



HAL
open science

On hyperexponential stabilization of double integrator in continuous and discrete time

Denis Efimov, Andrey Polyakov, Konstantin Zimenko, Jian Wang

► **To cite this version:**

Denis Efimov, Andrey Polyakov, Konstantin Zimenko, Jian Wang. On hyperexponential stabilization of double integrator in continuous and discrete time. IFAC 2023 - 22nd IFAC World Congress, Jul 2023, Yokohama, Japan. hal-04176126

HAL Id: hal-04176126

<https://inria.hal.science/hal-04176126v1>

Submitted on 2 Aug 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

On hyperexponential stabilization of double integrator in continuous and discrete time

Denis Efimov* Andrey Polyakov* Konstantin Zimenko**
Jian Wang***

* *Inria, Univ. Lille, CNRS, UMR 9189 - CRISTAL, F-59000 Lille, France*

** *ITMO University, 49 av. Kronverkskiy, 197101 Saint Petersburg, Russia*

*** *Hangzhou Dianzi University, Hangzhou, China*

Abstract: A time-varying state feedback is presented that guarantees stability with hyperexponential rate of convergence of all trajectories to the origin for a double integrator system. It is shown that this convergence property is uniform with respect to matched bounded external disturbances. An implicit time-discretization scheme of the closed-loop system is given, which preserves all main properties of the continuous-time counterpart, and in addition has bounded errors with respect to the measurement noises. Based on this discretization, for sampled-and-hold implementation, a modified linear time-varying state feedback is proposed providing an accelerated rate of convergence to the continuous-time plant. The efficiency of the suggested control is illustrated through numeric experiments.

1. INTRODUCTION

The problem of a state feedback design that ensures global asymptotic stability for a continuous-time linear or nonlinear dynamical system constitutes the base of the control engineering Utkin [1992], Isidori [1995], Khalil [2002], Chernous'ko et al. [2008]. The main important features of a stabilizing feedback include, 1) the rate of convergence to the origin of the trajectories, which can be, for example, exponential or finite-time Bhat and Bernstein [2000]; 2) robustness with respect to the exogenous disturbances, which is frequently understood in the input-to-state stability sense Dashkovskiy et al. [2011]; 3) optimization of the measurement noise sensitivity; and 4) a reliable discrete-time implementation that preserves the main characteristics of the closed-loop system in the continuous-time (this can be a complex issue for highly nonlinear stabilizers Efimov et al. [2017], Polyakov et al. [2019], see also chattering in sliding mode control systems Shtessel et al. [2014]).

In many applications there is a requirement on accelerated convergence of the trajectories to the equilibrium Efimov and Polyakov [2021], and due to mentioned complexity of discrete-time implementation of a nonlinear feedback, recently, the prescribed-time stability framework (a special case of fixed-time convergence Polyakov [2012]) gained popularity Song et al. [2017, 2018], Holloway and Krstic [2019], since it is based on utilization for control or estimation of linear dynamics with time-varying gains growing to infinity in a fixed time. This approach has an appealing advantage providing a uniform convergence

with respect to matched perturbations (similarly to sliding mode controls), but the price to pay is sensitivity to the measurement noises. There are also other kinds of accelerated convergence, among them we would like to mention the *hyperexponential* one Polyakov et al. [2015], Nekhoroshikh et al. [2022], which implies that the equilibrium can be reached asymptotically with a speed faster than in any exponentially converging dynamics.

In this note we are going to propose a linear time-varying feedback for a double integrator with convergence faster than any exponential, uniformly with respect to matched disturbances in the noise-free scenario (no knowledge of the perturbation's upper bound is required for the tuning), and admitting a simple digital implementation. The sensitivity with respect to measurement noises is evaluated. The gain selection guidelines are formulated using linear matrix inequalities, and it is demonstrated in simulations that the new control is performative even in the case of a slow sampling.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where \mathbb{R} is the set of real numbers, \mathbb{Z} is the set of integer numbers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.
- $|\cdot|$ denotes the absolute value in \mathbb{R} , $\|\cdot\|$ is used for the Euclidean norm on \mathbb{R}^n .
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $[t_0, t_1) \subset \mathbb{R}_+$ define the norm $\|d\|_\infty = \text{ess sup}_{t \geq 0} \|d(t)\|$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as \mathcal{L}_∞^m .
- For a sequence $d_k \in \mathbb{R}^m$, $k \in \mathbb{Z}_+$ define the norm $\|d\|_\infty = \sup_{k \in \mathbb{Z}_+} \|d_k\|$, and the set of sequences with $\|d\|_\infty < +\infty$ we further denote as \mathcal{L}_∞^m .
- A continuous function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K} if $\alpha(0) = 0$ and it is strictly increasing. The

* This work was partially supported by 111 project No. D17019 (China), by the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898, and by ECOS NORD Project M20M02. Section 5 is obtained with the aid of RSF under grant 21-71-10032 in ITMO University.

function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K}_∞ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{KL} if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s \in \mathbb{R}_+$.

- Denote the identity matrix of dimension $n \times n$ by I_n .
- The relation $P \prec 0$ ($P \succ 0$) means that a symmetric matrix $P \in \mathbb{R}^{n \times n}$ is negative (positive) definite, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimal and the maximal eigenvalues of such a matrix P .
- $\exp(1) = e$.

2. PRELIMINARIES

2.1 Uniform hyperexponential stability Efimov et al. [2022]

Consider a non-autonomous differential equation:

$$\dot{x}(t)/dt = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}_+, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the vector of external disturbances and $d \in \mathcal{L}_\infty^m$; $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuous function with respect to x , d and piecewise continuous with respect to t , $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}_+$. A solution of the system (1) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}_+$ and some $d \in \mathcal{L}_\infty^m$ is denoted as $X(t, t_0, x_0, d)$.

Definition 1. For a given set $\mathbb{D} \subset \mathcal{L}_\infty^m$, the origin is called *uniformly hyperexponentially stable* for the system (1) if it is uniformly globally asymptotically stable, and for any $\alpha > 0$ there exists $\rho_\alpha \in \mathcal{K}$ and $\beta_\alpha \in \mathcal{KL}$ such that for all $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$ and $d \in \mathbb{D}$, for all $t \geq t_0$:

$$\|X(t, t_0, x_0, d)\| \leq e^{-\alpha(t-t_0)} \rho_\alpha(\|x_0\|) + \beta_\alpha(\|d\|_\infty, t - t_0).$$

2.2 Auxiliary properties

We will use the following estimates:

Lemma 2. For all $\tau \geq 0$ it holds:

$$\begin{aligned} e^{-\tau} \int_0^\tau \frac{e^s}{2s+1} ds &\leq \frac{1}{\tau+1}, \\ e^{-\tau} \int_0^\tau \frac{e^s}{(2s+1)^2} ds &\leq \frac{1}{(\tau+1)^2}. \end{aligned} \quad (2)$$

3. PROBLEM STATEMENT

Consider a double integrator system with matched disturbances and noisy state measurements:

$$\dot{x}(t) = Ax(t) + b(u(t) + d(t)), \quad y(t) = x(t) + v(t), \quad (3)$$

where $x(t) \in \mathbb{R}^2$ is the state, $u(t) \in \mathbb{R}$ is the control to be synthesized, $d \in \mathcal{L}_\infty^1$ and $v \in \mathcal{L}_\infty^2$ are the exogenous disturbance and the measurement noise, respectively, $y(t) \in \mathbb{R}^2$ is the signal available for feedback realization,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is required to design a state feedback $u(t) = u(y(t), t)$ for this system, which stabilizes $x(t)$ at the origin with a hyperexponential rate of convergence uniformly in $d \in \mathbb{D} = \mathcal{L}_\infty^1$ while $\|v\|_\infty = 0$, and having a bounded regulation error for $v \in \mathcal{L}_\infty^2$.

4. CONTROL IN CONTINUOUS TIME

Let the required control for (3) be chosen in a linear time-varying form:

$$u(t) = K\Gamma(t)y(t), \quad \Gamma(t) = \text{diag}\{\varrho^2(t) \quad \varrho(t)\}^\top, \quad (4)$$

where $K = [K_1 \ K_2] \in \mathbb{R}^{1 \times 2}$ is the controller gain that will be selected later, $\varrho(t) = 1 + t$ is a strictly growing function of time.

Remark 3. Any other strictly growing integrable function of time $\varrho(t)$ can be used in (4), e.g., $\varrho(t) = ae^{at}$ or $\varrho(t) = (b + at)^\alpha$ for $a > 0$, $b > 0$ and $\alpha > 0$. The current choice is motivated by brevity of exposition.

After substitution of (4) in the right-hand side of the system (3), the closed-loop dynamics can be written as follows:

$$\dot{x}(t) = (A + bK\Gamma(t))x(t) + b(d(t) + K\Gamma(t)v(t)). \quad (5)$$

Theorem 4. Let there exist $P = P^\top \in \mathbb{R}^{2 \times 2}$, $U \in \mathbb{R}^{1 \times 2}$, $\gamma_1 > 0$ and $\gamma_2 > 0$ such that the linear matrix inequalities are verified:

$$\begin{aligned} P \succ 0, \quad Q \preceq 0, \\ Q = \begin{bmatrix} Q_{11} & bU & b & -P^{-1}c^\top c \\ U^\top b^\top & -\gamma_1 P^{-1} & 0 & 0 \\ b^\top & 0 & -\gamma_2 & 0 \\ -c^\top c P^{-1} & 0 & 0 & -2P^{-1} \end{bmatrix}, \\ Q_{11} = P^{-1}A^\top + AP^{-1} + U^\top b^\top + bU + P^{-1}, \end{aligned}$$

$c = [1 \ 0]$. Then for $K = UP$ and any $x(0) \in \mathbb{R}^2$, $d \in \mathcal{L}_\infty^1$ and $v \in \mathcal{L}_\infty^2$ in (5):

$$\begin{aligned} |x_1(t)| &\leq \mathcal{Q}(t, \|x(0)\|, \|v_1\|_\infty, \|v_2\|_\infty, \|d\|_\infty), \\ |x_2(t)| &\leq \varrho(t)\mathcal{Q}(t, \|x(0)\|, \|v_1\|_\infty, \|v_2\|_\infty, \|d\|_\infty), \end{aligned}$$

$$\begin{aligned} \mathcal{Q}(t, X, V_1, V_2, D) &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{t(t+2)}{4}} X \\ &+ \sqrt{\gamma_1} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \left(V_1 + \frac{\sqrt{2}V_2}{\sqrt{t(t+2)+2}} \right) \\ &+ \sqrt{\frac{2\gamma_2}{\lambda_{\min}(P)}} \frac{D}{t(t+2)+2} \end{aligned}$$

for all $t \geq 0$.

For $\|v\|_\infty = 0$ the result of the theorem implies uniform hyperexponential stability of the origin in (3), (4) with $d \in \mathcal{L}_\infty^1$. The class of disturbances $d \in \mathcal{L}_\infty^1$, for which the uniformity of the stability is kept, can be enlarged by ones growing *not faster than a linear* in time.

Note also that the given linear matrix inequalities are always feasible under the conditions of the theorem since the pair (A, b) is controllable.

Proof. Define new auxiliary variable $z(t) = \Lambda(t)x(t)$, where

$$\Lambda(t) = \text{diag}\{[1 \quad \varrho^{-1}(t)]^\top\},$$

whose dynamics takes the form:

$$\begin{aligned} \dot{z}(t) &= -\text{diag}\{[0 \quad \varrho^{-1}(t)]^\top\}z(t) + \Lambda(t)\dot{x}(t) \\ &= \varrho(t) \left((A_K - \Delta(t))z(t) + \varrho^{-2}(t)b(d(t) + K\Gamma(t)v(t)) \right), \\ A_K &= A + bK, \quad \Delta(t) = \text{diag}\{[0 \quad \varrho^{-2}(t)]^\top\}. \end{aligned}$$

Note that $z(0) = x(0)$. Finally, let us define a new time variable

$$\tau = \varphi(t) = \frac{t(t+2)}{2}, \quad t = \varphi(\tau)^{-1} = \sqrt{2\tau+1} - 1, \\ d\tau = (t+1)dt,$$

where $\varphi(\tau)^{-1}$ denotes the inverse of φ , then after the change of the time $t \rightarrow \tau$, the auxiliary variable $z(\tau)$ dynamics is reduced to an equivalent useful representation:

$$\frac{dz(\tau)}{d\tau} = \left(A_K - \tilde{\Delta}(\tau) \right) z(\tau) \quad (6) \\ + \frac{b}{2\tau+1} \left(d(\tau) + K\tilde{\Gamma}(\tau)v(\tau) \right),$$

where $d(\tau) = d(\varphi(\tau)^{-1})$ and $v(\tau) = v(\varphi(\tau)^{-1})$ are the external signals in the new time,

$$\tilde{\Delta}(\tau) = \Delta(\varphi(\tau)^{-1}) = \text{diag}\{[0 \quad (2\tau+1)^{-1}]^\top\},$$

$$\tilde{\Gamma}(\tau) = \Gamma(\varphi(\tau)^{-1}) = \text{diag}\{[2\tau+1 \quad \sqrt{2\tau+1}]^\top\}.$$

For analysis of stability properties in (6) let us choose a candidate Lyapunov function $V(z) = z^\top P z$ (that is positive definite since $P \succ 0$ due to the conditions of the theorem), whose derivative in the new time τ for the dynamics (6) can be written as follows:

$$\frac{dV(z(\tau))}{d\tau} = z(\tau)^\top \left(A_K^\top P + P A_K \right) z(\tau) \\ + 2z(\tau)^\top P \left(\frac{b}{2\tau+1} \left(d(\tau) + K\tilde{\Gamma}(\tau)v(\tau) \right) - \tilde{\Delta}(\tau)z(\tau) \right).$$

Due to the form of $\tilde{\Delta}(\tau)$ we have:

$$P\tilde{\Delta}(\tau) + \tilde{\Delta}(\tau)P + \frac{1}{2} \frac{c^\top c P c^\top c}{2\tau+1} \succ 0$$

for all $\tau \geq 0$ and $c = [1 \quad 0]$. Therefore,

$$\frac{dV(z(\tau))}{d\tau} \leq \begin{bmatrix} z(\tau) \\ \frac{\tilde{\Gamma}(\tau)}{2\tau+1}v(\tau) \\ \frac{d(\tau)}{2\tau+1} \end{bmatrix}^\top \tilde{Q} \begin{bmatrix} z(\tau) \\ \frac{\tilde{\Gamma}(\tau)}{2\tau+1}v(\tau) \\ \frac{d(\tau)}{2\tau+1} \end{bmatrix} - V(z(\tau)) \\ + \gamma_1 v^\top(\tau) \frac{\tilde{\Gamma}(\tau)}{2\tau+1} P \frac{\tilde{\Gamma}(\tau)}{2\tau+1} v(\tau) + \gamma_2 \left(\frac{d(\tau)}{2\tau+1} \right)^2, \\ \tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & PbK & Pb \\ K^\top b^\top P & -\gamma_1 P & 0 \\ -b^\top P & 0 & -\gamma_2 \end{bmatrix}, \\ \tilde{Q}_{11} = A_K^\top P + P A_K + \frac{c^\top c P c^\top c}{2} + P,$$

where γ_1, γ_2 are positive parameters according to formulation of the theorem. Next, let us apply a conventional transformation in order to separate the control gains:

$$T\tilde{Q}T = \hat{Q} + \frac{1}{2} \begin{bmatrix} P^{-1}c^\top c \\ 0 \\ 0 \end{bmatrix} P \begin{bmatrix} P^{-1}c^\top c \\ 0 \\ 0 \end{bmatrix}^\top,$$

where

$$T = \text{diag}\{P^{-1}, P^{-1}, 1\}, \\ \hat{Q} = \begin{bmatrix} \tilde{Q}_{11} & bU & b \\ U^\top b^\top & -\gamma_1 P^{-1} & 0 \\ -b^\top & 0 & -\gamma_2 \end{bmatrix}.$$

It is easy to see that the latter expression is the Schur complement of \hat{Q} , then the requirement of the theorem

$Q \preceq 0$ implies that $\tilde{Q} \preceq 0$ and passing to the time domain we get an estimate:

$$V(z(\tau)) \leq e^{-\tau} V(z(0)) + \gamma_2 e^{-\tau} \int_0^\tau e^s \left(\frac{\|d\|_\infty}{2s+1} \right)^2 ds \\ + \gamma_1 \lambda_{\max}(P) e^{-\tau} \int_0^\tau e^s \left(\|v_1\|_\infty^2 + \frac{\|v_2\|_\infty^2}{2s+1} \right) ds$$

for all $\tau \geq 0$. Using (2) we obtain:

$$V(z(\tau)) \leq e^{-\tau} V(z(0)) + \gamma_2 \left(\frac{\|d\|_\infty}{\tau+1} \right)^2 \\ + \gamma_1 \lambda_{\max}(P) \left(\|v_1\|_\infty^2 + \frac{\|v_2\|_\infty^2}{\tau+1} \right), \quad \tau \geq 0,$$

and in the original time (using $\tau = \varphi(t)$)

$$\|z(t)\| \leq \mathcal{Q}(t, \|z(0)\|, \|v_1\|_\infty, \|v_2\|_\infty, \|d\|_\infty)$$

for all $t \geq 0$. Since $z(t) = \Gamma(t)x(t)$, the desired estimate on the behavior of $x_1(t)$ and $x_2(t)$ can be derived for all $t \geq 0$ from this inequality.

5. DISCRETIZATION OF THE CONTROLLED SYSTEM

Two cases are investigated in this section. First, a consistent method of discretization of (5) is presented. Second, the sample-and-hold implementation of the control (4) for (3) is considered based on the previously derived discretization.

5.1 Implicit Euler discretization of (5)

Note that (5) is modeled by a linear time-varying system with external inputs $d(t)$ and $v(t)$. Since the time-varying control gain $K\Gamma(t)$ is strictly growing, the explicit Euler discretization cannot be used for all $t \geq 0$, however, the implicit one can be effectively applied Butcher [2008]. Let $h > 0$ be constant discretization step, denote by $t_k = hk$ for $k \in \mathbb{Z}_+$ the discretization time instants, then application of the implicit Euler discretization method to (4) gives for $k \in \mathbb{Z}_+$:

$$\xi_{k+1} = F^{-1}(t_{k+1}) (\xi_k + L(t_{k+1})v_{k+1} + hbd_{k+1}), \quad (7) \\ F(t) = I_2 - h(A + bK\Gamma(t)), \quad L(t) = hbK\Gamma(t),$$

where $\xi_k \in \mathbb{R}^2$ is an approximation of $x_k = x(t_k)$ (*i.e.*, $\xi_k \rightarrow x_k$ as $h \rightarrow 0$), $v_k = v(t_k)$ and $d_k = d(t_k)$. In this work we will assume that v_k and d_k take finite values with bounded norms as before, *i.e.*, $\|v\|_\infty = \sup_{k \in \mathbb{Z}_+} \|v_k\|$ and $\|d\|_\infty = \sup_{k \in \mathbb{Z}_+} |d_k|$ are well defined.

As we can conclude from these expressions, the discrete state transition matrix $F^{-1}(t)$ is nonsingular and element-wise bounded for all $t \geq 0$:

$$F^{-1}(t) = \frac{1}{\det\{F(t)\}} \begin{bmatrix} 1 - hK_2\varrho(t) & h \\ hK_1\varrho^2(t) & 1 \end{bmatrix},$$

$$\det\{F(t)\} = 1 - hK_2\varrho(t) - h^2K_1\varrho^2(t),$$

where $K_1 < 0$ and $K_2 < 0$ for a stabilizing control gain K . Moreover,

$$\lim_{t \rightarrow +\infty} F^{-1}(t) = \begin{bmatrix} 0 & 0 \\ -h^{-1} & 0 \end{bmatrix},$$

which implies that asymptotically the own dynamics of (7) approaches a static linear system with a nilpotent matrix of index 2. The measurement noise v gain matrix

$$F^{-1}(t)L(t) = \frac{h\varrho(t)}{\det\{F(t)\}} \begin{bmatrix} h \\ 1 \end{bmatrix} \begin{bmatrix} K_1\varrho(t) \\ K_2 \end{bmatrix}^\top$$

is also elementwise bounded with

$$\lim_{t \rightarrow +\infty} F^{-1}(t)L(t) = \begin{bmatrix} -1 & 0 \\ -h^{-1} & 0 \end{bmatrix},$$

which evaluates the asymptotic noise sensitivity. And, obviously, the effect of the disturbance d is still asymptotically annihilated by the control with

$$hF^{-1}(t)b = \frac{h}{\det\{F(t)\}} \begin{bmatrix} h \\ 1 \end{bmatrix}.$$

It remains to demonstrate that the hyperexponential rate of convergence is also preserved in (7).

Theorem 5. Let for given $\gamma > 0$, $\psi > 0$ and $\sigma > 0$ there exist $P = P^\top \in \mathbb{R}^{2 \times 2}$, $K \in \mathbb{R}^{1 \times 2}$ such that the matrix inequalities are verified:

$$\begin{aligned} P &> 0, (A + bK)P^{-1} + P^{-1}(A + bK)^\top \leq 0, \\ P^{-1} + \left(\frac{\psi^2 h^2}{(h^3 + \psi)^2} - 1 \right) bb^\top P^{-1} bb^\top &\geq 0, \\ (A + bK)P^{-1}(A + bK)^\top - \psi^{-1}P^{-1} \\ &- \gamma^{-1}bKK^\top b^\top - \sigma^{-1}bb^\top \geq 0. \end{aligned}$$

Then for any $\xi_0 \in \mathbb{R}^2$, $v \in L_\infty^2$ and $d \in L_\infty^1$ in (7):

$$\begin{aligned} \|\xi_k\|^2 &\leq \left(\frac{\psi}{h^2} \right)^k \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|\xi_0\|^2 \prod_{i=0}^{k-2} \varrho^{-2}(t_{i+1}) \\ &+ \frac{\varrho^2(t_k)}{\lambda_{\min}(P)} \sum_{i=0}^{k-1} \left(\frac{\psi}{h^2} \right)^{k-1-i} [\gamma(v_{1,i+1}^2 + \varrho^{-2}(t_{i+1})v_{2,i+1}^2) \\ &+ \sigma\varrho^{-4}(t_{i+1})d_{i+1}^2] \prod_{j=i+1}^{k-1} \varrho^{-2}(t_{j+1}) \end{aligned}$$

for all $k \geq 1$.

Note that the first two matrix inequalities of Theorem 5 are verified for P and K found in Theorem 4, the last matrix inequality is always satisfied in this case for some $\gamma > 0$, $\psi > 0$ and $\sigma > 0$. Then the only additional condition to check is the third inequality.

Proof. To investigate stability and the rate of convergence in (7), let us define a time-varying Lyapunov function candidate (the same was used before):

$$V_k = \xi_k^\top \Pi_k \xi_k, \quad \Pi_k = \Lambda(t_k)P\Lambda(t_k),$$

where $P = P^\top > 0$ has been introduced in the conditions of the theorem (it may be also chosen as in Theorem 4). For $\gamma > 0$, $\psi > 0$ and $\sigma > 0$, defining time-varying parameters $\psi_k = h^{-2}\psi\varrho^{-2}(t_{k+1})$ and $\sigma_k = \sigma\varrho^{-4}(t_{k+1})$ we obtain for all $k \in \mathbb{Z}_+$:

$$\begin{aligned} V_{k+1} - \psi_k V_k &= \begin{bmatrix} \xi_k \\ v_{k+1} \\ d_{k+1} \end{bmatrix}^\top Q_k \begin{bmatrix} \xi_k \\ v_{k+1} \\ d_{k+1} \end{bmatrix} \\ &+ \gamma v_{k+1}^\top \Lambda^2(t_{k+1})v_{k+1} + \sigma_k d_{k+1}^2 \end{aligned}$$

for

$$\begin{aligned} Q_k &= \begin{bmatrix} I_2 \\ L(t_{k+1}) \\ hb \end{bmatrix} (F^{-1}(t_{k+1}))^\top \Pi_{k+1} F^{-1}(t_{k+1}) \\ &\times \begin{bmatrix} I_2 \\ L(t_{k+1}) \\ hb \end{bmatrix}^\top - \begin{bmatrix} \psi_k \Pi_k & 0 & 0 \\ 0 & \gamma \Lambda^2(t_{k+1}) & 0 \\ 0 & 0 & \sigma_k \end{bmatrix}. \end{aligned}$$

We need to find the restrictions on P and the gains ψ_k , γ , σ_k such that $Q_k \leq 0$ for all $k \geq 0$. To this end, note that $-Q_k$ is the Schur complement of

$$\tilde{Q}_k = \begin{bmatrix} \psi_k \Pi_k & 0 & 0 & I_2 \\ 0 & \gamma \Lambda^2(t_{k+1}) & 0 & L^\top(t_{k+1}) \\ 0 & 0 & \sigma_k & hb^\top \\ I_2 & L(t_{k+1}) & hb & F(t_{k+1})\Pi_{k+1}^{-1}F^\top(t_{k+1}) \end{bmatrix},$$

then we seek for conditions guaranteeing $\tilde{Q}_k \succeq 0$, which calculating another Schur complement of \tilde{Q}_k are equivalent to

$$\begin{aligned} \hat{Q}_k &= \begin{bmatrix} \sigma_k & hb^\top \\ hb & F(t_{k+1})\Pi_{k+1}^{-1}F^\top(t_{k+1}) \end{bmatrix} \\ &- \begin{bmatrix} 0 & 0 \\ I_2 & L(t_{k+1}) \end{bmatrix} \begin{bmatrix} \psi_k^{-1}\Pi_k^{-1} & 0 \\ 0 & \gamma^{-1}\Lambda^{-2}(t_{k+1}) \end{bmatrix} \\ &\times \begin{bmatrix} 0 & I \\ 0 & L^\top(t_{k+1}) \end{bmatrix} = \begin{bmatrix} \sigma_k & hb^\top \\ hb & Z_k \end{bmatrix} \succeq 0, \\ Z_k &= F(t_{k+1})\Pi_{k+1}^{-1}F^\top(t_{k+1}) - \psi_k^{-1}\Pi_k^{-1} \\ &- \gamma^{-1}L(t_{k+1})\Lambda^{-2}(t_{k+1})L^\top(t_{k+1}). \end{aligned}$$

Finally, computing the Schur complement of \hat{Q}_k we get the condition to check:

$$Z_k - h^2\sigma_k^{-1}bb^\top \succeq 0.$$

Multiplying it from both sides on $\Lambda(t_{k+1})$ lead to the following matrix inequality:

$$\begin{aligned} 0 &\leq \Lambda(t_{k+1})(Z_k - h^2\sigma_k^{-1}bb^\top)\Lambda(t_{k+1}) \\ &= P^{-1} - \psi_k^{-1}\Lambda(t_{k+1})\Lambda^{-1}(t_k)P^{-1}\Lambda^{-1}(t_k)\Lambda(t_{k+1}) \\ &+ h^2\varrho^2(t_{k+1})(A + bK)P^{-1}(A + bK)^\top \\ &- h\varrho(t_{k+1})\left((A + bK)P^{-1} + P^{-1}(A + bK)^\top\right) \\ &- \gamma^{-1}h^2\varrho^2(t_{k+1})bKK^\top b^\top - h^2\sigma_k^{-1}\varrho^{-2}(t_{k+1})bb^\top. \end{aligned}$$

Recalling the definition of ψ_k , direct computations show that

$$\begin{aligned} P^{-1} - \psi_k^{-1}\Lambda(t_{k+1})\Lambda^{-1}(t_k)P^{-1}\Lambda^{-1}(t_k)\Lambda(t_{k+1}) \\ \succeq -\psi^{-1}h^2\varrho^2(t_{k+1})P^{-1} \end{aligned}$$

provided that

$$P^{-1} + \left(\frac{\psi^2 h^2}{(h^3 + \psi)^2} - 1 \right) bb^\top P^{-1} bb^\top \succeq 0.$$

Therefore, the property $Q_k \leq 0$ for all $k \geq 0$ follows from

$$\begin{aligned} h^2\varrho^2(t_{k+1})[(A + bK)P^{-1}(A + bK)^\top - \psi^{-1}P^{-1} \\ - h^{-1}\varrho^{-1}(t_{k+1})\left((A + bK)P^{-1} + P^{-1}(A + bK)^\top\right) \\ - \gamma^{-1}bKK^\top b^\top - \sigma^{-1}bb^\top] \succeq 0. \end{aligned}$$

If $(A + bK)P^{-1} + P^{-1}(A + bK)^\top \leq 0$, then the latter inequality can be simplified:

$$\begin{aligned} (A + bK)P^{-1}(A + bK)^\top - \psi^{-1}P^{-1} \\ - \gamma^{-1}bKK^\top b^\top - \sigma^{-1}bb^\top \succeq 0. \end{aligned}$$

Since all required matrix inequalities have been introduced in the formulation of the theorem, it has been shown above that

$$\begin{aligned} V_{k+1} &\leq \psi_k V_k + \gamma v_{k+1}^\top \Lambda^2(t_{k+1}) v_{k+1} + \sigma_k d_{k+1}^2 \\ &= \frac{\psi}{h^2} \varrho^{-2}(t_{k+1}) V_k + \gamma (v_{1,k+1}^2 + \varrho^{-2}(t_{k+1}) v_{2,k+1}^2) \\ &\quad + \sigma \varrho^{-4}(t_{k+1}) d_{k+1}^2 \end{aligned} \quad (8)$$

for all $k \in \mathbb{Z}_+$, which gives the required estimate.

As we can see from the obtained results, the convergence in the initial error is faster than any exponential and there is no asymptotic dependence in d , while the noise v gain admits a static linear upper bound.

5.2 Sample-and-hold implementation of (4)

Assume that in (3) the control $u(t) = u(t_k) = u_k$ stays constant on the interval $[t_k, t_{k+1})$ for all $k \in \mathbb{Z}_+$, where as before $t_k = kh$ with $h > 0$ being the sampling period. Then using the formula for solutions of a linear system we obtain for all $k \in \mathbb{Z}_+$:

$$x_{k+1} = A_h x_k + b_h u_k + \delta_k, \quad (9)$$

where

$$\begin{aligned} A_h &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, \quad b_h = h \begin{bmatrix} \frac{h}{2} \\ 1 \end{bmatrix}, \\ \delta_k &= \int_0^h \begin{bmatrix} h-s \\ 1 \end{bmatrix} d(t_k + s) ds, \end{aligned}$$

and $x_k = x(t_k)$ (it is not an approximation of solution in (3), but its exact value).

In order to find the values of u_k we look for a solution of the following minimization problem:

$$u_k(x_k) = \underset{u_k \in \mathbb{R}, \|v\|_\infty=0, \|d\|_\infty=0, \xi_k=x_k}{\arg \min} |x_{k+1} - \xi_{k+1}|,$$

i.e., in the perturbation-free case the control u_k has to ensure closeness of x_{k+1} to ξ_{k+1} from (7) having the same initial conditions with $\xi_k = x_k$. Hence, we look for a solution of the following equation:

$$A_h x_k + b_h u_k = F^{-1}(t_{k+1}) x_k,$$

which using the pseudo inverse of b_h takes the form:

$$\begin{aligned} u_k &= (b_h^\top b_h)^{-1} b_h^\top (F^{-1}(t_{k+1}) - A_h) x_k \\ &= \alpha_h \Upsilon(t_{k+1}) x_k, \end{aligned} \quad (10)$$

where $\alpha_h = \frac{2h^2+2}{h^2+4}$ (a constant parameter approaching 1 with $h \rightarrow 0$) and

$$\Upsilon(t) = \frac{\varrho^2(t)}{\det\{F(t)\}} [K_1 h K_1 + \varrho^{-1}(t) K_2].$$

It is easy to see that $\Upsilon(t)$ is bounded for all $t \geq 0$, and

$$\lim_{t \rightarrow +\infty} \Upsilon(t) = -h^{-1} [h^{-1} \ 1].$$

The closed-loop system (9), (10) takes the form:

$$x_{k+1} = (A_h + b_h \alpha_h \Upsilon(t_{k+1})) x_k + \alpha_h \Upsilon(t_{k+1}) v_k + \delta_k, \quad (11)$$

and considering the reconstruction error $\epsilon_k = x_k - \xi_k$ we get:

$$\begin{aligned} \epsilon_{k+1} &= F^{-1}(t_{k+1}) (\epsilon_k - L(t_{k+1}) v_{k+1} - h b d_{k+1}) \\ &\quad + \delta_k + \alpha_h \Upsilon(t_{k+1}) v_k + R(t_{k+1}) (\epsilon_k + \xi_k), \end{aligned}$$

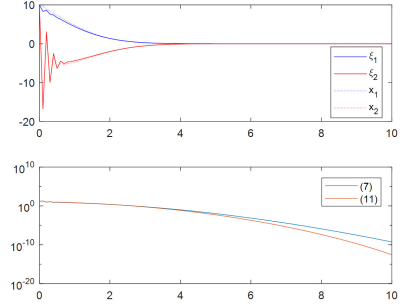


Fig. 1. The results of simulation in the noise-free case where

$$R(t) = \frac{h^2}{h^2 + 4} \frac{\varrho(t)}{\det\{F(t)\}} \begin{bmatrix} -2 \\ h \end{bmatrix} \begin{bmatrix} K_1 \varrho(t) \\ K_2 \end{bmatrix}^\top A_h$$

is a bounded matrix for all $t \geq 0$. It is easy to see that $R(t)$ has the same shape as $F^{-1}(t)L(t)$, then the influence of the error $R(t_{k+1})\epsilon_k$ is of the same order as for the noise $F^{-1}(t_{k+1})L(t_{k+1})v_{k+1}$, which has a linear time-varying asymptotic gain established previously. Hence, if γ is sufficiently small (it depends on the choice of K), then (11) recovers the properties of (7). Let us check this conjecture in simulation.

6. SIMULATIONS

Let us illustrate performance of the proposed hyperexponential stabilizer (4) in numeric experiments. For

$$K = [\ 10 \ 20]$$

there exist solutions of linear matrix inequalities from theorems 4 and 5. For $h = 0.1$, the results of simulations for $v_k = 0$ and $d_k = 0$ for all $k \in \mathbb{Z}_+$ are presented in Fig. 1. On the plots, the state trajectories and the state norm (in logarithmic scale) are shown versus the time for (7) and (11). Due to the shape of decay for the norm, we can conclude that in both cases the convergence to the origin is hyperexponential (it would be a straight line for the case of an exponential convergence of the state). To demonstrate robustness, take the perturbation (as it follows from Theorem 4, disturbance cannot grow faster than $\varrho(t)$):

$$d(t) = (\sqrt{t} + \sin(5t) + \text{rnd}(1)) / 300$$

and the noise

$$v(t) = 0.01 \begin{bmatrix} 1 - \text{rnd}(1) + \sin(15t) \\ \text{rnd}(1) + \cos(20t) - 1 \end{bmatrix},$$

where $\text{rnd}(1)$ denotes a uniformly distributed in the interval $[0, 1]$ random number, then the respective simulation results are given in Fig. 2. As we can conclude, both systems, (7) and (9), (10), demonstrate a similar behavior.

7. CONCLUSION

A new control law is proposed, which has a simple discrete-time implementation, guaranteeing a hyperexponential rate of convergence of the state trajectories to the origin uniformly in the external matched inputs in the noise-free case. It has also certain robustness with respect to

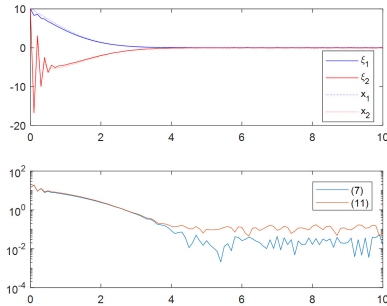


Fig. 2. The results of simulation with perturbations

the measurement noise. The tuning rules are formulated using feasible linear matrix inequalities. The results of simulation demonstrate a good performance of the proposed control, especially in the case of slow sampling. Extension to a higher order chain of integrators can be considered as a direction of future research.

REFERENCES

- Bhat, S. and Bernstein, D. (2000). Finite time stability of continuous autonomous systems. *SIAM J. Control Optim.*, 38(3), 751–766.
- Butcher, J.C. (2008). *Numerical Methods for Ordinary Differential Equations*. John Wiley & Sons, New York, 2nd edition.
- Chernous'ko, F.L., Ananievski, I.M., and Reshmin, S.A. (2008). *Control of Nonlinear Dynamical Systems*. Springer.
- Dashkovskiy, S., Efimov, D., and Sontag, E. (2011). Input to state stability and allied system properties. *Automation and Remote Control*, 72(8), 1579–1614.
- Efimov, D., Polyakov, A., Levant, A., and Perruquetti, W. (2017). Realization and discretization of asymptotically stable homogeneous systems. *IEEE Transactions on Automatic Control*, 62(11), 5962–5969.
- Efimov, D., Polyakov, A., Zimenko, K., and Wang, J. (2022). An exact robust hyperexponential differentiator. In *Proc. 61th IEEE CDC*. Cancun.
- Efimov, D. and Polyakov, A. (2021). Finite-time stability tools for control and estimation. *Foundations and Trends in Systems and Control*, 9(2-3), 171–364. doi:10.1561/26000000026.
- Holloway, J. and Krstic, M. (2019). Prescribed-time observers for linear systems in observer canonical form. *IEEE Transactions on Automatic Control*, 64(9), 3905–3912.
- Isidori, A. (1995). *Nonlinear control systems*. Communications and Control Engineering Series. Springer-Verlag, Berlin, third edition. doi:10.1007/978-1-84628-615-5.
- Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall, third edition.
- Nekhoroshikh, A., Efimov, D., Polyakov, A., Perruquetti, W., and Furtat, I. (2022). Hyperexponential and fixed-time stability of time-delay systems: Lyapunov-Razumikhin method. *IEEE Transactions on Automatic Control*. doi:10.1109/TAC.2022.3168487.
- Polyakov, A. (2012). Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8), 2106–2110.
- Polyakov, A., Efimov, D., and Brogliato, B. (2019). Consistent discretization of finite-time and fixed-time stable systems. *SIAM Journal of Control and Optimization*, 57(1), 78–103.
- Polyakov, A., Efimov, D., Perruquetti, W., and Richard, J. (2015). Implicit Lyapunov-Krasovski functionals for stability analysis and control design of time-delay systems. *IEEE Transactions on Automatic Control*, 60(12), 3344–3349. doi:10.1109/TAC.2015.2422451.
- Shtessel, Y., Edwards, C., Fridman, L., and Levant, A. (2014). *Sliding Mode Control and Observation*. Birkhauser.
- Song, Y., Wang, Y., Holloway, J., and Krstic, M. (2017). Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. *Automatica*, 83, 243–251.
- Song, Y., Wang, Y., and Krstic, M. (2018). Time-varying feedback for stabilization in prescribed finite time. *International Journal of Robust and Nonlinear Control*, 29(3), 618–633.
- Utkin, V.I. (1992). *Sliding Modes in Control Optimization*. Springer-Verlag, Berlin.