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# Homogeneous Unit Sliding Mode Control

Andrey Polyakov

**Abstract**—The methodology of the unit sliding mode control design (known since 1970s) for linear systems is revised based on the concept of the generalized homogeneity. The restriction about a consistency of the number of control inputs with the dimension of the sliding surface is eliminated. A simple procedure of control parameters tuning based on a known maximal magnitude of matched perturbations is developed. To deal with perturbations of unknown magnitude, a homogeneous sliding mode control with integral action is designed as well. The theoretical results are supported by numerical simulations.

## I. INTRODUCTION

The sliding mode control (SMC) seems to be the first robust control design methodology known since 1960s (see [38] and references therein). The unit control proposed in [12] is one of famous SMC algorithms (see [39], [6], [37]) supported with a very simple rule of control parameters tuning. In the SISO (Single Input Single Output) case, the unit control simply coincides with the conventional relay SMC. The difference can be discovered for linear MIMO (Multi-Input Multi-Output) systems. Indeed, the unit control has discontinuities only on a set being an intersection of several  $(n - 1)$ -dimensional linear surfaces, while the relay SMC is discontinuous on each such hyperplane. The discontinuity set of the unit control has a smaller dimension than the discontinuity set of the conventional relay SMC. The latter may simplify a practical implementation and a chattering attenuation of SMC system [34]. The main well-known restriction of the conventional SMC design for linear MIMO systems is a consistency of dimensions of the sliding surface and the control input. Namely, to implement the unit control for the linear MIMO system, the sliding surface  $\{x \in \mathbb{R}^n : Cx = 0\}$  must be of the dimension  $n - p$  with  $1 \leq p \leq m$  (i.e.,  $C \in \mathbb{R}^{p \times n}$ ), where  $n$  is a dimension of the state vector  $x \in \mathbb{R}^n$  and  $m$  is the number of control inputs. The first aim of this paper is eliminate this restriction and to develop a unit control design methodology ensuring sliding mode on a linear surface of any dimension less than  $n$ . For SISO case, this problem is treated by the so-called high order sliding mode (HOSM) algorithms (see [18], [34]). Design procedures of HOSM controllers are rather complicated and have to be supported with a non-trivial stability analysis (see, e.g. [4] and references therein). An extension of the HOSM control methodology to MIMO systems requires even more cumbersome constructions (see e.g., [28], [27], [10]).

The homogeneity is a dilation symmetry, which is well-known in mathematics due to the papers [44], [9], [16], [8]. A Lie symmetry in general and the dilation symmetry in particular is a feature of many physical process (see, e.g.

[24]). A lot of PDE models of mathematical physics are homogeneous in a generalized sense. Such systems have the so-called self-similar solutions known since 1940s (see, e.g., [35] for more details). The standard (or Euler) homogeneity is a symmetry of a mathematical object (a function, a set, a differential equation, etc) with respect to the standard dilation  $x \mapsto e^s x$  being simply a multiplication of a vector  $x$  by a positive scalar  $e^s$ , where  $s \in \mathbb{R}$ . The generalized dilation (see [44], [16]) is defined as  $x \mapsto \mathbf{d}(s)x$ , where  $\mathbf{d}(s)$  is a family of operators (parameterized by  $s \in \mathbb{R}$ ) having a topological characterization of a dilation (see [15]). The generalized homogeneity is widely utilized in control and systems theory for control design [3], [19], [25], [1], [43], state estimation [26], [1], [5], [20] and stability/controllability/robustness analysis [13], [33], [11], [14]. In fact, the conventional unit control is a standard homogeneous control algorithm (as explained in Section IV). This paper shows that using the generalized homogeneity, the methodology of the unit control can be adapted to sliding surfaces  $Cx = 0$  which may have any dimension less than  $n$ , but all other advantages of the unit control known before are preserved.

In the case of linear PI (Proportional-Integral) control, the integral action compensates a matched constant perturbation of unknown magnitude (and unknown bound). The super-twisting algorithm [17] is the most famous sliding mode control with an integral action, which have more advanced (with respect to the linear PI control) property. Namely, it may reject perturbations of unknown magnitude, which are (sufficiently) slowly varying Lipschitz continuous functions of time [22]. Based on results of [21], [41], a modification of the homogeneous unit control with an integral action having the property mentioned above is proposed. A preliminary version of this paper is presented at IFAC World Congress 2023 [31]. The journal version provides additionally the sliding mode control with integral action, proofs of all claims, new numerical simulations.

The paper is organized as follows. First, a problem statement, basic assumptions and the classical scheme of unit control design are presented. In Section III, some notions of homogeneity theory are briefly recalled. Homogeneous unit control is designed in Section IV for system with a known magnitude of perturbations. The sliding mode control for systems with unknown magnitude of perturbations is designed in Section V. The numerical example is given in Section VI. Finally, Section VII presents some conclusions.

*Notation.*  $\mathbb{R}$  is the field of reals;  $\|\cdot\|$  is a weighted Euclidean norm  $\mathbb{R}^n$ :  $\|x\| = \sqrt{x^\top P x}$  with  $x \in \mathbb{R}^n$ ,  $P = P^\top \in \mathbb{R}^{n \times n}$ ;  $P \succ 0$ , where  $P \succ 0$  means that the matrix  $P$  is positive definite;  $|x| = \sqrt{x^\top x}$  is the Euclidean norm in  $\mathbb{R}^n$ ;  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix;  $\text{rank } M$  denotes the rank of a matrix  $M$ ;  $\mathbf{0}$  is a zero element of a vector space.

## II. PROBLEM STATEMENT

Let us consider the linear MIMO system

$$\dot{x} = Ax + B(u + \gamma(t, x)), \quad t > 0, \quad x(0) = x_0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is the control input, the matrices  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are known and  $\gamma : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^m$  is an unknown locally integrable function, which models the matched system uncertainties and disturbances. The system (1) has Filippov solutions [7] for any locally bounded measurable feedback law  $u = u(t, x)$ .

*Assumption 1:* There exists a known constant  $\bar{\gamma} \geq 0$  :

$$|\gamma(t, x)| \leq \bar{\gamma}, \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}^n. \quad (2)$$

For the system (1) we consider the problem of the sliding mode control design (see [39], [6]). Namely, the **control aim** is to synthesize a feedback control law such that any trajectory of the closed-loop reaches a selected linear surface

$$Cx = \mathbf{0}, \quad C \in \mathbb{R}^{p \times n}, \quad \text{rank } C = p, \quad (3)$$

in a finite time and slides on it after then. The so-called unit control (see [12], [39], [6])

$$\tilde{u} = \hat{K}x + K \frac{Cx}{|Cx|}, \quad (4)$$

solves this problem provided that a)  $p = m$ , b) the matrix  $CB$  is invertible and c) the matrices  $\hat{K} \in \mathbb{R}^{m \times n}$  and  $K \in \mathbb{R}^{m \times p}$  are properly selected:  $\hat{K} = -(CB)^{-1}CA$  and  $K = -(\delta + |CB|\bar{\gamma})(CB)^{-1}$  with  $\delta > 0$ . This paper relaxes the restriction  $p = m$ ,  $\det(CB) \neq 0$  by the following mild assumption.

*Assumption 2:* There exists  $\tilde{K} \in \mathbb{R}^{m \times n}$  such that

$$C(A + B\tilde{K})(I_n - C^\top(CC^\top)^{-1}C) = \mathbf{0} \quad (5)$$

and the pair  $\{A_\sigma, B_\sigma\}$  is controllable, where

$$A_\sigma = C(A + B\tilde{K})C^\top(CC^\top)^{-1}, \quad B_\sigma = CB. \quad (6)$$

If  $CB$  is invertible then the latter assumption is always fulfilled for  $\tilde{K} = -(CB)^{-1}CA$ . However, this assumption is fulfilled in many other cases, for example, if  $C = I_n$  and the pair  $\{A, B\}$  is controllable. Assumption 2 simply means that the dynamics of the sliding variable  $\sigma = Cx$  is completely controllable. Indeed, below we show that, under Assumption 2, we have  $\dot{\sigma} = A_\sigma\sigma + B_\sigma(u - \tilde{K}x + \gamma)$ , so an influence of any other (possibly uncontrollable) component of the system state may be rejected by a control. Notice that the controllability of the pair  $\{A_\sigma, B_\sigma\}$  is necessary for a finite-time reachability of the sliding surface  $Cx = 0$  for any initial state  $x_0 \in \mathbb{R}^n$ .

*Remark 1:* Notice that  $CB \neq \mathbf{0}$  is necessary for controllability of  $\{A_\sigma, B_\sigma\}$ . To fulfill this condition, the original sliding surface  $Cx = 0$  can be extended artificially following the ideas of HOSM design [19]. For example, if  $CA^jB = \mathbf{0}$ ,  $j = 0, 1, \dots, r$  and  $CA^{r+1}B \neq \mathbf{0}$  then the new sliding variable can be defined as  $\sigma_{new} = \begin{pmatrix} C \\ CA \\ \dots \\ CA^r \end{pmatrix} x$ . To decrease a conservatism, the latter extension can be done in a component-wise manner (for each surface  $\sigma_i = 0$ ,  $i = 1, \dots, p$  separately). Obviously, the existence of the sliding mode on the surface  $\sigma_{new} = \mathbf{0}$  implies the existence of the sliding mode on the original surface  $\sigma = 0$ .

Assumption 2 can be easily checked by solving the linear algebraic equation (5) with respect to the variable  $\tilde{K}$ . The absence of solution would simply mean that the selected sliding surface is not appropriate for the unit SMC design. The sliding surface  $Cx = 0$  is usually designed such that the closed-loop system is, at least, exponentially stable. Any method of a stable sliding surface design (see, e.g., [39], [6], [34]) can be utilized taking into account Assumption 2.

The sliding mode control is known to be robust with respect to bounded matched disturbances. The Assumption 1 is conventional for the SMC theory (see, e.g., [39], [6], [34]). Therefore, the *first aim* is to design a unit sliding mode control under the classical Assumption 1 and the relaxed Assumption 2. To solve the problem, the term  $\frac{Cx}{|Cx|}$  in (4) is going to be modified using the concept of a generalized homogeneity.

The SMC design under Assumption 1 uses an estimate  $\tilde{\gamma}$  of the magnitude of perturbations. If the estimate is unknown then some information about a structure of the perturbation  $\gamma$  is required to design a robust control.

*Assumption 3:* There exists a known constant  $\bar{\gamma} \geq 0$  such that the unknown function  $\gamma$  admits the representation:

$$\gamma(t, x) = \gamma_0 + \tilde{\gamma}(t, x), \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \quad (7)$$

$$|\tilde{\gamma}(t, x)| \leq \bar{\gamma} \min\{|Cx|, 1\}, \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \quad (8)$$

where  $\gamma_0 \in \mathbb{R}^m$  is an unknown vector.

In this case, we assume that the uncertain function  $\gamma$  converges to an unknown constant  $\gamma_0$  as  $|Cx| \rightarrow 0$ . Below we show that the latter assumption can be relaxed allowing the constant  $\gamma_0$  to be a Lipschitz continuous function of time. In the latter case, Assumption 3 becomes similar to the case of the super-twisting SMC design (see e.g., [17], [22]). The *second aim* is to design SMC under Assumptions 2 and 3.

## III. PRELIMINARIES: LINEAR HOMOGENEITY

The linear dilation in  $\mathbb{R}^n$  is given by (see [29])

$$\mathbf{d}(s) = e^{sG_d} = \sum_{i=0}^{\infty} \frac{(sG_d)^i}{i!}, \quad s \in \mathbb{R}, \quad (9)$$

where  $G_d \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix<sup>1</sup> being a *generator of linear dilation*. The latter implies that  $\mathbf{d}$  satisfies the limit property,  $\|\mathbf{d}(s)x\| \rightarrow 0$  as  $s \rightarrow -\infty$  and  $\|\mathbf{d}(s)x\| \rightarrow +\infty$  as  $s \rightarrow +\infty$ , required for a group  $\mathbf{d}$  to be a dilation in  $\mathbb{R}^n$  (see, e.g., [16]). The linear dilation introduces an alternative norm topology in  $\mathbb{R}^n$  by means of the canonical homogeneous norm.

*Definition 1:* ([29]) The function  $\|\cdot\|_d : \mathbb{R}^n \mapsto \mathbb{R}_+$  given by  $\|x\|_d = 0$  for  $x = \mathbf{0}$  and

$$\|x\|_d = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (10)$$

is called the *canonical homogeneous norm* in  $\mathbb{R}^n$ , where  $\mathbf{d}$  is a linear monotone dilation<sup>2</sup>.

Notice that  $\|x\| = 1$  is equivalent to  $\|x\|_d = 1$ . Moreover,  $\|\mathbf{d}(-\ln \|x\|_d)x\| = 1$ , so the operator  $\mathbf{d}(-\ln \|x\|_d)x$  is the so-called  *$\mathbf{d}$ -homogeneous projector of  $x$  on the unit sphere*

<sup>1</sup>A matrix  $G_d \in \mathbb{R}^{n \times n}$  is anti-Hurwitz if  $-G_d$  is Hurwitz.

<sup>2</sup>A dilation in  $\mathbb{R}^n$  is monotone if for any  $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  the function  $s \mapsto \|\mathbf{d}(s)x\|$ ,  $s \in \mathbb{R}$  is strictly increasing.

[30, page 159]. For the standard dilation  $\mathbf{d}(s) = e^s I_n, s \in \mathbb{R}$  we have  $\|\cdot\| = \|\cdot\|_{\mathbf{d}}$  and  $\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x = \frac{x}{\|x\|}$  is the standard homogeneous projector of  $x$  on the unit sphere.

*Theorem 1 ([29]):* If  $\|x\| = \sqrt{x^\top P x}$  with a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  satisfying  $P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, P \succ 0$  then  $\mathbf{d}$  is monotone and the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is continuous on  $\mathbb{R}^n$ , smooth on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ ,

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \frac{\|x\|_{\mathbf{d}} x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x}, \quad \forall x \neq \mathbf{0}. \quad (11)$$

Moreover,  $\underline{\sigma}(\|x\|) \leq \|x\|_{\mathbf{d}} \leq \bar{\sigma}(\|x\|), \forall x \in \mathbb{R}^n$ , with

$$\underline{\sigma}(r) = \begin{cases} r^{1/\alpha} & \text{if } r \geq 1, \\ r^{1/\beta} & \text{if } r < 1, \end{cases} \quad \bar{\sigma}(r) = \begin{cases} r^{1/\beta} & \text{if } r \geq 1, \\ r^{1/\alpha} & \text{if } r < 1, \end{cases}$$

where  $\alpha = 0.5 \lambda_{\max}(P^{1/2} G_{\mathbf{d}} P^{-1/2} + P^{-1/2} G_{\mathbf{d}}^\top P^{1/2}) > 0$  and  $\beta = 0.5 \lambda_{\min}(P^{1/2} G_{\mathbf{d}} P^{-1/2} + P^{-1/2} G_{\mathbf{d}}^\top P^{1/2}) > 0$ .

Below the canonical homogeneous norm is utilized as a Lyapunov function for a homogeneous unit control design.

*Definition 2 ([16], [3]):* A vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  (resp. a function  $h : \mathbb{R}^n \mapsto \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if  $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s) f(x)$  (resp.  $h(\mathbf{d}(s)x) = e^{\mu s} h(x)$ ), for all  $x \in \mathbb{R}^n, s \in \mathbb{R}$ .

If  $f$  is  $\mathbf{d}$ -homogeneous of degree  $\mu$  then solutions of  $\dot{x} = f(x)$  are symmetric ([16], [3]):  $x(e^{-\mu s} t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(t, x_0)$ , where  $x(t, z)$  denotes a solution with  $x(0) = z$ . By definition, the system  $\dot{x} = f(x)$  is  $\mathbf{d}$ -homogeneous if the vector field is  $\mathbf{d}$ -homogeneous. The homogeneity degree specifies the convergence rate of a stable homogeneous system.

*Theorem 2:* ([3], [25]) Let a piece-wise continuous vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$ . If the system  $\dot{x} = f(x)$  is locally asymptotically stable then it is globally uniformly finite-time stable for  $\mu < 0$ .

Any homogeneous sliding mode control implicitly uses this result for ensuring a finite-time reaching of a sliding surface.

The homogeneous control systems are robust (Input-to-State Stable) with respect to a large class of perturbations [14], [1].

#### IV. HOMOGENEOUS UNIT CONTROL

The conventional unit control (4) has two components. The meaning of each component can be explained in the context of the homogeneity theory as follows : *the linear component is aimed at a homogenization of the unperturbed dynamics of the sliding variable  $\sigma = Cx$ ,  $C \in \mathbb{R}^{p \times n}$ , while the second (stabilizing) component is discontinuous and homogeneous of negative degree to guarantee finite-time stability of the sliding variable in the perturbed case.* Indeed, for (1), (4) we have

$$\dot{\sigma} = (CA + CB\hat{K})x + CBK \frac{\sigma}{|\sigma|} + CB\gamma.$$

Since, in the conventional case,  $p = m$  and  $\det(CB) \neq 0$  then in order to eliminate the linear term in the above equation, the matrix  $\hat{K}$  is traditionally selected as  $\hat{K} = -(CB)^{-1}CA$ . In the latter case, the unperturbed system  $\dot{\sigma} = CBK \frac{\sigma}{|\sigma|}$  is, obviously, standard homogeneous of negative degree ( $\mu = -1$ ). Moreover, the perturbed system ( $\gamma \neq 0$ ) is globally uniformly finite-time stable under proper selection of control parameters (e.g.,  $K = -(\delta + |CB|\bar{\gamma})(CB)^{-1}$ ,  $\delta > 0$ ). We follow this homogeneity-based interpretation in order to design a similar two component feedback law for the case

$p \neq m$ . Namely, the linear component should homogenize (in a generalized sense) the dynamics of the sliding variable, while the discontinuous one should be uniformly bounded and generalized homogeneous of degree  $\mu = -1$ . Moreover, the dynamics of the sliding variable has to be globally uniformly finite-time stable even in the perturbed case.

Notice that a linear vector field  $\sigma \mapsto \tilde{A}\sigma$  with  $\tilde{A} \in \mathbb{R}^{p \times p}$  may be standard homogeneous of non-zero degree if and only if  $\tilde{A} = \mathbf{0}$ . This explains the structure of the matrix  $\hat{K} = -(CB)^{-1}CA$  in the classical unit control (4). To relax this condition, we use the generalized homogeneity. On the one hand, according to [30, Lemma 7.5] and [43, Lemma 1]), a linear vector field  $\sigma \mapsto \tilde{A}\sigma$  is  $\mathbf{d}$ -homogeneous of non-zero degree if and only if the matrix  $\tilde{A}$  is nilpotent. On the other hand, the operator  $\frac{\sigma}{|\sigma|}$  is the standard homogeneous projector on the unit sphere (*that is why the feedback law (4) is called the unit control*). Such a projector for a dilation  $\mathbf{d}$  is defined as  $\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma$ , where  $\|\cdot\|_{\mathbf{d}}$  is a homogeneous norm.

Following the above explanations we define the control law:

$$u = \hat{K}x + K\mathbf{d}(-\ln \|Cx\|_{\mathbf{d}})Cx, \quad \hat{K} = \tilde{K} + K_0C, \quad (12)$$

where  $Cx = \mathbf{0}$  is a selected sliding surface, the matrix  $\tilde{K} \in \mathbb{R}^{m \times n}$  is defined in Assumption 2, the matrix  $K_0 \in \mathbb{R}^{m \times p}$  is required for  $\mathbf{d}$ -homogenization of a linear component of the dynamics of the sliding variable  $\sigma = Cx$ , the matrix  $K \in \mathbb{R}^{m \times p}$  has to be properly designed for the finite-time stabilization of  $\sigma$  to zero,  $\mathbf{d}$  is a linear dilation (9) in  $\mathbb{R}^p$  and  $\|\cdot\|_{\mathbf{d}}$  is a canonical homogeneous norm to be defined. Since, by definition,  $\|\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma\| = 1$  then (12) is a unit control as well. However, its design is based on the concept of linear homogeneity, so, in order distinguish the conventional and suggested algorithms, we call the feedback law (12) by *the homogeneous unit control*.

If  $\tilde{K}$  satisfies Assumption 2 then using the identity  $C(I_n - C^\top(CC^\top)^{-1}C) = \mathbf{0}$  we derive

$$CA + CB\tilde{K} = C(A + B\tilde{K})(I_n - C^\top(CC^\top)^{-1}C) + C(A + B\tilde{K})C^\top(CC^\top)^{-1}C = A_\sigma C$$

where  $A_\sigma = C(A + B\tilde{K})C^\top(CC^\top)^{-1}$ . The sliding variable  $\sigma = Cx$  of the closed-loop system (1), (12) satisfies

$$\dot{\sigma} = (A_\sigma + B_\sigma K_0)\sigma + B_\sigma K\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma + B_\sigma \gamma, \quad (13)$$

where  $B_\sigma = CB$ . Since the pair  $\{A_\sigma, B_\sigma\}$  is assumed to be controllable then the matrix  $A_0 = A_\sigma + B_\sigma K_0$  can be designed nilpotent by means of a proper selection of  $K_0$  and the system (13) can be homogeneously stabilized to zero in a finite time by a proper selection of the gain  $K$ .

*Theorem 3:* If Assumptions 1 and 2 are fulfilled for  $\bar{\gamma} \geq 0$  and  $C \in \mathbb{R}^{p \times m}$  then 1) the linear algebraic equation

$$A_\sigma G_0 - G_0 A_\sigma + B_\sigma Y_0 = A, \quad G_0 B_\sigma = \mathbf{0} \quad (14)$$

always has a solution  $Y_0 \in \mathbb{R}^{m \times p}$ ,  $G_0 \in \mathbb{R}^{p \times p}$ ; moreover, for any solution of (14), the matrix

$$G_{\mathbf{d}} = I_n - G_0 \quad (15)$$

is anti-Hurwitz, the matrix

$$A_0 = A_\sigma - B_\sigma Y_0 G_{\mathbf{d}}^{-1} \quad (16)$$

is nilpotent and satisfies the identity

$$A_0 G_d = (G_d - I_n) A_0, \quad G_d B_\sigma = B_\sigma; \quad (17)$$

2) the linear algebraic system

$$\begin{aligned} (A_0 + \rho G_d)X + X(A_0 + \rho G_d)^\top + B_\sigma Y + Y^\top B_\sigma^\top + \bar{\gamma} X &= -\bar{\gamma} B_\sigma B_\sigma^\top, \\ G_d X + X G_d^\top > 0, \quad X = X^\top > 0 \end{aligned} \quad (18)$$

has a solution  $X \in \mathbb{R}^{n \times n}$ ,  $Y \in \mathbb{R}^{m \times n}$  for any  $\rho > 0$ ;

3) the system (13) is globally uniformly finite-time stable and  $\sigma(t) = \mathbf{0}$ ,  $\forall t \geq \|\sigma(0)\|_d / \rho$ , provided that  $\mathbf{d}(s) = e^{s G_d}$ ,  $s \in \mathbb{R}$ ,

$$K_0 = -Y_0 G_d^{-1}, \quad K = Y X^{-1}, \quad (19)$$

and  $\|\cdot\|_d$  is the canonical homogeneous norm induced by the weighted Euclidean norm  $\|\sigma\| = \sqrt{\sigma^\top X^{-1} \sigma}$ .

*Proof:* The feasibility of (14) (as well as property of  $G_0$ ) is studied in [43], [23], but the feasibility of (18) is proven in [32] and refined in [43]. Let us prove the finite-time stability of (13). Using (11), (12) we derive

$$\frac{d\|\sigma(t)\|_d}{dt} = \|\sigma\|_d \frac{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} \mathbf{d}(-\ln \|\sigma\|_d) \dot{\sigma}}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} G_d \mathbf{d}(-\ln \|\sigma\|_d) \sigma}.$$

The identity (17) means that [30, Lemma 7.5]  $A_0 \mathbf{d}(s) = e^{-s} \mathbf{d}(s) A_0$ ,  $\mathbf{d}(s) B = e^s B$ ,  $\forall s \in \mathbb{R}$ . Hence, we conclude

$$\begin{aligned} \frac{d\|\sigma(t)\|_d}{dt} &= \frac{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} (A_0 + B_\sigma K) \mathbf{d}(-\ln \|\sigma\|_d) \sigma}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} G_d \mathbf{d}(-\ln \|\sigma\|_d) \sigma} \\ &\quad + \frac{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} B_\sigma \gamma}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} G_d \mathbf{d}(-\ln \|\sigma\|_d) \sigma}. \end{aligned}$$

Since for  $z = \mathbf{d}(-\ln \|\sigma\|_d) \sigma$  we have  $z^\top X^{-1} (A_0 + B_\sigma K) z = z^\top X^{-1} (A_0 X + B_\sigma Y) X^{-1} z = \frac{1}{2} z^\top X^{-1} (A_0 X + X A_0^\top + B_\sigma Y + Y^\top B_\sigma^\top) X^{-1} z$  and  $z^\top X^{-1} G_d z = \frac{1}{2} z^\top (X^{-1} G_d + G_d^\top X^{-1}) z$  then using (18) we derive

$$\begin{aligned} \frac{d\|\sigma(t)\|_d}{dt} &= -\rho \frac{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) (X^{-1} G_d + G_d^\top X^{-1}) \mathbf{d}(-\ln \|\sigma\|_d) \sigma}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) (X^{-1} G_d + G_d^\top X^{-1}) \mathbf{d}(-\ln \|\sigma\|_d) \sigma} \\ &\quad - \frac{\bar{\gamma} \sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} \mathbf{d}(-\ln \|\sigma\|_d) \sigma}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) (X^{-1} G_d + G_d^\top X^{-1}) \mathbf{d}(-\ln \|\sigma\|_d) \sigma} \\ &\quad - \frac{\bar{\gamma} \sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} B_\sigma B_\sigma^\top X^{-1} \mathbf{d}(-\ln \|\sigma\|_d) \sigma}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) (X^{-1} G_d + G_d^\top X^{-1}) \mathbf{d}(-\ln \|\sigma\|_d) \sigma} \\ &\quad + \frac{2 \sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} B_\sigma \gamma}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) (X^{-1} G_d + G_d^\top X^{-1}) \mathbf{d}(-\ln \|\sigma\|_d) \sigma}. \end{aligned}$$

For  $\bar{\gamma} > 0$  and  $\xi = \left( \frac{\gamma/\sqrt{\bar{\gamma}}}{\sqrt{\bar{\gamma}} B_\sigma^\top X^{-1} \mathbf{d}(-\ln \|\sigma\|_d) \sigma} \right)$  we have

$$\begin{aligned} 0 \leq \xi^\top \begin{pmatrix} I_m & -I_m \\ -I_m & I_m \end{pmatrix} \xi &= \frac{|\gamma|^2}{\bar{\gamma}} - 2 \sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} B_\sigma \gamma \\ &\quad + \bar{\gamma} \sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} B_\sigma B_\sigma^\top X^{-1} \mathbf{d}(-\ln \|\sigma\|_d) \sigma. \end{aligned}$$

Therefore, using  $\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) X^{-1} \mathbf{d}(-\ln \|\sigma\|_d) \sigma = 1$  (see the definition of  $\|\cdot\|_d$ ) we derive

$$\begin{aligned} \frac{d\|\sigma(t)\|_d}{dt} &= -\rho - \frac{\bar{\gamma} - |\gamma|^2 / \bar{\gamma}}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) (X^{-1} G_d + G_d^\top X^{-1}) \mathbf{d}(-\ln \|\sigma\|_d) \sigma} \\ &\quad - \frac{\xi^\top \begin{pmatrix} I_m & -I_m \\ -I_m & I_m \end{pmatrix} \xi}{\sigma^\top \mathbf{d}^\top(-\ln \|\sigma\|_d) (X^{-1} G_d + G_d^\top X^{-1}) \mathbf{d}(-\ln \|\sigma\|_d) \sigma}. \end{aligned}$$

So,  $\frac{d\|\sigma(t)\|_d}{dt} \leq -\rho$  and  $\sigma(t) = 0$  for  $t \geq \|\sigma(0)\|_d / \rho$ .  $\blacksquare$

The homogeneous unit control (12) is discontinuous only on the sliding surface  $Cx = 0$  and rejects bounded matched perturbations  $|\gamma(t, x)| \leq \bar{\gamma}$ .

**Algorithm** of the homogeneous unit control design:

- Step 1: given  $C \in \mathbb{R}^{p \times m}$  solve the linear algebraic equation (5) with respect to the variable  $\tilde{K} \in \mathbb{R}^{m \times n}$ .

- Step 2: check the controllability of the pair  $\{A_\sigma, B_\sigma\}$ , where  $A_\sigma \in \mathbb{R}^{p \times p}$  and  $B_\sigma \in \mathbb{R}^{p \times n}$  are given by (6).
- Step 3: solve the linear algebraic equation (14) with respect to  $Y_0 \in \mathbb{R}^{m \times p}$  and  $G_0 \in \mathbb{R}^{p \times p}$ .
- Step 4: solve the linear matrix inequality (18) with respect to  $Y \in \mathbb{R}^{m \times p}$  and  $X \in \mathbb{R}^{p \times p}$ .
- Step 5: compute the parameters  $K_0 \in \mathbb{R}^{m \times p}$ ,  $K \in \mathbb{R}^{m \times p}$  and the homogeneous unit control (12) by the formulas (19), (15), (9), (10) with  $\|\sigma\| = \sqrt{\sigma^\top X^{-1} \sigma}$ .

*Remark 2 (On sliding motion equation):* Using the equivalent control method (see [39]) for  $t \geq \|Cx(0)\|_d$  we derive the sliding motion equation in the form:

$$\dot{x}(t) = (A + B\tilde{K})x(t), \quad Cx(t) = \mathbf{0}, \quad (20)$$

where  $\tilde{K} \in \mathbb{R}^{m \times n}$  is defined in Assumption 2. If the control aim is to stabilize the system (1) then the sliding surface  $Cx = 0$  has to be selected such that the system (20) is globally asymptotically stable. The simplest condition to  $C$  in this case is given by the following matrix inequality

$$P(A + B\tilde{K}) + (A + B\tilde{K})^\top P - \beta C^\top C < 0, \quad P > 0, \quad \beta > 0. \quad (21)$$

Considering the Lyapunov function  $V = x^\top P x$  we derive

$$\begin{aligned} \dot{V} &= 2x^\top P(A + B\tilde{K})x \\ &= x^\top (P(A + B\tilde{K}) + (A + B\tilde{K})^\top P - \beta C^\top C)x < 0, \end{aligned}$$

where the identity  $Cx = 0$  and the inequality (21) are utilized on the last step. To design a stabilizing homogeneous unit control (12) for the system (1) it is sufficient to find a matrix  $C$  satisfying Assumption 2 and the inequality (21).

*Remark 3 (On chattering attenuation):* Following the conventional approach to the chattering reduction (see, e.g., [39]), the discontinuous term in (12) can be "smoothed" using the saturation function:

$$u_\alpha = \hat{K}x + K \mathbf{d}(-\ln \max\{\alpha, \|Cx\|_d\}) Cx, \quad \alpha \geq 0 \quad (22)$$

coincides with (12) for  $\alpha = 0$ . On the one hand, the parameter  $\alpha > 0$  simply defines an attractive set of the sliding variable  $\sigma$ . Indeed, in the view of Theorem 3 for the closed-loop system (1), (22) we have  $\frac{d\|\sigma(t)\|_d}{dt} \leq -\rho$  if  $\|\sigma(t)\|_d \geq \alpha$ . The latter means that the sliding variable  $\sigma$  converges to the homogeneous ball  $\|\sigma\|_d \leq \alpha$  in a finite time. On the other hand, the nonlinear term in (22) becomes linear  $K \mathbf{d}(-\ln \alpha) Cx$  for  $\|x\|_d \leq \alpha$ , so the feedback law  $u_\alpha$  is continuous on  $\mathbb{R}^n$  if  $\alpha > 0$ . In practice the parameter  $\alpha$  can be utilized for an attenuation of the chattering caused by discontinuity of SMC.

*Remark 4 (On computation of  $\|\cdot\|_d$ ):* Since the canonical homogeneous norm is defined implicitly by the formula (10), a computational algorithm is required for practical implementation of the homogeneous unit control given by (12). In [29], [30, Chapter 8], [42] a scheme for an approximation of  $\|\cdot\|_d$  by an explicit homogeneous function is presented. Issues of numerical on-line computation of  $\|\cdot\|_d$  are studied in [32] based on a bisection method. This numerical method is shown to be very efficient in practice (see e.g., [40]).

*Remark 5 (On exactness of the reaching time estimate):* If  $\bar{\gamma} = 0$  and  $\rho = 1$  then  $T(x_0) = \|Cx_0\|_d$  is the exact reaching time of the sliding surface  $Cx = 0$  by trajectory of

the closed loop system (1), (12). Indeed, repeating the proof of Theorem 3 we derive  $\frac{d\|\sigma(t)\|_{\mathbf{d}}}{dt} = -1$ , i.e.,  $\sigma(t) \neq \mathbf{0}$  for all  $t \in [0, \|\sigma(0)\|_{\mathbf{d}}]$  and  $\sigma(t) = \mathbf{0}$  for all  $t \geq \|\sigma(0)\|_{\mathbf{d}}$ . The reaching time can be tuned by the parameter  $\rho > 0$ . The larger  $\rho$ , faster the reaching of the sliding surface (see the estimate of the reaching time in Theorem 3). Theoretically, the reaching time may be adjusted arbitrary small. However, selection of a too large  $\rho$  may lead to a large feedback gains and a large chattering in practice. In practice,  $\rho$  could be one more tuning parameter for the chattering attenuation.

## V. HOMOGENEOUS SMC WITH INTEGRAL ACTION

Inspired by [21], [41], let us consider the control law with integral action

$$\begin{aligned} u(x) &= \hat{K}x + \sqrt{\|\sigma\|_{\mathbf{d}}} K \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma + \int_0^t u_i d\tau, \\ u_i &= \frac{K_i \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma}{q(\sigma)}, \quad \sigma = Cx. \quad K_i \in \mathbb{R}^{m \times p}, \end{aligned} \quad (23)$$

where  $\hat{K} = \tilde{K} + K_0 C$ ,  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ ,  $G_{\mathbf{d}} = I_n - 0.5G_0$ ,  $K = YX^{-1}$ ,  $P = X^{-1}$ ,  $K_0$  and  $G_0$  are defined in Theorem 3,

$$q(\sigma) = \sigma^{\top} \mathbf{d}^{\top}(-\ln \|\sigma\|_{\mathbf{d}}) (PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top} P) \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma.$$

In this case, under Assumption 2 and 3 the dynamics of the sliding variable admits the representation

$$\begin{aligned} \dot{\sigma} &= A_0 \sigma + B_{\sigma} \left( \|\sigma\|_{\mathbf{d}}^{\frac{1}{2}} K \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma + \eta + \tilde{\gamma} \right), \\ \dot{\eta} &= u_i(\sigma) \end{aligned} \quad (24)$$

where the nilpotent matrix  $A_0$  is defined in Theorem 3,  $\eta = \gamma_0 + \int_0^t u_i d\tau$ . Since  $\|\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma\| = 1$  then  $0 < \beta \leq q(\sigma) \leq \alpha$ , where the constants  $\alpha, \beta$  are defined in Theorem 1. The latter means that a kind of the unit control is utilized to define the integral action  $u_i$ . Notice that if  $p = 1$  then  $A_{\sigma}, B_{\sigma} \in \mathbb{R}$ ,  $\hat{K}, K, K_i \in \mathbb{R}$ ,  $P = \frac{1}{2}$ ,  $G_{\mathbf{d}} = 1$ ,  $\mathbf{d}(s) = e^s$ ,  $\sigma = x$ ,  $\|\sigma\|_{\mathbf{d}} = |\sigma|$  and  $\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma = \text{sign}(\sigma)$ , and the control (23) becomes

$$u = \hat{K}x + K \sqrt{|\sigma|} \text{sign}(\sigma) + K_i \int_0^t \text{sign}(\sigma(\tau)) d\tau.$$

Therefore, in the single-input case the control law (23) is the super-twisting algorithm.

*Theorem 4:* Let Assumptions 2 and 3 be fulfilled for  $C \in \mathbb{R}^{p \times n}$  and  $\tilde{\gamma} > 0$ . If the parameters  $K_0, G_0$  are defined as in Theorem 3 and  $G_{\mathbf{d}} = I_n - 0.5G_0$  then the system of linear matrix inequalities is

$$\begin{aligned} (A_0 + \rho G_{\mathbf{d}})X + X(A_0 + \rho G_{\mathbf{d}})^{\top} + B_{\sigma} Y + Y^{\top} B_{\sigma}^{\top} + \tilde{\gamma} X &= -\tilde{\gamma} B_{\sigma} B_{\sigma}^{\top}, \\ G_{\mathbf{d}} X + X G_{\mathbf{d}}^{\top} &\succ X, \quad I_n \succeq X \succ 0 \end{aligned} \quad (25)$$

feasible for any  $\tilde{\gamma} \geq 0$ . For  $K = YX^{-1}$ ,  $K_i = -QB_{\sigma}^{\top}P$ ,  $0 \prec Q = Q^{\top} \in \mathbb{R}^{m \times m}$ , the unperturbed system (24) is  $\tilde{\mathbf{d}}$ -homogeneous of degree  $-1/2$  with respect to the dilation in  $\mathbb{R}^{p+m}$  defined as follows

$$\tilde{\mathbf{d}}(s) = \begin{pmatrix} \mathbf{d}(s) & \mathbf{0} \\ \mathbf{0} & e^{\frac{1}{2}s} I_m \end{pmatrix}, \quad s \in \mathbb{R};$$

and the system (24) is globally uniformly finite-time stable.

*Proof:* Denoting  $\tilde{G}_{\mathbf{d}} = I_n - G_0$  and  $Y = Y_1 + 0.5\rho B_{\sigma}^{\top}$  and the system (25) without  $X \succeq I_n$  can be rewritten as

$$\begin{aligned} (A_0 + \frac{1}{2}\rho \tilde{G}_{\mathbf{d}})X + X(A_0 + \frac{1}{2}\rho \tilde{G}_{\mathbf{d}})^{\top} + B_{\sigma} Y_1 + Y_1^{\top} B_{\sigma}^{\top} + (\rho + \tilde{\gamma})X &= \\ -(\rho + \tilde{\gamma}) B_{\sigma} B_{\sigma}^{\top}, \\ 0.5(\tilde{G}_{\mathbf{d}} X + X \tilde{G}_{\mathbf{d}}^{\top}) &\succ 0, \quad X \succ 0 \end{aligned}$$

The above system of LMIs is feasible (see Theorem 3) for any  $\tilde{\gamma} \geq 0$  and any  $\rho > 0$ . Making the change of variable  $Y_1 = \tilde{Y} - 0.5(\tilde{\gamma} + \rho) B_{\sigma}^{\top}$  we derive the equivalent feasible system of LMIs:

$$\begin{aligned} (A_0 + \frac{1}{2}\rho \tilde{G}_{\mathbf{d}})X + X(A_0 + \frac{1}{2}\rho \tilde{G}_{\mathbf{d}})^{\top} + B_{\sigma} \tilde{Y} + \tilde{Y}^{\top} B_{\sigma}^{\top} + (\rho + \tilde{\gamma})X &= 0, \\ \tilde{G}_{\mathbf{d}} X + X \tilde{G}_{\mathbf{d}}^{\top} &\succ 0, \quad X \succ 0 \end{aligned}$$

If the pair  $X$  and  $\tilde{Y}$  is a solution of the above LMI then for any  $\lambda > 0$  the pair  $\lambda X$  and  $\lambda \tilde{Y}$  is also a solution of the latter system of LMIs as well. Hence, for a sufficiently small  $\lambda > 0$  we derive  $I_n \succeq \lambda X$ . Therefore, the system (25) is feasible.

For the selected matrix  $G_{\mathbf{d}}$  we have  $A_0 G_{\mathbf{d}} = (-0.5I_n + G_{\mathbf{d}})A_0$ ,  $G_{\mathbf{d}} B_{\sigma} = B_{\sigma}$ ,  $A_0 \mathbf{d}(s) = e^{-\frac{1}{2}s} \mathbf{d}(s) A_0$ ,  $\mathbf{d}(s) B_{\sigma} = e^s B_{\sigma}$ , and  $f(\tilde{\mathbf{d}}(s)\zeta) = e^{-\frac{1}{2}s} \tilde{\mathbf{d}}(s) f(\zeta)$ , where

$$f(\zeta) = \begin{pmatrix} A_0 \sigma + B_{\sigma} \|\sigma\|_{\mathbf{d}}^{\frac{1}{2}} K \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma + B_{\sigma} \eta \\ u_i(\sigma) \end{pmatrix}, \quad \zeta = \begin{pmatrix} \sigma \\ \eta \end{pmatrix}.$$

Let us consider the Lyapunov function candidate

$$V = \|\sigma\|_{\mathbf{d}} + \eta^{\top} Q^{-1} \eta.$$

Using  $\mathbf{d}$ -homogeneity we derive

$$\begin{aligned} \dot{V} &= \|\sigma\|_{\mathbf{d}} \frac{\sigma^{\top} \mathbf{d}^{\top}(-\ln \|\sigma\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \dot{\sigma}}{\sigma^{\top} \mathbf{d}^{\top}(-\ln \|\sigma\|_{\mathbf{d}}) X^{-1} G_{\mathbf{d}} \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma} + 2\eta^{\top} Q^{-1} \dot{\eta} \\ &= \sqrt{\|\sigma\|_{\mathbf{d}}} \frac{\sigma^{\top} \mathbf{d}^{\top}(-\ln \|\sigma\|_{\mathbf{d}}) X^{-1} (A_0 + B_{\sigma} K) \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma}{0.5q(\sigma)} \\ &\quad + \frac{\sigma^{\top} \mathbf{d}^{\top}(-\ln \|\sigma\|_{\mathbf{d}}) X^{-1} B_{\sigma} (\tilde{\gamma} + \eta)}{0.5q(\sigma)} + 2\eta^{\top} Q^{-1} \dot{\eta} \\ &= -\rho \|\sigma\|_{\mathbf{d}}^{\frac{1}{2}} + \frac{2\pi^{\top} P B_{\sigma} \frac{\tilde{\gamma}}{\sqrt{\|\sigma\|_{\mathbf{d}}}} - \tilde{\gamma} - \tilde{\gamma} \pi^{\top} X^{-1} B_{\sigma} B_{\sigma}^{\top} X^{-1} \pi}{q(\sigma)}, \end{aligned}$$

where  $\pi = \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}}) \sigma$  is a homogeneous projection of  $\sigma$  on the unit sphere  $\|\sigma\| = 1$ .

If  $|\tilde{\gamma}| \leq \tilde{\gamma} \sqrt{\|\sigma\|_{\mathbf{d}}}$  then repeating the proof of Theorem 3 we derive  $\dot{V} \leq -\rho \|\sigma\|_{\mathbf{d}}^{0.5}$ . Hence, we conclude that the set  $\{\zeta \in \mathbb{R}^{p+m} : V(\zeta) \leq 1\}$  is a strictly positively invariant set of the system (24). The latter guarantees global asymptotic stability of the system (see, [3]), but the negative homogeneity degree implies the finite-time stability.

To complete the proof we just need to show that  $\min\{|\sigma|^2, 1\} \leq \|\sigma\|_{\mathbf{d}}$ . Indeed, if  $|\sigma| \geq 1$  then we need to show that  $\|\sigma\|_{\mathbf{d}} \geq 1$  or equivalently  $\sigma^{\top} X^{-1} \sigma \geq 1$ . The latter inequality holds since  $\sigma^{\top} \sigma$  is assumed to be greater than 1 but  $X^{-1} \succeq I_n$  due to  $I_n \succeq X$  (see LMI (25)). In the case  $|\sigma| \leq 1$  we need to prove that  $|\sigma|^2 \leq \|\sigma\|_{\mathbf{d}}$ . Since  $X^{-1} \succeq I_n$  then  $|\sigma| \leq \|\sigma\|$ . In this case, for the function  $\underline{\sigma}$  defined in Theorem 1 we have  $\underline{\sigma}(|\sigma|) \leq \underline{\sigma}(\|\sigma\|) \leq \|\sigma\|_{\mathbf{d}}$ . The inequality  $G_{\mathbf{d}} X + X G_{\mathbf{d}}^{\top} \succ X$  (see (25)) guarantees that  $\beta \geq 0.5$ , where the parameter  $\beta$  defines the function  $\underline{\sigma}$  (see Theorem 1). So, for  $|\sigma| \leq 1$  we have  $|\sigma|^2 \leq |\sigma|^{1/\beta} \leq \|\sigma\|_{\mathbf{d}}$ . ■

According the proven theorem, the homogeneous control (23) with integral action enforces a sliding mode in the system (1) on the surface  $Cx = 0$  in a finite time provided that

the uncertain function  $\gamma$  satisfies Assumption 3. The known parameter  $\bar{\gamma}$  is utilized for the tuning of the controller.

*Corollary 1:* If an unknown vector  $\gamma_0$  in Assumption 3 is replaced with a Lipschitz continuous function  $\gamma_0 : \mathbb{R} \mapsto \mathbb{R}^m$ , i.e.,  $\exists \bar{\gamma}_0 > 0 : |\dot{\gamma}_0(t)| \leq \bar{\gamma}_0$  almost everywhere, then Theorem 4 remains valid provided that  $\bar{\gamma}_0$  is sufficiently small.

*Proof:* For  $\dot{\gamma}_0 = 0$  Theorem 4 guarantees global uniform asymptotic stability of the system  $\dot{\zeta} = f(\zeta)$ , where  $f$  and  $\zeta$  are defined in the proof of Theorem 4. Then by Zubov-Rosier theorem ([44], [33], [2]) there exists a  $\mathbf{d}$ -homogeneous Lyapunov function  $V \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$  of degree 1 and a constant  $c_0 > 0$ :  $\dot{V} = \frac{\partial V}{\partial \zeta} f(\zeta) \leq -c_0$ ,  $\forall \zeta \neq \mathbf{0}$ . If  $\dot{\gamma}_0 \neq \mathbf{0}$  for the same Lyapunov function we have  $\dot{V} = \frac{\partial V}{\partial \zeta} (f(\zeta) + \begin{pmatrix} \mathbf{0} \\ \gamma_0 \end{pmatrix}) \leq -c_0 + \bar{\gamma}_0 \left| \frac{\partial V}{\partial \zeta} \right|$ . Since  $V$  is a  $\mathbf{d}$ -homogeneous function of degree 1 then  $\left| \frac{\partial V}{\partial \zeta} \right|$  is a  $\mathbf{d}$  homogeneous function of degree 0. Taking into account  $V \in C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$  we conclude that  $\left| \frac{\partial V}{\partial \zeta} \right|$  is bounded on the unit sphere and on  $\mathbb{R}^n$  (due to the zero homogeneity degree). Therefore, the system (24) remains finite-time stable provided that  $\bar{\gamma}_0$  is sufficiently small. ■

This corollary relaxes Assumption 3 and shows that the homogeneous sliding mode control with integral action can also reject (sufficiently) slowly-varying matched perturbations of an unknown magnitude. This repeats the property of the super-twisting algorithm.

## VI. NUMERICAL EXAMPLE

Let us consider the system with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & -1 & 3 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The pair  $\{A, B\}$  is not controllable, so the first two components of the state vector cannot be affected by the control. However, the selected matrix  $C$  means that the sliding mode needs to be enforced in the last three components of the state vector simultaneously. Since the dimensions of  $C$  and  $B$  are not consistent then we cannot use the conventional scheme of the unit control design. Let us design the homogeneous unit control according to Theorem 3. Using (5) we derive  $\tilde{K} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$ , so  $A_\sigma = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{pmatrix}$ ,  $B_\sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ . The pair  $\{A_\sigma, B_\sigma\}$  is controllable, so Assumption 2 is fulfilled. A solution of (14) is given by  $G_0 = \begin{pmatrix} -1 & 0 & 0 \\ -0.5 & 0 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}$ ,  $Y_0 = \begin{pmatrix} -0.5 & -2.5 & -0.5 \\ -1.5 & 1.5 & -3.5 \end{pmatrix}$ , so the generator of dilation  $\mathbf{d}$  is given by

$$G_{\mathbf{d}} = \begin{pmatrix} 2 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix}, \text{ and } K_0 = \begin{pmatrix} 0 & 0 & 0.5000 & -2.5000 & -0.5000 \\ 0 & 0 & -1.5000 & 1.5000 & -3.5000 \end{pmatrix}.$$

### A. Homogeneous unit control

First, let us consider the disturbance-free case  $\gamma \equiv \mathbf{0}$ . Solving the linear system (18) for  $\bar{\gamma} = 0$  and  $\rho = 1$  we derive

$$K = \begin{pmatrix} 4.75 & -2 & 1 \\ -4.75 & 1 & -2 \end{pmatrix}, \quad P = \begin{pmatrix} 4.6 & -1.2 & 1.2 \\ -1.2 & 0.5538 & -0.2462 \\ 1.2 & -0.2462 & 0.5538 \end{pmatrix}.$$

The numerical simulation of the closed-loop system (1), (12) with  $x_0 = (1, 0, 2, 0, 1)^\top$  was done in MATLAB using the explicit Euler method with the constant step  $h = 0.001$ . The

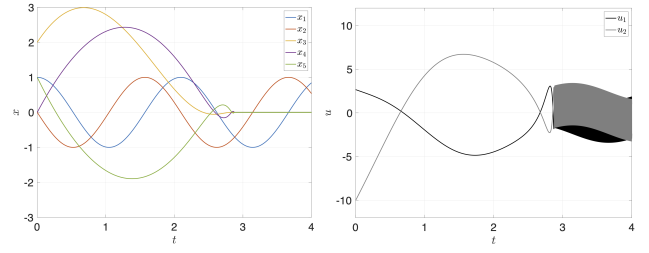


Fig. 1. The system with homogeneous unit control (disturbance-free case)

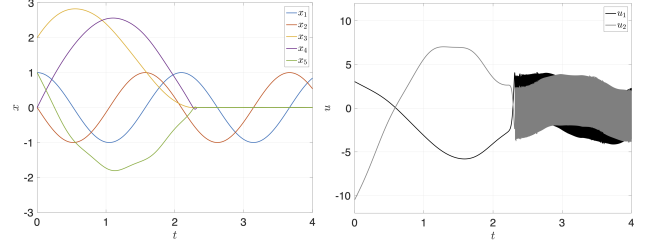


Fig. 2. The system with homogeneous unit control (perturbed case)

simulation results for the disturbance-free case ( $\gamma \equiv \mathbf{0}$ ) are depicted on Fig. 1. In the disturbance-free case, the canonical homogeneous norm defines the reaching time. Indeed, according to Remark 5 the reaching time of the trajectory of the closed loop system is  $\|C x_0\|_{\mathbf{d}} \approx 2.8860$ . The numerical simulations confirm this (up to the computation precision). The homogeneous unit control design in the perturbed case uses Assumption 1 to define the maximum magnitude of the matched perturbations. Solving the linear system (18) for  $\bar{\gamma} = 0.5$  we derive the control parameters

$$K = \begin{pmatrix} 5.8379 & -2.5103 & 1.1938 \\ -5.8379 & 1.1938 & -2.5103 \end{pmatrix}, P = \begin{pmatrix} 5.3112 & -1.3265 & 1.3265 \\ -1.3265 & 0.5413 & -0.2751 \\ 1.3265 & -0.2751 & 0.5413 \end{pmatrix}.$$

According to Theorem 3 the homogeneous unit control rejects bounded matched disturbances  $\gamma$  of the magnitude  $\bar{\gamma} = 0.5$ . The numerical simulations for the system with the matched disturbances modeled by  $\gamma(t) = 0.5(\sin(10t), \cos(10t))^\top$  are presented on Fig. 2. To demonstrate the efficiency of the chattering attenuation scheme suggested in Remark 3, we present numerical simulations for the homogeneous unit control with the saturated canonical homogeneous norm (see the formula (22)). The simulation results for the system with matched perturbations and the saturation parameter  $\alpha = 0.002$  are depicted in Fig. 3.

### B. Homogeneous SMC with integral action

Selection of the parameters of the homogeneous SMC control with integral action is inspired by Theorem 4:

$$K = \begin{pmatrix} 4.4043 & -2.2933 & 0.9462 \\ -4.4043 & 0.9462 & -2.2933 \end{pmatrix}, K_1 = \begin{pmatrix} 1.3174 & -0.6732 & 0.2849 \\ -1.3174 & 0.2849 & -0.6732 \end{pmatrix}$$

with  $\bar{\gamma} = 0.5$ . The simulation for homogeneous SMC with integral action with  $\gamma_0 = 1$  and  $\tilde{\gamma} = |x_2|0.5(\sin(10t), \cos(10t))^\top$  are depicted on Fig. 4. In contrast to the homogeneous unit control, the homogeneous SMC with integral action does not have a large (visible) chattering even without use of any chattering attenuation technique. This repeats the observation for the super-twisting controller in the single input case [17], [22].

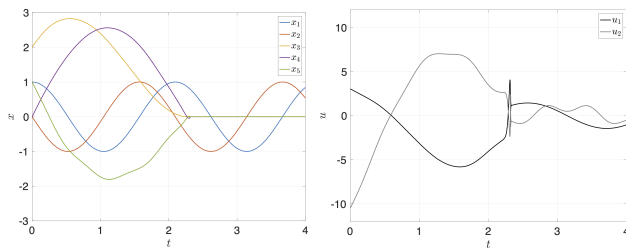


Fig. 3. The system with homogeneous unit control and chattering attenuation

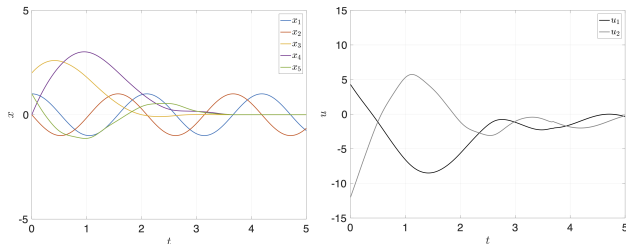


Fig. 4. The system with homogeneous SMC with integral action

## VII. CONCLUSIONS

A methodology of a unit (sliding mode) control design has been revised. It is shown that a linear sliding surface  $Cx = 0$  can be selected independently of the number of control inputs provided that the dynamics of the sliding variable  $\sigma = Cx$  is controllable. The design is essentially based on the so-called homogeneous norm, which defines the homogeneous projector on the unit sphere. The mentioned homogeneous projector forms the discontinuous component of the unit control (in both classical and revised cases), while the linear (continuous) component of the unit control is utilized (as before) for a homogenization of the dynamics of the sliding variable. A possible extension of the super-twisting algorithm to the multi-input case is proposed as well. The simplicity of the control design and parameters tuning is demonstrated on example. An extension of the presented methodology to infinite dimensional unit control systems (see, e.g. [36]) is the interesting problem for the future research.

## VIII. DEDICATION

This paper is dedicated to the memory of Prof. Vadim Utkin who passed away recently and who inspired me with the presented revision of the unit sliding mode control method.

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