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# Long time behaviour for electromagnetic waves in dissipative Lorentz media

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## Abstract

A very general class of models for describing the propagation of waves in dispersive electromagnetic media is provided by generalized Lorentz models [1]. In this work, we study the long time behaviour of the solutions of the dissipative version of these models.

## 1 Introduction

We are interested in Maxwell's equations

$$\begin{cases} \varepsilon_0 \partial_t \mathbf{E} - \nabla \times \mathbf{H} + \varepsilon_0 \partial_t \mathbf{P} = 0 \\ \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} + \mu_0 \partial_t \mathbf{M} = 0, \end{cases} \quad (1.1)$$

where  $\mathbf{E}$  and  $\mathbf{H}$  are, respectively, the electric and magnetic fields, while  $\mathbf{P}$  (resp.  $\mathbf{M}$ ) is the electric polarization (resp. magnetization). In generalized dissipative Lorentz media, these are related to the electromagnetic field via a system of ODE's (the constitutive laws of the medium)

$$\begin{cases} \mathbf{P} = \sum_{j=1}^{N_e} \Omega_{e,j}^2 \mathbf{P}_j, \quad \mathbf{M} = \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \mathbf{M}_\ell, \\ \partial_t^2 \mathbf{P}_j + \alpha_{e,j} \partial_t \mathbf{P}_j + \omega_{e,j}^2 \mathbf{P}_j = \mathbf{E}, \\ \partial_t^2 \mathbf{M}_\ell + \alpha_{m,\ell} \partial_t \mathbf{M}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell = \mathbf{H}, \end{cases} \quad (1.2)$$

where  $(\mathbf{P}_j, \partial_t \mathbf{P}_j, \mathbf{M}_\ell, \partial_t \mathbf{M}_\ell)$  vanish at  $t = 0$ . In (1.2),  $(\Omega_{e,j}, \Omega_{m,\ell}) > 0$  while  $(\alpha_{e,j}, \alpha_{m,\ell}) \geq 0$  and  $(\omega_{e,j}, \omega_{m,\ell}) > 0$ . Our goal is to analyze the long time behaviour of the solution of the Cauchy problem associated to (1.1, 1.2) and any initial data  $(\mathbf{E}_0, \mathbf{H}_0)$  under the so-called weak dissipative assumption

$$\sum_{j=1}^{N_e} \alpha_{e,j} + \sum_{\ell=1}^{N_m} \alpha_{m,\ell} > 0. \quad (1.3)$$

## 2 Main results

In what follows  $f \lesssim g$  means  $f \leq C g$ , for some constant  $C > 0$  independent of  $x, t, \mathbf{E}_0, \mathbf{H}_0$ .

Let  $\mathcal{E}(t)$  be the electromagnetic energy

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^3} (\varepsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2)(x, t) dx \quad (2.1)$$

The generic result (cf. Remark 3) is as follows

**Theorem 1** *For  $\mathbf{E}_0$  and  $\mathbf{H}_0$  in  $L^2(\mathbb{R}^3)^3$  such that  $\nabla \cdot \mathbf{E}_0 = \nabla \cdot \mathbf{H}_0 = 0$ , the energy  $\mathcal{E}(t)$  tends to 0 when  $t \rightarrow +\infty$ . Moreover if for some integers  $m \geq 0$  and  $p \geq 0$ ,*

$$\begin{cases} (\mathbf{E}_0, \mathbf{H}_0) \in H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3)^3, \\ |x|^p (\mathbf{E}_0, \mathbf{H}_0) \in L^1(\mathbb{R}^3)^3 \times L^1(\mathbb{R}^3)^3, \\ \int_{\mathbb{R}^3} x^\alpha \mathbf{E}_0 dx = 0, \int_{\mathbb{R}^3} x^\alpha \mathbf{H}_0 dx = 0, \quad \forall |\alpha| < p \end{cases} \quad (2.2)$$

one has a polynomial decay rate

$$\mathcal{E}(t) \leq \frac{C_r^m(\mathbf{E}_0, \mathbf{H}_0)}{t^m} + \frac{C_i^p(\mathbf{E}_0, \mathbf{H}_0)}{t^{p+\frac{3}{2}}} \quad (2.3)$$

where the above constants satisfy

$$\begin{cases} C_r^m(\mathbf{E}_0, \mathbf{H}_0) \lesssim \|\mathbf{E}_0\|_{H^m(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{H^m(\mathbb{R}^3)}^2, \\ C_i^p(\mathbf{E}_0, \mathbf{H}_0) \lesssim \|\mathbf{E}_0\|_{L^{1,p}(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{L^{1,p}(\mathbb{R}^3)}^2 \end{cases}$$

and  $\|u\|_{L^{1,p}(\mathbb{R}^3)} := \|(1 + |x|^p) u\|_{L^1(\mathbb{R}^3)}$ .

**Remark 2** *The upper bounds in (2.3) are sharp in the sense that similar lower bounds can be obtained for well chosen initial data.*

**Remark 3** *The statement of the theorem has to be slightly modified if one of the  $\alpha_{e,j}$  vanishes while  $\alpha_{\ell,m} = 0$  for all  $m$  (or vice versa). More precisely, with the same assumptions, one has to replace in the estimate (2.3)  $t^{-m}$  by  $t^{-m/2}$ .*

## 3 Comparison with the literature

The law (1.2) enters a more general class of non local (in time) constitutive laws of the form

$$\begin{cases} \mathbf{P}(x, \cdot) = \chi_e \star \mathbf{E}(x, \cdot), \\ \mathbf{M}(x, \cdot) = \chi_m \star \mathbf{H}(x, \cdot) \end{cases} \quad (3.1)$$


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where  $(\chi_e, \chi_m)$  are time convolution causal kernels. The long time behaviour of solutions of (1.1, 3.1) has been investigated in many papers such as in [2] for bounded domains of propagation (using abstract results from semi-group theory). In these works, polynomial stability, i.e. time decay estimates of the type

$$\mathcal{E}(t) \lesssim (1+t)^{-p}, \quad \text{for some } p > 0, \quad (3.2)$$

is proven under direct assumptions on the kernels  $(\chi_e, \chi_m)$ . In the case of Lorentz media, the assumptions used in [2] are satisfied only under the strong dissipativity assumption, i. e.

$$\alpha_{e,j}, \alpha_{m,\ell} > 0, \quad \forall j, \ell, \quad (3.3)$$

Another major difference is in the method of proof: we use here a more physically oriented method based on a modal/spectral approach.

#### 4 Method of proof

One first writes the problem as a generalized Schrödinger equation of the form

$$\frac{d\mathcal{U}}{dt} + i\mathcal{A}(\nabla)\mathcal{U} = 0 \quad (4.1)$$

where  $\mathcal{U} := (\mathbf{E}, \mathbf{H}, \mathbf{P}_j, \partial_t \mathbf{P}_j, \mathbf{M}_\ell, \partial_t \mathbf{M}_\ell) \in \mathbb{R}^N$  with  $N = 3(2 + 2N_e + 2N_m)$  and  $\mathcal{A}(\nabla)$  is a first order differential operator in space. We apply the space Fourier transform

$$\mathcal{U}(x, t) \longrightarrow \mathbf{U}(\mathbf{k}, t),$$

so that  $\mathbf{U}(\mathbf{k}, t)$  satisfies

$$\frac{d\mathbf{U}}{dt}(\mathbf{k}, t) + i\mathcal{A}(\mathbf{k})\mathbf{U}(\mathbf{k}, t) = 0, \quad (4.2)$$

where  $\{\mathcal{A}(\mathbf{k}), \mathbf{k} \in \mathbb{R}^3\}$  is a family of *non-normal*  $N \times N$  matrices. We next derive a priori estimates for  $\mathbf{U}(\mathbf{k}, t)$  before coming back to space domain via Plancherel's theorem.

We have developed two approaches to obtain estimates in the  $\mathbf{k}$ -space.

(I) *Via frequency dependent Lyapunov functions.*

This approach is *more direct* but *limited* to the strict dissipativity assumption (3.3).

We construct a  $|\mathbf{k}|$ -dependent and positive quadratic functional  $\mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U})$  such that

$$\frac{d}{dt} \mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U}) + \Phi(|\mathbf{k}|) \mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U}) \leq 0,$$

for some function  $\Phi(r) > 0$  ( $r \in \mathbb{R}_*^+$ ). Then combining Grönwall's lemma with a careful examination of the behaviour of  $\Phi(|\mathbf{k}|)$  for small and large values of  $|\mathbf{k}|$  leads to (2.3).

(II) *Via spectral decomposition.*

We use the spectrum  $\{\omega_n(\mathbf{k})\}$  of  $\mathcal{A}(\mathbf{k})$  which is made of the solutions of a *dispersion relation* that satisfy  $\text{Im } \omega_n(\mathbf{k}) < 0$ . We then use the associated spectral decomposition of  $\mathcal{A}(\mathbf{k})$  to represent the solution  $\mathbf{U}(\mathbf{k}, t)$  of (4.2) as

$$\mathbf{U}(\mathbf{k}, t) = \sum e^{-i\omega_n(\mathbf{k})t} P_n(\mathbf{k}, t) \mathbf{U}_0(\mathbf{k}), \quad (4.3)$$

where the operators  $P_n(\mathbf{k}, t)$  are directly related to the spectral projectors of  $\mathcal{A}(\mathbf{k})$ . We then split the analysis in three parts:

- (i) estimates for low (space) frequencies  $|\mathbf{k}| < m$ ,
- (ii) estimates for high frequencies  $|\mathbf{k}| > M$ ,
- (iii) estimates for mid-frequencies  $m \leq |\mathbf{k}| < M$ .

While step (iii) provides a uniform (in  $\mathbf{k}$ ) exponential decay, the polynomial decay results from the estimates (i) and (ii) which rely on

- (a) the asymptotic behaviour of each  $\omega_n(\mathbf{k})$  for small and large values of  $|\mathbf{k}|$ ,
- (b) uniform bounds for the operators  $P_n(\mathbf{k}, t)$ .

For the above, we use the implicit function theorem and the holomorphic functional calculus.

#### References

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