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# Analysis of homogeneous systems with distributed delay using averaging approach

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**Abstract:** The conditions for stability of the zero solution are formulated for homogeneous dynamical systems of non-zero degree with distributed delays. It is assumed that if the disturbances are absent and the non-delayed state substitutes the delayed one, then the obtained nominal system is globally asymptotically stable. In such a case it is proven that in the disturbance-free scenario the zero solution is locally asymptotically stable for positive degree and practically globally asymptotically stable for negative degree. Using averaging tools, the influence of time-varying disturbances is investigated and the respective stability margins are derived. Finally, the obtained theoretical findings are illustrated by a mechanical example.

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## 1. INTRODUCTION

For dynamical systems, the stability analysis is the main research problem, while investigations of influence of delays become popular nowadays due to significant growth of the control and estimation applications in the internet of things and cyber-physical systems [Karafyllis et al., 2016, Hetel et al., 2017]. For time-delay systems the conditions of stability can be obtained with the use of Lyapunov–Krasovskii (LK) or Lyapunov–Razumikhin (LR) approaches [Fridman, 2014]. The first method utilizes LK functionals (LKF) [Pepe et al., 2017, Efimov and Fridman, 2020], while the second one is based on conventional LR functions, which however may be easier to apply.

Complex communication networks lead to appearance of distributed delays [Anan’evskii and Kolmanovskii, 1989, Feng and Lam, 2012, Feng et al., 2020], and there exist extensions of LK and LR methods for this case [Xie et al., 2001, Solomon and Fridman, 2013]. Presence of external perturbations requires analysis of their outcomes and evaluation of admissible upper bounds of the disturbances in terms of the delayed state (*i.e.*, estimation of the asymptotic gains in terms of the input-to-state stability (ISS) [Teel, 1998, Fridman et al., 2008]). If the perturbations belong to the class of periodic or almost periodic signals, then the averaging method can

be efficiently utilized for the robustness quantification [Bogoliubov and Mitropolsky, 1961, Khapaev, 1993].

In this work, the homogeneous dynamical systems are considered. This class of models has a lot of useful properties in the delay-free case (the systems with zero delay described by ordinary differential equations) [Efimov and Polyakov, 2021], and there are also extensions to infinite-dimensional systems [Efimov et al., 2014, 2016]. It has been demonstrated that if the delay-free counterparts are asymptotically stable, then for any delay value the homogeneous dynamics are *locally asymptotically stable* at the origin for positive degree [Aleksandrov and Zhabko, 2012, 2014, Efimov et al., 2014] or *practically globally asymptotically stable* (we will also refer this property as *uniform ultimate boundedness* (UUB) of the solutions) for negative homogeneity degree [Zimenko et al., 2017]. In above works, the LR method is used, and the LK approach application can be found in [Aleksandrov et al., 2015, Efimov and Aleksandrov, 2021].

This note is the conference version of Aleksandrov et al. [2023], where the robust stability conditions are extended to a class of homogeneous systems containing distributed delays. Both cases, with the use and without of the averaging approach, are considered. The proofs, which are excluded due to space limitations, are based on application of Lyapunov–Krasovskii and Lyapunov–Razumikhin methods, and can be found in Aleksandrov et al. [2023].

The outline of this work is as follows. Preliminaries are given in Section 2. The considered analysis problem is described in Section 3. The general stability and

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robustness results are formulated in sections 4 and 5. Using the averaging approach, the respective stability results are presented in Section 6. An illustrative example is shown in Section 7.

## 2. PRELIMINARIES

The real numbers are denoted by  $\mathbb{R}$ ,  $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ , and  $|s|$  is an absolute value for  $s \in \mathbb{R}$ . Euclidean norm for a vector  $x \in \mathbb{R}^n$  is defined as  $\|x\|$ . We denote by  $C([a, b], \mathbb{R}^n)$ ,  $[a, b] \subset \mathbb{R}$  the Banach space of continuous functions  $\phi : [a, b] \rightarrow \mathbb{R}^n$  with the uniform norm  $\|\phi\|_C = \sup_{a \leq \zeta \leq b} \|\phi(\zeta)\|$ .

A sequence of integers,  $m, m+1, \dots, n$  for  $m < n$  is further denoted by  $\overline{m, n}$ .

A continuous function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also radially unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is a strictly decreasing to zero for any fixed  $r \in \mathbb{R}_+$ .

The standard definitions of stability and related properties for time-delay systems can be found in [Hale, 1977, Kolmanovskii and Myshkis, 1999, Fridman, 2014], and for delay-free dynamics in [Khalil, 2002].

### 2.1 Useful inequalities

The *Young's inequality* claims that for any  $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+$  and  $\gamma > 0$ ,  $\delta > 0$  [Bacciotti and Rosier, 2001]:

$$\mathbf{a}^\gamma \mathbf{b}^\delta \leq \frac{1}{p} \mathbf{a}^{\gamma p} + \frac{p-1}{p} \mathbf{b}^{\frac{\delta p}{p-1}}$$

for any  $p > 1$ , while *Hölder's inequality* for any  $f, g : I \rightarrow \mathbb{R}$  with  $I \subset \mathbb{R}$  ensures that [Bacciotti and Rosier, 2001]:

$$\int_I |f(s)g(s)| ds \leq \left( \int_I |f(s)|^p ds \right)^{\frac{1}{p}} \left( \int_I |g(s)|^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}}$$

for any  $p > 1$ .

Using the properties of homogeneous functions the following result can be obtained [Efimov and Aleksandrov, 2021]:

*Lemma 1.* Let  $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+$  and  $\ell > 0$ ,  $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma > 0$ ,  $\delta > 0$  be given, then

$$\mathbf{a}^\alpha + \mathbf{b}^\beta - \ell \mathbf{a}^\gamma \mathbf{b}^\delta \geq 0$$

provided that

- 1)  $\max\{\mathbf{a}^\alpha, \mathbf{b}^\beta\} \geq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$  and  $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} < 1$ ,
- 2)  $\max\{\mathbf{a}^\alpha, \mathbf{b}^\beta\} \leq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$  and  $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1$ .

### 2.2 Homogeneity

For any  $r_i > 0$ ,  $i = \overline{1, n}$  and  $\lambda > 0$ , define the *vector of weights*  $\mathbf{r} = [r_1, \dots, r_n]$  and the *dilation matrix*  $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$ ;  $r_{\min} = \min_{i=\overline{1, n}} r_i$  and  $r_{\max} = \max_{i=\overline{1, n}} r_i$ .

*Definition 2.* [Zubov, 1958, Efimov and Polyakov, 2021] The function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is called  *$\mathbf{r}$ -homogeneous*, if for any  $x \in \mathbb{R}^n$  the relation

$$h(\Lambda_r(\lambda)x) = \lambda^\nu h(x)$$

holds for some  $\nu \in \mathbb{R}$  and all  $\lambda > 0$ .

The vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called  *$\mathbf{r}$ -homogeneous*, if for any  $x \in \mathbb{R}^n$  the relation

$$f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda)f(x)$$

holds for some  $\nu \geq -r_{\min}$  and all  $\lambda > 0$ .

In both cases, the constant  $\nu$  is called the *degree of homogeneity*.

For any  $x \in \mathbb{R}^n$  and  $\varpi \geq r_{\max}$ , a *homogeneous norm* can be defined as follows

$$\|x\|_r = \left( \sum_{i=1}^n |x_i|^{\varpi/r_i} \right)^{1/\varpi}.$$

For all  $x \in \mathbb{R}^n$ , its Euclidean norm  $\|x\|$  is related with the homogeneous one:

$$\underline{\sigma}_r(\|x\|_r) \leq \|x\| \leq \bar{\sigma}_r(\|x\|_r)$$

for some  $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$  [Efimov et al., 2018]. In the following, due to this ‘‘equivalence’’, stability analysis with respect to the norm  $\|x\|$  can be substituted with analysis for the norm  $\|x\|_r$ . The homogeneous norm has an important property: it is  *$\mathbf{r}$ -homogeneous of degree 1*, that is  $\|\Lambda_r(\lambda)x\|_r = \lambda\|x\|_r$  for all  $x \in \mathbb{R}^n$  and  $\lambda > 0$ .

## 3. STATEMENT OF THE PROBLEM

Consider a system without disturbances

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^t G(x(s)) ds, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$ , vector fields  $F(x)$  and  $G(x)$  are continuous for  $x \in \mathbb{R}^n$  and  *$\mathbf{r}$ -homogeneous of the degree  $\nu$*  with respect to weights  $\mathbf{r} = (r_1, \dots, r_n)$ ,  $\nu + r_{\min} > 0$ ;  $\tau = \text{const} > 0$  is the delay; the initial functions for (1) belong to the space  $C([-\tau, 0], \mathbb{R}^n)$ . Denote by  $x_t$  the restriction of a solution  $x(t)$  to the segment  $[t - \tau, t]$ , *i.e.*,  $x_t : \xi \mapsto x(t + \xi)$ ,  $\xi \in [-\tau, 0]$ . The system (1) admits the zero solution.

In Section 4, our aim is to derive the conditions of *asymptotic stability* for the zero solution (there exists  $\beta \in \mathcal{KL}$  such that  $\|x(t)\| \leq \beta(\|x_0\|_C, t)$  for all  $t \geq 0$  and  $\|x_0\|_C < \bar{\rho}$  for some  $\bar{\rho} > 0$ ), or we look for *UUB* conditions in (1) (there exist  $\beta \in \mathcal{KL}$  and  $\bar{\rho} > 0$  such that  $\|x(t)\| \leq \beta(\|x_0\|_C, t) + \bar{\rho}$  for all  $t \geq 0$  and  $x_0 \in C([-\tau, 0], \mathbb{R}^n)$ ), depending on positive or negative sign of the homogeneity degree  $\nu$  of  $F$  and  $G$ , respectively. Further, we will study, first, the impact on the dynamics of (1) of general non-stationary perturbations (in Section 5). Next, the influence of disturbances admitting averaging (*e.g.*, periodic or almost periodic) with a nonlinear gain containing distributed delay for all signs of homogeneity degree are investigated in Section 6.

The results of Section 6 are obtained with the aid of time-delay approach to averaging (see [Fridman and Zhang, 2020, Zhang and Fridman, 2022]).

## 4. BASIC STABILITY AND UUB CONDITIONS

To start, we assume asymptotic stability of the origin for the delay-free counterpart of (1), which is obtained after substitution of  $x(t)$  in place of  $x(s)$ :

*Assumption 1.* The zero solution of the auxiliary delay-free system

$$\dot{x}(t) = F(x(t)) + \tau G(x(t)), \quad x(t) \in \mathbb{R}^n, \quad t \geq 0,$$

is asymptotically stable.

*Remark 3.* In the case of linear system (1) with  $F(x) = Ax$  and  $G(x) = A_d x$ , where  $A$  and  $A_d \in \mathbb{R}^{n \times n}$ , Assumption 1 means that  $A + \tau A_d$  is Hurwitz, which guarantees the stability of the system provided that an additional condition holds (see Section 3.8 in [Fridman, 2014]). Surprisingly, for homogeneous systems with  $\nu \neq 0$ , Assumption 1 appears to be sufficient for stability or UUB of (1) (this will be demonstrated in Theorem 5 below).

*Remark 4.* It is well known (see [Zubov, 1958, Rosier, 1992]) that, under Assumption 1, the delay-free system admits a Lyapunov function  $V(x)$  with the following properties:

- (i)  $V(x)$  is twice continuously differentiable for  $x \in \mathbb{R}^n$ ;
- (ii)  $V(x)$  is positive definite;
- (iii)  $V(x)$  is  $\mathbf{r}$ -homogeneous of the degree  $\mu > 2r_{\max}$ ;
- (iv) the function  $(\partial V(x)/\partial x)^\top (F(x) + \tau G(x))$  is negative definite.

There is no constructive approach for design such a Lyapunov function  $V(x)$  for a concrete system, but for the analysis below only its existence is used.

In [Aleksandrov et al., 2015, Efimov and Aleksandrov, 2021], an approach to the so-called complete-type LKF's construction for homogeneous systems with discrete delays was developed. In the present contribution, we will extend this approach to homogeneous systems with distributed delays.

*Theorem 5.* Let Assumption 1 be fulfilled,

- (i) if  $\nu > 0$ , then the zero solution of (1) is asymptotically stable;
- (ii) if  $\nu < 0$ , then solutions of (1) are UUB.

Note that the results of this work deal with non-zero homogeneity degree  $\nu$ , which means that the linear systems (that are homogeneous with zero degree  $\nu = 0$  [Efimov and Polyakov, 2021] and mostly studied in the literature [Fridman, 2014]) are not considered.

## 5. STABILITY OF THE PERTURBED SYSTEM WITH DISTRIBUTED DELAY

Next, consider a perturbed system

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^t G(x(s))ds + \int_{t-\tau}^t R(s, x(s))ds, \quad (2)$$

where vector function  $R(t, x)$  is continuous for  $t + \tau \geq 0$  and  $x \in \mathbb{R}^n$ . Here we do not introduce any additional constraint on the form of dependence of the perturbation  $R$  in the time argument  $t$  (this will be done in the next sections).

*Assumption 2.* The inequalities

$$|R_i(t, x)| \leq a|x|_r^{r_i+\lambda}, \quad i = \overline{1, n},$$

are valid for  $t + \tau \geq 0$ ,  $x \in \mathbb{R}^n$ , where  $a > 0$ ,  $\lambda + r_{\min} > 0$  and  $R_i(t, x)$  are components of the vector function  $R(t, x)$ .

This kind of upper estimates for  $R$  can be satisfied if, for example, for each fixed value of  $t$  the vector field  $R$  is  $\mathbf{r}$ -homogeneous of the degree  $\lambda$ .

*Theorem 6.* Let assumptions 1 and 2 be fulfilled, then

- (i) for  $\nu > 0$ ,  $\lambda > \nu$  the zero solution of (2) is asymptotically stable;
- (ii) for  $\nu < 0$ ,  $\lambda < \nu$  solutions of (2) are UUB.

In the next section we will consider systems with non-stationary perturbations of a special form.

## 6. STABILITY AND UUB ANALYSIS VIA AVERAGING

In many cases the time dependence of perturbation has periodic or almost periodic nature, which can be effectively used in the analysis improving the results of Theorem 6.

### 6.1 Application of the Lyapunov–Krasovskii approach

Consider a perturbed system of the form

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^t G(x(s))ds + B\left(\frac{t}{\varepsilon}\right) \int_{t-\tau}^t Q(x(s))ds, \quad (3)$$

where  $\varepsilon$  is a positive parameter, the matrix  $B(t) \in \mathbb{R}^{n \times q}$  is continuous for  $t \in (-\infty, +\infty)$ , vector function  $Q: \mathbb{R}^n \rightarrow \mathbb{R}^q$  is continuous.

*Assumption 3.* Let  $\|B(t)\| \leq M$  for all  $t \in \mathbb{R}$ ,  $M = \text{const} > 0$ .

*Remark 7.* Under Assumption 3, there exists a number  $\delta \in [0, M]$  such that

$$\frac{1}{\varepsilon} \left\| \int_{t-\varepsilon}^t B\left(\frac{\theta}{\varepsilon}\right) d\theta \right\| \leq \delta \quad (4)$$

for  $t \geq 0$ ,  $\varepsilon > 0$ . For instance,  $\delta = 0$  if all elements of  $B(\cdot)$  are zero-mean 1-periodic signals.

*Remark 8.* Under Assumption 3, the estimate

$$\frac{1}{\varepsilon} \left\| \int_{t-\varepsilon}^t (\theta + \varepsilon - t) B\left(\frac{\theta}{\varepsilon}\right) d\theta \right\| \leq \frac{1}{2} \varepsilon M$$

holds for  $t \geq 0$ ,  $\varepsilon > 0$ .

*Assumption 4.* For any fixed  $t \in \mathbb{R}$ ,  $B(t)Q(x)$  is  $\mathbf{r}$ -homogeneous function of  $x$  of the degree  $\sigma > -r_{\min}$ .

Under this hypothesis, Assumption 2 is satisfied for  $R(t, x) = B(t)Q(x)$  with  $\lambda = \sigma$ .

*Theorem 9.* Let assumptions 1, 3, 4 be fulfilled. Then one can choose positive numbers  $\varepsilon_0$  and  $\delta_0$  such that if  $0 < \varepsilon < \varepsilon_0$  and constant  $\delta$  in the estimate (4) satisfies the condition  $0 \leq \delta < \delta_0$ , then

- (i) for  $\nu > 0$ ,  $\sigma \geq \nu$  the zero solution of (3) is asymptotically stable;
- (ii) for  $\nu < 0$ ,  $\sigma \leq \nu$  solutions of (3) are UUB.

*Remark 10.* It is worth noticing that we were not able to prove Theorem 9 on the basis of the LR approach.

### 6.2 Application of the LR approach

Next, consider a perturbed system of the form

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^t G(x(s))ds + \int_{t-\tau}^t B\left(\frac{s}{\varepsilon}\right) Q(x(s))ds, \quad (5)$$

where again  $\varepsilon$  is a positive parameter, the matrix  $B(t)$  is continuous for  $t \in \mathbb{R}$ , and vector function  $Q(x)$  is continuous for  $x \in \mathbb{R}^n$  having similar dimensions.

In comparison with (3), in (5) the time-varying averaging term  $B$  is distributed. If appearance of  $Q$  is originated by a control, then in (3) the time-varying perturbation influences the control channel, while in (5) it is injected in the sensor.

*Theorem 11.* Let assumptions 1, 3, 4 be fulfilled. Then there exist positive numbers  $\varepsilon_0$  and  $\delta_0$  such that if  $0 < \varepsilon < \varepsilon_0$  and constant  $\delta$  in the estimate (4) satisfies the condition  $0 \leq \delta < \delta_0$ , then

(i) for  $\nu > 0$ ,  $\sigma \geq \nu$  the zero solution of (5) is asymptotically stable;

(ii) for  $\nu < 0$ ,  $\sigma \leq \nu$  solutions of (5) are UUB.

*Remark 12.* It is worth noticing that we could not prove Theorem 11 on the basis of the LK approach.

In this section, due to additional hypotheses imposed on the form of time dependence of the perturbations, in theorems 9 and 11, the degree of perturbation  $\sigma$  can be bigger/smaller or equal to the degree of the delay-free system  $\nu$ , while without these restrictions, in Theorem 6, only strict inequalities are allowed between  $\lambda$  (an analogue of  $\sigma$ ) and  $\nu$ .

## 7. EXAMPLE

Assume that a mechanical system is modeled by the equation

$$\ddot{x}(t) + \frac{\partial W(\dot{x}(t))}{\partial \dot{x}} = u(t) + d(t),$$

where  $x(t) \in \mathbb{R}^n$ ,  $W(\dot{x})$  is a continuously differentiable for  $\dot{x} \in \mathbb{R}^n$  positive definite homogeneous of the degree  $\lambda + 1 > 1$  function with respect to the standard dilation (*i.e.*,  $\mathbf{r} = (1, \dots, 1)$ ), which represents the dissipative forces in the system,  $\lambda \in (0, 2)$ ;  $u(t) \in \mathbb{R}^n$  is the control input and  $d(t) \in \mathbb{R}^n$  is the disturbance. Let the control be taken in the form a position functional feedback:

$$u(t) = -\frac{\partial \Pi(x(t))}{\partial x} - \int_{t-\tau}^t \frac{\partial \tilde{\Pi}(x(s))}{\partial x} ds,$$

where  $\Pi(x)$  and  $\tilde{\Pi}(x)$  are continuously differentiable for  $x \in \mathbb{R}^n$  homogeneous of the degree  $\rho + 1 > 1$  functions with respect to the standard dilation, chosen such that the function  $\Pi(x) + \tau \tilde{\Pi}(x)$  is positive definite. Take the parameter  $\rho = \frac{\lambda}{2-\lambda}$ . The term  $\partial \Pi(x(t))/\partial x$  corresponds to the position feedback or contribution of potential forces, whereas the term  $\int_{t-\tau}^t \partial \tilde{\Pi}(x(s))/\partial x ds$  can be interpreted as an integral part of a PID-like regulator [Formal'sky, 1997, Radaideh and Hayajneh, 2002], which can be introduced to reduce oscillations in the steady-state mode. Then the closed-loop system can be rewritten as follows:

$$\ddot{x}(t) + \frac{\partial W(\dot{x}(t))}{\partial \dot{x}} + \frac{\partial \Pi(x(t))}{\partial x} + \int_{t-\tau}^t \frac{\partial \tilde{\Pi}(x(s))}{\partial x} ds = d(t). \quad (6)$$

The system (6) is a vector type Rayleigh equation [Rayleigh, 1945] describing the dynamics of perturbed mechanical systems with dissipative and potential forces.

For this choice of  $\rho$ , the corresponding first-order system

$$\dot{x}(t) = y(t),$$

$$\dot{y}(t) = -\frac{\partial W(y(t))}{\partial \dot{x}} - \frac{\partial \Pi(x(t))}{\partial x} - \int_{t-\tau}^t \frac{\partial \tilde{\Pi}(x(s))}{\partial x} ds$$

is  $\mathbf{r}$ -homogeneous of the degree  $(\rho - 1)/2$  with respect to the dilation  $\mathbf{r} = (r_1, \dots, r_{2n})$ , where  $r_i = 1$  for  $i = \overline{1, n}$ , and  $r_i = (\rho + 1)/2$  for  $i = n + 1, 2n$ . Since the function  $\Pi(x) + \tau \tilde{\Pi}(x)$  is positive definite, it is well-known that the zero solution of the auxiliary system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = -\frac{\partial W(y(t))}{\partial \dot{x}} - \frac{\partial \Pi(x(t))}{\partial x} - \tau \frac{\partial \tilde{\Pi}(x(t))}{\partial x}$$

is asymptotically stable [Rouche et al., 1977].

Applying Theorem 5, we obtain that for  $d(t) \equiv 0$  for all  $t \geq 0$ :

(i) for  $\rho > 1$  the equilibrium position  $x = \dot{x} = 0$  of (6) is asymptotically stable;

(ii) for  $\rho < 1$  solutions of (6) are UUB.

Next, consider the associated perturbed system with  $d(t) = B\left(\frac{t}{\varepsilon}\right) \int_{t-\tau}^t Q(x(s))ds$ :

$$\begin{aligned} \ddot{x}(t) + \frac{\partial W(\dot{x}(t))}{\partial \dot{x}} + \frac{\partial \Pi(x(t))}{\partial x} + \int_{t-\tau}^t \frac{\partial \tilde{\Pi}(x(s))}{\partial x} ds \\ - B\left(\frac{t}{\varepsilon}\right) \int_{t-\tau}^t Q(x(s))ds = 0, \end{aligned} \quad (7)$$

where as before we assume that  $\varepsilon$  is a positive parameter, the matrix  $B(t)$  is continuous and bounded for  $t \in \mathbb{R}$ , vector function  $Q(x)$  is continuous for  $x \in \mathbb{R}^n$  and homogeneous of the order  $\sigma > 0$  with respect to the standard dilation. Such a perturbation  $d$  can be used to reflect time variations in the control gains. Using Theorem 9, we obtain that there exist positive numbers  $\varepsilon_0$  and  $\delta_0$  such that if  $0 < \varepsilon < \varepsilon_0$  and constant  $\delta$  in the estimate (4) satisfies the condition  $0 \leq \delta < \delta_0$ , then

(i) for  $\sigma \geq \rho > 1$  the zero solution of (7) is asymptotically stable;

(ii) for  $\sigma \leq \rho < 1$  solutions of (7) are UUB.

Let us make the simulations for  $n = 1$ ,  $\tau = 1$ ,  $\lambda = 1.5$  and, hence,  $\rho = 3$ , then

$$W(\dot{x}) = \frac{h_1}{2.5} |\dot{x}|^{2.5}, \quad \Pi(x) = \frac{h_2}{4} x^4, \quad \tilde{\Pi}(x) = \frac{h_3}{4} x^4,$$

where  $h_i > 0$ ,  $i = 1, 2, 3$  are constant characterizing the respective forces and control gains. Take  $\sigma = \rho$ :

$$Q(x) = h_4 x^3, \quad h_4 \in \mathbb{R},$$

and select

$$B(t) = \sin\left(\frac{t}{2\pi}\right),$$

as a periodic signal with zero mean (it implies that  $\delta = 0$ ), which satisfy all conditions of Theorem 9. The closed-loop system (7) should possess local asymptotic stability property of the zero solution. The results of simulation (the explicit Euler method was used with the discretization step 0.01) for

$$h_1 = 1, h_2 = 1, h_3 = 2, h_4 = 4, \varepsilon = 0.5$$

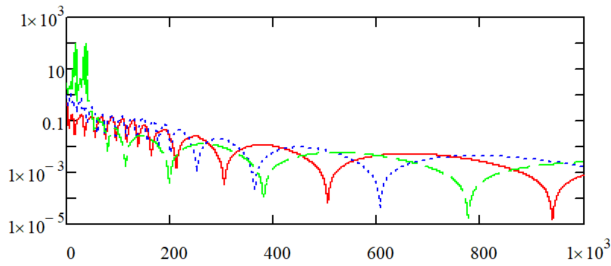


Fig. 1. Behavior of  $\sqrt{|x(t)|^2 + |\dot{x}(t)|^2}$  in logarithmic scale for  $t \in [0, 10^3]$  and three different initial conditions

and initial conditions  $x(s) = \text{const}$  (for three different values of the constant),  $\dot{x}(s) = 0$  for  $s \in [-\tau, 0]$  are shown in Fig. 1 confirming the theoretical findings.

## 8. CONCLUSION

For a nonlinear system with distributed delay and a homogeneous right-hand side, several conditions of robust stability in the presence of time-varying perturbations multiplied by a nonlinear functional gain are derived. In Theorem 5 it is proven that the zero solution is either locally asymptotically stable (for positive homogeneity degree) or practically globally asymptotically stable (for negative degree). Next, Theorem 6 describes generic robustness properties of the system without using the averaging approach (without assuming certain periodicity of the perturbations). Theorems 9, 11 under some restrictions on the time-varying perturbations improve the result of Theorem 6 using the averaging techniques (the homogeneity degree of the perturbation can be equal to one of the system itself). To show efficiency of the approach, a model of mechanical system described by a vector Rayleigh equation is considered.

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