



HAL
open science

Nonlinear Adaptive Observers for an SIS System Counting Primo-infections

Marcel Fang, Pierre-Alexandre Bliman, Denis Efimov, Rosane Ushirobira

► **To cite this version:**

Marcel Fang, Pierre-Alexandre Bliman, Denis Efimov, Rosane Ushirobira. Nonlinear Adaptive Observers for an SIS System Counting Primo-infections. 22nd World Congress of the International Federation of Automatic Control, Jul 2023, Yokohama, Japan. hal-04063936v2

HAL Id: hal-04063936

<https://inria.hal.science/hal-04063936v2>

Submitted on 20 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Nonlinear Adaptive Observers for an SIS System Counting Primo-infections

Marcel Fang* Pierre-Alexandre Bliman* Denis Efimov**
Rosane Ushirobira**

* Sorbonne Université, Université Paris Cité, CNRS, Laboratoire
Jacques-Louis Lions, UMR 7598, Inria, équipe Mamba, Paris, France
** Inria, Université de Lille, CNRS, UMR 9189 - CRISTAL, Lille, France

Abstract: Observation and identification are important issues for the practical use of compartmental models of epidemic dynamics. Usually, the state and parameters of the epidemic model are evaluated based on the number of infected individuals (the prevalence) or the newly infected cases (the incidence). Other estimation techniques, for example, based on the exploitation of the proportion of primo-infected individuals (easily retrievable data), are rarely considered. We are thus interested in a general question: may the measure of the number of primo-infected individuals and the prevalence improve simultaneous state and parameter estimation? In this paper, we design a nonlinear adaptive observer for a simple infection model with waning immunity and consequent reinfections to answer this question. The practical asymptotic stability of the estimation errors is then proved using the Lyapunov function method. Finally, the convergence of the observer is illustrated in simulations.

Keywords: Adaptive estimation; SIS model; Lyapunov method

1. INTRODUCTION

In mathematical epidemiology, compartmental models are popular and well-studied tools for modeling epidemic processes. Many research works in mathematical epidemiology are usually interested in directly analyzing and simulating these epidemic processes. However, such a utilization implies some knowledge of the existence and reliability of parameter estimates beforehand. Equivalently, the state estimation of epidemiological systems is a subject of vital interest for the survey and control of epidemics. Thus, the inverse problem of simultaneously estimating states and parameters is essential in the direct problem for modeling epidemics.

Although identification and observation are well-studied issues in many fields, this is still not the case in mathematical epidemiology, where only a small amount of literature is available, and among them most are only interested in the parameters identification of epidemic models even if state estimation is equally significant for understanding and monitoring epidemic processes. An interested reader can refer, for example, to Niazzi and Johansson (2022), Hamelin et al. (2020) for common observer design of generic epidemiological model, but also Alanwar et al. (2022), Bliman and D’Avila Barros (2017) for observer design with set-based approach. In the present paper, we focus on joint estimation of the state and parameters of a susceptible-infected-susceptible (SIS) epidemic model counting primo-infections. The usual approach for estimating states and parameters is based on the measurement of prevalence (the number of infected individuals) or incidence (the number of newly infected cases), see Bliman et al. (2018), Ushirobira et al. (2019). However, in case of epidemic with reinfections, other data of interest may be available as the number of primo-

infections. Thus the main interest here is to see if an additional measurement of primo-infected, which is rarely considered but easy to obtain data, could improve the observation and identification of epidemiological systems. The model has been analyzed in a previous paper by Fang and Bliman (2022), where in particular, the algebraic observability and identifiability were derived under the condition that measurement of primo-infected is available. Moreover, a class of non-linear observers for the system is presented in Fang et al. (2022) where the stability is deduced using both copositive and Lur’e-type Lyapunov functions and techniques based on linear matrix inequalities (LMI).

In the continuity of the previous papers, we are now interested in the design of an adaptive observer for the SIS model. In Section 2, we recall the studied model and its transformation through a change of coordinates into a suitable system for observer design proposed by Fang et al. (2022). This new system is then recast for the adaptive observer design. Estimates’ convergence of the proposed adaptive observer is proven in Section 3, and numerical simulations are computed in Section 4.

1.1 Notation

- The sets of real and nonnegative real numbers are denoted by \mathbb{R} and \mathbb{R}_+ respectively.
- Euclidean norm for a vector $x \in \mathbb{R}^n$ is defined as $|x|$.
- For a Lebesgue measurable function of time $d : [a, b] \rightarrow \mathbb{R}^m$ define the norm $\|d\|_{[a,b]} = \text{ess sup}_{t \in [a,b]} |d(t)|$, then $\|d\|_\infty = \|d\|_{[0, +\infty)}$ and the space of d with $\|d\|_{[a,b]} < +\infty$ ($\|d\|_\infty < +\infty$) we further denote as $\mathcal{L}_{[a,b]}^m$ (\mathcal{L}_∞^m).
- The definitions of classes \mathcal{K} , \mathcal{K}_∞ , and \mathcal{KL} are standard and can be found in Sontag and Wang (2000).

* The financial support of MODCOV19 is acknowledged.

2. MODELING

2.1 A SIS model with primo-infections

The studied SIS model introduced in Fang and Bliman (2022); Fang et al. (2022) has the following state-space representation:

$$\dot{S} = \mu - \beta SI - \mu S + \gamma I, \quad \dot{I} = \beta SI - (\mu + \gamma)I, \quad (1a)$$

$$\dot{S}_1 = \mu - \beta S_1 I - \mu S_1, \quad \dot{I}_1 = \beta S_1 I - (\mu + \gamma)I_1, \quad (1b)$$

$$y = (\alpha I \quad \alpha I_1)^\top, \quad (1c)$$

where the four state variables $(S(t), I(t), S_1(t), I_1(t)) \in (0,1)^4$, $t \in \mathbb{R}_+$, represent the proportions of susceptible, infected, never infected susceptible and primo-infected individuals, respectively; the coefficients $\mu, \beta, \gamma > 0$ are the natural mortality rate, the contact rate, and the recovery rate; the output $y \in [0,1]^2$ provides the measurement of a given portion $\alpha \in [0,1]$ of the infected individuals and of the primo-infected individuals. We assume that this portion, which corresponds to the infected individuals detected by the Public health system (e.g., the symptomatic cases), is identical for the primo-infected and the global infected populations. As testified by formula (1a), the constant total population $S + I$ is normalized to 1. The basic reproduction number of this system is

$$\mathcal{R}_0 := \frac{\beta}{\mu + \gamma}, \quad (2)$$

which is assumed to be greater than 1 so that convergence to the (unique, positive) endemic equilibrium occurs.

For an observer design, an alternative state-space representation is introduced in Fang et al. (2022) by computing the following change of variables:

$$z = S + I, \quad s_1 = \ln(S_1), \quad \iota = \ln(I), \quad z_1 = S_1 + I_1.$$

This yields the following equations:

$$\dot{z} = \mu - \mu z, \quad (3a)$$

$$\dot{\iota} = \beta z - \beta(e^\iota - 1) - (\beta + \mu + \gamma), \quad (3b)$$

$$\dot{z}_1 = \mu - (\mu + \gamma)z_1 + \gamma(e^{s_1} - 1) + \gamma, \quad (3c)$$

$$\dot{s}_1 = \mu(e^{-s_1} - 1) - \beta(e^\iota - 1) - \beta, \quad (3d)$$

$$y_1 = \alpha(e^\iota - 1) + \alpha, \quad y_2 = \alpha z_1 - \alpha(e^{s_1} - 1) - \alpha. \quad (3e)$$

In the rest of the paper, we consider that the mortality μ is a known parameter, which is not a costly assumption as the life expectancy of the population is often available. We also suppose that the measured portion of the infected individuals α is also a known quantity (identified in advance), thus for simplicity we fix $\alpha = 1$ in the rest of the paper.

2.2 State-space representation for adaptive observer design

We rewrite the system (3) in a more suitable way for estimation of β and γ , by taking advantage of the knowledge of y_1, y_2 :

$$\dot{z} = \mu - \mu z \quad (4a)$$

$$\dot{\iota} = \beta z - \beta y_1 - (\mu + \gamma) \quad (4b)$$

$$\dot{z}_1 = \mu - \mu z_1 - \gamma y_2 \quad (4c)$$

$$\dot{s}_1 = \mu(e^{-s_1} - 1) - \beta y_1 \quad (4d)$$

$$y_1 = e^\iota, \quad y_2 = z_1 - e^{s_1}. \quad (4e)$$

The proposed adaptive observer has the form:

$$\dot{\hat{z}} = \mu - \mu \hat{z} \quad (5a)$$

$$\dot{\hat{\iota}} = \hat{\beta} \hat{z} - \hat{\beta} y_1 - (\mu + \hat{\gamma}) + L_1(y_1 - \hat{y}_1) \quad (5b)$$

$$\dot{\hat{z}}_1 = \mu - \mu \hat{z}_1 - \hat{\gamma} y_2 + L_2(y_2 - \hat{y}_2) \quad (5c)$$

$$\dot{\hat{s}}_1 = \mu(e^{-\hat{s}_1} - 1) - \hat{\beta} y_1 \quad (5d)$$

$$\dot{\hat{\beta}} = f(\hat{x}, y, \hat{y}), \quad \dot{\hat{\gamma}} = g(\hat{x}, y, \hat{y}) \quad (5e)$$

$$\hat{y}_1 = e^{\hat{\iota}}, \quad \hat{y}_2 = \hat{z}_1 - e^{\hat{s}_1} \quad (5f)$$

where $\hat{x}(t) = [\hat{z}(t) \quad \hat{\iota}(t) \quad \hat{z}_1(t) \quad \hat{s}_1(t) \quad \hat{\beta}(t) \quad \hat{\gamma}(t)]^\top \in \mathbb{R}^6$ is the observer state vector composed of estimates of the state $[z \quad \iota \quad z_1 \quad s_1]^\top$ and parameters β, γ in (4); $y = [y_1 \quad y_2]^\top$, $\hat{y} = [\hat{y}_1 \quad \hat{y}_2]^\top$ are respectively the system and observer output vectors; f and g are scalar functions describing adaptation rules for the estimates of parameters that will be determined below. Denoting

$$e_1 := z - \hat{z}, \quad e_2 := \iota - \hat{\iota}, \quad e_3 := z_1 - \hat{z}_1,$$

$$e_4 := s_1 - \hat{s}_1, \quad \theta_1 := \beta - \hat{\beta}, \quad \theta_2 := \gamma - \hat{\gamma},$$

the error dynamics between system (4) and adaptive observer (5) fulfills the following equations:

$$\dot{e}_1 = -\mu e_1, \quad (6a)$$

$$\dot{e}_2 = \beta e_1 + \hat{z} \theta_1 - \theta_1 y_1 - \theta_2 - L_1(y_1 - \hat{y}_1), \quad (6b)$$

$$\dot{e}_3 = -\mu e_3 - \theta_2 y_2 - L_2(y_2 - \hat{y}_2), \quad (6c)$$

$$\dot{e}_4 = \mu(e^{-s_1} - e^{-\hat{s}_1}) - \theta_1 y_1, \quad (6d)$$

$$\dot{\theta}_1 = -\dot{\hat{\beta}} = -f, \quad \dot{\theta}_2 = -\dot{\hat{\gamma}} = -g. \quad (6e)$$

All over the paper, as the system is normalized, we may assume that $z(0) = \hat{z}(0) = 1$, so that $z(t) = \hat{z}(t) = 1$ and $e_1 \equiv 0$, thanks to (4a), (5a). Thus the dynamics of z and \hat{z} can be ignored.

3. ADAPTIVE OBSERVER DESIGN

The main goal is to find functions f, g such that the observer is (practically) asymptotically stable, and for this goal, the theory of input-to-state stability (ISS) will be used. In subsection 3.1, we start by recalling the notions of ISS and ISS-Lyapunov function. Then in subsection 3.2, a candidate Lyapunov function with absolute and quadratic terms is proposed, and the parameter adjustment laws f, g are designed so that the candidate function fulfills the requirement of an ISS-Lyapunov function.

3.1 Stability notions

We recall first the well-known concept of input-to-state stability, introduced by Sontag and Wang (1995). To this end, consider a system

$$\dot{x}(t) = h(x(t), u(t)), \quad (7)$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$, $u \in \mathcal{L}_\infty^m$, $h: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Lipschitz continuous function.

Definition 1. The system (7) is called ISS if there exist a \mathcal{KL} function κ and \mathcal{K}_∞ function λ such that one has

$$\|x(t)\| \leq \kappa(\|x(0)\|, t) + \lambda(\|u\|_\infty),$$

for any $t \geq 0$, any $x(0) \in \mathbb{R}^n$ and any $u \in \mathcal{L}_\infty^m$.

ISS stability is often proved by exhibiting the existence of an ISS-Lyapunov function.

Definition 2. A smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is called an ISS-Lyapunov function of (7) if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty, \chi \in \mathcal{K}$ and a positive-definite function Υ , such that:

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n \quad (8)$$

and $\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m$ it holds:

$$|x| \geq \chi(|u|) \Rightarrow \nabla V(x) \cdot h(x, u) \leq -\Upsilon(|x|). \quad (9)$$

The following theorem, also from Sontag and Wang (1995), links the ISS property to the existence of an ISS-Lyapunov function.

Theorem 3. The system (7) is ISS if and only if it admits an ISS-Lyapunov function.

The ISS property can also be used to establish practical asymptotic stability of (7) if the input u is constant, (i.e., it represents the system parameters or some uniform bounds on disturbing terms). Consequently, assume we have proven that the system (6) is ISS and denote its state vector by $e = [e_1 \dots e_4 \theta_1 \theta_2]^\top$, then $\|e(t)\|$ is ultimately bounded (thus, the error's norm converges to a bounded set in a neighborhood of 0). The observer design should be done in such a way that the domain of convergence can be made arbitrarily small.

3.2 Stability through Lyapunov function with absolute and quadratic terms

Despite the ISS theory has been recalled above for state variable in \mathbb{R}^n , it can be equivalently used for a state belonging to invariant cones (e.g., in \mathbb{R}_+^n) or locally for the norm of the state and the input bounded by a constant. Denoting

$$\rho_1 := y_1 - \hat{y}_1, \quad \rho_2 := y_2 - \hat{y}_2 \quad (10)$$

we aim to prove that an ISS-Lyapunov function for (6) can be selected as follows:

$$V := |e_1| + b|e_2| + c|\rho_2| + \frac{1}{2}(de_4^2 + \eta_1^{-1}\theta_1^2 + \eta_2^{-1}\theta_2^2),$$

where b, c, d, η_1 , and η_2 are positive parameters defined later. Let us compute first the time derivatives of ρ_1, ρ_2 :

$$\begin{aligned} \dot{\rho}_1 &= (\dot{e}^e - \dot{\hat{e}}^e) \\ &= (\beta z - \beta y_1 - \mu - \gamma)e^e - (\hat{\beta}\hat{z} - \hat{\beta}y_1 - \mu - \hat{\gamma} + L_1\rho_1)e^{\hat{e}} \\ &= \theta_1 z y_1 + \hat{\beta}y_1 e_1 + \hat{\beta}\hat{z}\rho_1 - \theta_1 y_1 e^e - \hat{\beta}y_1 e^e + \hat{\beta}\hat{y}_1 e^e - \mu\rho_1 \\ &\quad - \theta_2 y_1 - \hat{\gamma}y_1 + \hat{\gamma}\hat{y}_1 - L_1\hat{y}_1\rho_1 \\ &= (\hat{\beta}\hat{z} - \hat{\beta}y_1 - \hat{\gamma} - \mu - L_1\hat{y}_1)\rho_1 + \theta_1 z e^e + \hat{\beta}e_1 e^e \\ &\quad - \theta_1 y_1 e^e - \theta_2 e^e \\ &= (\hat{\beta}\hat{z} - \hat{\beta}y_1 - \hat{\gamma} - \mu - L_1\hat{y}_1)\rho_1 - \theta_2 y_1 + \theta_1 y_1 (1 - y_1) \\ \dot{\rho}_2 &= \dot{e}_3 - (\dot{s}_1 e^{s_1} - \dot{\hat{s}}_1 e^{\hat{s}_1}) \\ &= -\mu e_3 - \theta_2 y_2 - L_2 \rho_2 - [(\mu(e^{-s_1} - 1) - \beta y_1)e^{s_1} \\ &\quad - (\mu(e^{-\hat{s}_1} - 1) - \hat{\beta}y_1)e^{\hat{s}_1}] \\ &= -(\mu + L_2)\rho_2 - \theta_2 y_2 + \beta y_1 (e^{s_1} - e^{\hat{s}_1}) + \theta_1 y_1 e^{\hat{s}_1} \end{aligned}$$

We deduce the following algebraic relation based on derivatives

$$\theta_1(1 - y_1) - \theta_2 = \frac{\dot{\rho}_1 - (\hat{\beta}\hat{z} - \hat{\beta}y_1 - \hat{\gamma} - \mu - L_1\hat{y}_1)\rho_1}{y_1} =: \Pi$$

In particular, notice that all components on the right-hand side of formula above are known. Thus the computation shows that

the function Π , which includes terms of interest θ_1, θ_2 that cannot be directly measured, is nonetheless accessible through outputs derivative using the right-hand side expressions and provided that y_1 is non-zero (which is the case as $\mathcal{R}_0 > 1$). Therefore these quantities can be included in the design of f and g of the observer for the purpose of enhancing parameters estimation. The derivative $\dot{\rho}_1$ can be estimated online using different methods (see Reichhartinger et al. (2018)). The time derivative of V along the trajectories is

$$\begin{aligned} \dot{V} &= \dot{e}_1 \operatorname{sgn}(e_1) + b\dot{e}_2 \operatorname{sgn}(e_2) + c\dot{\rho}_2 \operatorname{sgn}(\rho_2) + de_4 \dot{e}_4 \\ &\quad + \eta_1^{-1}\theta_1 \dot{\theta}_1 + \eta_2^{-1}\theta_2 \dot{\theta}_2 \\ &= -\mu|e_1| + b\operatorname{sgn}(e_2)[\beta e_1 + \hat{z}\theta_1 - y_1\theta_1 - \theta_2 - L_1(e^e - e^{\hat{e}})] + c\operatorname{sgn}(\rho_2)[-(\mu + L_2)\rho_2 - \theta_2 y_2 \\ &\quad + \beta y_1 (e^{s_1} - e^{\hat{s}_1}) + \theta_1 y_1 e^{\hat{s}_1}] + d\mu e_4 (e^{-s_1} - e^{-\hat{s}_1}) \\ &\quad - de_4 y_1 \theta_1 + \eta_1^{-1}\theta_1 \dot{\theta}_1 + \eta_2^{-1}\theta_2 \dot{\theta}_2 \\ &= -\mu|e_1| + b\operatorname{sgn}(e_2)[\beta e_1 - L_1(e^e - e^{\hat{e}})] - de_4 y_1 \theta_1 \\ &\quad + d\mu e_4 (e^{-s_1} - e^{-\hat{s}_1}) + c\operatorname{sgn}(\rho_2)[-(\mu + L_2)\rho_2 \\ &\quad + \beta y_1 (e^{s_1} - e^{\hat{s}_1})] \\ &\quad + \eta_1^{-1}\theta_1 (\dot{\theta}_1 + \eta_1 [b\operatorname{sgn}(e_2)(\hat{z} - y_1) + c\operatorname{sgn}(\rho_2)y_1 e^{\hat{s}_1}]) \\ &\quad + \eta_2^{-1}\theta_2 (\dot{\theta}_2 - \eta_2 [b\operatorname{sgn}(e_2) + c\operatorname{sgn}(\rho_2)y_2]). \end{aligned}$$

The design of $\dot{\hat{\beta}}, \dot{\hat{\gamma}}$ and consequently $\dot{\theta}_1, \dot{\theta}_2$ is then done in a way to cancel the product terms of θ_1, θ_2 in brackets appearing above in the two last lines. As announced we will also include measurable or estimated signal Π in order to introduce the errors θ_1, θ_2 , into the dynamics of $\hat{\beta}, \hat{\gamma}$ for parameters estimation. Let the functions f, g be as in (11), with $k_1, k_2 > 0$:

$$\begin{aligned} f(\hat{x}, y, \hat{y}) &= \eta_1 \left(b(\hat{z} - y_1) \operatorname{sgn}(\rho_1) + c y_1 e^{\hat{s}_1} \operatorname{sgn}(\rho_2) + k_1 \frac{\Pi}{1 - y_1} \right) \quad (11) \\ g(\hat{x}, y, \hat{y}) &= \eta_2 (-b \operatorname{sgn}(\rho_1) - c y_2 \operatorname{sgn}(\rho_2) - k_2 \Pi) \end{aligned}$$

Let us prove now that V is an ISS-Lyapunov function for the system (6)-(11).

In the computations below we will use the facts that $\iota < 0, s_1 < 0, z_1 \in (0, 1), \beta > 0$ and $\gamma > 0$, hence, $y_1 \in (0, 1)$ and we can naturally take $\hat{z}_1(0) \in (0, 1), \hat{\iota}(0) < 0$ and $\hat{s}_1(0) < 0$.

Noting that $\operatorname{sgn}(\rho_1) = \operatorname{sgn}(e_2)$ and using Young's inequality, for any $\varepsilon_1 > 0$ we have:

$$\begin{aligned} \dot{V} &\leq -(\mu - b\beta)|e_1| - bL_1|e^e - e^{\hat{e}}| - d\mu|e_4||e^{-s_1} - e^{-\hat{s}_1}| \\ &\quad + c\beta y_1 |e^{s_1} - e^{\hat{s}_1}| |\operatorname{sgn}(\rho_2)| + dy_1 \left(\frac{\varepsilon_1}{2} \theta_1^2 + \frac{1}{2\varepsilon_1} e_4^2 \right) \\ &\quad - c(\mu + L_2)|\rho_2| - k_1 \frac{\Pi}{1 - y_1} \theta_1 + k_2 \Pi \theta_2 \\ &\leq -(\mu - b\beta)|e_1| - bL_1|e^e - e^{\hat{e}}| - d\mu|e_4||e^{-s_1} - e^{-\hat{s}_1}| \\ &\quad + c\beta y_1 |e^{s_1} - e^{\hat{s}_1}| |\operatorname{sgn}(\rho_2)| + dy_1 \left(\frac{\varepsilon_1}{2} \theta_1^2 + \frac{1}{2\varepsilon_1} e_4^2 \right) \\ &\quad - c(\mu + L_2)|\rho_2| - k_1 \theta_1^2 - k_2 \theta_2^2 \\ &\quad + \theta_1 \theta_2 \left[k_2(1 - y_1) + \frac{k_1}{1 - y_1} \right] \end{aligned}$$

Due to presence of negative feedback terms in θ_1 and θ_2 introduced by Π into the adaptation laws, these error variables

stay bounded provided that all other variables possess this property. Assume that this is the case. Since $\hat{\beta}(0) > 0$ by tuning η_1 and η_2 , there exist $\underline{\beta}, \bar{\beta}, \underline{\gamma}, \bar{\gamma} \in \mathbb{R}_+$ such that $\hat{\beta} \in (\underline{\beta}, \bar{\beta})$, $\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma})$ (this bound can be shown by applying Grönwall's inequality on $[\hat{\beta} \ \hat{\gamma}]^\top$). For $\hat{\beta} > 0$ we get that $\hat{s}_1 < 0$. Then, thanks to the Mean value theorem, we obtain

$$|e^{s_1} - e^{\hat{s}_1}| \leq |e_4| \leq |e^{-s_1} - e^{-\hat{s}_1}|.$$

Therefore,

$$\begin{aligned} \dot{V} \leq & -(\mu - b\beta)|e_1| - bL_1|e^e - e^{\hat{e}}| - c(\mu + L_2)|\rho_2| \\ & - d(\mu - \frac{1}{2\varepsilon_1})|e_4|^2 + c\beta y_1|e^{s_1} - e^{\hat{s}_1}| - k_2\theta_2^2 \\ & - \left(k_1 - \frac{d\varepsilon_1}{2}\right)\theta_1^2 + |\theta_1||\theta_2| \left[k_2 + \frac{k_1}{1-y_1}\right] \end{aligned}$$

Moreover, as $\hat{\beta} \in (\underline{\beta}, \bar{\beta})$ and $\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma})$, there exists $\bar{\theta}_1, \bar{\theta}_2$ such that $|\theta_1| < \bar{\theta}_1, |\theta_2| < \bar{\theta}_2$. Secondly, it is known that $\sup y_1 < 1$ (see Hethcote (1989)), thus $\frac{1}{1-y_1} < k_3$ for some $k_3 > 0$.

Finally, notice that $|e^{s_1} - e^{\hat{s}_1}| < 1$, which leads to the estimate:

$$\begin{aligned} \dot{V} \leq & -(\mu - b\beta)|e_1| - bL_1|e^e - e^{\hat{e}}| - c(\mu + L_2)|\rho_2| \quad (12) \\ & - d(\mu - \frac{1}{2\varepsilon_1})|e_4|^2 - k_2\theta_2^2 - \left(k_1 - \frac{d\varepsilon_1}{2}\right)\theta_1^2 + \Delta \end{aligned}$$

where $\Delta = c\beta + \bar{\theta}_1\bar{\theta}_2[k_2 + k_1k_3]$ is the constant bias. Fixing the parameters k_1 and k_2 of the observer, let us select ε_1, b, d as $\varepsilon_1 > \frac{1}{2\mu}, d < 2\frac{k_1}{\varepsilon_1}, b < \frac{\mu}{\bar{\beta}}$, such that the only remaining positive term Δ in \dot{V} is proportional to free-tuning gains k_1, k_2 and c . Note that the condition $b < \frac{\mu}{\bar{\beta}}$ can be ignored as $|e_1| = 0$. The values of tuning parameters $c > 0, d > 0$ present in the Lyapunov function can be taken arbitrarily small and thus the same for k_1 and k_2 also. Consequently, for any $\bar{\theta}_1, \bar{\theta}_2$ (which depend on $\underline{\beta}, \bar{\beta}, \underline{\gamma}, \bar{\gamma}$ proportional to the domain of attraction of the observer) there exist k_1, k_2, b, d, c such that V can be used as an ISS-Lyapunov function with regard to the input Δ . Finally, the assumption on boundedness of all variables in the system is satisfied, confirming that for a suitable choice of parameters $\hat{s}_1 < 0$.

We have thus proved the following result.

Theorem 4. There exist positive values of $L_1, L_2, \eta_1, \eta_2, b, c, \mu, \beta, \gamma, k_1$ and k_2 such that the system (6)-(11) is practically locally asymptotically stable (locally ISS with respect to $\Delta(c, k_1, k_2)$), where the domain of convergence can be made arbitrarily small.

The theorem shows that the present adaptive observer is best suited for estimating the SIS system with a small contact rate, as the system (6)-(11) is ISS with regard to the coefficient β . In practice, numerical simulations show that the estimation is accurate for $\beta < 1 \text{ days}^{-1}$, including most diseases.

4. NUMERICAL SIMULATIONS

4.1 Parameters tuning

The ISS property obtained in Theorem 4 suggests that one natural strategy in tuning the parameters is to minimize as much as possible the values of k_1 and k_2 . However, these parameters

are multiplier coefficients of estimation errors in the adjustment laws f, g , and lowering too much their values may theoretically affect the observer's convergence velocity. Thus there is a trade-off between the size of the ultimate bounded set to where the observer converges (thus the precision of the estimation) and the speed of convergence. This issue is important as the system subject to the observation itself usually converges to its steady state relatively quickly, and in this case we have a more practical and efficient way of estimating parameters by analyzing only the steady state itself (see Fang and Bliman (2022)). Notice also that k_1 is multiplied by a constant k_3 in (12) which could be relatively large, thus k_1 should be taken smaller than k_2 .

For setting the gains L_1, L_2 , we can see that the output error multiplied by L_1 in (6b) is a direct estimation of e_2 which is not the case for the output errors multiplied by L_2 in (6c) with regard to e_3 . Thus $L_1(y_1 - \hat{y}_1)$ is a way more efficient output injection for estimation than the others and the gains should respect the ordering $L_1 > L_2$.

4.2 Simulation

For illustration purposes, we present numerical simulations of the adaptive observer computed in original coordinates. We fix the known parameter as $\mu^{-1} = 70$ years and assume that the parameters to be estimated are $\beta = 0.2 \text{ days}^{-1}, \gamma = \frac{1}{12} \text{ days}^{-1}$. We also choose the gain values as $L_1 = 10, L_2 = 1$. Moreover, we fix $\eta_1 = \eta_2 = 1, k_1 = 1 \times 10^{-5}, k_2 = 1 \times 10^{-3}$ and $b = c = 1$. The initial state of the observer is taken randomly in a set verifying the conditions $\hat{S}(0) + \hat{I}(0) = 1, \hat{S}_1(0) \leq \hat{S}(0), \hat{I}_1(0) \leq \hat{I}(0)$. Notice that the values of $S(0), I(0), I_1(0)$ are accessible directly, but it is practically more interesting to choose different values for the initial state of the observer in order to add excitation to the system. Simulations are performed in Matlab with ODE solver ode45, and the results are shown in the figures below. The convergence of the estimated states \hat{I} (and thus \hat{S}) and \hat{I}_1 to their actual values is fast and usually happens in the first to third week starting from the first measurement, which is expected as these states are directly accessible from output measurement. Convergence of the parameters is also quick, and we can expect to retrieve a precise estimation with two significant figures after the second or third week. Conversely, we may obtain a relatively reliable estimate of S_1 , usually starting from the thirty to forty day since the first measurement, this lower efficiency is expected as there is no output injection for the estimation of \hat{s}_1 . Last, Figures 7 and 8 show the 2-norm of parameter and state estimation errors in a semi-log scale.

5. CONCLUSION

We have proposed in this paper a nonlinear adaptive observer for a SIS system measuring primo-infections. The practical asymptotic stability of the estimation errors is obtained, and the used Lyapunov function includes absolute values of the components of the error and the quadratic terms. Numerical simulations illustrate the convergence of the estimates of the proposed adaptive observer to a neighborhood of ideal values. In the proposed design, the coefficient α , representing the detection rate, is assumed to be known, and extending the adaptive observer to the case when α is also unknown can be considered a direction of future research.

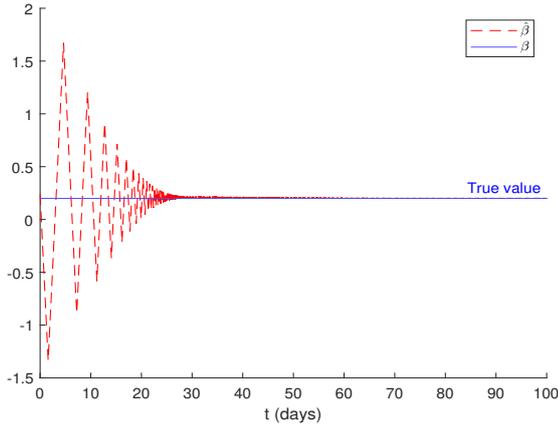


Fig. 1. Estimation of β

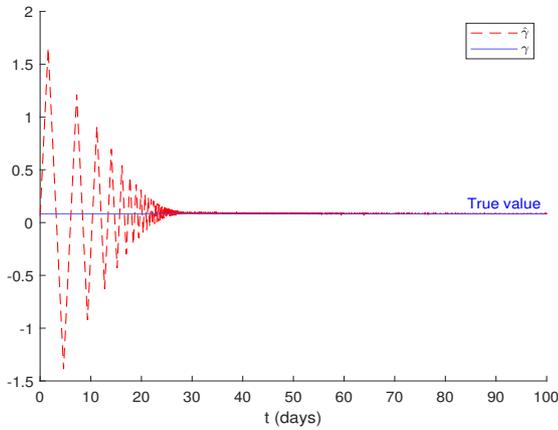


Fig. 2. Estimation of γ

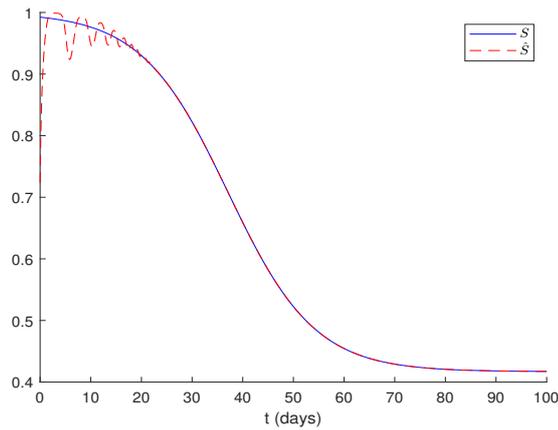


Fig. 3. Estimation \hat{S} of the susceptible population S

REFERENCES

- Alanwar, A., Niazi, M.U.B., and Johansson, K.H. (2022). Data-driven set-based estimation of polynomial systems with application to SIR epidemics. In *2022 European Control Conference (ECC)*, 888–893.
- Bliman, P.A. and D'Avila Barros, B. (2017). *Interval Observers for SIR Epidemic Models Subject to Uncertain Seasonality*, 31–39. Springer International Publishing, Cham.

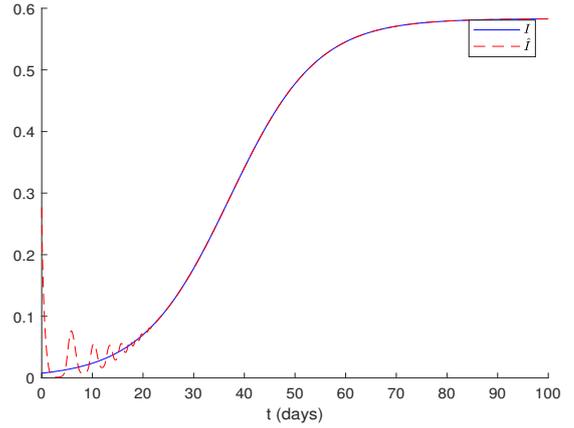


Fig. 4. Estimation \hat{I} of the susceptible population I

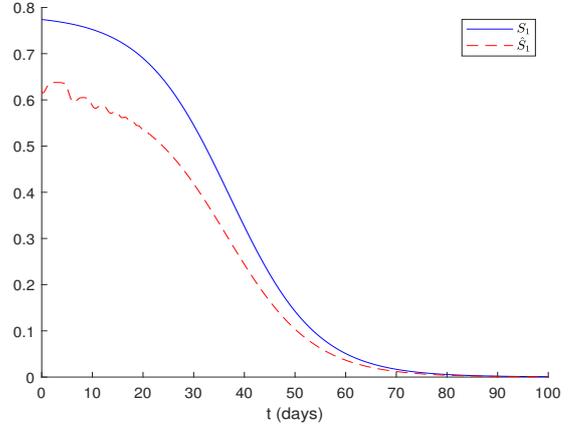


Fig. 5. Estimation \hat{S}_1 of the susceptible population S_1

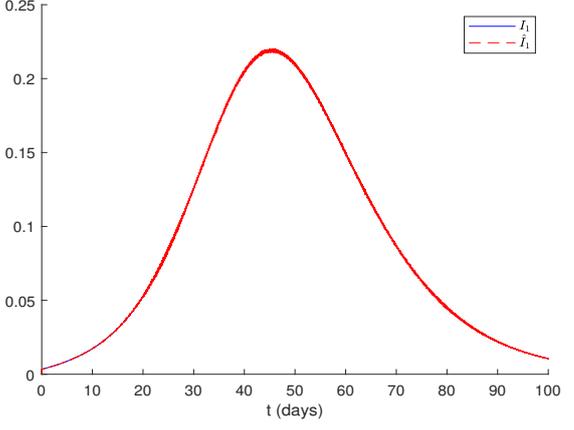


Fig. 6. Estimation \hat{I}_1 of the susceptible population I_1

- Bliman, P.A., Efimov, D., and Ushirobira, R. (2018). A class of nonlinear adaptive observers for SIR epidemic model. In *2018 European Control Conference (ECC)*, 1–6.
- Fang, M. and Bliman, P.A. (2022). Modelling, analysis, observability and identifiability of epidemic dynamics with reinfections. In *Proc. 20th European Control Conference*.
- Fang, M., Bliman, P.A., Efimov, D., and Ushirobira, R. (2022). A class of nonlinear state observers for an SIS system counting primo-infections. In *61st IEEE Conference on Decision*

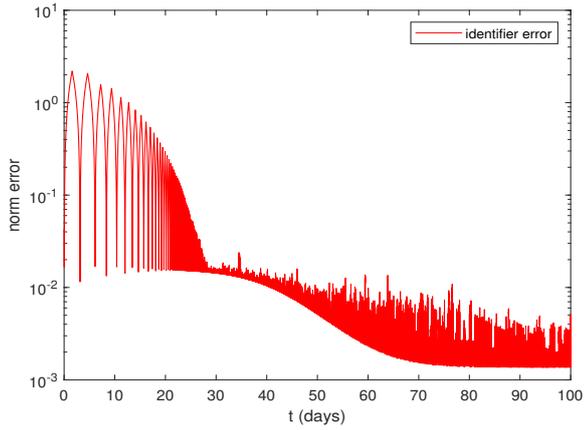


Fig. 7. 2-norm of parameter estimation error

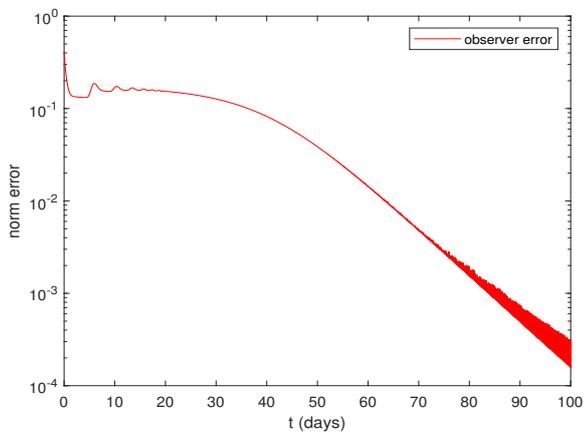


Fig. 8. 2-norm of state estimation error

and Control. Cancún, Mexico.

Hamelin, F., Iggidr, A., Rapaport, A., and Sallet, G. (2020).

Observability, identifiability and epidemiology – a survey.

Hethcote, H.W. (1989). Three basic epidemiological models.

Niazi, M.U.B. and Johansson, K.H. (2022). Observer design for the state estimation of epidemic processes.

Reichhartinger, M., Efimov, D., and Fridman, L. (2018). Special issue on differentiators. *International Journal of Control*, 91(9), 1980–1982.

Sontag, E. and Wang, Y. (2000). Lyapunov characterizations of input to output stability. *SIAM Journal on Control and Optimization*, 39(1), 226–249.

Sontag, E.D. and Wang, Y. (1995). On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5), 351–359.

Ushirobira, R., Efimov, D., and Bliman, P.A. (2019). Estimating the infection rate of a SIR epidemic model via differential elimination. In *2019 18th European Control Conference (ECC)*, 1170–1175.