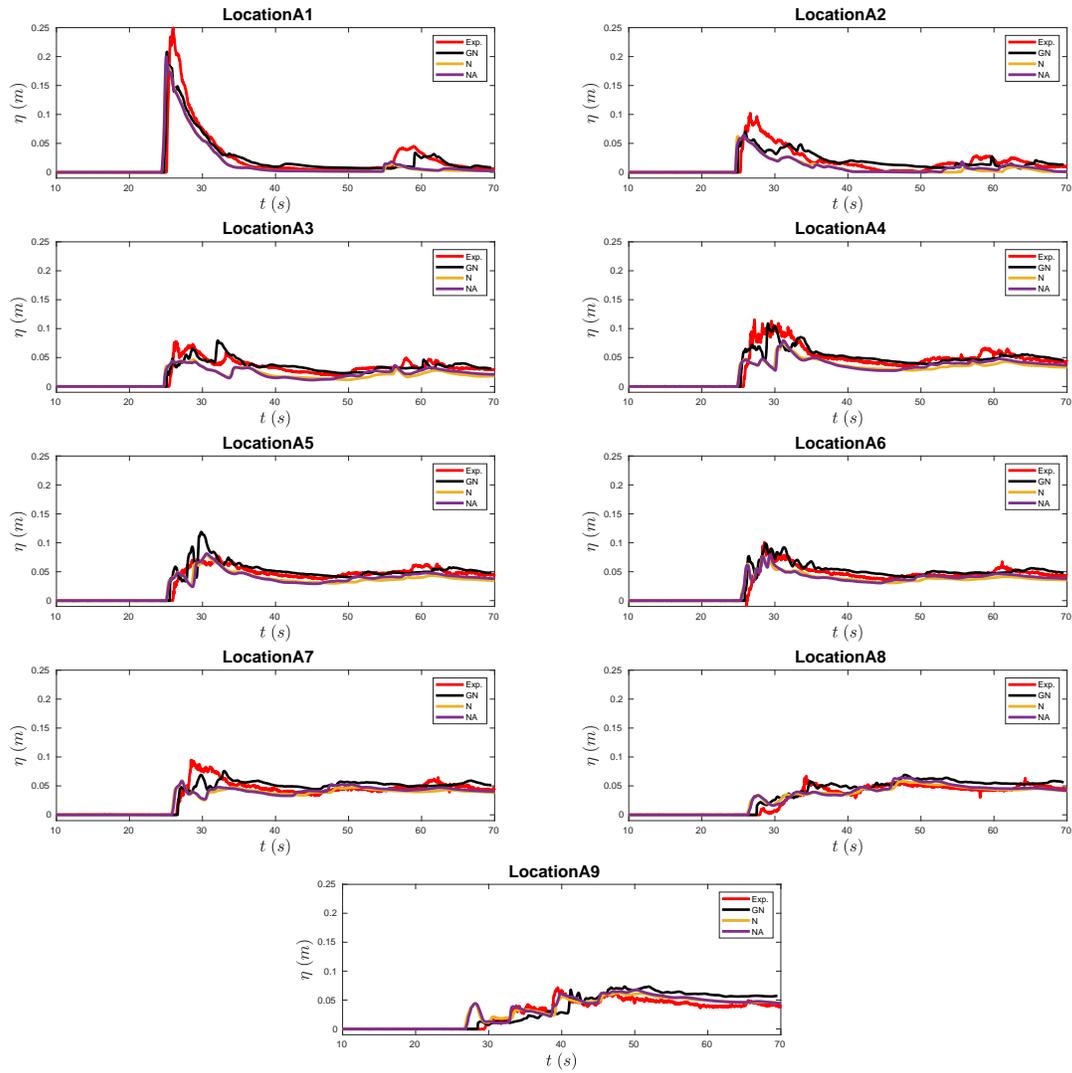


Figure 26: Free surface elevation measured on wave gauges at location A

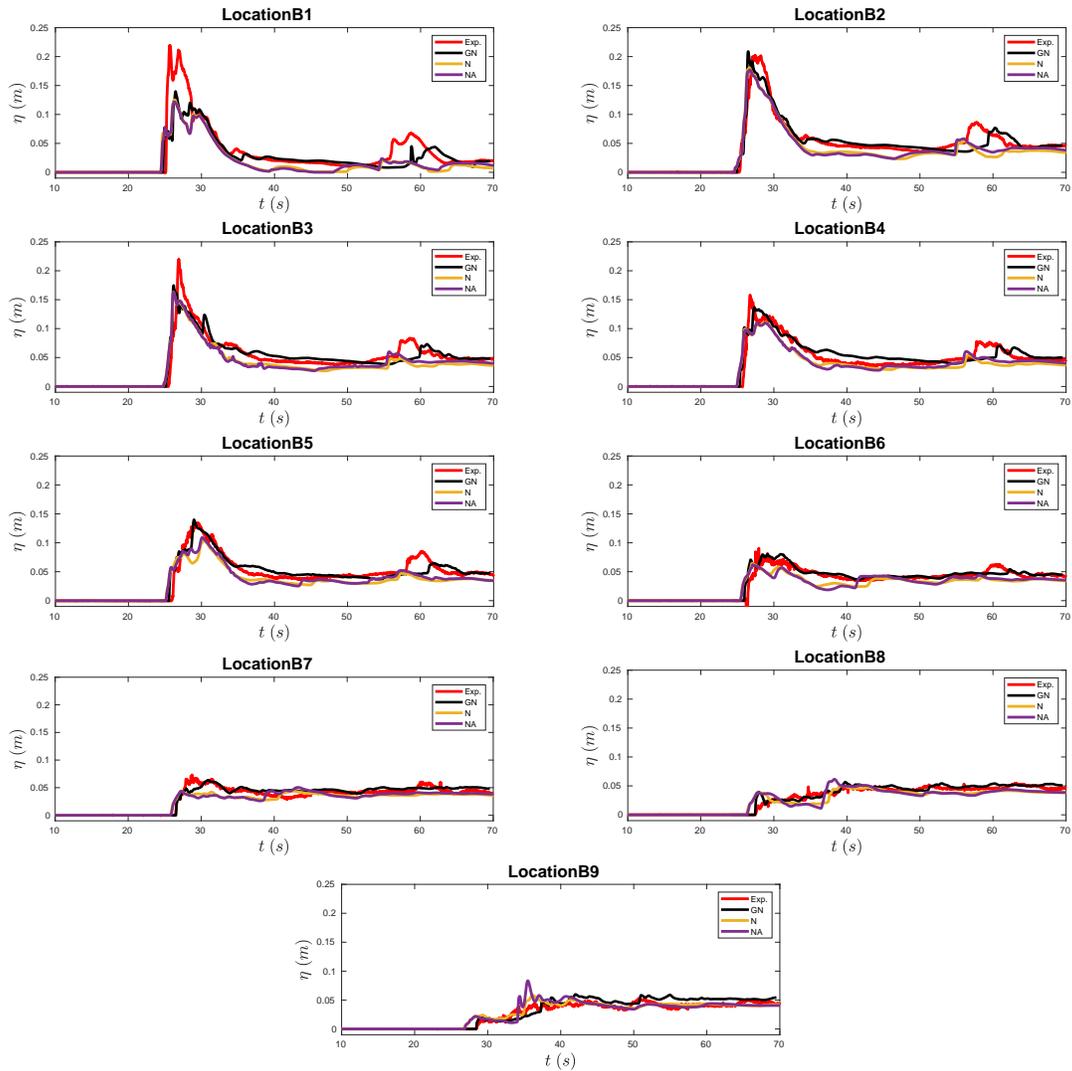


Figure 27: Free surface elevation measured on wave gauges at location B

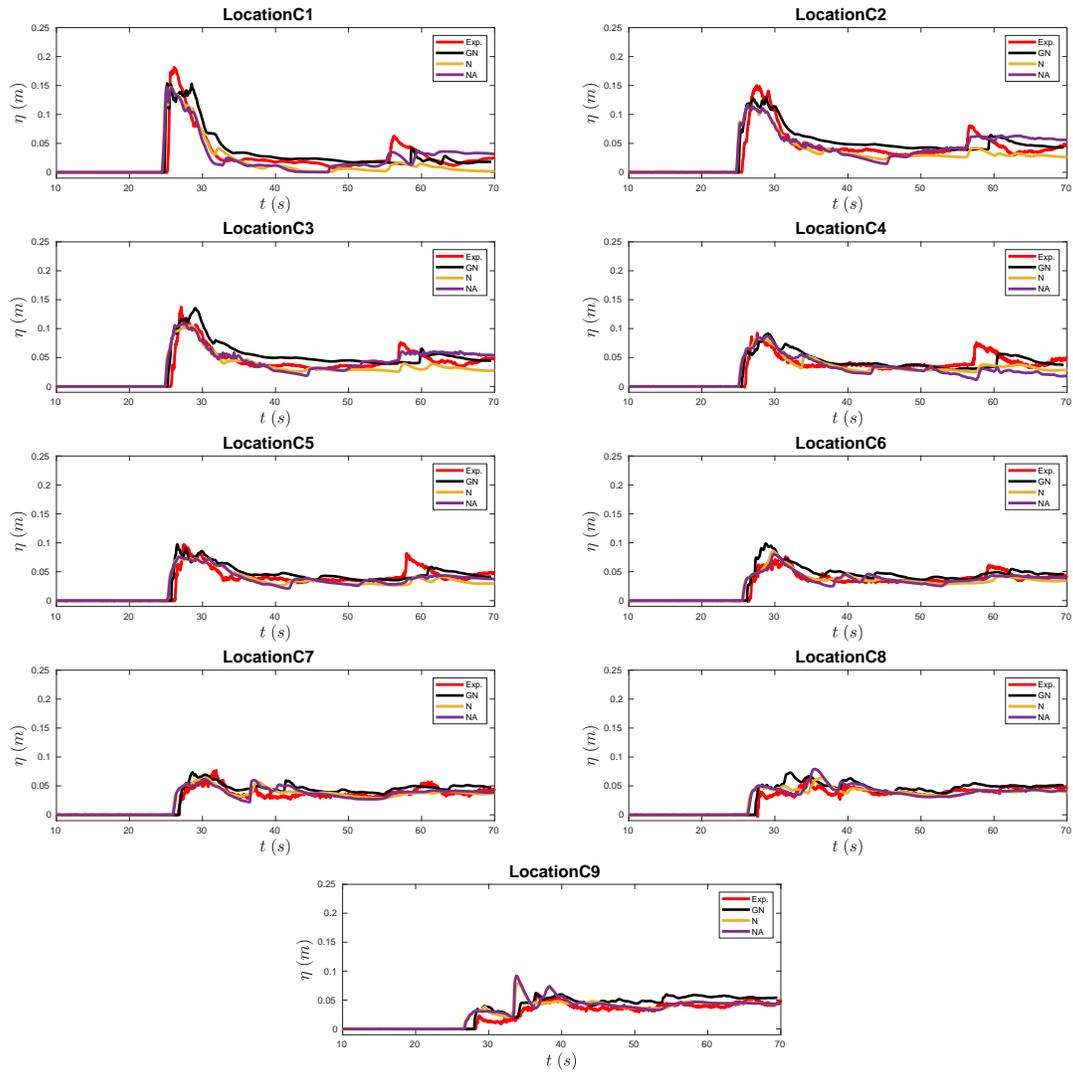


Figure 28: Free surface elevation measured on wave gauges at location C

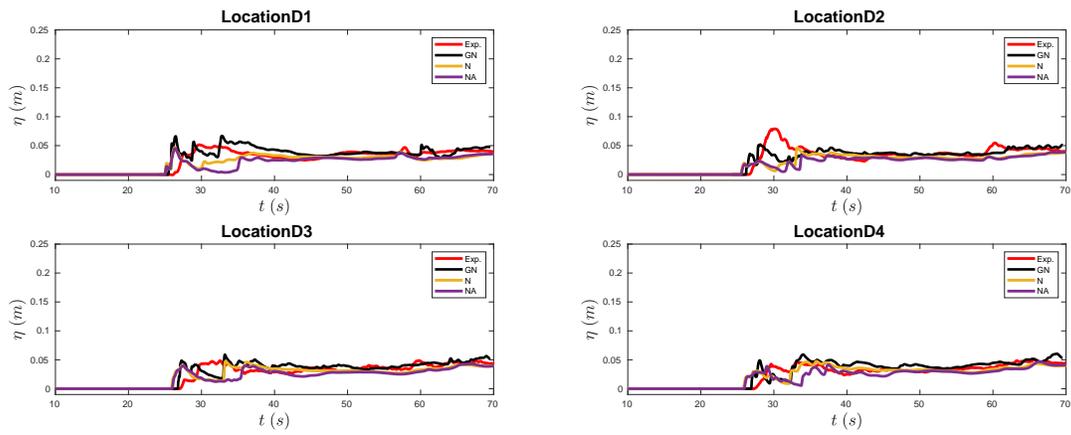


Figure 29: Free surface elevation measured on wave gauges at location D