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ARTICLE TYPE

On ISS-Lyapunov Functions for Mechanical Systems[†]

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Summary

For two canonical models of mechanical systems with disturbances presented by Rayleigh- and Liénard-type equations, several designs of Lyapunov functions are investigated. Under given restrictions, the existence of these functions implies global or local input-to-state stability property for the systems. The proposed stability conditions are constructive, which is demonstrated in applications.

KEYWORDS:

Input-to-state stability, Lyapunov function, Mechanical systems

1 | INTRODUCTION

Analysis of stability of dynamical systems is an important and popular subject of research^{1,2,3,4}. There are different kinds of stability and domains of validity, global or local, that should be characterized in applications. Local conditions can be easier to obtain, while global conditions provide more information about the system, especially in the presence of parametric or signal uncertainty.

The most general, necessary and sufficient, conditions of various stability concepts are formulated with the use of the Lyapunov function (LF) method⁵. Usually, a LF defines a sort of distance with respect to an object of interest (*e.g.*, invariant set or an equilibrium), and its time derivative should be negative on the trajectories of the studied system. Such a LF is called strict if its derivative is negative definite with respect to the same object, and discovering a strict LF is more difficult but advantageous⁶, since it allows to evaluate the rates of convergence, sensitivity and robustness characteristics. The principal weak point of the method is the absence of constructive guidelines for designing these LFs for general nonlinear systems, then different canonical forms are used with established structures of LFs.

One of the most popular concepts of robust stability in the presence of external perturbations and modeling uncertainties is given in the framework of input-to-state stability (ISS)^{7,8}. This property implies global asymptotic stability of the origin in the noise-free case, and guarantees asymptotic convergence of all trajectories to a neighborhood of the equilibrium, whose size is proportional to the amplitude of perturbation. The equivalent conditions of ISS are also formulated with the use of strict LFs, and all features of the method are valid in such a case also: local or global analysis, with difficulties in selection of a LF for given nonlinear dynamical system. The choice of LF in ISS analysis is even more tricky than for a conventional global stability, and a LF can be suitable for unperturbed system, but it may fail to quantify the robustness.

Following⁹, where the options for ISS-LFs were discussed for one-degree of freedom nonlinear pendulums, in the present work the problem of finding LFs to check global or local ISS property is considered for two classes of multidimensional mechanical systems with exogenous perturbations. Several existing designs of LFs are revisited and generalized in the studied context.

The outline of this work is as follows. After introduction of the preliminaries in Section 2, the problem statement is given in Section 3. The syntheses of LFs for two investigated classes of mechanical systems are presented in sections 4 and 5. The results of applications are summarized in Section 6.

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Notation

- \mathbb{R} denotes the field of real numbers, and \mathbb{R}^n is the n -dimensional Euclidean space, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$;
- the Euclidean norm $\|\cdot\|$ will be used for vectors;
- for a (Lebesgue) measurable function $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $[t_0, t_1] \subseteq \mathbb{R}_+$ define the norm $\|d\|_{[t_0, t_1]} = \text{ess sup}_{t \in [t_0, t_1]} \|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0, +\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as \mathcal{L}_∞^m (the set of essentially bounded measurable functions);
- a continuous function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K} if $\alpha(0) = 0$ and the function is strictly increasing; the function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K}_∞ if $\alpha \in \mathcal{K}$ and it is increasing to infinity; a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{KL} if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s \in \mathbb{R}_+$;
- $I \in \mathbb{R}^{n \times n}$ denotes the identity matrix, $\text{diag}(v) \in \mathbb{R}^{n \times n}$ is a diagonal matrix having $v \in \mathbb{R}^n$ on the main diagonal;
- for a symmetric matrix $P \in \mathbb{R}^{n \times n}$, the relations $P > 0$ or $P \geq 0$ mean that it is positive definite or semi-definite, respectively; $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ correspond to the maximal and minimal eigenvalues of P .

2 | PRELIMINARIES

For any $a, b \in \mathbb{R}^n$ and $q > 1$ the Young's inequality states:

$$a^\top b \leq \frac{\|a\|^q}{q} + (q-1) \frac{\|b\|^{\frac{q}{q-1}}}{q},$$

and its generalized version is given in the following lemma:

Lemma 1. ^{10, Lemma 17} For any $\alpha \in \mathcal{K}_\infty$ and $p > 0$,

$$ab \leq \alpha^{-1}(pa)a + p^{-1}\alpha(b)b, \quad \forall a, b \geq 0.$$

Consider a nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)), \quad t \geq 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$, and $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ is a locally Lipschitz (or Hölder) continuous function, $f(0, 0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty^m$, define the corresponding solutions by $X(t, x_0, d)$ for any $t \geq 0$ for which the solution exists.

The theory of ISS is well developed^{7,8}:

Definition 1. The system (1) is called *ISS*, if there are functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$\|X(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_{[0, t]}) \quad \forall t \geq 0$$

for any input $d \in \mathcal{L}_\infty^m$ and any $x_0 \in \mathbb{R}^n$. The function γ is called *nonlinear asymptotic gain*. The system is called *locally ISS* if the above inequality is satisfied for $\|d\|_\infty \leq \rho$ and $\|x_0\| \leq \rho$ for some $\rho \in \mathbb{R}_+$.

Definition 2. A smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called *ISS-LF* for the system (1) if there are $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\theta \in \mathcal{K}$ such that

$$\begin{aligned} \alpha_1(\|x\|) &\leq V(x) \leq \alpha_2(\|x\|), \\ DV(x)f(x, d) &\leq \theta(\|d\|) - \alpha_3(\|x\|) \end{aligned}$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$. Such a function V is called *local ISS-LF* if the above inequalities are verified only for $\|d\| \leq \rho$ and $\|x\| \leq \rho$ for some $\rho \in \mathbb{R}_+$.

Note that an ISS-LF can also satisfy the following equivalent condition for some $\chi \in \mathcal{K}$:

$$\|x\| > \chi(\|d\|) \Rightarrow DV(x)f(x, d) \leq -\alpha_3(\|x\|)$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$.

Theorem 1. ⁴ The system (1) is (locally) ISS if and only if it admits a (local) ISS-LF.

If $d(t) = 0$ for all $t \geq 0$, then ISS property is reduced to the conventional global asymptotic stability of the origin, while an ISS-LF becomes a strict LF, whose existence is necessary and sufficient for the asymptotic stability⁴.

3 | STATEMENT OF THE PROBLEM

In⁹, nonlinear oscillators of the form

$$\ddot{x} + \mu(\dot{x}) + k(x) = d$$

were considered for $x, \dot{x}, d \in \mathbb{R}$, where it was assumed that $r\mu(r) > 0$, $rk(r) > 0$ for $r \neq 0$. Sufficient conditions of the existence of global strict LFs (GSLFs) for the case $d = 0$ were obtained, and conditions of ISS were derived for $d \neq 0$ with the aid of the constructed functions.

In this paper, we will consider a similar problem for more general models of mechanical systems of the following form:

$$\ddot{x} + \frac{\partial W(\dot{x})}{\partial \dot{x}} + \frac{\partial G(x)}{\partial x} = d, \quad (2)$$

$$\ddot{x} + \frac{\partial F(x)}{\partial x} \dot{x} + \frac{\partial G(x)}{\partial x} = d, \quad (3)$$

where $x \in \mathbb{R}^n$ and $(x^\top, \dot{x}^\top)^\top \in \mathbb{R}^{2n}$ is the state vector, $d \in \mathbb{R}^n$ is the external input, $d \in \mathcal{L}_\infty^n$; $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $G : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuously differentiable functions of their arguments. Here G represents the potential energy, while W and F can be used to characterize the friction profiles. The system (2) is a multi-degree of freedom example of Rayleigh-type equation, which is frequently used for modeling of mechanical, electrical and acoustical systems¹¹. The system (3) corresponds to a vector Liénard equation (a popular example is van der Pol oscillator)¹².

For the case $d = 0$ it is required to find a GSLF for the systems (2) and (3) under different additional hypotheses imposed on the functions W , G and F , and next to establish the conditions of ISS with respect to the input d .

4 | ISS ANALYSIS FOR THE FIRST SYSTEM

Several designs are given below, which are based on different properties of the functions W and G .

4.1 | Linear Potential Forces

In⁹, the case with $k(r) = kr$ for $k > 0$ has been studied, with an additional restriction that $\mu'(r) > 0$ for all $r \in \mathbb{R}$. For such a case, the following GSLF was constructed:

$$V(x, \dot{x}) = k^2 x^2 + k \dot{x}^2 + kx\mu(\dot{x}) + \frac{1}{2}\mu^2(\dot{x}).$$

To generalize this scenario, consider the system (2) under the following auxiliary hypotheses:

Assumption 1. Let $G(x) = \frac{1}{2}x^\top Cx$, where $C = C^\top > 0$, while W be twice continuously differentiable and such that

$$\begin{aligned} \dot{x}^\top \frac{\partial W(\dot{x})}{\partial \dot{x}} &> 0, \quad \forall \dot{x} \in \mathbb{R}^n \setminus \{0\}; \\ \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} &> 0, \quad \forall \dot{x} \in \mathbb{R}^n. \end{aligned} \quad (4)$$

The conditions of this assumption imply that the potential energy has minimum at the origin (a prerequisite for stability according to Lagrange-Dirichlet theorem¹³). Then the equation describing dynamics of the system can be rewritten as follows:

$$\ddot{x} + \frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx = d,$$

and the condition (4) implies that the friction (or velocity) forces are dissipative.

Choose a GSLF candidate in the form:

$$V = \frac{1}{2}\dot{x}^\top \dot{x} + \frac{1}{2}x^\top Cx + \frac{1}{2}\dot{x}^\top C\dot{x} + \frac{1}{2} \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)^\top \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right), \quad (5)$$

where the first two terms represent the full energy of the system. The direct computations for $d = 0$ show that

$$\dot{V} = -\dot{x}^\top \frac{\partial W(\dot{x})}{\partial \dot{x}} - \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)^\top \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)$$

hence, (5) is a GSLF for the system (2) under Assumption 1. The additional hypotheses needed to establish ISS in such a scenario are given below:

Theorem 2. Consider the system (2) under Assumption 1, then it is locally ISS. Let additionally

$$0 < \underline{S} + \underline{\rho} \|\dot{x}\|^\sigma I \leq \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} \leq \bar{S} + \bar{\rho} \|\dot{x}\|^{\bar{\sigma}} I, \quad \forall \dot{x} \in \mathbb{R}^n,$$

where $\underline{S}, \bar{S} \in \mathbb{R}^{n \times n}$ are symmetric matrices, $0 \leq \underline{\sigma} \leq \bar{\sigma} < \underline{\sigma} + 2$ and $0 < \underline{\rho} \leq \bar{\rho}$, then the system is ISS.

Proof. Clearly, LF (5) is positive definite and radially unbounded. The time derivative of the LF (5) calculated for the system (2) has the form:

$$\begin{aligned} \dot{V} &= -\dot{x}^\top \frac{\partial W(\dot{x})}{\partial \dot{x}} - \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)^\top \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right) \\ &\quad + \dot{x}^\top (I + C)d + \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)^\top \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} d. \end{aligned}$$

Due to Assumption 1 $\frac{\partial W(0)}{\partial \dot{x}} = 0$, and using the lower bound imposed on the second derivative of W in the theorem we obtain:

$$\begin{aligned} \dot{x}^\top \frac{\partial W(\dot{x})}{\partial \dot{x}} &= \dot{x}^\top \left(\frac{\partial W(0)}{\partial \dot{x}} + \int_0^1 \frac{\partial^2 W(s\dot{x})}{\partial \dot{x}^2} d s \dot{x} \right) = \dot{x}^\top \int_0^1 \frac{\partial^2 W(s\dot{x})}{\partial \dot{x}^2} d s \dot{x} \\ &\geq \dot{x}^\top \left(\int_0^1 \underline{S} + \underline{\rho} \|s\dot{x}\|^\sigma I d s \right) \dot{x} = \dot{x}^\top \underline{S} \dot{x} + \underline{\rho} \frac{\|\dot{x}\|^{\sigma+2}}{\underline{\sigma} + 1}, \quad \forall \dot{x} \in \mathbb{R}^n. \end{aligned}$$

Hence, by applying Young's inequality for $q = 2$:

$$\begin{aligned} \dot{x}^\top (I + C)d &= \dot{x}^\top \underline{S}^{0.5} \underline{S}^{-0.5} (I + C)d \leq \frac{1}{2} (\dot{x}^\top \underline{S} \dot{x} + d^\top (I + C) \underline{S}^{-1} (I + C)d), \\ \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)^\top \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} d &\leq \frac{1}{2} \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)^\top \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right) \\ &\quad + \frac{1}{2} d^\top (\bar{S} + \bar{\rho} \|\dot{x}\|^{\bar{\sigma}} I) d, \end{aligned}$$

and next for $q = \frac{\sigma+2}{\bar{\sigma}}$:

$$\|\dot{x}\|^{\bar{\sigma}} \|d\|^2 = (c \|\dot{x}\|^{\bar{\sigma}}) (c^{-1} \|d\|^2) \leq c^{\frac{\sigma+2}{\bar{\sigma}}} \bar{\rho} \frac{\|\dot{x}\|^{\sigma+2}}{\underline{\sigma} + 2} + (\underline{\sigma} + 2 - \bar{\sigma}) \frac{\|d\|^2}{\underline{\sigma} + 2} c^{-\frac{\sigma+2}{\bar{\sigma}}}$$

for any $c > 0$, and by taking $c = \left(\frac{\underline{\rho}}{\bar{\rho} \underline{\sigma} + 1} \right)^{\frac{\bar{\sigma}}{\sigma+2}}$ for $\bar{\sigma} \neq 0$ and $c = 1$ otherwise we derive:

$$\begin{aligned} \dot{V} &\leq -\frac{1}{2} \dot{x}^\top \frac{\partial W(\dot{x})}{\partial \dot{x}} - \frac{1}{2} \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right)^\top \underline{S} \left(\frac{\partial W(\dot{x})}{\partial \dot{x}} + Cx \right) \\ &\quad + \frac{1}{2} d^\top \left((I + C) \underline{S}^{-1} (I + C) + \bar{S} \right) d + \frac{\bar{\rho}}{2} (\underline{\sigma} + 2 - \bar{\sigma}) \frac{\|d\|^2}{\underline{\sigma} + 2} c^{-\frac{\sigma+2}{\bar{\sigma}}}, \end{aligned}$$

which implies that V is an ISS-LF (Definition 2) and the system (2) is ISS (by Theorem 1). For local analysis, a local existence of \underline{S} and \bar{S} with $\underline{\sigma} = \bar{\sigma} = 0$ follows for any twice continuously differentiable W from Assumption 1. \square

4.2 | Linear Dissipative Forces

In⁹ also the case $\mu(r) = \mu r$ with $\mu > 0$ was studied, and for this case, the following GSLF was constructed:

$$V(x, \dot{x}) = \frac{1}{2}\mu^2 x^2 + \mu x \dot{x} + \dot{x}^2 + 2 \int_0^x k(r) dr.$$

This analysis can be generalized by considering again the system (2) under the following hypotheses:

Assumption 2. Let $W(x) = \frac{1}{2}\dot{x}^\top \mathbb{W} \dot{x}$, where $\mathbb{W} = \mathbb{W}^\top > 0$, while G be continuously differentiable, $G(0) = 0$ and such that

$$G(x) > 0, \quad x^\top \frac{\partial G(x)}{\partial x} > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

This assumption formulates the same properties as Assumption 1, considering nonlinear potential forces and linear dissipative ones (in Assumption 1 the potential forces are linear). Then the system equation can be rewritten as follows:

$$\ddot{x} + \mathbb{W} \dot{x} + \frac{\partial G(x)}{\partial x} = d,$$

and using the approach proposed in¹⁴, let us select a LF in the form:

$$V = h \left(\frac{1}{2} \dot{x}^\top \dot{x} + G(x) \right) + \frac{1}{2} x^\top \mathbb{W} x + x^\top \dot{x}, \quad (6)$$

where $h > 0$ is a design parameter, then

$$\dot{V} = -\dot{x}^\top (h\mathbb{W} - I) \dot{x} - x^\top \frac{\partial G(x)}{\partial x}.$$

Hence, (6) is GSLF for a sufficiently big value of h for the system (2) under Assumption 2. Let us demonstrate that a mild modification is needed to get ISS:

Theorem 3. Consider the system (2) under Assumption 2, then it is locally ISS. Let additionally

$$x^\top \frac{\partial G(x)}{\partial x} \geq 2\eta(\|x\|)\|x\|, \quad \forall x \in \mathbb{R}^n,$$

for some $\eta \in \mathcal{K}_\infty$, then the system is ISS.

Proof. Let $h > 0$ be selected to ensure that

$$\frac{h}{2}\mathbb{W} - I > 0,$$

which is always possible due to Assumption 2. First, let us demonstrate positive definiteness of LF (6), to this end note that

$$V \geq \frac{1}{2} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^\top \begin{bmatrix} \mathbb{W} & I \\ I & hI \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

and the condition of positive definiteness of this quadratic form is that

$$h\mathbb{W} - I > 0,$$

which is valid due to the choice of h . Clearly, under the imposed restriction, the LF (6) is also radially unbounded. The derivative of this LF calculated for the system (2) under introduced hypotheses takes the form (using Lemma 1):

$$\begin{aligned} \dot{V} &= -x^\top \frac{\partial G(x)}{\partial x} - \dot{x}^\top (h\mathbb{W} - I) \dot{x} + x^\top d + h\dot{x}^\top d \\ &\leq -\frac{1}{2}x^\top \frac{\partial G(x)}{\partial x} - \dot{x}^\top \left(\frac{h}{2}\mathbb{W} - I \right) \dot{x} + \eta^{-1}(\|d\|)\|d\| + \frac{h}{2}d^\top \mathbb{W}^{-1}d, \end{aligned}$$

which implies ISS property. And for local ISS property with $\|x\| \leq \rho$, $\|\dot{x}\| \leq \rho$ for some $\rho > 0$ we have

$$\dot{V} \leq -x^\top \frac{\partial G(x)}{\partial x} - \dot{x}^\top (h\mathbb{W} - I) \dot{x} + \rho\|d\| + h\rho\|d\|,$$

which implies locally the desired stability characteristics since $x^\top \frac{\partial G(x)}{\partial x} > 0$ for all $x \in \mathbb{R}^n$ by Assumption 2. \square

4.3 | An extension of the previous cases

The proposed LFs, (5) and (6), were selected based on physical meaning of forces presented in the system (2) dynamics. Applying the approach developed in¹⁵, the following generalized sufficient condition of ISS can be formulated in the form of linear matrix inequalities (in such a case both, potential and dissipative, forces can be nonlinear):

Theorem 4. Consider (2) satisfying the hypotheses:

i) There exist matrices $C_i \in \mathbb{R}^{p_i \times n}$ and functions $\sigma_i \in \mathcal{K}_\infty$, $i = 1, 2$ such that

$$G(x) \geq \sigma_1(\|C_1 x\|), \quad \forall x \in \mathbb{R}^n,$$

$$W(\dot{x}) \geq \sigma_2(\|C_2 \dot{x}\|), \quad \forall \dot{x} \in \mathbb{R}^n.$$

ii) There exist matrices $Q, J, K \in \mathbb{R}^{n \times n}$, $0 \leq R = R^\top, U = U^\top \in \mathbb{R}^{n \times n}$ and $0 < L = L^\top \in \mathbb{R}^{n \times n}$ such that

$$\dot{x}^\top R \dot{x} \leq 2\dot{x}^\top Q \frac{\partial W(\dot{x})}{\partial \dot{x}}, \quad \forall \dot{x} \in \mathbb{R}^n;$$

$$\frac{\partial G(x)^\top}{\partial x} U \frac{\partial G(x)}{\partial x} \leq 2x^\top J \frac{\partial G(x)}{\partial x}, \quad x^\top L x \leq 2x^\top K \frac{\partial G(x)}{\partial x}, \quad \forall x \in \mathbb{R}^n.$$

iii) There exist matrices $0 \leq P = P^\top = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$, $0 < \Phi = \Phi^\top \in \mathbb{R}^{n \times n}$, $\Omega \in \mathbb{R}^{n \times n}$, $\Psi \in \mathbb{R}^{n \times n}$ and scalars $\ell > 0$, $\alpha > 0$, $\beta > 0$, $\gamma_j > 0$ for $j = 1, 2, 3$, $\mu_1, \mu_2 \in \mathbb{R}$ satisfying the following system of linear matrix inequalities:

$$\Xi = \begin{bmatrix} P - \begin{bmatrix} \mu_1 C_1^\top C_1 & 0 \\ 0 & \mu_2 C_2^\top C_2 \end{bmatrix} > 0, & \Xi \leq 0, \\ \begin{bmatrix} -\gamma_3 L + \alpha I & P_{11} & 0 & \gamma_2 J + \gamma_3 K - P_{12} & -P_{12} & P_{12} \\ P_{11} & 2P_{12} - \gamma_1 R + \beta I & 0 & \ell I - P_{22} & \gamma_1 Q - P_{22} & P_{22} \\ 0 & 0 & -\Psi^\top - \Psi & -\Psi^\top - \Omega & -\Psi^\top & \Psi^\top \\ \gamma_2 J^\top + \gamma_3 K^\top - P_{12} & \ell I - P_{22} & -\Psi - \Omega^\top & -\Omega - \Omega^\top - \gamma_2 U & -\alpha I - \Omega^\top & \Omega^\top \\ -P_{12} & \gamma_1 Q^\top - P_{22} & -\Psi & -\alpha I - \Omega & -2\alpha I & \alpha I \\ P_{12} & P_{22} & \Psi & \Omega & \alpha I & -\Phi \end{bmatrix} \end{bmatrix}.$$

Then the system (2) is ISS, and it is locally ISS if the properties stated in the parts (i) and (ii) hold locally.

Proof. Consider a LF candidate:

$$V = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^\top P \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + 2\alpha W(\dot{x}) + 2\ell G(x), \quad (7)$$

which is positive definite due to the constraint $P > \begin{bmatrix} \mu_1 C_1^\top C_1 & 0 \\ 0 & \mu_2 C_2^\top C_2 \end{bmatrix}$ verified for some $\mu_1, \mu_2 \in \mathbb{R}$ and Finsler's lemma.

The time derivative of this LF, under the introduced hypotheses and using the descriptor method, can be written as follows:

$$\begin{aligned} \dot{V} &= \begin{bmatrix} x \\ \dot{x} \\ \frac{\partial G(x)}{\partial x} \\ \frac{\partial W(\dot{x})}{\partial \dot{x}} \\ d \end{bmatrix}^\top \begin{bmatrix} 0 & P_{11} & -P_{12} & -P_{12} & P_{12} \\ P_{11} & 2P_{12} & \ell I - P_{22} & -P_{22} & P_{22} \\ -P_{12} & \ell I - P_{22} & 0 & -\alpha I & 0 \\ -P_{12} & -P_{22} & -\alpha I & -2\alpha I & \alpha I \\ P_{12} & P_{22} & 0 & \alpha I & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \frac{\partial G(x)}{\partial x} \\ \frac{\partial W(\dot{x})}{\partial \dot{x}} \\ d \end{bmatrix} \\ &\quad - 2 \left(\frac{\partial G(x)^\top}{\partial x} \Omega^\top + \dot{x}^\top \Psi^\top \right) \left(\ddot{x} + \frac{\partial W(\dot{x})}{\partial \dot{x}} + \frac{\partial G(x)}{\partial x} - d \right) \\ &\leq \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \frac{\partial G(x)}{\partial x} \\ \frac{\partial W(\dot{x})}{\partial \dot{x}} \\ d \end{bmatrix}^\top \Xi \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ \frac{\partial G(x)}{\partial x} \\ \frac{\partial W(\dot{x})}{\partial \dot{x}} \\ d \end{bmatrix} - \alpha x^\top x - \beta \dot{x}^\top \dot{x} + d^\top \Phi d \\ &\leq -\alpha x^\top x - \beta \dot{x}^\top \dot{x} + d^\top \Phi d, \end{aligned}$$

which implies that V is an ISS-LF. That was necessary to prove. \square

Note that (6) is in the form of LF (7). This result can be further extended to cover the case of LF (5) under corresponding hypotheses on W and $\frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2}$.

5 | ISS ANALYSIS OF THE SECOND SYSTEM

In¹⁶, the global asymptotic stability of the origin for a scalar Lienard equation was analyzed:

$$\ddot{x} + \varphi(x)\dot{x} + f(x) = 0,$$

where $x, \dot{x} \in \mathbb{R}$, $\varphi(x) > 0$ for all $x \in \mathbb{R}$ and $xf(x) > 0$ for all $x \neq 0$. For this dynamical system a GSLF was suggested:

$$V = (\dot{x} + \int_0^x \varphi(s)ds)^2 + 4 \int_0^x f(s)ds + \dot{x}^2$$

such that

$$\dot{V} = -2f(x) \int_0^x \varphi(s)ds - 2\varphi(x)\dot{x}^2.$$

The system (3) represents a vector extension of the Lienard equation, for which we will assume the following:

Assumption 3. Let $G(0) = 0$ and

$$G(x) > 0, \quad F^\top(x) \frac{\partial G(x)}{\partial x} > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

while

$$\frac{\partial F(x)}{\partial x} + \frac{\partial F(x)^\top}{\partial x} > 0, \quad \forall x \in \mathbb{R}^n.$$

As before, this assumption introduces the necessary requirements on potential and dissipative forces implying stability of the origin. Let us take a candidate LF:

$$V = \frac{1}{2} \dot{x}^\top \dot{x} + 2G(x) + \frac{1}{2} (\dot{x} + F(x))^\top (\dot{x} + F(x)), \quad (8)$$

whose derivative can be presented as

$$\dot{V} = -F^\top(x) \frac{\partial G(x)}{\partial x} - \dot{x}^\top \frac{\partial F(x)}{\partial x} \dot{x}$$

and, therefore, if (8) is radially unbounded, then it is a GSLF for (3). The conditions of ISS are given below:

Theorem 5. Consider the system (3) under Assumption 3, then it is locally ISS. Let additionally

$$0 < 4\Sigma \leq \frac{\partial F(x)}{\partial x} + \frac{\partial F(x)^\top}{\partial x}, \quad F^\top(x) \frac{\partial G(x)}{\partial x} \geq 2\psi(\|F(x)\|)\|F(x)\|, \quad \forall x \in \mathbb{R}^n,$$

where $\Sigma \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $\psi \in \mathcal{K}_\infty$, then the system is ISS.

Proof. Under the introduced conditions, the LF candidate (8) is radially unbounded ($F(x)$ possesses such a property as a function of x , indeed, using the Mean value theorem we get: $x^\top F(x) = x^\top F(0) + \frac{1}{2} x^\top \int_0^1 \left(\frac{\partial F(sx)}{\partial x} + \frac{\partial F(sx)^\top}{\partial x} \right) ds x \geq x^\top F(0) + 2x^\top \Sigma x$), and using Lemma 1 its time derivative for the system (3) can be written as follows:

$$\begin{aligned} \dot{V} &= 2\dot{x}^\top d + F^\top(x)d - \dot{x}^\top \frac{\partial F(x)}{\partial x} \dot{x} - F^\top(x) \frac{\partial G(x)}{\partial x} \\ &\leq -\dot{x}^\top \Sigma \dot{x} - \psi(\|F(x)\|)\|F(x)\| + d^\top \Sigma^{-1} d + \psi^{-1}(\|d\|)\|d\|, \end{aligned}$$

which implies that V is an ISS-LF in this case. For local analysis with $\|x\| \leq \rho$, $\|\dot{x}\| \leq \rho$ for some $\rho > 0$ we have

$$\dot{V} \leq (2\rho + \varsigma)\|d\| - \dot{x}^\top \frac{\partial F(x)}{\partial x} \dot{x} - F^\top(x) \frac{\partial G(x)}{\partial x},$$

where $\varsigma = \sup_{\|x\| \leq \rho} \|F(x)\|$, and the conditions of Assumption 3 are enough for local verification of the properties from Definition 2. \square

6 | APPLICATIONS

6.1 | Mass-spring systems

The nonlinear oscillations in n degrees of freedom systems have been widely studied in the literature^{17,18,19,20,21,22,23} due to their numerous applications in various engineering domains (*e.g.*, coupled vibrating systems, elastic beams with springs, vibration in milling machines, micro-electro-mechanical systems (MEMS), suspension in different transports, *etc.*).

Assumption 1 is naturally satisfied in this kind of dynamics with $G(x) = \frac{1}{2}x^\top Cx$, where a symmetric positive definite matrix C represents the spring interconnections, while the function W defines the admissible friction. Let

$$W(\dot{x}) = \frac{1}{2}\dot{x}^\top S\dot{x} + \phi\|\dot{x}\|^{2\alpha},$$

where $S = S^\top > 0$, $\phi > 0$ and $\alpha \geq 2$, then it introduces usual, linear in velocities, restoring forces proportional to S , with approximations of high-order terms weighted by ϕ . Hence,

$$\frac{\partial W(\dot{x})}{\partial \dot{x}} = (S + 2\alpha\phi\|\dot{x}\|^{2\alpha-2}I)\dot{x}, \quad \frac{\partial^2 W(\dot{x})}{\partial \dot{x}^2} = S + 2\alpha\phi\|\dot{x}\|^{2\alpha-2}I + 4(\alpha-1)\phi\|\dot{x}\|^{2\alpha-4}\dot{x}\dot{x}^\top,$$

that verifying conditions of Assumption 1 and Theorem 2 results in $\underline{S} = S$, $\overline{S} = S + 4(\alpha-1)\phi I$, $\underline{\rho} = 2\alpha\phi$, $\overline{\rho} = (6\alpha-4)\phi$ and $\underline{\sigma} = \overline{\sigma} = 2\alpha - 2$, then the system (2) is ISS.

Example 1. Let $n = 2$ and

$$C = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}, \quad S = \begin{bmatrix} k_4 & 0 \\ 0 & k_5 \end{bmatrix}, \quad \phi > 0, \quad \alpha = 2,$$

where $k_i > 0$ for all $i = 1, \dots, 5$. The equations of the system (2) can be rewritten as follows in this case:

$$\begin{aligned} \ddot{x}_1 &= -k_1x_1 - k_3(x_1 - x_2) - (k_4 + 2\alpha\phi\|\dot{x}\|^2)\dot{x}_1 + d_1, \\ \ddot{x}_2 &= -k_2x_2 - k_3(x_2 - x_1) - (k_5 + 2\alpha\phi\|\dot{x}\|^2)\dot{x}_2 + d_2, \end{aligned}$$

and the amplitude of total velocity of the system modifies the friction gain.

The conditions of Assumption 2 can also be easily checked in systems with nonlinear characteristics of the springs. Let $W(x) = \frac{1}{2}x^\top \mathbb{W}x$ with $\mathbb{W} = \mathbb{W}^\top > 0$ and take

$$G(x) = \frac{1}{2}(x^\top \mathbb{L}x + x^\top Z^\top Zx + \sigma x^\top Z^\top \text{diag}^2(Zx)Zx),$$

where $\mathbb{L} \in \mathbb{R}^{n \times n}$ is a nonnegative definite diagonal matrix, $Z \in \mathbb{R}^{n-1 \times n}$ is the spring relation matrix between masses, $\sigma > 0$. Hence,

$$\frac{\partial G(x)}{\partial x} = (\mathbb{L} + Z^\top Z + 2\sigma Z^\top \text{diag}^2(Zx)Z)x,$$

and provided that $\mathbb{L} + Z^\top Z > 0$ the conditions of Assumption 2 and Theorem 3 are satisfied for $\eta(s) = 0.5\lambda_{\min}(\mathbb{L} + Z^\top Z)s$, therefore, the system (2) is ISS.

Example 2. Let $n = 2$ and

$$\mathbb{L} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad Z = \sqrt{k_3} \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad \mathbb{W} = \begin{bmatrix} k_4 & 0 \\ 0 & k_5 \end{bmatrix}, \quad \sigma > 0,$$

where $k_i > 0$ for all $i = 1, \dots, 5$. The equations of the system (2) can be rewritten as follows in this case with a nonlinear stiffness²⁰:

$$\begin{aligned} \ddot{x}_1 &= -k_1x_1 - k_3(x_1 - x_2) - 2\sigma k_3^2(x_1 - x_2)^3 - k_4\dot{x}_1 + d_1, \\ \ddot{x}_2 &= -k_2x_2 - k_3(x_2 - x_1) - 2\sigma k_3^2(x_2 - x_1)^3 - k_5\dot{x}_2 + d_2. \end{aligned}$$

Since LF (6) used in Theorem 3 is an example of LF (7) from Theorem 4, then the above example illustrates also the latter theorem. Note that its important advantage is that non-quadratic functions W and G can be considered.

Example 3. Note that the sufficient part of Theorem 1 holds true even for a locally Lipschitz continuous ISS-LF \mathcal{V} , while f can be discontinuous under the conventional regularity restrictions²⁴. Let $n = 2$ and

$$W(\dot{x}) = \sum_{i=1}^2 \left(r_i |\dot{x}_i| + \frac{r_{i+2}}{3} |\dot{x}_i|^3 \right), \quad G(x) = \sum_{i=1}^2 k_i x_i^2 + \frac{k_3}{2} (x_1 - x_2)^2 + \frac{2k_4}{3} |x_1 - x_2|^{1.5},$$

for some $r_i > 0, k_i > 0$ with $i = 1, \dots, 4$. It is straightforward to verify that the conditions of the part (ii) in Theorem 4 are satisfied for:

$$Q = I, R = 4 \begin{bmatrix} \sqrt{r_1 r_3} & 0 \\ 0 & \sqrt{r_2 r_4} \end{bmatrix}; K = \frac{I}{2}, L = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix};$$

$$U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, J = 4 \max\{k_1, k_2\} I.$$

The equations of the system (2) can be presented as follows with a non-integer elastic connection²²:

$$\begin{aligned} \ddot{x}_1 &= -k_1 x_1 - k_3(x_1 - x_2) - k_4 \sqrt{|x_1 - x_2|} \text{sign}(x_1 - x_2) \\ &\quad - r_1 \text{sign}(\dot{x}_1) - r_3 |\dot{x}_1| \dot{x}_1 + d_1, \\ \ddot{x}_2 &= -k_2 x_2 - k_3(x_2 - x_1) - k_4 \sqrt{|x_2 - x_1|} \text{sign}(x_2 - x_1) \\ &\quad - r_2 \text{sign}(\dot{x}_2) - r_4 |\dot{x}_2| \dot{x}_2 + d_2. \end{aligned}$$

6.2 | Coupled Lienard systems

Consider coupled systems in the form (3) with

$$G(x) = \frac{1}{2} x^\top C x, F(x) = Sx + \rho \|x\|^2 x$$

with $C = C^\top \in \mathbb{R}^{n \times n}$ and $S = S^\top \in \mathbb{R}^{n \times n}$ defining the structure of interconnection between subsystems, $\rho > 0$. Then the conditions of Theorem 5 are verified provided that

$$C > 0, SC > 0,$$

and the system is ISS.

Example 4. Let $n = 2$ and

$$C = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}, S = C, \rho = 1$$

for $k_i > 0$ with $i = 1, \dots, 4$, which ensure the needed conditions. The equations of the system (2) can be rewritten in the form:

$$\begin{aligned} \ddot{x}_1 &= -k_1 x_1 - k_3(x_1 - x_2) - (k_1 + k_3 + 3x_1^2 + x_2^2) \dot{x}_1 \\ &\quad - (2x_1 x_2 - k_3) \dot{x}_2 + d_1, \\ \ddot{x}_2 &= -k_2 x_2 - k_3(x_2 - x_1) - (k_2 + k_3 + 3x_2^2 + x_1^2) \dot{x}_2 \\ &\quad - (2x_1 x_2 - k_3) \dot{x}_1 + d_2. \end{aligned}$$

7 | CONCLUSION

For two canonical forms of mechanical systems, (2) and (3), several designs of GSLF and ISS-LF are investigated, and the respective stability conditions are formulated. It is demonstrated in the examples that these conditions can be verified in different practical scenarios. An extension to synthesis of LF for finite-time/fixed-time stability analysis can be considered as a direction of future research.

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