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Konstantin Zimenko [a], Andrey Polyakov [b,a], Denis Efimov [b,a]

[a] *Faculty of Control Systems and Robotics, ITMO University, 49 Kronverkskiy av., 197101 Saint Petersburg, Russia*

[b] *Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France*

Abstract

The paper develops control algorithms for a class of affine nonlinear systems using the so-called canonical homogeneous representation. It is demonstrated that such a representation exists for any homogeneous vector field bounded on the unit sphere. It is shown that canonical homogeneous representation is useful for LMI-based control design and stability analysis of nonlinear systems. Theoretical results are supported by numerical examples.

1 Introduction

Homogeneous systems are widely used in stability analysis, nonlinear control and observer design (see, for example, [1]-[25]). Additional interest to homogeneity (homogenization) based approaches consists in their practically important robustness (see [1]-[4]), e.g., Input-to-State Stability, non-Lipschitz disturbance rejection, etc.

Stability (stabilizability) analysis for homogeneous systems is the subject of many studies. In [9], [15] it is shown that for a stable (weighted) homogeneous system there exists a corresponding homogeneous Lyapunov function. The papers [12], [16] present conditions for the existence of stabilizing homogeneous feedbacks. Note that for stable homogeneous systems rates of convergence (exponential, finite-time, nearly fixed-time) can be determined based on the sign of homogeneity degree (see, e.g., [5], [6], [7]). The paper [17] develops the result [15] for the class of *generalized* homogeneous systems. It is shown that any generalized homogeneous system is diffeomorphic to a standard homogeneous one, i.e., there exists a change of coordinates, which transforms an asymptotically stable (generalized) homogeneous system to a quadratically stable one. In some particular cases (mostly for linear and close to linear systems, e.g. [25]) these results jointly with implicit Lyapunov function approach allow to formulate useful stability (stabilizability) conditions in terms of linear matrix inequalities (LMIs).

In order to obtain simple LMI-based stability (stabilizability) conditions and control tuning algorithms the embedding of a nonlinear system into a linear differential inclusion can be used (e.g., see [26], [27], [28]). The similar idea was utilized in this paper to introduce a canonical representation for homogeneous systems. Two algorithms for derivation of the canonical homogeneous representation are presented for smooth (differentiable) and non-smooth vector fields, respectively. In the latter case, the boundedness on the unit sphere of the homogeneous vector field is required. Based on this representation a stability analysis method and two control design algorithms with LMI-based stability (stabilizability) conditions are proposed for a class of non-linear homogeneous systems. These results are extended for some non-homogeneous systems as well.

The preliminary results of this paper are discussed at IFAC World Congress [18] and European Control Conference [23]. The key differences are as follows:

- the proofs of all claims are given;
- new LMI-based stability conditions for non-linear homogeneous systems are obtained (Lemma 4);
- new schemes simplifying control design are presented (Remarks 4, 8, 9, Algorithms 1, 2, 3);
- new numerical examples demonstrating efficiency of the proposed control design schemes are considered.

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Corresponding author K. Zimenko.

Email addresses: `konstantin.zimenko@itmo.ru` (Konstantin Zimenko), `andrey.polyakov@inria.fr` (Andrey Polyakov), `denis.efimov@inria.fr` (Denis Efimov).

Notation: $\mathbb{R}_+ = \{x \in \mathbb{R}\colon x > 0\}$, where $\mathbb{R}$ is the field of real numbers; the symbol $\overline{1,m}$ is used to denote a sequence of integers $1,...,m$; $\mathrm{diag}\{a_i\}_{i=1}^n$ is the diagonal matrix with the elements a_i on the main diagonal; $\mathcal{C}^n(X,Y)$ is the set of continuously differentiable (at least up to the order n) functions $X \to Y$, where X and Y are open subsets of $\mathbb{R}^n$; $e_s(i) = \Big(\underbrace{0 \cdots 0 \overbrace{1}^{i\text{th}} 0 \cdots 0}_{s \text{ components}}\Big)^T \in \mathbb{R}^s$, $s \geq 1$; $\mathtt{1}_n$ is the vector of dimension n with all elements equal 1; the inequality $P > 0$ ($P \geq 0$) means that a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$ is positive definite (positive semi-definite); $\|\cdot\|$ denotes a norm; $\mathbb{S} = \{x \in \mathbb{R}^n\colon \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

2 Problem statement

Consider the system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \tag{1}$$

where $x \in \mathbb{R}^n$ are states, $u = (u_1, ..., u_m)^T \in \mathbb{R}^m$ are controls, $f_0(0) = 0$, f_0 and f_i are homogeneous (in the sense explained below) or almost homogeneous (presented by product of a homogeneous vector field and a bounded scalar function). The state vector x is assumed to be measured.

The main goal is to derive a LMI-based stability condition for the open-loop system ($u_i = 0$) and to propose LMI-based design of a feedback control algorithm for the system (1). In order to obtain constructive stability conditions and control tuning algorithms, a specific canonical representation of a homogeneous system needs to be obtained using the so-called convex embedding (see, e.g., [26, Chapter 4]).

3 Preliminaries

3.1 Stability Notions

Consider the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $f(0) = 0$.

Definition 1 [1], [22], [7] *A set $M \subset \mathbb{R}^n$ is said to be globally* ***finite-time attractive*** *for (2) if any solution $x(t, x_0)$ of (2) reaches M in some finite time moment $t = T(x_0)$ and remains there for all $t \geq T(x_0)$, where $T\colon \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the so-called settling-time function. It is* ***fixed-time attractive*** *if in addition the settling-time function $T(x_0)$ is globally bounded by some number $T_{\max} > 0$. If $M = \{0\}$ is globally finite/fixed-time attractive and Lyapunov stable then the origin is said to be* ***globally finite/fixed-time stable****. The origin is said to be* ***nearly fixed-time stable*** *if it is globally Lyapunov stable and any neighborhood of the origin is fixed-time attractive.*

Theorem 1 [22], [19] *Suppose there exist a positive definite $\mathcal{C}^1$ function V defined on an open neighborhood of the origin $D \subset \mathbb{R}^n$ and real numbers $C > 0$ and $\sigma \geq 0$, such that for $x \in D\backslash\{0\}$ the condition $\dot{V}(x) \leq -CV^\sigma(x)$ is true for the system (2). Then depending on the value σ the origin is stable with different types of convergence:*

- *if $\sigma = 1$, the origin is asymptotically stable;*
- *if $0 \leq \sigma < 1$, the origin is finite-time stable and $T(x_0) \leq \frac{1}{C(1-\sigma)} V^{1-\sigma}(x_0)$;*
- *if $\sigma > 1$ the origin is nearly fixed-time stable and for every $\varepsilon \in \mathbb{R}_+$, the set $B = \{x \in D\colon V(x) \leq \varepsilon\}$ is fixed-time attractive with $T_{\max} = \frac{1}{C(\sigma-1)\varepsilon^{\sigma-1}}$.*

If $D = \mathbb{R}^n$ and V is radially unbounded, then the mentioned stability properties of the system (2) hold globally.

3.2 'Universal' Stabilizer for Control Affine Systems

Consider a system in the form (1), where $f_0, f_i \in \mathcal{C}^\infty$, and $f_0(0) = 0$. Let this system admit a control-Lyapunov function V that is, a smooth, proper, and positive definite function $\mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ so that

$$\inf_{u\in\mathbb{R}^m} \left\{ \frac{\partial V(x)}{\partial x} f_0(x) + \frac{\partial V(x)}{\partial x} \sum_{i=1}^{m} u_i f_i(x) \right\} < 0, \quad \forall x \neq 0. \tag{3}$$

Definition 2 [29] *The control-Lyapunov function V satisfies the small control property if for each $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some $u \in \mathbb{R}^m$ with $\|u\| < \varepsilon$ such that*

$$\frac{\partial V(x)}{\partial x} f_0(x) + \frac{\partial V(x)}{\partial x} \sum_{i=1}^{m} u_i f_i(x) < 0.$$

Definition 3 [29] *Let $k : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping, smooth on $\mathbb{R}^n \setminus \{0\}$ and $k(0) = 0$. This is a smooth feedback stabilizer provided that the closed-loop system $\dot{x} = f_0(x) + \sum_{i=1}^m k_i(x) f_i(x)$ is globally asymptotically stable, where $k = (k_1, ..., k_m)^T$.*

Theorem 2 [29] *Let V be a smooth control-Lyapunov function V. Then*

(a) there exists a smooth feedback stabilizer k;
(b) if V satisfies the small control property, then k can be chosen to be also continuous at the origin, e.g., in the form

$$k(x) = \begin{cases} \frac{a(x)+\sqrt{a^2(x)+|B(x)|^4}}{|B(x)|^2} B(x), & \text{if } |B(x)| \neq 0, \\ 0, & \text{if } |B(x)| = 0, \end{cases}$$

where $a(x) = \frac{\partial V(x)}{\partial x} f_0(x)$, $B(x) = (b_1, ..., b_m)^T$, $b_i(x) = \frac{\partial V(x)}{\partial x} f_i(x)$ *for* $i = \overline{1, m}$.

3.3 Generalized Homogeneity

Homogeneity is a certain invariance of a mathematical object with respect to a group of transformations called dilations. In this paper we deal with the one-parameter group $\mathbf{d}(s) : \mathbb{R}^n \to \mathbb{R}^n$ of linear dilations given by $\mathbf{d}(s) = e^{G_{\mathbf{d}} s} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}$, $s \in \mathbb{R}$, where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz matrix (i.e., $-G_{\mathbf{d}}$ is Hurwitz) called the generator of the dilation.

Definition 4 [1] *The dilation* $\mathbf{d}$ *is said to be strictly monotone if* $\exists \beta > 0 : \|\mathbf{d}(s)\| < e^{\beta s}$ *for* $s \leq 0$.

Theorem 3 [17] *If* $\mathbf{d}$ *is a dilation in* $\mathbb{R}^n$*, then it is strictly monotone with respect to the norm* $\|x\|_P = \sqrt{x^T P x}$ *for* $x \in \mathbb{R}^n$ *and* P *satisfying*

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^T P > 0, \quad P = P^T > 0. \tag{4}$$

Theorem 3 shows that any dilation $\mathbf{d}$ is strictly monotone if $\mathbb{R}^n$ is equipped with the weighted Euclidean norm $\|x\|_P = \sqrt{x^T P x}$ provided that the matrix P satisfies (4).

Definition 5 [13] *A vector field* $f : \mathbb{R}^n \to \mathbb{R}^n$ *(a function* $h : \mathbb{R}^n \to \mathbb{R}$*) is said to be* $\mathbf{d}$*-homogeneous of degree* $\nu \in \mathbb{R}$ *if*

$$f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \tag{5}$$

$$(\textit{resp. } h(\mathbf{d}(s)x) = e^{\nu s}h(x), \quad \forall x \in \mathbb{R}^n, \ \forall s \in \mathbb{R}.) \tag{6}$$

Let $\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ (respectively $\mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$) be the set of vector fields $\mathbb{R}^n \to \mathbb{R}^n$ (respectively functions $\mathbb{R}^n \to \mathbb{R}$) satisfying the identity (5) (respectively (6)), which are continuous on $\mathbb{R}^n \setminus \{0\}$. Let $\deg_{\mathbb{F}_{\mathbf{d}}}(f)$ (respectively $\deg_{\mathbb{H}_{\mathbf{d}}}(f)$) denote the homogeneity degree of $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ (respectively $h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$).

A special case of homogeneous function is a homogeneous norm [11], [16], [17]: a continuous positive definite $\mathbf{d}$-homogeneous function of degree 1. For monotone dilations we define the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ as

$$\|x\|_{\mathbf{d}} = e^{s_x}, \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, x \neq 0 \tag{7}$$

and, by continuity, we assign $\|0\|_{\mathbf{d}} = 0$ (see, [17] for more details). Notice that (7) also implies

$$\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x\| = 1. \tag{8}$$

Lemma 1 [17] *If* $\mathbf{d}$ *is a strictly monotone dilation, then*

- *the homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *is Lipschitz continuous outside the origin;*
- *if the norm* $\|\cdot\|$ *is smooth outside the origin, then the homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *is also smooth outside of the origin and*

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \left. \frac{\|x\|_{\mathbf{d}} \left.\frac{\partial \|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x}}{\left.\frac{\partial \|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x} G_{\mathbf{d}}\mathbf{d}(-s)x} \right|_{s=\ln \|x\|_{\mathbf{d}}}. \tag{9}$$

Lemma 2 [17] *The vector field* $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *is Lipschitz continuous (smooth) on* $\mathbb{R}^n \setminus \{0\}$ *if and only if it satisfies a Lipschitz condition (it is smooth) on the unit sphere* $\mathbb{S}$*, provided that* $\mathbf{d}$ *is strictly monotone on* $\mathbb{R}^n$.

If a function (or a vector field) is smooth, then homogeneity is inherited by its derivatives in a certain way. The following lemma is an extension of the well-known Euler's homogeneous function theorem to the case of linear homogeneity.

Lemma 3 [1, pages 211, 212], [17, Corollary 2], [18] *If* $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n) \cap \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ *and* $h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$*, then*

$$e^{\deg_{\mathbb{H}_{\mathbf{d}}}(h)s} \frac{\partial h(x)}{\partial x} = \left.\frac{\partial h(z)}{\partial z}\right|_{z=\mathbf{d}(s)x} \mathbf{d}(s), \tag{10}$$

$$e^{\deg_{\mathbb{F}_{\mathbf{d}}}(f)s} \mathbf{d}(s) \frac{\partial f(x)}{\partial x} = \left.\frac{\partial f(z)}{\partial z}\right|_{z=\mathbf{d}(s)x} \mathbf{d}(s), \tag{11}$$

$$\frac{\partial h(x)}{\partial x} G_{\mathbf{d}} x = \deg_{\mathbb{H}_{\mathbf{d}}}(h)h(x), \tag{12}$$

$$\frac{\partial f(x)}{\partial x} G_{\mathbf{d}} x = (\deg_{\mathbb{F}_{\mathbf{d}}}(f)I_n + G_{\mathbf{d}})f(x) \tag{13}$$

for $x \in \mathbb{R}^n \setminus \{0\}$ *and* $s \in \mathbb{R}$.

Due to topological equivalence of $\mathbf{d}$-homogeneous system to a standard homogeneous one [17], all results about stability and robustness of standard and weighted homogeneous systems hold for $\mathbf{d}$-homogeneous systems as well. For example, the convergence rate of homogeneous systems can be assessed by its homogeneity degree:

Theorem 4 [19] *An asymptotically stable* $\mathbf{d}$*-homogeneous system* $\dot{x} = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^n$ *is finite-time stable if* $\deg_{\mathbb{F}_{\mathbf{d}}}(f) < 0$ *(nearly fixed-time stable if* $\deg_{\mathbb{F}_{\mathbf{d}}}(f) > 0$*).*

The homogeneity provides many other advantages to analysis and design of nonlinear control system. For example, homogeneous control systems are Input-to-State Stable (ISS) with respect to measurement noise and additive disturbances (see, for example, [1], [2], [3], [4]).

3.4 Convex embedding

Definition 6 [31, pages 10, 12] *A subset* S *of* $\mathbb{R}^n$ *is said to be* ***convex*** *if* $(1-\lambda)x + \lambda y \in S$ *whenever* $x \in S$,

$y \in S$ and $\lambda \in (0,1)$. *The* ***convex hull*** *of a given set S may be defined as the intersection of all convex sets containing S and is denoted as co S.*

Theorem 5 (Carathéodory's theorem) [32, page 62] *For any bounded closed set $S \subset \mathbb{R}^n$ any point $x \in$ co S can be represented in the form $x = \sum_{i=0}^{k} \alpha_i x_i$, where $x_i \in S$, $\alpha_i \in \mathbb{R}_+ \cup \{0\} : \sum_{i=0}^{k} \alpha_i = 1$ and $k \leq n$.*

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, where $\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}$, $i, j = \overline{1, n}$. Let us show how the convex embedding concept can be used in order to represent the matrix A as a convex combination. Let us define the set $\mathcal{A} = \left\{a = (a_{11}, ..., a_{1n}, ..., a_{nn}) : \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}, i, j = \overline{1, n}\right\}$. The set $\mathcal{A} \subset \mathbb{R}^{n^2}$ is a bounded convex domain of which the set of vertices is defined by $\mathcal{V}_a = \left\{\vartheta = (\vartheta_{11}, ..., \vartheta_{1n}, ..., \vartheta_{nn}) : \vartheta_{jl} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}, i, j = \overline{1, n}\right\}$. Then applying the Carathéodory's theorem, for any $a \in \mathcal{A}$ we obtain $a = \sum_{i=0}^{k} \alpha_i a_i$, where $a_i \in \mathcal{V}_a$, $\sum_{i=0}^{k} \alpha_i = 1$, $k \leq n^2$ or $A = \sum_{i=0}^{k} \alpha_i A_i$, where $A_i \in \mathcal{V}_A$, $\mathcal{V}_A = \left\{\vartheta = \begin{pmatrix} \vartheta_{11} & \cdots & \vartheta_{1n} \\ \vdots & \ddots & \vdots \\ \vartheta_{n1} & \cdots & \vartheta_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n} : \vartheta_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}, i, j = \overline{1, n}\right\}$. In [26], [28] the convex embedding concept was used for embedding of a differentiable nonlinear system into a linear differential inclusion.

4 Convex embedding for homogeneous systems

4.1 Convex Embedding for Differentiable Homogeneous Systems

Let us consider a system in the form

$$\dot{x}(t) = f(x(t)), \quad t \geq 0 \tag{14}$$

where $x(t) \in \mathbb{R}^n$, $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$, $\deg_{\mathbb{F}_{\mathbf{d}}}(f) = \nu \in \mathbb{R}$, f is locally bounded.

Let $f \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. For $i, j = \overline{1, n}$ define the bounds

$\overline{g}_{ij} = \sup_{y \in \mathbb{R}^n : \|y\|=1} \left.\frac{\partial f_i(z)}{\partial z_j}\right|_{z=y}$, $\underline{g}_{ij} = \inf_{y \in \mathbb{R}^n : \|y\|=1} \left.\frac{\partial f_i(z)}{\partial z_j}\right|_{z=y}$, that are always exist due to $f \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$. Denote the set of vertices defined by

$$\mathcal{V}_g = \left\{\vartheta = \begin{pmatrix} \vartheta_{11} & \cdots & \vartheta_{1n} \\ \vdots & \ddots & \vdots \\ \vartheta_{n1} & \cdots & \vartheta_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n} : \vartheta_{ij} \in \{\underline{g}_{ij}, \overline{g}_{ij}\}, i, j = \overline{1, n}\right\}. \tag{15}$$

Proposition 1 *Let the matrix $\nu I_n + G_{\mathbf{d}}$ be invertible. There exist $\alpha_i : \mathbb{S} \to \mathbb{R}_+ \cup \{0\} : \sum_{i=1}^{N} \alpha_i(x) = 1$ and $M_i \in \mathcal{V}_g$, $i = 1, ..., N \leq 2^{n^2}$ such that every trajectory $x(t)$ of (14) is also a trajectory of*

$$\dot{x} = \|x\|_{\mathbf{d}}^{\nu} \sum_{i=1}^{N} \alpha_i(p(x)) \mathbf{d}(\ln \|x\|_{\mathbf{d}}) A_i p(x), \quad t \geq 0 \tag{16}$$

as long as $x(t) \neq 0$, where $A_i = (\nu I_n + G_{\mathbf{d}})^{-1} M_i G_{\mathbf{d}}$ and $p(x) = \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$ is the so-called homogeneous projection of the vector x to the unit sphere.

Proof. Due to $\nu I_n + G_{\mathbf{d}}$ is invertible then by (13) we have

$$f(x) = (\nu I_n + G_{\mathbf{d}})^{-1} \frac{\partial f(x)}{\partial x} G_{\mathbf{d}} x \tag{17}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. Consider (11) with $s = -\ln \|x\|_{\mathbf{d}}$ then one can obtain

$$\begin{aligned} \frac{\partial f(x)}{\partial x} &= e^{-\nu s} \mathbf{d}(-s) \left.\frac{\partial f(z)}{\partial z}\right|_{z=\mathbf{d}(s)x} \mathbf{d}(s) \\ &= e^{\nu \ln \|x\|_{\mathbf{d}}} \mathbf{d}(\ln \|x\|_{\mathbf{d}}) \left.\frac{\partial f(z)}{\partial z}\right|_{z=p(x)} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) \\ &= \|x\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|x\|_{\mathbf{d}}) \left.\frac{\partial f(z)}{\partial z}\right|_{z=p(x)} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) \end{aligned}$$

Substituting the obtained expression for $\frac{\partial f(x)}{\partial x}$ to (17) and taking into account $(\nu I_n + G_{\mathbf{d}})^{-1} \mathbf{d}(\ln \|x\|_{\mathbf{d}}) = \mathbf{d}(\ln \|x\|_{\mathbf{d}}) (\nu I_n + G_{\mathbf{d}})^{-1}$ and $\mathbf{d}(-\ln \|x\|_{\mathbf{d}}) G_{\mathbf{d}} = G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})$ we have

$$\begin{aligned} f(x) &= (\nu I_n + G_{\mathbf{d}})^{-1} \frac{\partial f(x)}{\partial x} G_{\mathbf{d}} x \\ &= \|x\|_{\mathbf{d}}^{\nu} (\nu I_n + G_{\mathbf{d}})^{-1} \mathbf{d}(\ln \|x\|_{\mathbf{d}}) \left.\frac{\partial f(z)}{\partial z}\right|_{z=p(x)} \\ &\quad \times \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) G_{\mathbf{d}} x \\ &= \|x\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|x\|_{\mathbf{d}}) (\nu I_n + G_{\mathbf{d}})^{-1} \left.\frac{\partial f(z)}{\partial z}\right|_{z=p(x)} G_{\mathbf{d}} p(x) \end{aligned}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. Taking into account that

$$\left.\frac{\partial f(z)}{\partial z}\right|_{z=p(x)} = \sum_{j=1}^{n} \sum_{l=1}^{n} Q_{jl} \left.\frac{\partial f_j(z)}{\partial z_l}\right|_{z=p(x)}$$

with $Q_{jl} = e_n(j) e_n^T(l)$, for $\theta_{jl} \in [\underline{g}_{jl}, \overline{g}_{jl}]$ we obtain

$$\begin{aligned} f(x) &= \|x\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|x\|_{\mathbf{d}}) (\nu I_n + G_{\mathbf{d}})^{-1} \\ &\quad \times \left(\sum_{j=1}^{n} \sum_{l=1}^{n} Q_{jl} \left.\frac{\partial f_j(z)}{\partial z_l}\right|_{z=p(x)} \right) G_{\mathbf{d}} p(x) \\ &= \|x\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|x\|_{\mathbf{d}}) (\nu I_n + G_{\mathbf{d}})^{-1} \\ &\quad \times \left(\sum_{j=1}^{n} \sum_{l=1}^{n} Q_{jl} \theta_{jl} \right) G_{\mathbf{d}} p(x). \end{aligned}$$

Finally, using Carathéodory theorem we can represent the term $\sum_{j=1}^{n} \sum_{l=1}^{n} Q_{jl} \theta_{jl}(z) \in co\ \mathcal{V}_g$ as $\sum_{i=1}^{N} \alpha_i(z) M_i$

with $M_i \in \mathcal{V}_g$. Then all trajectories of (14) are also trajectories of the system (16). $\blacksquare$

Example 1 Consider the system (14) with $f(x) = \begin{pmatrix} -x_1x_3 \\ -x_1 \\ x_2 \end{pmatrix}$ that is $\mathbf{d}$-homogeneous of degree 1, where $G_{\mathbf{d}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a generator of the dilation. According to Proposition 1 the system can be represented in the form (16) with $\|x\| = \sqrt{x^Tx}$, $M_i \in \begin{pmatrix} \{-1,1\} & 0 & \{-1,1\} \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $A_i = \begin{pmatrix} \{-0.75,0.75\} & 0 & \{-0.25,0.25\} \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ for $i = \overline{1,4}$.

In [18] the representation of the function f in the form like (16) was obtained by another constructions. Comparing with [18] the given result allows to obtain vertices A_i significantly closer to each other, that may simplify the stability analysis and control design for essentially nonlinear homogeneous systems.

Example 2 Consider the system (14) with $f(x) = \begin{pmatrix} x_2x_3 \\ x_3 \\ 0 \end{pmatrix}$ that is $\mathbf{d}$-homogeneous with negative degree $\deg_{\mathbb{F}_{\mathbf{d}}} = -1$ and $G_{\mathbf{d}} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Applying Proposition 1 with $\|x\| = \sqrt{x^Tx}$ one can obtain the representation (16), where $A_i = \begin{pmatrix} 0 & \{-0.6,0.6\} & \{-0.4,0.4\} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $i = \overline{1,4}$.

Another advantage with respect to [18] is that the differentiability at the origin is not required. However, the condition $f \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ also restricts sufficiently the class of homogeneous functions for representation in the form (16). This condition, however, can be relaxed as shown below.

4.2 Convex Embedding for Non-Smooth Homogeneous Systems

Let $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ in (14) be locally bounded on $\mathbb{R}^n$ but possibly non-differentiable on a unit sphere and let $P \in \mathbb{R}^{n\times n}$ satisfy (4). Define the bounds

$$\overline{q}_{ij} = \sup_{y\in\mathbb{R}^n:\|y\|_P=1} \frac{e_n^T(i)f(y)y_j}{y^TPG_{\mathbf{d}}y}, \qquad \underline{q}_{ij} = \inf_{y\in\mathbb{R}^n:\|y\|_P=1} \frac{e_n^T(i)f(y)y_j}{y^TPG_{\mathbf{d}}y}$$

and denote the corresponding set of vertices $\mathcal{V}_q$ as above (see, the formula (15)) using the sets $\{\underline{q}_{ij}, \overline{q}_{ij}\}$ instead of $\{\underline{g}_{ij}, \overline{g}_{ij}\}$.

Proposition 2 *There exist* $\alpha_i : \mathbb{S} \to \mathbb{R}_+ \cup \{0\}$: $\sum_{i=1}^N \alpha_i(x) = 1$ *and* $M_i \in \mathcal{V}_q$, $i = 1, ..., N \le 2^{n^2}$ *such that every trajectory* $x(t)$ *of (14) is also a trajectory of (16) as long as* $x(t) \neq 0$, where $A_i = M_iPG_{\mathbf{d}}$.

Proof. Since P satisfies (4), then the dilation $\mathbf{d}$ is strictly monotone on $\mathbb{R}^n$ equipped with the norm $\|x\|_P = \sqrt{x^TPx}$. Note that by (5), for $s = -\ln\|x\|_{\mathbf{d}}$ we have

$$\begin{aligned} f(x) &= \|x\|_{\mathbf{d}}^{\nu}\mathbf{d}(\ln\|x\|_{\mathbf{d}})f(p(x)) \\ &= \|x\|_{\mathbf{d}}^{\nu}\mathbf{d}(\ln\|x\|_{\mathbf{d}})\mathrm{diag}\{e_n^T(i)f(p(x))\}_{i=1}^n \mathbf{1}_n. \end{aligned}$$

Then due to $\left.\frac{\partial\|z\|_{\mathbf{d}}}{\partial z}\right|_{z=p(x)} G_{\mathbf{d}}p(x) = 1$ (see (10) and (12)) and $\left.\frac{\partial\|z\|_{\mathbf{d}}}{\partial z}\right|_{z=p(x)} = \frac{z^TP}{z^TPG_{\mathbf{d}}z}$ (see (9)) we obtain

$$\begin{aligned} f(x) &= \|x\|_{\mathbf{d}}^{\nu}\mathbf{d}(\ln\|x\|_{\mathbf{d}})\mathrm{diag}\{e_n^T(i)f(p(x))\}_{i=1}^n \mathbf{1}_n \\ &= \|x\|_{\mathbf{d}}^{\nu}\mathbf{d}(\ln\|x\|_{\mathbf{d}}) \begin{bmatrix} \left.\frac{e_n^T(1)f(z)z^T}{z^TPG_{\mathbf{d}}z}\right|_{z=p(x)} \\ \vdots \\ \left.\frac{e_n^T(n)f(z)z^T}{z^TPG_{\mathbf{d}}z}\right|_{z=p(x)} \end{bmatrix} PG_{\mathbf{d}}p(x). \end{aligned}$$

Finally, using convex embedding procedure on the unit sphere we finish the proof. $\blacksquare$

Remark 1 If one can represent the system as $\dot{x} = \sum_{i=1}^q f_i(x)$ with $f_i \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$, $q > 1$, then in order to obtain vertices closer to each other the given results (e.g., based on Proposition 1 and/or Proposition 2) can be used for each term independently. In this case the representation (16) stays true with $N \le q2^{n^2}$, $\sum_{i=1}^N \alpha_i(x) = q$. In the case $q = 2$ and $f_1 = Cx$ the representation (16) stays true with $N \le 2^{n^2}$, $\sum_{i=1}^N \alpha_i(x) = 1$ and $A_i = C + \tilde{A}_i$, where $\tilde{A}_i = (\nu I_n + G_{\mathbf{d}})^{-1}\tilde{M}_iG_{\mathbf{d}}$ ($\tilde{A}_i = \tilde{M}_iPG_{\mathbf{d}}$) and $\tilde{M}_i$ are vertices obtained from representation of f_2 based on Proposition 1 (Proposition 2).

Example 3 Consider the system $\dot{x} = \begin{pmatrix} x_1-x_2+|x_2|^{1/2} \\ x_1-x_2+|x_2|^{1/2} \end{pmatrix}$ that is $\mathbf{d}$-homogeneous of degree $-1/2$, where $G_{\mathbf{d}} = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$. Let us rewrite the system as $\dot{x} = f_1(x) + f_2(x)$, where $f_1(x) = \begin{pmatrix} x_1-x_2 \\ x_1-x_2 \end{pmatrix}$, $f_2(x) = \begin{pmatrix} |x_2|^{1/2} \\ |x_2|^{1/2} \end{pmatrix}$. Then using the results of Proposition 2 for representing $f_2(x)$ with $\|x\| = \sqrt{x^Tx}$ the following matrices can be obtained $M_i = \begin{pmatrix} \{-1.502,1.502\} & \{-1.235,1.235\} \\ \{-1.502,1.502\} & \{-1.235,1.235\} \end{pmatrix}$, $i = \overline{1,16}$, and according to Remark 1 the system under consideration can be represented as

$$\dot{x} = \|x\|_{\mathbf{d}}^{\nu} \sum_{i=1}^{16} \alpha_i(p(x))\mathbf{d}(\ln\|x\|_{\mathbf{d}}) \left(\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + A_i \right) p(x),$$

where $\sum_{i=1}^{16} \alpha_i(x) = 1$, $A_i = M_iPG_{\mathbf{d}}$ and $\nu = -0.5$.

4.3 Generalization to Some Non-Homogeneous Systems

The results of Propositions 1, 2 can be used for some non-homogeneous dynamics representation in the form (16).

Consider the system

$$\dot{x} = b(x)f(x), \tag{18}$$

where $b : \mathbb{R}^n \to \mathbb{R}$ is bounded on $\mathbb{R}^n$ and $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is bounded on $\mathbb{S}$, $f(0) = 0$.

Define the bounds

$$\overline{p}_{ij} = \sup_{x\in\mathbb{R}^n} b(x)\frac{e_n^T(i)f(z)z_j}{z^T PG_{\mathbf{d}} z}\Big|_{z=p(x)},$$
$$\underline{p}_{ij} = \inf_{x\in\mathbb{R}^n} b(x)\frac{e_n^T(i)f(z)z_j}{z^T PG_{\mathbf{d}} z}\Big|_{z=p(x)}$$

and denote the corresponding set of vertices $\mathcal{V}_p$ (if $f(x)$ is differentiable on the unit sphere define the bounds as $\overline{p}_{ij} = \sup_{x\in\mathbb{R}^n} b(x)\frac{\partial f_i(z)}{\partial z_j}\Big|_{z=p(x)}$, $\underline{p}_{ij} = \inf_{x\in\mathbb{R}^n} b(x)\frac{\partial f_i(z)}{\partial z_j}\Big|_{z=p(x)}$).

Corollary 1 *Every trajectory $x(t)$ of (18) is also a trajectory of (16) as long as $x(t) \neq 0$, where A_i is defined as in Proposition 2 (Proposition 1), provided that $f(x)$ is bounded on the unit sphere (resp., $f(x)$ is differentiable on the unit sphere).*

Proof. The proof is a straightforward consequence of Proposition 2 (resp., Proposition 1). ∎

Remark 2 If one can represent the system under consideration

$$\dot{x} = \sum_{i=1}^{M} b_i(x)f_i(x), \tag{19}$$

with bounded $b_i : \mathbb{R}^n \to \mathbb{R}$, $f_i \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$, $M \leq n$, then according to Remark 1 in order to obtain vertices closer to each other the given results can be used for each term independently (see Example 4). Similarly, the systems $\dot{x} = \mathrm{diag}\{b_i(x)\}_{i=1}^n f(x)$ with bounded $b_i : \mathbb{R}^n \to \mathbb{R}$, $f_i \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ can be considered under the commutativity conditions that

$$\mathrm{diag}\{b_i(x)\}_{i=1}^n \mathbf{d}(\ln\|x\|_{\mathbf{d}}) = \mathbf{d}(\ln\|x\|_{\mathbf{d}})\mathrm{diag}\{b_i(x)\}_{i=1}^n. \tag{20}$$

Example 4 Consider the system $\dot{x} = \begin{pmatrix} \cos(x_1)(x_2x_3^2+x_2^2) \\ x_1 \\ x_2+x_3^2 \end{pmatrix}$. Note that $f(x) = \begin{pmatrix} x_2x_3^2+x_2^2 \\ x_1 \\ x_2+x_3^2 \end{pmatrix}$ is $\mathbf{d}$-homogeneous of degree 1, where $G_{\mathbf{d}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and for $b_1(x) = \cos(x_1)$, $b_2(x) = b_3(x) = 1$ the condition (20) is satisfied. Then according to Proposition 1 for $\|x\| = \sqrt{x^TPx}$, $P = \begin{pmatrix} 16.59 & 57.72 & 66.66 \\ 57.72 & 276.57 & 362.98 \\ 66.66 & 362.98 & 574.75 \end{pmatrix}$ one can represent the system under consideration in the form (16) with $A_i \in \begin{pmatrix} 0 & \{-0.0629, 0.0630\} & \{-0.0023, 0.0023\} \\ 1 & 0 & 0 \\ 0 & 1 & \{-0.0759, 0.0759\} \end{pmatrix}$, $i = \overline{1,8}$. Note that the choice of the norm (matrix P in $\|\cdot\|_P$) significantly affects the location of vertices, that should be taken into account for control design based on the use of the canonical representation (see Section 5).

5 Stability Analysis and Control Design

5.1 LMI-based Stability Conditions

The system representation (16) can be useful for stability analysis, where stability conditions are presented in the form of LMIs. Consider the system (14) represented in the canonical form (16).

Lemma 4 *Let $\nu < 0$ (resp., $\nu > 0$). If $H \in \mathbb{R}^{n\times n}$, $\eta, \delta \in \mathbb{R}_+$, $\theta_{\min}, \theta_{\max} \in \mathbb{R}$ satisfy the system of LMIs*

$$\begin{aligned}
&\theta_{\min}(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP) \leq H - HG_{\mathbf{d}} - G_{\mathbf{d}}^TH \leq \theta_{\max}(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP),\\
&(\theta_{\max}P + H)\, A_i + A_i^T\,(\theta_{\max}P + H) \leq -\delta P,\\
&(\theta_{\min}P + H)\, A_i + A_i^T\,(\theta_{\min}P + H) \leq -\delta P,\\
&\eta P \geq H > 0, \quad \theta_{\max} \geq \theta_{\min} > 0
\end{aligned} \tag{21}$$

for $i = \overline{1,N}$, then the system (16) is finite-time (resp., nearly fixed-time) stable.

Proof. Let $\nu \geq -1$. Choose a candidate Lyapunov function in the form

$$V(x) = \|x\|_{\mathbf{d}}x^T\mathbf{d}^T(-\ln\|x\|_{\mathbf{d}})H\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x. \tag{22}$$

Then we have:

$$\begin{aligned}
\dot{V} \;=\;& p^T(x)Hp(x)\frac{2\|x\|_{\mathbf{d}}p^T(x)P\mathbf{d}(-\ln\|x\|_{\mathbf{d}})}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)p(x)}\dot{x}\\
&+2\|x\|_{\mathbf{d}}p^T(x)H\left(-\|x\|_{\mathbf{d}}^{-1}G_{\mathbf{d}}p(x)\frac{\partial\|x\|_{\mathbf{d}}}{\partial x}+\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\right)\dot{x}\\
=\;& 2\|x\|_{\mathbf{d}}\frac{p^T(x)Hp(x)}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)p(x)}p^T(x)P\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\dot{x}\\
&+2\|x\|_{\mathbf{d}}p^T(x)H\left(-2G_{\mathbf{d}}p(x)\frac{p^T(x)P}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)p(x)}+I_n\right)\\
&\times\,\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\dot{x}\\
=\;& 2\|x\|_{\mathbf{d}}s^T\left(\frac{p^T(x)\left(H-HG_{\mathbf{d}}-G_{\mathbf{d}}^TH\right)p(x)}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)p(x)}P+H\right)\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\dot{x}\\
=\;& 2\|x\|_{\mathbf{d}}^{1+\nu}p^T(x)\left(\frac{p^T(x)\left(H-HG_{\mathbf{d}}-G_{\mathbf{d}}^TH\right)p(x)}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)p(x)}P+H\right)\\
&\times\,\sum_{i=1}^N\alpha_i(p(x))A_ip(x)\\
=\;& 2\|x\|_{\mathbf{d}}^{1+\nu}p^T(x)\left(\theta(p(x))P+H\right)\sum_{i=1}^N\alpha_i(p(x))A_ip(x),
\end{aligned}$$

where $\frac{p^T(x)\left(H-HG_{\mathbf{d}}-G_{\mathbf{d}}^TH\right)p(x)}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)p(x)} = \theta(p(x)) \in [\theta_{\min}, \theta_{\max}]$.

Using convex embedding one can show that

$$\begin{aligned}
&(\theta(p(x))P+H)\sum_{i=1}^N\alpha_i(p(x))A_i\\
&\quad+\sum_{i=1}^N\alpha_i(p(x))A_i^T\,(\theta(p(x))P+H)\leq-\delta P,
\end{aligned} \tag{23}$$

if the inequalities

$$\begin{aligned}(\theta_{\max}P+H)\,A_i+A_i^T\,(\theta_{\max}P+H)\le-\delta P,\\(\theta_{\min}P+H)\,A_i+A_i^T\,(\theta_{\min}P+H)\le-\delta P,\end{aligned}\tag{24}$$

are satisfied for $i=\overline{1,N}$. Finally, if (23) is satisfied then we have

$$\dot V\le-\delta\|x\|_{\mathbf d}^{1+\nu}\le-\frac{\delta}{\eta^{1+\nu}}V^{1+\nu}.\tag{25}$$

In the case $\nu<-1$ choose the candidate Lyapunov function in the form $\tilde V=V^{\mu}$, where $\mu\ge-\nu$. Then according to (25) we can obtain $\dot{\tilde V}\le-\frac{\mu\delta}{\eta^{1+\nu}}\tilde V^{\frac{\mu+\nu}{\mu}}$.

Finally, according to Theorem 1 the system is finite-time (nearly fixed-time) stable if $\nu<0$ ($\nu>0$). ∎

Remark 3 According to Theorem 1 the following estimates of the settling time function can be obtained:

- for $\nu<0$ the origin is finite-time stable and the settling time function is bounded as $T(x_0)\le-\frac{\eta^{1+\nu}V_0^{-\nu}}{\delta\nu}$, where $V_0=V(x_0)$;
- for $\nu>0$ the system is nearly fixed-time stable and for any $\varepsilon\in\mathbb R_+$, the set $B=\{x\in\mathbb R^n\colon V(x)\le\varepsilon\}$ is fixed-time attractive with $T_{\max}=\frac{\eta^{1+\nu}}{\delta\nu}\varepsilon^{-\nu}$.

Remark 4 Note that the matrix P cannot be used as a solution of the inequalities (21) while it is used for representation of $f(x)$ in the form (16), i.e., the matrices A_i depend on P satisfying (4). In this regard, feasibility of (21) may depend on the choice of the matrix P, then the following heuristic algorithm can be aplied:

Algorithm 1

`INITIALIZATION:` `take` $P=I_n$ `;` `compute matrices` A_i `;`
`STEP:`
`While (21) is not feasible`
  `Solve the LMI`

$$X>0,\qquad A_iX+XA_i^T+\delta X<0,\ i=\overline{1,N}\tag{26}$$

  `for some` $\delta\in\mathbb R_+$ `with respect to` $X\in\mathbb R^{n\times n}$ `.`
  `take` $P=X^{-1}$ `;`
  `recompute matrices` A_i `;`
`end`

Example 5 Consider the double integrator system $\begin{pmatrix}\dot x_1\\\dot x_2\end{pmatrix}=\begin{pmatrix}x_2\\u\end{pmatrix}$ with the finite-time stabilizing control [24]

$$\begin{aligned}u(x)&=c^{l-\mu}(x)K\begin{pmatrix}c^{-l-\mu}(x)&0\\0&c^{-l}(x)\end{pmatrix}x,\\c(x)&=\left(|x_1|^{\frac{2l+2\mu+0.1}{l+\mu}}+|x_2|^{\frac{2l+2\mu+0.1}{l}}\right)^{\frac{1}{2l+2\mu+0.1}},\\K&=\begin{pmatrix}-12&-6\end{pmatrix},\quad\mu\in(0,1],\quad l\in\mathbb R_+.\end{aligned}$$

The system is homogeneous of degree $-\mu$. In the stability analysis of such systems, LMI-based conditions are often satisfied for a sufficiently small μ only (i.e. for the case the control is close to the linear one) or for a sufficiently big l (which can lead to a conservative estimate of $T(x_0)$). Applying Proposition 1 for the system representation in the form (16) and using Lemma 4 with Algorithm 1 it can be shown that the closed-loop system is finite-time stable, in particular, for $\mu=1$ and $l=4.5$ with the estimation of the settling time $T(x_0)\le4.925$ for $x_0=(0\ 1)^T$ (according to simulation $T(x_0)\approx1.65$). Applying the result of [24] for the same parameters the obtained estimation of the settling time is too conservative: $T(x_0)\le1128.5$.

5.2 LMI-based Design of Sontag's Universal Controller

In this section, we demonstrate how the canonical form can be utilized for control design of the system (1), where f_0, f_i are homogeneous functions. Using the representation (16) the control for (1) can be obtained based on the scheme of a 'universal' stabilizing control design given in the work [29]. Let us define the matrices $A_{i,j}$, $i=\overline{0,m}$, $j=\overline{1,N}$, and $\alpha_{i,j}(p(x))$, such that f_i in (1) can be represented as

$$f_i(x)=\sum_{j=1}^{N_i}\alpha_{i,j}(p(x))\|x\|_{\mathbf d}^{\deg_{\mathbb F_{\mathbf d}}(f_i)}\mathbf d(\ln\|x\|_{\mathbf d})A_{i,j}p(x),\tag{27}$$

where $N_i\le2^{n^2}$.

Theorem 6 *Let* $\deg_{\mathbb F_{\mathbf d}}(f_0)\ge-1$, *and* $\deg_{\mathbb F_{\mathbf d}}(f_0)\ge\deg_{\mathbb F_{\mathbf d}}(f_i)$ *for* $i=\overline{1,m}$. *If the system of LMIs*

$$PA_{0,j}+A_{0,j}^TP<\sum_{i=1}^{m}\tau_i(PA_{i,l_i}+A_{i,l_i}^TP),\ j=\overline{1,N_0},\ l_i=\overline{1,N_i}\tag{28}$$

is feasible for some $\tau_i\in\mathbb R$, *then the control in the form*

$$u_i(x)=\begin{cases}-\|x\|_{\mathbf d}^{\nu_i}b_i(x)\frac{a(x)+\sqrt{a^2(x)+|B(x)|^4}}{|B(x)|^2}, & \text{if}\quad |B(x)|\ne0,\\0, & \text{if}\quad |B(x)|=0,\end{cases}\tag{29}$$

where $i=\overline{1,m}$, $\nu_i=\deg_{\mathbb F_{\mathbf d}}(f_0)-\deg_{\mathbb F_{\mathbf d}}(f_i)$, $B(\cdot)=(b_1(\cdot),...,b_m(\cdot))^T$, *and*

$$\begin{aligned}a(x)&=\frac{2p^T(x)Pf_0(p(x))}{p^T(x)(PG_{\mathbf d}+G_{\mathbf d}^TP)p(x)},\\b_i(x)&=\frac{2p^T(x)Pf_i(p(x))}{p^T(x)(PG_{\mathbf d}+G_{\mathbf d}^TP)p(x)},\end{aligned}$$

globally finite-time (nearly fixed-time) stabilizes the system (1) at the origin if $\deg_{\mathbb F_{\mathbf d}}(f_0)<0$ ($\deg_{\mathbb F_{\mathbf d}}(f_0)>0$).

Proof. Assume that $V=\|x\|_{\mathbf d}$ is a candidate control-Lyapunov function. Since $\|x\|_{\mathbf d}=e^s\colon\|\mathbf d(-s)x\|=1$,

$$\begin{aligned}\frac{\partial\|\mathbf d(-s)x\|}{\partial s}&=-\left.\frac{\partial\|z\|}{\partial z}\right|_{z=\mathbf d(-s)x}G_{\mathbf d}\mathbf d(-s)x\\&=-\tfrac12\|\mathbf d(-s)x\|^{-1}x^T\mathbf d^T(-s)\left(PG_{\mathbf d}+G_{\mathbf d}^TP\right)\mathbf d(-s)x,\\\frac{\partial\|\mathbf d(-s)x\|}{\partial x}&=\|\mathbf d(-s)x\|^{-1}x^T\mathbf d^T(-s)P\mathbf d(-s),\end{aligned}$$

then implying $\frac{\partial s}{\partial x} = -\left[\frac{\partial\|\mathbf{d}(-s)x\|}{\partial s}\right]^{-1}\frac{\partial\|\mathbf{d}(-s)x\|}{\partial x}$ (by means of Implicit Function Theorem [30] for the function $s\colon \mathbb{R}^n \to \mathbb{R}$ implicitly defined by $\|\mathbf{d}(-s)x\| = 1$) we obtain

$$\begin{aligned}\tfrac{\partial V(x)}{\partial x} f_0(x) &= \|x\|_{\mathbf{d}}^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f_0)} a(x),\\ \tfrac{\partial V(x)}{\partial x} f_i(x) &= \|x\|_{\mathbf{d}}^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f_i)} b_i(x), \quad i = \overline{1,m},\end{aligned}$$

and for $x \neq 0$ we have

$$\begin{aligned}&\tfrac{\partial V(x)}{\partial x} f_0(x) + \tfrac{\partial V(x)}{\partial x}\textstyle\sum_{i=1}^m u_i f_i(x)\\ &\quad = -\|x\|_{\mathbf{d}}^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f_0)}\sqrt{a^2(x) + |B(x)|^4}\end{aligned}$$

and the right-hand side of this inequality is negative provided that the condition

$$|B(x)| = 0 \Rightarrow a(x) < 0 \tag{30}$$

is satisfied, which according to (3) follows from the assumption that this V is a control-Lyapunov function. Let us verify this hypothesis.

Since $\mathbf{d}$ is strictly monotone according to Theorem 3 we have $PG_{\mathbf{d}} + G_{\mathbf{d}}^T P > 0$. Then due to the function f_i can be represented as (27), when we substitute f_i from (27) into the expressions of $a(x)$ and $b_i(x)$, then the condition (30) takes the form

$$\begin{gathered}\mathcal{S} \supseteq \cap_i Q_i, \quad i = \overline{1,m},\\ Q_i = \{x \in \mathbb{S} : x^T \textstyle\sum_{j=1}^N \alpha_{i,j}(x)\left(PA_{i,j} + A_{i,j}^T P\right)x = 0\},\\ \mathcal{S} = \{x \in \mathbb{S} : x^T \textstyle\sum_{j=1}^N \alpha_{0,j}(x)\left(PA_{0,j} + A_{0,j}^T P\right)x < 0\}.\end{gathered} \tag{31}$$

Then applying S-procedure, the condition (31) is satisfied if the system of LMIs (28) is feasible for some $\tau_i \in \mathbb{R}$, $i = \overline{1,m}$. Finally, since the controller (29) homogenizes the closed-loop system with degree $\deg_{\mathbb{F}_{\mathbf{d}}}(f_0)$, then according to Theorem 4 the closed-loop system is nearly fixed-time (finite-time) stable, if $\deg_{\mathbb{F}_{\mathbf{d}}}(f_0) > 0$ ($\deg_{\mathbb{F}_{\mathbf{d}}}(f_0) < 0$). ■

Remark 5 If in Theorem 6 the strict inequality $\deg_{\mathbb{F}_{\mathbf{d}}}(f_0) > \deg_{\mathbb{F}_{\mathbf{d}}}(f_i)$ is satisfied, then the control-Lyapunov function satisfies the small control property and the control (29) is continuous at 0.

Remark 6 The system of LMI (28) is just a sufficient condition to satisfy (30). In general, since

$$\begin{aligned}a(x) &= \tfrac{2p^T(x)Pf_0(p(x))}{p^T(x)(PG_{\mathbf{d}} + G_{\mathbf{d}}^T P)p(x)},\\ b_i(x) &= \tfrac{2p^T(x)Pf_i(p(x))}{p^T(x)(PG_{\mathbf{d}} + G_{\mathbf{d}}^T P)p(x)}, \quad i = \overline{1,m},\end{aligned}$$

the condition (30) can be numerically checked using a numerical grid on a sphere. Explicitly calculating $a(x)$ and $B(x)$ we can obtain that

$$\tfrac{\partial V(x)}{\partial x} f_0(x) + \tfrac{\partial V(x)}{\partial x}\textstyle\sum_{i=1}^m u_i f_i(x) \le -CV^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f_0)},$$

where $C = \min_{y \in \mathbb{S}}\sqrt{a^2(y) + |B(y)|^4}$. Then, utilizing Theorem 1 we can get estimates for settling-time functions.

Example 6 Consider the system (1) with $x \in \mathbb{R}^2$, $m = 2$, where $f_0 = \begin{pmatrix} -|x_1|^{1/2}x_2^2 \\ 0 \end{pmatrix}$, $f_1 = \begin{pmatrix} x_2^2 + x_1 \\ x_2 \end{pmatrix}$, $f_2 = \begin{pmatrix} x_2^3 \\ x_1 + x_2^2 \end{pmatrix}$. In this system f_0, f_1 and f_2 are $\mathbf{d}$-homogeneous of degrees 1, 0 and 1, respectively, where $\mathbf{d}$ is strictly monotone dilation equipped with the norm $\|x\| = \sqrt{x^T x}$, and $G_{\mathbf{d}} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the generator of the dilation. For these functions, using Proposition 1 we have representations in the form (27) with the matrices $A_{0i} = \begin{pmatrix} \{-0.438, 0.438\} & \{-0.437, 0.437\} \\ 0 & 0 \end{pmatrix}$, $A_{1j} = \begin{pmatrix} 1 & \{-1,1\} \\ 0 & 1 \end{pmatrix}$, $A_{2l} = \begin{pmatrix} \{0,2\} & \{-0.333, 0.333\} \\ 1 & \{-1,1\} \end{pmatrix}$, where $i = \overline{1,4}$, $j = \overline{1,2}$, $l = \overline{1,8}$. For these matrices the system (28) is feasible with $P = I_2$, $\tau_1 = 13.8175$, $\tau_2 = 0.9915$, i.e., the closed-loop system with the control (29) is nearly fixed-time stable. Numerical simulations (Fig. 1) have been done using explicit Euler method with the step $h = 10^{-3}$ and $x_0 = \begin{pmatrix} 5 & -1 \end{pmatrix}^T$. The results of simulation for five different initial conditions with using the logarithmic scale are shown in Fig. 2 in order to demonstrate nearly fixed-time convergence rate (e.g., in accordance with Theorem 1 and Remark 6 the set $B = \{x \in \mathbb{R}^2 : V(x) \le 1\}$ is fixed-time attractive with $T_{\max} = 4$).

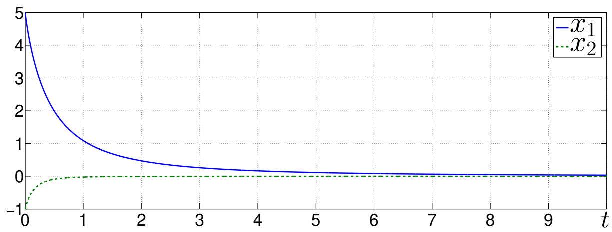

Fig. 1. System states x versus time

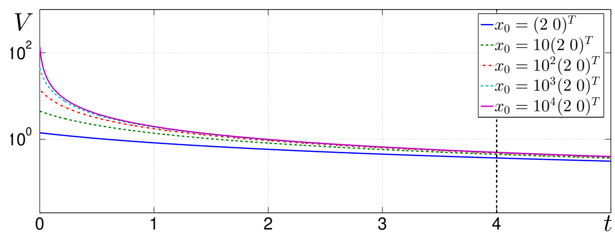

Fig. 2. Evolution of the Lyapunov function V for different initial values

5.3 *Homogeneous control for $f_i = const$, $i \ge 1$*

Now, let us demonstrate how the canonical form can be utilized for control design of the system

$$\dot{x} = f(x) + Bu, \tag{32}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $B \in \mathbb{R}^{n\times m}$, and f can be represented in the form

$$f(x) = \|x\|_{\mathbf{d}}^{\nu} \sum_{i=1}^{N} \alpha_i(p(x))\mathbf{d}(\ln\|x\|_{\mathbf{d}})A_i p(x), \tag{33}$$

with corresponding matrices $A_i \in \mathbb{R}^{n\times n}$, $i = \overline{1,N}$, a generator $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$, $\nu \in \mathbb{R}$, where the dilation $\mathbf{d}$ is strictly monotone on $\mathbb{R}^n$ equipped with the norm $\|x\|_P = \sqrt{x^T P x}$, for $P > 0$ satisfying (4).

Theorem 7 *Let $G_{\mathbf{d}}B = \gamma B$, where $\gamma \in \mathbb{R} : \gamma + \nu \geq 0$. Let $K \in \mathbb{R}^{m\times n}$, $H \in \mathbb{R}^{n\times n}$, $\eta, \delta \in \mathbb{R}_+$ be a solution of the bilinear matrix inequalities (BMIs)*

$$\begin{aligned}
&\eta P \geq H > 0,\\
&\theta_{\min}(PG_{\mathbf{d}}+G_{\mathbf{d}}^T P) \leq H - HG_{\mathbf{d}} - G_{\mathbf{d}}^T H \leq \theta_{\max}(PG_{\mathbf{d}}+G_{\mathbf{d}}^T P),\\
&(\theta_{\max}P+H)(A_i+BK)+(A_i+BK)^T(\theta_{\max}P+H) \leq -\delta P,\\
&(\theta_{\min}P+H)(A_i+BK)+(A_i+BK)^T(\theta_{\min}P+H) \leq -\delta P,
\end{aligned} \tag{34}$$

for $i = \overline{1,N}$ and some $\theta_{\min}, \theta_{\max} \in \mathbb{R}$, then the system (32) with

$$u(x) = \|x\|_{\mathbf{d}}^{\gamma+\nu} K p(x) \tag{35}$$

is finite-time (nearly fixed-time) stable if $\nu < 0$ ($\nu > 0$).

Proof. Let us show that the equality

$$\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\|x\|_{\mathbf{d}}^{\nu+\gamma} BKp(x) = \|x\|_{\mathbf{d}}^{\nu} BKp(x) \tag{36}$$

holds. To do this, it is enough to show that

$$(e^{\gamma s}\mathbf{d}(-s) - I_n)BK = 0. \tag{37}$$

Since $(e^{\gamma s}\mathbf{d}(-s) - I_n)BK = \sum_{i=1}^{+\infty} \frac{s^i(\gamma I_n - G_{\mathbf{d}})^i}{i!} BK$, the equality (37) holds if $(G_{\mathbf{d}} - \gamma I_n)BK = 0$ which is implied by $G_{\mathbf{d}}B = \gamma B$. Then, using (37) it is easy to show that (36) holds.

Let $\nu \geq -1$. Choose a candidate Lyapunov function in the form (22). Due to (36) we have that

$$\mathbf{d}(-\ln\|x\|_{\mathbf{d}})\dot{x} = \|x\|_{\mathbf{d}}^{\nu} \left(\sum_{i=1}^{N} \alpha_i(p(x))A_i + BK\right) p(x). \tag{38}$$

Then following the proof of Lemma 4 and applying (38) we have

$$\begin{aligned}
\dot{V} &= 2\|x\|_{\mathbf{d}}^{1+\nu} p^T(x) \left(\frac{p^T(x)\big(H - HG_{\mathbf{d}} - G_{\mathbf{d}}^T H\big)p(x)}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^T P)p(x)} P + H\right)\\
&\quad \times \left(\textstyle\sum_{i=1}^{N} \alpha_i(p(x))A_i + BK\right) p(x)\\
&= 2\|x\|_{\mathbf{d}}^{1+\nu} p^T(x)\left(\theta(p(x))P + H\right)\\
&\quad \times \left(\textstyle\sum_{i=1}^{N} \alpha_i(p(x))A_i + BK\right) p(x),
\end{aligned}$$

where $\frac{p^T(x)\big(H - HG_{\mathbf{d}} - G_{\mathbf{d}}^T H\big)p(x)}{p^T(x)(PG_{\mathbf{d}}+G_{\mathbf{d}}^T P)p(x)} = \theta(p(x)) \in [\theta_{\min}, \theta_{\max}]$.

Using convex embedding one can show that

$$\begin{aligned}
&(\theta(p(x))P + H)\left(\textstyle\sum_{i=1}^{N} \alpha_i(p(x))A_i + BK\right)\\
&+ \left(\textstyle\sum_{i=1}^{N} \alpha_i(p(x))A_i + BK\right)^T (\theta(p(x))P + H) \leq -\delta P,
\end{aligned} \tag{39}$$

if the inequalities

$$\begin{aligned}
&(\theta_{\max}P+H)(A_i+BK)+(A_i+BK)^T(\theta_{\max}P+H) \leq -\delta P,\\
&(\theta_{\min}P+H)(A_i+BK)+(A_i+BK)^T(\theta_{\min}P+H) \leq -\delta P,
\end{aligned}$$

are satisfied for $i = \overline{1,N}$. If (39) is satisfied then we have

$$\dot{V} \leq -\delta\|x\|_{\mathbf{d}}^{1+\nu} \leq -\frac{\delta}{\eta^{1+\nu}} V^{1+\nu}. \tag{40}$$

Similarly to the proof of Lemma 4, in the case $\nu < -1$ choose the candidate Lyapunov function in the form $\tilde{V} = V^{\mu}$, where $\mu \geq -\nu$. Then according to (40) we can obtain $\dot{\tilde{V}} \leq -\frac{\mu\delta}{\eta^{1+\nu}} \tilde{V}^{\frac{\mu+\nu}{\mu}}$. Finally, applying Theorem 1 one can show that the feedback law (35) stabilizes the system (32) with different types of convergence. ■

Remark 7 For $\nu < 0$ the closed-loop system (32), (35) is finite-time stable with the following settling time estimate $T(x_0) \leq -\frac{\eta^{1+\nu}V_0^{-\nu}}{\delta\nu}$, where $V_0 = V(x_0)$. For $\nu > 0$ the closed-loop system is nearly fixed-time stable and for any $\varepsilon \in \mathbb{R}_+$, the set $B = \{x \in \mathbb{R}^n : V(x) \leq \varepsilon\}$ is fixed-time attractive with $T_{\max} = \frac{\eta^{1+\nu}}{\delta\nu}\varepsilon^{-\nu}$.

Remark 8 Analogously to Remark 4 and Algorithm 1, feasibility of BMI (34) may depend on the choice of the matrix P, and the following LMI-based heuristic algorithm can be applied:

Algorithm 2

`INITIALIZATION:` `take` $P = I_n$, $K = (0\ 0\ 0)$;
`compute matrices` A_i;
`STEP:`
`While LMI (34) is not feasible with respect to` $H \in \mathbb{R}^{n\times n}$, $\theta_{\min}, \theta_{\max} \in \mathbb{R}$, $\eta, \delta \in \mathbb{R}_+$
`Solve the LMI system`

$$\begin{aligned}
&A_i X + X A_i^T + BY + Y^T B^T + \delta X < 0,\ i = \overline{1,N},\\
&G_{\mathbf{d}}X + XG_{\mathbf{d}}^T > 0, \quad X > 0
\end{aligned} \tag{41}$$

`for some` $\delta \in \mathbb{R}_+$ `with respect to` $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$.
`take` $P = X^{-1}$, $K = YP$;
`recompute matrices` A_i;
`end`

Note that if Algorithm 2 (Algorithm 1) does not find a solution, it is advisable to reduce $\delta \in \mathbb{R}_+$. The logic of these algorithms can be augmented by the maximum admissible number of iterations with stop afterwards avoiding an infinite loop.

Example 7 Consider the system (32) with $f(x)$ as in Example 4 and $B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. The condition $G_{\mathbf{d}}B = \gamma B$ is satisfied for $\gamma = 3$. Applying Algorithm 2 for $\eta = 0.2, \delta = 0.1$ we obtain $K = \begin{pmatrix} -5.5055 & -15.8387 & -16.3807 \end{pmatrix}$, and the closed-loop system is nearly fixed-time stable due to $\nu = 1$. Numerical simulations (Fig. 3-5) have been done using explicit Euler method with the step $h = 10^{-3}$. Due to homogeneity the closed-loop system is ISS with respect to measurement noise and additive disturbances. Fig. 4 shows the results of simulation with measurement band limited noise of power 10^{-6} and additive disturbances $d = \begin{pmatrix} 0.5\sin(x_1)\cos(x_2) & 0 & 0 \end{pmatrix}^T$. The results of simulation for three different initial conditions using the logarithmic scale of V are shown in Fig. 5 in order to demonstrate nearly fixed-time convergence rate (e.g., in accordance with Remark 7 the set $B = \{x \in \mathbb{R}^3 : V(x) \leq 1\}$ is fixed-time attractive with $T_{\max} = 0.4$).

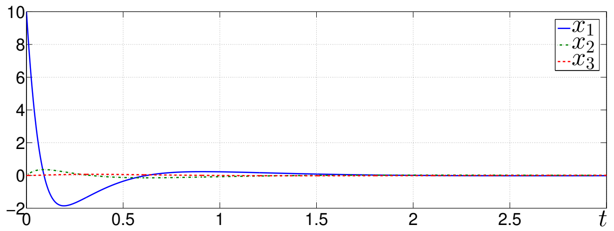

Fig. 3. System states x versus time

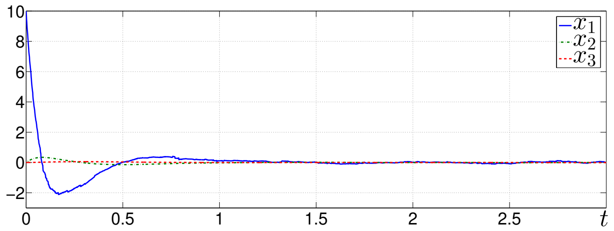

Fig. 4. System states x versus time in the presence of measurement noises and disturbances

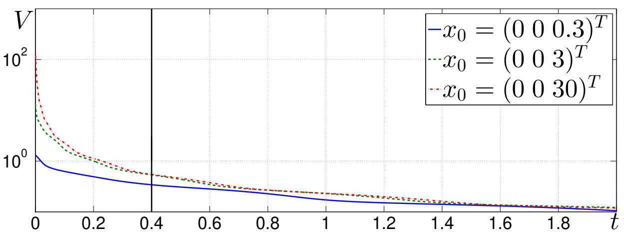

Fig. 5. Evolution of the Lyapunov function V for different initial values

Note, that the given control is a generalized version of [25]. Indeed, if $f(x)$ in (32) is linear, then the proposed result coincides with [25].

Remark 9 Note that the function $\|x\|_{\mathbf{d}}$ is defined implicitly by (8). In order to realize the control (29) or (35) the bisection method may be utilized:

Algorithm 3

INITIALIZATION: $\|x\|_{\mathbf{d}} = 1$; $a = \epsilon_{\min}$; $b = 1$;
STEP :
If $x_i^T \mathbf{d}(-\ln b)^T P \mathbf{d}(-\ln b) x_i > 1$ then $a = b$; $b = 2b$;
elseif $x_i^T \mathbf{d}(-\ln a)^T P \mathbf{d}(-\ln a) x_i < 1$ then
$b = a$; $a = \max\left\{\frac{a}{2}, \epsilon_{\min}\right\}$;
else $c = \frac{a+b}{2}$;
If $x_i^T \mathbf{d}(-\ln c)^T P \mathbf{d}(-\ln c) x_i < 1$ then $b = c$;
else $a = \max\{\epsilon_{\min}, c\}$;
endif;
endif;
$\|x_i\|_{\mathbf{d}} = b$;

Here $\epsilon \in \mathbb{R}_+$ is a minimal admissible value of $\|x_i\|_{\mathbf{d}}$. If STEP is applied recurrently many times to the same vector x_i then it allows to localize the unique positive root of the equation $\|\mathbf{d}(-\ln \|x_i\|_{\mathbf{d}})x\| = 1$ as follows $a < \|x_i\|_{\mathbf{d}} < b$, and next, to improve the latter estimate using the bisection method.

6 Conclusions

The paper presents methods for representation of homogeneous systems in the canonical form (16). These methods were extended for some non-homogeneous systems as well. It is demonstrated that the representation in the canonical form allows to present stability conditions in the form of LMIs even for sufficiently nonlinear systems. In the paper, the stabilizing control algorithms for affine in control nonlinear systems are presented utilizing the representation in the canonical form. Moreover, due to homogeneity property the proposed controls are robust (in ISS sense) to a large class of disturbances and measurement noises. Numerical examples demonstrate effectiveness of the proposed control algorithms.

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