



HAL
open science

A Note on the Complexity of Maximizing Temporal Reachability via Edge Temporalisation of Directed Graphs

Alkida Balliu, Filippo Brunelli, Pierluigi Crescenzi, Dennis Olivetti, Laurent Viennot

► **To cite this version:**

Alkida Balliu, Filippo Brunelli, Pierluigi Crescenzi, Dennis Olivetti, Laurent Viennot. A Note on the Complexity of Maximizing Temporal Reachability via Edge Temporalisation of Directed Graphs. 2023. hal-04054527

HAL Id: hal-04054527

<https://inria.hal.science/hal-04054527>

Preprint submitted on 31 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Note on the Complexity of Maximizing Temporal Reachability via Edge Temporalisation of Directed Graphs

Alkida Balliu¹, Filippo Brunelli^{*2}, Pierluigi Crescenzi¹, Dennis Olivetti¹, and Laurent Viennot^{*2}

¹GSSI, I-67100 L'Aquila, Italy

²Inria, Irif, Université de Paris, F-75013 Paris, France

March 31, 2023

Abstract

A temporal graph is a graph in which edges are assigned a time label. Two nodes u and v of a temporal graph are connected one to the other if there exists a path from u to v with increasing edge time labels. We consider the problem of assigning time labels to the edges of a digraph in order to maximize the total reachability of the resulting temporal graph (that is, the number of pairs of nodes which are connected one to the other). In particular, we prove that this problem is NP-hard. We then conjecture that the problem is approximable within a constant approximation ratio. This conjecture is a consequence of the following graph theoretic conjecture: any strongly connected directed graph with n nodes admits an out-arborescence and an in-arborescence that are edge-disjoint, have the same root, and each spans $\Omega(n)$ nodes.

Keywords: temporal graph; temporal path; time assignment; temporal reachability.

1 Introduction

Temporal graphs have received increasing attention over the last two decades [11, 12, 15, 19] and have been defined in several different ways [1, 10, 13, 2, 6, 5, 14] (see [3] for a classification of temporal graphs). Here, we say that a *temporal graph* $G = (V, \mathbb{E})$ is a list \mathbb{E} of *temporal edges* (u, v, t) , where $u, v \in V$ are two *nodes* of the graph (called, respectively, *tail* and *head* of the temporal edge) and

^{*}This work was partly supported by the French National Research Agency (ANR) through the projects Multimod (ANR-17-CE22-0016) and Tempogral (ANR-22-CE48-0001).

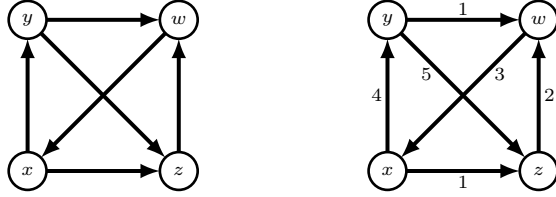


Figure 1: A digraph D (left), and an edge temporalisation of D (right). The temporal reachability of the resulting temporal graph is 16, since each node is temporally reachable from any other node.

t is the *appearing time* of the temporal edge. For each temporal edge (u, v, t) , we can traverse the edge starting from u at time t and arrive in v at time $t + 1$, which is the *arrival time* of the temporal edge.

We study a network optimisation problem related to the notion of reachability in temporal graphs. Given a temporal graph G , a (temporal) *path* from a node u to a node v is a sequence e_1, e_2, \dots, e_k of temporal edges such that the tail of e_1 is u , the head of e_k is v , and, for any i with $1 < i \leq k$, the tail of e_i is equal to the head of e_{i-1} and the appearing time of e_i is greater than the appearing time of e_{i-1} . The *temporal reachability* of G is the number of pairs of nodes u and v such that v is *temporally reachable* from u , that is, there exists a temporal path from u to v . The MAXIMUM REACHABILITY EDGE TEMPORALISATION (MRET) problem consists of, given a directed graph (in short, digraph) $D = (V, E)$, find an *edge temporalisation* $\tau : E \rightarrow \mathbf{N}$ such that the temporal reachability of the resulting temporal graph is maximized. For example, let us consider the digraph shown in the left part of Figure 1. In the right part of the figure, we show an edge temporalisation of a digraph D with four nodes, such that the temporal reachability of the resulting temporal graph is equal to 16, which is clearly the maximum possible temporal reachability.

The MRET problem restricted to undirected graphs has been studied in [9], where it is shown that the problem of deciding whether the resulting temporal graph is temporally connected (that is, for any two nodes u and v , v is temporally reachable from u) is NP-complete (clearly, this implies that the MRET problem restricted to undirected graphs is NP-hard). It is also easy to see that the MRET problem restricted to undirected connected graphs can be approximated within a constant approximation ratio, since this simply requires to look for a “centroid” in a spanning tree as a temporalisation where half of the nodes can reach the other half can then easily be computed. Note, however, that temporalising a symmetric digraph is not equivalent to temporalising an undirected graph as different times can be assigned to an edge (u, v) and the symmetric edge (v, u) .

In the following, we prove that the MRET problem is NP-hard, even when restricted to strongly connected digraphs. We will conclude by conjecturing

that the problem is approximable within a constant approximation ratio, and suggesting a graph theory conjecture which could be used to prove the conjecture concerning the approximability of the MRET problem.

Problems similar to the one considered in this paper have already been analysed [13, 16, 7, 8, 17, 20, 18, 4]. For instance, in [17] the authors propose two cost minimization parameters for temporal network design (that is, the maximum number of appearing times of an edge and the total number of appearing times of all edges), and they study the problem of optimizing these parameters subject to some connectivity constraint.

2 Hardness result

The next result shows that there is no polynomial-time algorithm solving the MRET problem, unless P is equal to NP. In the following, we will refer to edge temporalisations as schedules, that is, as orderings of the edges of the digraph. Indeed, one can easily transform an edge temporalisation τ into an edge temporalisation τ' , where all time labels are pairwise distinct and where the total reachability according to τ is preserved. We then note that the total reachability according to τ' depends only on the ordering of the edges according to their time label. Given a digraph $D = (V, E)$ and a schedule S , a node v is said to be S -reachable from a node u if it is temporally reachable in the temporal graph G induced by D and the temporalisation τ_S that assigns appearing time i to the i th edge of S for $i \in [|E|]$. The set of nodes S -reachable from a node u is denoted as $\mathcal{R}_G(u)$. The S -reachability of D is defined as the temporal reachability of the temporal graph induced by τ_S .

Theorem 1. *The MRET problem is NP-hard, even if the digraph D is strongly connected.*

Proof. We reduce 3-SAT to MRET as follows. Let us consider a 3-SAT formula Φ , with n variables x_1, \dots, x_n and m clauses c_1, \dots, c_m . Without loss of generality we will assume that each variable appears positive in at least one clause and negative in at least one clause. We first define the *unweighted* digraph $D = (V, E)$ as the union of the following gadgets (see Figure 2).

Variable gadgets For each variable x_i of Φ , V contains the nodes t_i^1, t_i^2, f_i^1 , and f_i^2 and E contains the edges $(t_i^1, f_i^2), (f_i^2, f_i^1), (f_i^1, t_i^2)$, and (t_i^2, t_i^1) .

Clause gadgets For each clause c_j , V contains the nodes c_j^1 and c_j^2 . If the literal x_i appears in c_j , E contains the edges (c_j^1, t_i^1) and (t_i^2, c_j^2) , while if the literal $\neg x_i$ appears in c_j , E contains the edges (c_j^1, f_i^1) and (f_i^2, c_j^2) . Moreover, for each two clauses c_h and c_j with $h \neq j$, E contains the edge (c_j^1, c_h^2) (see the dashed edges in the figure). Finally, for each clause c_j , V also contains the nodes d_j^i and e_j^i , for $i \in [K]$ (the value of K will be specified later in the proof), and E contains the edges (d_j^i, c_j^1) and (c_j^2, e_j^i) , for $i \in [K]$.

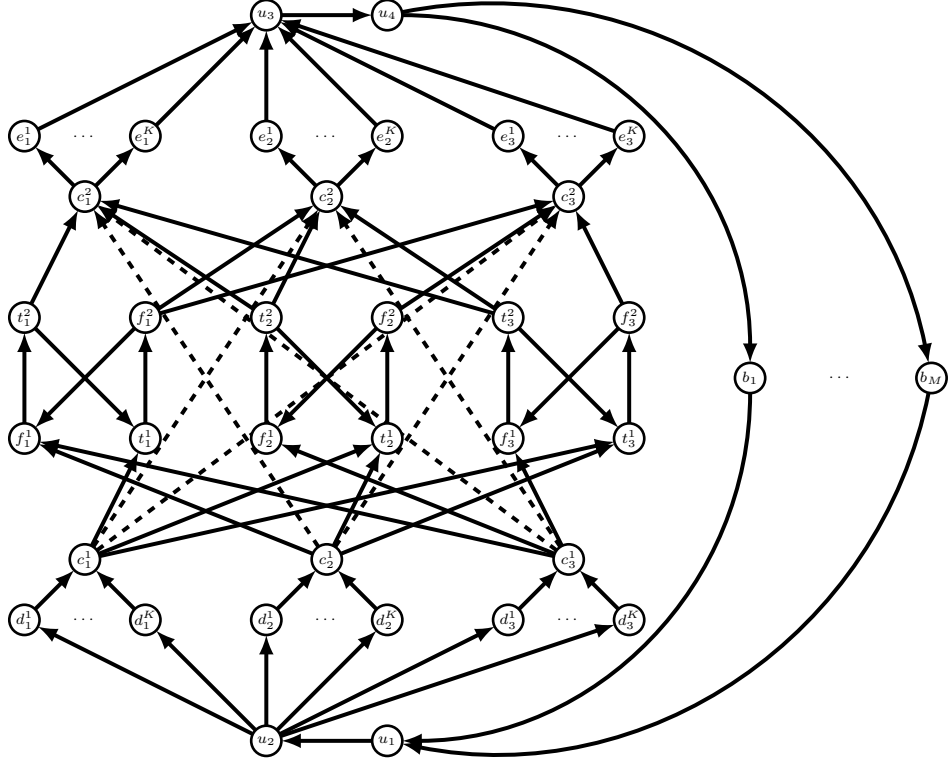


Figure 2: An example of the reduction of 3-SAT to MRET. The 3-SAT formula is $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$.

Block gadget V contains the nodes u_1, u_2, u_3 , and u_4 , and the nodes b_i , for $i \in [M]$ (the value of M will be specified later in the proof). E contains the edges $\{(b_i, u_1) : i \in [M]\}$, (u_1, u_2) , $\{(u_2, d_j^l) : j \in [m], l \in [K]\}$, $\{(e_j^l, u_3) : j \in [m], l \in [K]\}$, (u_3, u_4) , and $\{(u_4, b_i) : i \in [M]\}$.

Note that D is strongly connected. Indeed, let us consider the cycles

$$C_{i,j,l,p} = \langle u_1, u_2, d_j^l, c_j^1, t_i^1, f_i^2, f_i^1, t_i^2, c_j^2, e_j^l, u_3, u_4, b_p, u_1 \rangle,$$

where $j \in [m]$, i is such that x_i is a literal of the clause c_j , $l \in [K]$, and $p \in [M]$, and the cycles

$$C_{i,j,l,p} = \langle u_1, u_2, d_j^l, c_j^1, f_i^1, t_i^2, t_i^1, f_i^2, c_j^2, e_j^l, u_3, u_4, b_p, u_1 \rangle,$$

where $j \in [m]$, i is such that $\neg x_i$ is a literal of the clause c_j , $l \in [K]$, and $p \in [M]$. The union of these cycles contains each node in V , and each of these cycles contains node u_1 . This proves the strong connectivity of D .

In the following, B denotes the set $\{b_i : i \in [M]\}$ and H denotes the set of nodes which do not belong to the block gadget, that is, $H = V \setminus (\{u_1, u_2, u_3, u_4\} \cup B)$ (note that $|V| = M + |H| + 4$ and that $|H| = 2(K + 1)m + 4n$).

Activation of pairs of nodes in a variable gadget Consider a variable x_i and the associated variable gadget and a schedule S of the 4 edges associated to the variable gadget. We say that S *activates* the pair (t_i^1, t_i^2) (respectively, (f_i^1, f_i^2)) if t_i^2 (respectively, f_i^2) is S -reachable from t_i^1 (respectively, f_i^1) within the gadget, that is (t_i^1, f_i^2) , (f_i^2, f_i^1) , and (f_i^1, t_i^2) (respectively, (f_i^1, t_i^2) , (t_i^2, t_i^1) , (t_i^1, f_i^2)) are scheduled in that order. Note that no schedule can activate both (t_i^1, t_i^2) and (f_i^1, f_i^2) as the edge (t_i^1, f_i^2) is scheduled either before or after the edge (f_i^1, t_i^2) .

Constructing a schedule from a satisfying assignment. Suppose that there is an assignment α that satisfies Φ , and let us consider the following schedule S . First we schedule the edges in $\{(b_i, u_1) : i \in [M]\}$ (in any arbitrary order), then the edge (u_1, u_2) , and then the edges in $\{(u_2, d_j^i) : j \in [m], i \in [K]\}$ (in any arbitrary order). Then we schedule the edges $\{(d_j^i, c_j^1) : j \in [m], i \in [K]\}$ (in any arbitrary order), and, then, the edges going out from the nodes c_j^1 , for $j \in [m]$ (in any arbitrary order). Then, for each $i \in [n]$, if $\alpha(x_i) = \text{TRUE}$, we schedule the edges (t_i^1, f_i^2) , (f_i^2, f_i^1) , (f_i^1, t_i^2) , and (t_i^2, t_i^1) in this order (thus activating (t_i^1, t_i^2)). Otherwise (that is, $\alpha(x_i) = \text{FALSE}$), we schedule the edges (f_i^1, t_i^2) , (t_i^2, t_i^1) , (t_i^1, f_i^2) , and (f_i^2, f_i^1) in this order (thus activating (f_i^1, f_i^2)). Then we schedule all the edges entering the nodes c_j^2 , for $j \in [m]$ (in any arbitrary order), and then all the edges going out from the nodes c_j^2 , for $j \in [m]$ (in any arbitrary order). Finally, we schedule the edges in $\{(e_j^l, u_3) : j \in [m], l \in [K]\}$ (in any arbitrary order), then the edge (u_3, u_4) , and all the edges in $\{(u_4, b_i) : i \in [M]\}$ (in any arbitrary order). Let G be the temporal graph induced by D and the schedule S .

First observe that, for any clause c_j with $j \in [m]$, there exists a literal that satisfies c_j according to the assignment α . Let x_i (respectively, $\neg x_i$) be a literal satisfying c_j . Since (t_i^1, t_i^2) (respectively, (f_i^1, f_i^2)) is activated, there exists a temporal path from c_j^1 to c_j^2 that goes through variable gadget corresponding to x_i . This means that $c_j^2 \in \mathcal{R}_G(c_j^1)$ and that, for $l, l' \in [K]$, $e_j^{l'} \in \mathcal{R}_G(d_j^l)$. We now prove a lower bound \mathbb{L} on the S -reachability by showing a lower bound on the number of nodes temporally reachable from each possible source.

- For any $v \in V$ and for $i \in [M]$, $v \in \mathcal{R}_G(b_i)$. This adds $M(M + |H| + 4)$ to \mathbb{L} .
- For $i \in [4]$ and for $j \in [M]$, $b_j \in \mathcal{R}_G(u_i)$. Moreover, $\{u_1, u_2, u_3, u_4\} \cup H \subseteq \mathcal{R}_G(u_1)$, $\{u_2, u_3, u_4\} \cup H \subseteq \mathcal{R}_G(u_2)$, $u_3, u_4 \in \mathcal{R}_G(u_3)$, and $u_4 \in \mathcal{R}_G(u_4)$. This adds $4M + 2|H| + 10$ to \mathbb{L} .

- For $j, h \in [m]$, $i, l \in [K]$, and $p \in [M]$, $c_h^2, e_h^l \in \mathcal{R}_G(d_j^i)$ (because of the above observation) and $b_p \in \mathcal{R}_G(d_j^i)$. This adds $Km(M + Km + m)$ to \mathbb{L} .
- For $j, h \in [m]$, $l \in [K]$, and $i \in [M]$, $c_h^2, e_h^l, b_i \in \mathcal{R}_G(c_j^1)$. This adds $m(M + Km + m)$ to \mathbb{L} .
- For $i \in [n]$, there exists $j \in [m]$ such that c_j is satisfied by $\alpha(x_i)$. Hence, for $p \in [2]$, $l \in [K]$, and $h \in [M]$, $e_j^l, b_h \in \mathcal{R}_G(t_i^p)$ and $e_j^l, b_h \in \mathcal{R}_G(f_i^p)$. This adds $4n(M + K)$ to \mathbb{L} .
- For $j \in [m]$, $l \in [K]$, and $h \in [M]$, $e_j^l, b_h \in \mathcal{R}_G(c_j^2)$. This adds $m(M + K)$ to \mathbb{L} .
- For $j \in [m]$, $l \in [K]$, and $h \in [M]$, $b_h \in \mathcal{R}_G(e_j^l)$. This adds MKm to \mathbb{L} .

Thus, the S -reachability is at least

$$\begin{aligned} \mathbb{L} = & M(M + |H| + 4) + (4M + 2|H| + 10) + Km(M + Km + m) \\ & + m(M + Km + m) + 4n(M + K) + m(M + K) + MKm. \end{aligned}$$

Bounding reachability when Φ is not satisfiable. Let us set M equal to any value greater than $(|H| + 5)^2$. We now prove that, if there exists no truth-assignment satisfying the formula Φ , then no schedule S can have S -reachability greater than or equal to \mathbb{L} . First notice that if S assigns to the edge (u_3, u_4) a starting time smaller than the starting time assigned to (u_1, u_2) , then the S -reachability is less than \mathbb{L} . This is because, in this case, for $i, j \in [M]$ with $i \neq j$, b_j is not S -reachable from b_i . Hence, the S -reachability is bounded by $\mathbb{U}_1 = M(|H| + 4 + 1) + (|H| + 4)(M + |H| + 4)$: this would happen if, for each node $v \notin B$, $\mathcal{R}_G(v) = V$. Since $\mathbb{L} > M^2$, $\mathbb{U}_1 = M(|H| + 4 + 1) + (|H| + 4)(M + |H| + 4) = 2M(|H| + 4) + (|H| + 4)^2 + M < M(|H| + 5)^2$, and $M > (|H| + 5)^2$, it holds that $\mathbb{L} > \mathbb{U}_1$. We can then focus on schedules that assign to the edge (u_1, u_2) a starting time smaller than the starting time assigned to the edge (u_3, u_4) . Let S be such a schedule and let G be the temporal graph induced by D and S . We now prove an upper bound \mathbb{U}_2 on the S -reachability by giving an upper bound on the nodes reachable from each possible source. Observe that, for any two nodes u and v , v might belong to $\mathcal{R}_G(u)$ only if in D there exists a path from u to v that does not include the edge (u_3, u_4) before the edge (u_1, u_2) .

- For $i \in [M]$, $|\mathcal{R}_G(b_i)| \leq |V|$. This adds $M(M + |H| + 4)$ to \mathbb{U}_2 .
- For $i \in [2]$, $|\mathcal{R}_G(u_i)| \leq |V|$, while $|\mathcal{R}_G(u_3)| \leq M + 3$ and $|\mathcal{R}_G(u_4)| \leq |V| = M + |H| + 4$.
- For $j \in [m]$ and $i \in [K]$, in the best case $\mathcal{R}_G(d_j^i)$ contains d_j^i, c_j^1 , the 12 nodes corresponding to the three variables appearing in c_j , and the nodes in $\{c_h^2 : h \in [m]\} \cup \{e_h^l : h \in [m], l \in [K]\} \cup \{u_1, u_3, u_4\} \cup B$, yielding $|\mathcal{R}_G(d_j^i)| \leq M + Km + m + 17$. However, we can show that there exists an index j^* such that, for $l, l' \in [K]$, $e_{j^*}^{l'} \notin \mathcal{R}_G(d_{j^*}^l)$, implying that the

d -nodes add at most $Km(M + Km + m + 17) - K^2$ to \mathbb{U}_2 . For defining j^* , we consider the following truth-assignment α : for any variable x_i with $i \in [n]$, $\alpha(x_i) = \text{TRUE}$ if (t_i^1, f_i^2) is scheduled before (f_i^1, t_i^2) , otherwise $\alpha(x_i) = \text{FALSE}$. Note that if $\alpha(x_i) = \text{TRUE}$ (respectively, $\alpha(x_i) = \text{FALSE}$) we know that S does not activate (f_i^1, f_i^2) (respectively, (t_i^1, t_i^2)). Since the formula Φ is not satisfiable there exists $j^* \in [m]$ such that c_{j^*} is not satisfied by α . Let x_i (respectively, $\neg x_i$) be a literal in c_{j^*} . Since c_{j^*} is not satisfied by α , we that $\alpha(x_i) = \text{FALSE}$ (respectively, $\alpha(x_i) = \text{TRUE}$) and that (t_i^1, t_i^2) (respectively, (f_i^1, f_i^2)) is not activated. It is thus impossible to reach $c_{j^*}^2$ from $c_{j^*}^1$ through the variable gadget of x_i . On the other hand, in all the other walks in D that connect $c_{j^*}^1$ to $c_{j^*}^2$ the edge (u_3, u_4) appears before the edge (u_1, u_2) . Hence, $c_{j^*}^2 \notin \mathcal{R}_G(c_{j^*}^1)$ and $e_{j^*}^{l'} \notin \mathcal{R}_G(d_{j^*}^l)$ for $l, l' \in [K]$.

- For $j \in [m]$, in the best case $\mathcal{R}_G(c_j^1)$ contains c_j^1 , the 12 nodes corresponding to the three variables appearing in clause c_j , and the nodes in $\{c_h^2 : h \in [m]\} \cup \{e_h^l : h \in [m], l \in [K]\} \cup \{u_1, u_3, u_4\} \cup B$. This adds $m(M + Km + m + 16)$ to \mathbb{U}_2 .
- For $i \in [n]$ and for $j \in [2]$, in the best case $\mathcal{R}_G(t_i^j)$ and $\mathcal{R}_G(f_i^j)$ contain the corresponding four variable nodes and the nodes in $\{c_h^2 : h \in [m]\} \cup \{e_h^l : h \in [m], l \in [K]\} \cup \{u_1, u_3, u_4\} \cup B$. This adds $4n(M + Km + m + 7)$ to \mathbb{U}_2 .
- For $j \in [m]$, in the best case $\mathcal{R}_G(c_j^2)$ contains c_j^2 and the nodes in $\{e_j^l : l \in [K]\} \cup \{u_1, u_3, u_4\} \cup B$. This adds $m(M + K + 4)$ to \mathbb{U}_2 .
- For $j \in [m]$ and $i \in [K]$, in the best case $\mathcal{R}_G(e_j^i)$ contains e_j^i and the nodes in $\{u_1, u_3, u_4\} \cup B$. This adds with $Km(M + 4)$ to \mathbb{U}_2 .

In summary,

$$\begin{aligned} \mathbb{U}_2 &= M(M + |H| + 4) + (4M + 3|H| + 15) + (Km(M + Km + m + 17) - K^2) \\ &\quad + m(M + Km + m + 16) + 4n(M + Km + m + 7) + m(M + K + 4) \\ &\quad + Km(M + 4). \end{aligned}$$

We have that $\mathbb{L} - \mathbb{U}_2 = -|H| - 5K^2 - 21Km - 4n(K(m - 1) + m + 7) - 20m = K^2 - 23Km - 4n(K(m - 1) + m + 8) - 22m - 5 > K^2 - Knm(23 + 4(1 + 1 + 8) + 22 + 5) = K^2 - 90Knm$ using $K, n, m \geq 1$. Let us set K equal to any value greater than or equal to $91nm$. We then have $K^2 > 90Knm$ and, thus, $\mathbb{L} > \mathbb{U}_2$. That is, the S -reachability has to be smaller than \mathbb{L} .

Conclusion. We have thus proved that the formula Φ is satisfiable if and only if there exists a schedule S such that the S -reachability of D is at least \mathbb{L} . This completes the proof of the theorem. \square \square

3 Conclusion and open problems

In this paper, we have considered MRET problem, that is, the problem of assigning appearing times to the edges of a digraph in order to maximize the total reachability of the resulting temporal graph. We have proved that this problem is NP-hard, even when the digraph is strongly connected. We conjecture that the MRET problem can be approximated within a constant approximation ratio. In particular, we conjecture that any strongly connected digraph admits an edge temporalisation with temporal reachability at least equal to $c \cdot n^2$ for some constant $c > 0$. One way to prove such a statement would be to prove the following interesting graph theory conjecture.

Almost Spanning Two Rooted-Arborescences conjecture (ASTRA). *Any strongly connected digraph admits an out-arborescence and an in-arborescence that are edge-disjoint, have the same root, and each spans $\Omega(n)$ nodes.*

Note that it is not difficult to prove that the root of the two arborescences mentioned in the ASTRA conjecture cannot be any node in the graph. For example, let us consider the graph shown in Figure 3. In this case, the node x_1 cannot be the common root of the two arborescences, since the only in-arborescence and the only out-arborescence with root x_1 share the edge (x_2, y_2) , so that one of the two arborescences cannot include more than one node (of course, this example can be generalized to any even number of nodes).

Note also that the ASTRA conjecture is false if we require that the two arborescences span at least $\frac{n}{3-\epsilon}$ nodes, for any positive constant ϵ . For example, consider the digraph $G = (V, E)$ shown in Figure 4, where, for some integer parameter $k > 0$, the set of nodes is $V = \{x, y, x_1, y_1, x_2, y_2, x_3, y_3\} \cup \{z_{i,j} \mid 1 \leq i \leq 3, 1 \leq j \leq k\}$, and the set of edges is $E = \{(x, y)\} \cup \{(y, x_i), (x_i, y_i), (y_i, x), (y_i, z_{i,1}), (z_{i,k}, x_i) \mid 1 \leq i \leq 3\} \cup \{(z_{i,j}, z_{i,j+1}) \mid 1 \leq i \leq 3, 1 \leq j < k\}$. Observe that the total number of nodes is $n = 3k + 8$. Let us first give an upper bound on the minimum between the amount of nodes in the in-arborescence and in the out-arborescence in the case where the root is not x nor y . For any such node, either the in-arborescence or the out-arborescence can contain at most $k + 3$ nodes, since the edge (x, y) can be in one arbores-

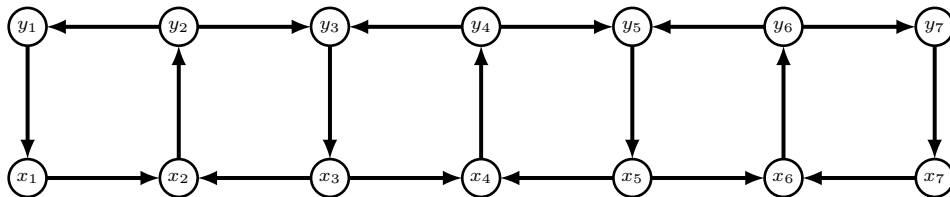


Figure 3: An example of a digraph for which only some nodes can be roots of two arborescences each spanning $\Omega(n)$ nodes.

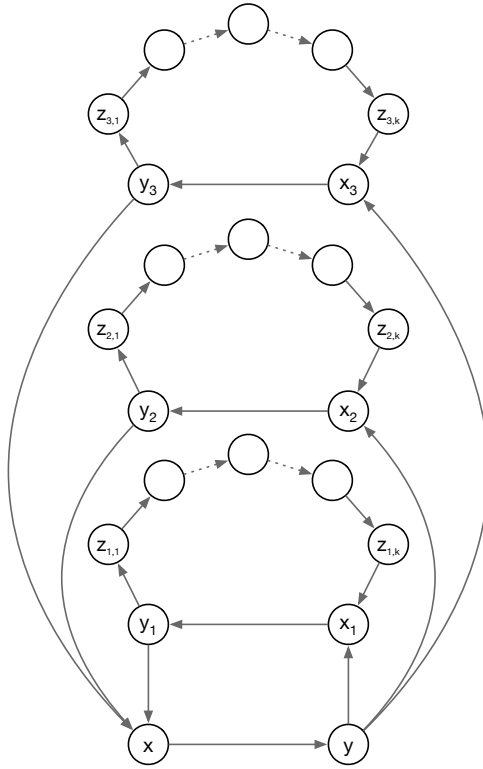


Figure 4: An example of a digraph for which there are no two edge-disjoint arborescences with a common root and each spanning more than $n/3 + c$ nodes, for some positive constant c .

cence only. Consider now the case in which either x or y is the root. Let us suppose that the root is x (the other case can be analysed in a similar way). Since, for each $i = 1, 2, 3$, the edge (x_i, y_i) can be in one arborescence only, then either the in-arborescence or the out-arborescence rooted at x can contain at most $n - 2k = k + 8$ nodes. We thus obtained that, in all cases, either the in-arborescence or the out-arborescence is upper bounded by $k + 8 = n/3 + O(1)$.

References

- [1] Kenneth A. Berman. Vulnerability of scheduled networks and a generalization of menger's theorem. *Networks*, 28(3):125–134, 1996.
- [2] S. Bhadra and A. Ferreira. Complexity of connected components in evolving graphs and the computation of multicast trees in dynamic networks. In *ADHOC-NOW*, pages 259–270, 2003.

- [3] F. Brunelli, P. Crescenzi, and L. Viennot. On computing pareto optimal paths in weighted time-dependent networks. *Information Processing Letters*, 168:106086, 2021.
- [4] Filippo Brunelli, Pierluigi Crescenzi, and Laurent Viennot. Maximizing reachability in a temporal graph obtained by assigning starting times to a collection of walks. *Networks*, pages 1–27, 2002.
- [5] A. Casteigts, P. Flocchini, W. Quattrociocchi, and N. Santoro. Time-varying graphs and dynamic networks. *IJPEDS*, 27(5):387–408, 2012.
- [6] Eddie Cheng, Jerrold W Grossman, and Marc J Lipman. Time-stamped graphs and their associated influence digraphs. *Discrete Applied Mathematics*, 128(2):317–335, 2003.
- [7] F. Corò, G. D’Angelo, and Y. Velaj. Recommending links to maximize the influence in social networks. In *IJCAI*, pages 2195–2201, 2019.
- [8] J.A. Enright, K. Meeks, G.B. Mertzios, and V. Zamaraev. Deleting edges to restrict the size of an epidemic in temporal networks. In *MFCs*, pages 57:1–57:15, 2019.
- [9] F. Göbel, J. Orestes Cerdeira, and H.J. Veldman. Label-connected graphs and the gossip problem. *Discrete Mathematics*, 87(1):29–40, 1991.
- [10] F. Harary and G. Gupta. Dynamic graph models. *Mathematical and Computer Modelling*, 25(7):79–87, 1997.
- [11] P. Holme and J. Saramäki. Temporal networks. *Physics Reports*, 519(3):97–125, 2012.
- [12] Petter Holme and Jari Sarämäki. *Temporal Networks*. Springer, Berlin, Heidelberg, 2013.
- [13] D. Kempe, J. Kleinberg, and A. Kumar. Connectivity and inference problems for temporal networks. *Journal of Computer and System Sciences*, 64(4):820 – 842, 2002.
- [14] M. Latapy, T. Viard, and C. Magnien. Stream graphs and link streams for the modeling of interactions over time. *Social Netw. Analys. Mining*, 8(1):61:1–61:29, 2018.
- [15] Naoki Masuda and Renaud Lambiotte. *A Guide to Temporal Networks*. World Scientific, Singapore, 2016.
- [16] G.B. Mertzios, O. Michail, I. Chatzigiannakis, and P.G. Spirakis. Temporal network optimization subject to connectivity constraints. In *ICALP*, pages 657–668, 2013.

- [17] G.B. Mertzios, O. Michail, and P.G. Spirakis. Temporal network optimization subject to connectivity constraints. *Algorithmica*, 81(4):1416–1449, 2019.
- [18] George B. Mertzios, Hendrik Molter, Malte Renken, Paul G. Spirakis, and Philipp Zschoche. The complexity of transitively orienting temporal graphs. In *MFCS*, volume 202 of *LIPICs*, pages 75:1–75:18, 2021.
- [19] O. Michail. An introduction to temporal graphs: An algorithmic perspective. *Internet Mathematics*, 12(4):239–280, 2016.
- [20] Hendrik Molter, Malte Renken, and Philipp Zschoche. Temporal reachability minimization: Delaying vs. deleting. In *MFCS*, volume 202 of *LIPICs*, pages 76:1–76:15, 2021.