On finite-time stability analysis of homogeneous vector fields with multiplicative perturbations

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Abstract

The problem of finite-time and fixed-time stability analysis is considered for a class of nonlinear systems $\dot{x} = H(x)b(x)$, where H is homogeneous of a degree $\nu \neq 0$ and b is bounded. It is shown that under certain conditions on b the asymptotic stability of the system implies its finite-time stability for $\nu < 0$ or nearly fixed-time stability for $\nu > 0$. The so-called homogeneous extensions are utilized for the analysis. The results are generalized to a class of systems $\dot{x} = \sum_{i=0}^{p} H_i(x)b_i(x)$, where for every $i = 0, 1, ..., p,\ p \in \mathbb{N}$, H_i are homogeneous of some (possibly different) degrees and b_i satisfy certain boundedness conditions. An example of a mechanical system is presented in the last section to illustrate the obtained results.

Keywords : *Homogeneous extension, asymptotic stability, finite-time stability, fixed-time stability.*

1 Introduction

Finite-time stability and stabilization could be useful if there exist limitations on a convergence time to a set-point or to a reference trajectory for a control system. The notion of finite-time stability of nonlinear systems was introduced in [1] and [2]. It combines an asymptotic stability and a finite-time attractivity of an equilibrium. The particular case of finite-time stability, when the convergence time is bounded uniformly on initial conditions, was called the fixed-time stability [3]. The Lyapunov function method is the conventional tool for finite-time and fixed-time stability analysis (see [1, 3, 4, 2, 5, 6, 7, 8]). Notice that the corresponding analysis of differential inclusions can also be based on Lyapunov functions [9]. Another way of analysis of convergence rates is based on the theory of homogeneous systems.

Homogeneity is a certain symmetry of an object (a set, a function, a vector field, etc), when it remains invariant with respect to a class of transformations called dilations. For example, a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is standard homogeneous if $f(\lambda x) = \lambda^{\nu+1} f(x), \forall x \in \mathbb{R}^n, \forall \lambda > 0$, where the real number ν is the homogeneity degree. The feature of homogeneous systems is the equivalence of local and global properties. For example, local asymptotic stability always imply the global one. Homogeneity also simplifies the finite-time stability analysis. Indeed, in [10, page 110] it was shown that any asymptotically stable standard homogeneous system of negative degree is finite-time stable. In [5] this result is extended to the most generalized concept of homogeneity called the geometric homogeneity (see [11, 12]). Notice that the same property holds for homogeneous differential inclusions [13, 14] and homogeneous evolution equations in Banach spaces [15]. One of important results in this context is the existence of a homogeneous Lyapunov function for any asymptotically stable homogeneous system [12].

In this paper we study finite-time stability and nearly fixed-time stability of a non-homogeneous system of the form $\dot{x} = \sum_{i=0}^{p} H_i(x)b_i(x),\ p \in \mathbb{N}$, where H_i are homogeneous matrix-valued functions (in a certain sense) and b_i are bounded vector valued functions. We show that under certain conditions on the functions b_i, the finite-time or the nearly fixed-time stability of the system can be derived from its asymptotic stability and the homogeneity of H_i. To simplify the corresponding analysis we use the so-called homogeneous extensions, which were introduced originally

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for evolution systems in Banach spaces [16]. Notice that the homogeneous extension can be defined even when the homogeneous approximation [17] does not exist.

A preliminary version of this paper has been presented without proofs in [18]. The present paper extends the mentioned results to a more general class of non-homogeneous systems $\dot{x} = \sum_{i=0}^{p} H_i(x)b_i(x)$, and study an example of a mechanical system to illustrate the theoretical conclusions.

The paper is organized as follows. In the next section, we introduce the notation, the definitions of finite-time/fixed-time stability and homogeneity. We define also the homogeneous extension of a non-homogeneous continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. In Section III we give a sufficient condition to guarantee that the homogeneous extension of f preserves the asymptotic stability. Section IV uses the homogeneous extensions for finite-time/fixed-time stability analysis. The obtained results are illustrated on a numerical example. Finally, some conclusions are presented.

2 Preliminaries

2.1 Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.
- S denotes the unit sphere $||x|| = 1$ in the Euclidian space $\mathbb{R}^n$.
- $\mathcal{M}_{m,n}$ is the set of all $m \times n-$matrices over the field of real numbers, and it forms a normed vector space with the matrix norm $\|A\|_{\mathcal{M}_{m,n}} = \sup_{x \in S} \|Ax\|$ induced by a vector norm $\|\cdot\|$. When $m = n$ we write $\mathcal{M}_n$ instead of $\mathcal{M}_{n,n}$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity.
- The notation $\nabla V(x) = \left(\frac{\partial V(x)}{\partial x_1}, \cdots, \frac{\partial V(x)}{\partial x_n}\right)^T$ stands for the first derivative of a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ at point $x \in \mathbb{R}^n$.
- For $V : \mathbb{R}^n \to \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, the notation $\langle \nabla V(x), f(x) \rangle = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x) f_i(x)$ stands for the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at point $x \in \mathbb{R}^n$.
- $C^p(\Omega, \mathbb{R})$ denotes the space of continuous functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ which have p continuous derivatives and $C^\infty(\Omega, \mathbb{R})$ denotes the space of smooth functions.

2.2 Finite-time and nearly fixed-time stability

Let us consider the ordinary differential equation (ODE):

$$\begin{cases} \dot{x}(t) & = f(x(t)), \quad t \geq 0, \\ x(0) & = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ and $x_0 \in \mathbb{R}^n$ is the initial condition at $t = 0$. Let the vector field f ensure the existence of solutions of (1) in the forward time. Let $x(t, x_0)$ denote a solution of the system (1).

Definition 2.1 *[1],[19],[20] The origin of the system* (1) *is*

- ***Lyapunov stable*** *if there exist a nonempty open neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin and a function $\alpha \in \mathcal{K}_\infty$ such that $\|x(t, x_0)\| \leq \alpha(\|x_0\|)$ for all $t \geq 0$ and all solutions $x(t, x_0)$ of the system* (1) *with $x_0 \in \mathcal{V}$;*
- ***asymptotically stable*** *if it is Lyapunov stable and $\|x(t, x_0)\| \to 0$ as $t \to +\infty$ for all solutions $x(t, x_0)$ of the system* (1) *with $x_0 \in \mathcal{V}$;*

- ***finite-time stable*** *if it is Lyapunov stable and finite-time attractive, i.e. there exists a function* $T:\mathcal{V}\longrightarrow\mathbb{R}_+$ *(called settling-time function) such that*

$$x(t,x_0)=0,\quad \forall t\geq T(x_0),$$

for all solutions $x(t,x_0)$ *of the system* (1) *with* $x_0\in\mathcal{V}$*;*

- ***uniformly finite-time stable*** *if it is finite-time stable and there exists a settling time function locally bounded on* $\mathcal{V}$*;*
- ***fixed-time stable*** *if it is finite-time stable and there exists a settling-time function* T *uniformly bounded on* $\mathcal{V}$*;*
- ***nearly fixed-time stable*** *if it is Lyapunov stable and any neighborhood of the origin is fixed-time attractive, i.e., for any neighborhood* $M\subset\mathbb{R}^n$ *of the origin there exists a positive number* $T_M>0$ *such that*

$$x(t,x_0)\in M,\quad \forall t\geq T_M,$$

for all solutions $x(t,x_0)$ *of the system* (1) *with* $x_0\in\mathcal{V}$.

If $\mathcal{V}=\mathbb{R}^n$, then the origin of (1) is globally Lyapunov, asymptotically, (uniformly) finite-time and (nearly) fixed-time stable, respectively.

The **near fixed-time stability** ensures the convergence of systems solutions to any neighborhood of the origin in fixed-time. In this case, $\sup\limits_M T_M$ may be infinite, while in the case of the fixed-time stability the latter supremum is always finite.

2.3 Homogeneity

Homogeneity is an invariance with respect to a group of transformations called dilations.

Definition 2.2 *[11] A family of mappings* $\mathbf{d}(s):\mathbb{R}^n\to\mathbb{R}^n$ *with* $s\in\mathbb{R}$ *is said to be a dilation group in* $\mathbb{R}^n$ *if*

- *Group property:* $\boldsymbol{d}(0)=I_n$, $\boldsymbol{d}(t+s)=\boldsymbol{d}(t)\boldsymbol{d}(s)$, $t,s\in\mathbb{R}$.
- *Limit property:* $\lim\limits_{s\longrightarrow-\infty}\|\boldsymbol{d}(s)x\|=0$ *and* $\lim\limits_{s\longrightarrow+\infty}\|\boldsymbol{d}(s)x\|=+\infty$ *for all* $x\neq 0$.

The so-called geometric dilations studied in [11], [12], [5] are assumed to be generated by smooth vector fields. Below we deal only with the so-called **linear geometric dilations** [21] defined as follows

$$\mathbf{d}(s)=e^{G_{\mathbf{d}}s}=\sum_{i=0}^{+\infty}\frac{s^iG_{\mathbf{d}}^i}{i!},$$

where $G_{\mathbf{d}}\in\mathcal{M}_n$ is an anti-Hurwitz[1] matrix known as the generator of the dilation group $\mathbf{d}$. It is well known that $\frac{d}{ds}\mathbf{d}(s)=G_{\mathbf{d}}\mathbf{d}(s)=\mathbf{d}(s)G_{\mathbf{d}}$ for all $s\in\mathbb{R}$.

Definition 2.3 *[11] A vector field* $f:\mathbb{R}^n\to\mathbb{R}^n$ *(resp., a function* $h:\mathbb{R}^n\to\mathbb{R}$*) is said to be* $\boldsymbol{d}$*-homogeneous if there exists a number* $\nu\in\mathbb{R}^n$ *(called the homogeneity degree) such that for all* $s\in\mathbb{R}$ *and all* $x\in\mathbb{R}^n$ *we have*

$$e^{-\nu s}\boldsymbol{d}(-s)f(\boldsymbol{d}(s)x)=f(x), \tag{2}$$

(resp., $e^{-\nu s}h(\boldsymbol{d}(s)x)=h(x)$*).*

Any linear vector field is $\mathbf{d}$-homogeneous of degree 0 with respect to the group $\mathbf{d}(s)=e^sI$.

The following lemma provides a useful comparison between homogeneous functions.

Lemma 2.1 *[5] Suppose that* $V_1,V_2:\mathbb{R}^n\to\mathbb{R}_+$ *are continuous* $\mathbf{d}-$*homogeneous functions of degrees* $l_1>0$ *and* $l_2>0$*, respectively, and* V_1 *is positive definite. Then, for every* $x\in\mathbb{R}^n$*,*

$$a_1V_1^{\frac{l_2}{l_1}}(x)\leq V_2(x)\leq a_2V_1^{\frac{l_2}{l_1}}(x), \tag{3}$$

with $a_1=\min\limits_{\{z:\ V_1(z)=1\}}V_2(z)$ *and* $a_2=\max\limits_{\{z:\ V_1(z)=1\}}V_2(z)$.

If the origin of the system (1) is asymptotically stable and f is $\mathbf{d}$-homogeneous of negative degree then the origin of (1) is globally uniformly finite-time stable [5].

[1] A matrix is anti-Hurwitz if all its eigenvalues are placed in the right complex half-plane.

2.4 Homogeneous extension

Differential inclusions appear as models of dynamical systems which do not satisfy the classical assumptions of regularity. For example, a regularization of a discontinuous system of the form (1) proposed by Filippov [22] consists in replacing (1) with a suitable differential inclusion

$$\begin{cases} \dot{x}(t) \in F(x(t)), \ t > 0, \\ x(0) = x_0, \end{cases} \tag{4}$$

where $x(t), x_0 \in \mathbb{R}^n$ and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. The notion of solution obtained in this way depends on the construction of the set-valued map F (see [23]). For example, the Filippov solution of a discontinuous system (1) is a strong solution[2] of (4) with

$$F(x) = \cap_{r>0} \cap_{\mu(N)=0} \overline{\text{co}} \left\{ f(B(x,r) \setminus N) \right\},$$

where μ is the Lebesgue measure of $\mathbb{R}^n$, $\overline{\text{co}}$ denotes the closure of the convex hull and $B(x,r)$ denotes a ball in $\mathbb{R}^n$ of the radius $r > 0$ cantered at $x \in \mathbb{R}^n$. If the set-valued mapping $F(x)$ is upper semicontinuous and nonempty-, compact- and convex-valued then (4) has strong solutions [22]. Notice that the differential inclusions may have strong solutions even if F do not satisfy the above assumptions.

Let us introduce the definition of homogeneous extension [16] which allows homogeneity-based methods of analysis to be applied for non-homogeneous systems.

Definition 2.4 *A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be $\mathbf{d}$-homogeneous extension of a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ with a degree $\nu \in \mathbb{R}$ if $f(x) \in F(x)$ for any $x \in \mathbb{R}^n$ and F is $\mathbf{d}$-homogeneous of the degree ν.*

Such extensions appear, for example, as a result of Filippov regularization of a homogeneous discontinuous vector field (see [13]). In this paper we deal with the $\mathbf{d}$-homogeneous extensions given by [16]

$$F(x) = \bigcup_{s\in\mathbb{R}} \{ e^{-\nu s} \mathbf{d}(-s) f(\mathbf{d}(s)x) \}, \ x \in \mathbb{R}^n, \tag{5}$$

where $\mathbf{d}$ is a linear dilation and $\nu \in \mathbb{R}$. The set-valued map F given above is, obviously, a $\mathbf{d}$-homogeneous extension of the vector field f and

$$e^{\nu\tau} \mathbf{d}(\tau) F(x) = F(\mathbf{d}(\tau)x), \ \tau \in \mathbb{R}, \ x \in \mathbb{R}^n. \tag{6}$$

Such an extension exists even for functions which do not have a homogeneous approximation[3]. Therefore, to study its stability and a convergence rate, the homogeneous extension can be used. For example, the right-hand side of the system

$$\dot{x}(t) = -\left(2 + \cos\left(\frac{1}{x(t)}\right)\right) x^{1/3}(t), \ t \geq 0. \tag{7}$$

does not have a homogeneous approximation neither at zero nor at infinity, but it has a homogeneous extension $\dot{x}(t) \in F(x(t)) = -x^{1/3}(t)\,[1,3]\,, \ t \geq 0$, which allow us to study the finite-time stability using the homogeneity of set-valued mappings.

Remark 2.1 *We do not need to prove the existence of strong solutions for the differential inclusion $\dot{x} \in F(x)$ with F given by (5) provided that f is a continuous vector field. Indeed, since $f(x) \in F(x)$ then solutions of the system (1) belongs to a set of the solutions of the differential inclusion $\dot{x} \in F(x)$.*

The stability definitions for differential inclusions are literary the same as for the differential equations (Definition 2.1).

3 Asymptotic stability of homogeneous extensions

3.1 Homogeneous vector fields with multiplicative perturbations

Let the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ of the system (1) satisfy the following assumptions

$\mathcal{H}_1$ f be continuous on $\mathbb{R}^n$ and $f(x) = H(x)b(x)$ for all $x \in \mathbb{R}^n$;

[2] An absolutely continuous function $x : I \subset \mathbb{R}_+ \to \mathbb{R}^n$ is said to be a strong solution of (4) if it satisfies (4) almost everywhere on I.

[3] We refer the reader to [17] for more details about homogeneous approximations.

$\mathcal{H}_2$ a mapping $H : \mathbb{R}^n \to \mathcal{M}_{n,m}$ is continuous and $\mathbf{d}$-homogeneous of a degree $\nu \in \mathbb{R}$, (i.e., $e^{-\nu s}\mathbf{d}(-s)H(\mathbf{d}(s)x) = H(x), \forall x \in \mathbb{R}^n$ and $\forall s \in \mathbb{R}$);

$\mathcal{H}_3$ a function $b : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly bounded

$$\sup_{x\in\mathbb{R}^n} \|b(x)\| < +\infty$$

and continuous on $\mathbb{R}^n \setminus \{0\}$;

$\mathcal{H}_4$ the origin of the system (1) is globally asymptotically stable.

Assumption $\mathcal{H}_4$ implies that 1) $f(0) = 0$; 2) $b(x)$ does not belong to the kernel of $H(x)$, $\forall x \in \mathbb{R} \setminus \{0\}$ and 3) $H(x) = 0 \Leftrightarrow x = 0$. Recall [19] that, in the view of Kurzweil's converse Lyapunov theorem, any globally asymptotically stable system (1) with a continuous vector field f admits a smooth positive definite and proper[4] (or, equivalently, radially unbounded [24]) Lyapunov function. We use this fact in order to derive a sufficient condition allowing an expansion of the global asymptotic stability of the system (1) to its homogeneous extension (4), (5).

Theorem 3.1 *Let the system* (1) *satisfy the assumptions* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$, *and there exist a positive definite proper Lyapunov function* $V \in C^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R}_+) \cap C(\mathbb{R}^n, \mathbb{R}_+)$ *for the system* (1) *such that*

$$\sup_{y\neq 0} \|b(y) - b(x)\| \leq -\beta \frac{\langle \nabla V(x), f(x)\rangle}{\|H^\top(x)\nabla V(x)\|}, \tag{8}$$

for any $x \in \tilde{S}$ *and some constant* $\beta \in (0, 1)$, *where* $\tilde{S} = \{x \in \mathbb{R}^n : V(x) = 1\}$.

Then the differential inclusion (4) *with* F *given by* (5) *is globally asymptotically stable and there exists a positive definite radially unbounded* $\mathbf{d}$*-homogeneous Lyapunov function* $V_\mathbf{d} \in C^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R}_+) \cap C(\mathbb{R}^n, \mathbb{R}_+)$ *of degree* 1 *such that*

$$\sup_{h\in F(x)} \langle \nabla V_\mathbf{d}(x), h\rangle < -W_\mathbf{d}(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}\,. \tag{9}$$

where the positive definite function $W_\mathbf{d} : \mathbb{R}^n \to \mathbb{R}_+$ *is* $\mathbf{d}$*-homogeneous of the degree* $1 + \nu$.

Proof of Theorem 3.1: The assumptions $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ imply that the homogeneous extension is defined as follows $F(x) = \bigcup\limits_{s\in\mathbb{R}} \{H(x)b(\mathbf{d}(s)x)\}$ and the set $F(x)$ is bounded, $\sup\limits_{h\in F(x)} \langle \nabla V(x), h\rangle < +\infty$ for every $x \in \mathbb{R}^n$.

Let us consider the set $K_{\gamma_1,\gamma_2} = \{x \in \mathbb{R}^n : \gamma_1 \leq V(x) \leq \gamma_2\}$ and prove that for some parameters $\gamma_1 \in [1/2, 1)$ and $\gamma_2 \in (1, 3/2]$ sufficiently close to 1 we have $\sup_{h\in F(y)}\langle \nabla V(y), h\rangle \leq -c$, $\forall y \in K_{\gamma_1,\gamma_2}$ or, equivalently,

$$\langle \nabla V(y), H(y)b(\mathbf{d}(s)y)\rangle \leq -c,\ \forall y \in K_{\gamma_1,\gamma_2},\ \forall s \in \mathbb{R},$$

where $c := -0.5(1 - \beta) \max\limits_{x\in\tilde{S}} \langle \nabla V(x), f(x)\rangle > 0$. Indeed, using the Cauchy-Schwarz inequality and (8) we derive

$$\begin{array}{rcl} \langle \nabla V(x), H(x)[b(z) - b(x)]\rangle & \leq & \|\nabla V(x)H(x)\|\|b(z) - b(x)\| \\ & \leq & -\beta\,\langle \nabla V(x), f(x)\rangle\,, \end{array} \tag{10}$$

for all $z \in \mathbb{R}^n$ and $x \in \tilde{S}$. Hence, for any $x \in \tilde{S}$, any $s \in \mathbb{R}$ and any $\Delta \in \mathbb{R}^n : \Delta \neq -x$ we have

$$\begin{array}{rl} & \langle \nabla V(x + \Delta), H(x + \Delta)b(\mathbf{d}(s)(x + \Delta))\rangle \\ = & \langle H^\top(x + \Delta)\nabla V(x+\Delta) - H^\top(x)\nabla V(x), b(\mathbf{d}(s)(x+\Delta))\rangle + \\ & \langle \nabla V(x), f(x)\rangle + \langle \nabla V(x), H(x)[b(\mathbf{d}(s)(x+\Delta)) - b(x)]\rangle \\ \leq & \|H^\top(x + \Delta)\nabla V(x + \Delta) - H^\top(x)\nabla V(x)\|\bar{b} + \\ & \langle\ \nabla V(x), f(x)\rangle - \beta\,\langle \nabla V(x), f(x)\rangle \\ \leq & -2c + \|H^\top(x + \Delta)\nabla V(x + \Delta) - H^\top(x)\nabla V(x)\|\bar{b}, \end{array}$$

where $\bar{b} := \sup\limits_{x\in\mathbb{R}^n} \|b(x)\|$. On the one hand, since V is proper then the set K_{γ_1,γ_2} is compact. On the other hand, since the vector field $x \to g(x) := H^\top(x)\nabla V(x)$ is continuous on $\mathbb{R}^n\backslash\{0\}$ then by Heine-Cantor theorem it is uniformly continuous[5] on the compact set $K_{1/2,3/2}$ with some modulus of continuity $\omega \in \mathcal{K}$. Notice that

[4] A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is proper if an inverse image of any compact set is a compact set.

[5] A vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is said to be uniformly continuous on a set $\Omega \subset \mathbb{R}^n$ if there exists the so-called modulus of continuity $\omega \in \mathcal{K}$ such that $\|g(x) - g(y)\| \leq \omega(\|x - y\|)$ for all $x, y \in \Omega$.

g is uniformly continuous on $K_{\gamma_1,\gamma_2} \subset K_{1/2,3/2}$ with the same modulus of continuity. For any $x \in \tilde{S}$ and any $\Delta \in \mathbb{R}^n : x + \Delta \in K_{1/2,3/2}$ we have

$$\|H^\top(x+\Delta)\nabla V(x+\Delta) - H^\top(x)\nabla V(x)\|\bar{b} \leq \bar{b}\omega(\|\Delta\|).$$

Let $\delta > 0$ be such that $\bar{b}\omega(\delta) \leq c$. Such $\delta > 0$ always exists due to $\omega \in \mathcal{K}$. Let $\gamma_1 \in [1/2, 1)$ and $\gamma_2 \in (1, 3/2]$ be selected sufficiently close to 1 such that

$$K_{\gamma_1,\gamma_2} \subset \{x + \Delta \in \mathbb{R}^n : x \in \tilde{S} \text{ and } \|\Delta\| \leq \delta\}.$$

Such γ_1 and γ_2 always exist due to uniform continuity of the vector field g on $K_{1/2,3/2}$, which implies the convergence $K_{\gamma_1,\gamma_2} \to \tilde{S}$ as $|1-\gamma_1| + |\gamma_2 - 1| \to 0$ in the Hausdorff metric (see e.g. [22]).

The rest part of the proof uses the scheme suggested in [12]. We consider the $\mathbf{d}$-homogeneous function $V_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}$ defined as follows

$$V_{\mathbf{d}}(x) = \int_{-\infty}^{+\infty} e^{-s} a(V(\mathbf{d}(s)x))ds$$

where $a : \mathbb{R} \to \mathbb{R}$ is an arbitrary $a \in C^\infty$ function such that $a(\rho) = 0$ for $\rho \leq \gamma_1$, $a(\rho) = 1$ for $r \geq \gamma_2$ and $a'(\rho) > 0$ for $\rho \in (\gamma_1, \gamma_2)$. The function $V_{\mathbf{d}}$ is continuous on $\mathbb{R}^n$, continuously differentiable on $\mathbb{R}^n\backslash\{0\}$ and $\mathbf{d}$-homogeneous of the degree 1. Moreover, for any $x \in \mathbb{R}^n\backslash\{0\}$ we have

$$\sup_{h\in F(x)} \langle \nabla V_{\mathbf{d}}(x), h\rangle = \sup_{h\in F(x)} \int_{-\infty}^{+\infty} e^{-s} a'(V(\mathbf{d}(s)x))\langle \nabla V(\mathbf{d}(s)x), \mathbf{d}(s)h\rangle ds \leq$$

$$\int_{-\infty}^{+\infty} e^{-s} a'(V(\mathbf{d}(s)x)) \sup_{h\in F(x)} \langle \nabla V(\mathbf{d}(s)x), \mathbf{d}(s)h\rangle ds \leq$$

$$\int_{-\infty}^{+\infty} e^{-s} a'(V(y)) \sup_{\tilde{h}: e^{-\nu s}\mathbf{d}(-s)\tilde{h}\in F(\mathbf{d}(-s)y)} \langle \nabla V(y), e^{-\nu s}\tilde{h}\rangle ds,$$

where the following notation $y = \mathbf{d}(s)x$ and $\tilde{h} = e^{\nu s}\mathbf{d}(s)h$ is utilized. Since $a'(V(y)) = 0$ for $y \notin K_{\gamma_1,\gamma_2}$ and $\sup_{h\in F(y)}\langle \nabla V(y), h\rangle \leq -c$ for $y \in K_{\gamma_1,\gamma_2}$, then using $\mathbf{d}$-homogeneity of F we derive

$$\sup_{h\in F(x)} \langle \nabla V_{\mathbf{d}}(x), h\rangle \leq \int_{-\infty}^{+\infty} e^{-(\nu+1)s} a'(V(\mathbf{d}(s)x)) \sup_{\tilde{h}\in F(y)} \langle \nabla V(y), \tilde{h}\rangle ds \leq$$

$$-c\int_{-\infty}^{+\infty} e^{-(\nu+1)s} a'(V(\mathbf{d}(s)x))ds = -W_{\mathbf{d}}(x).$$

The function $W_{\mathbf{d}}$ is, obviously, positive definite and $\mathbf{d}$-homogeneous of the degree $\nu + 1$. The proof is complete. ■

Remark 3.1 *The most sophisticated condition of Theorem 3.1 is* (8). *This condition gives a maximal upper bound of the admissible deviation of $b(\cdot)$ with respect to its values on the sphere $\tilde{S}$. In the right-hand side of* (8)*, only the values of b from the sphere are used, and the right-hand side itself is a constant proportional to $0 < \beta < 1$. Then the condition* (8) *evaluates how far $b(y)$ is varying with respect to $b(x)$, where $y \in \mathbb{R}^n \setminus \{0\}$ and $x \in \tilde{S}$, respectively. Obviously, if $b(\cdot)$ is a constant function, then the left-hand side is zero and the condition* (8) *is always satisfied (in such a case f is a homogeneous function). Similarly, for a scalar function b the condition (8) can be omitted.*

Remark 3.2 *If the assumptions $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ hold and in addition if the function b satisfies the condition*

$$\sup_{y\neq 0} \|b(y) - b(x)\| \leq \beta\alpha^*\|b(x)\|, \tag{11}$$

for all $x \in \tilde{S}$, where $\alpha^* = \inf_{x\in\tilde{S}} \left\langle \frac{-H^\top(x)\nabla V(x)}{\|H^\top(x)\nabla V(x)\|}, \frac{b(x)}{\|b(x)\|} \right\rangle$. Then, the results of Theorem 3.1 holds.

The following result is an extension of Theorem 3.1 to the system (1), where f is a homogeneous vector fields with different degrees and multiplicative perturbations.

Corollary 3.1 *Let us assume that the system* (1) *is globally asymptotically stable and : $f : x \in \mathbb{R}^n \to f(x) = \sum_{i=0}^{p} H_i(x)b_i(x) \in \mathbb{R}^n,\ p \in \mathbb{N}$, is continuous on $\mathbb{R}^n$ where*

- *the matrix-valued functions $H_i : \mathbb{R}^n \to \mathcal{M}_{n,m_i}$ are continuous and $\boldsymbol{d}$-homogeneous of degree $\nu_i \in \mathbb{R}$ with $\nu_0 \leq \nu_1 \leq ... \leq \nu_p$;*
- *the functions $b_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ are continuous on $\mathbb{R}^n\backslash\{0\}$ and there exists $\nu \in \mathbb{R}$ such that function $x \to b(x) := \left(\begin{smallmatrix} b_0(x) \\ \cdots \\ b_p(x) \end{smallmatrix}\right)$ satisfies the following condition*

$$0 < \inf_{x\neq 0, s\in\mathbb{R}} \|\Lambda(s)b(\boldsymbol{d}(s)x)\| \leq \sup_{x\neq 0, s\in\mathbb{R}} \|\Lambda(s)b(\boldsymbol{d}(s)x)\| < +\infty,$$

$$\Lambda(s) = \begin{pmatrix} e^{(\nu_0-\nu)s}I_{m_0} & 0 & \cdots & 0 \\ 0 & e^{(\nu_1-\nu)s}I_{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{(\nu_p-\nu)s}I_{m_p} \end{pmatrix}.$$

If there exist a Lyapunov function $V \in C^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R}_+) \cap C(\mathbb{R}^n, \mathbb{R}_+)$ for the system (1) *and a number $0 < \beta < 1$ such that*

$$\sup_{y\neq 0, s\in\mathbb{R}} \|\Lambda(s)b(\mathbf{d}(s)y) - b(x)\| \leq -\frac{\beta\langle\nabla V(x), f(x)\rangle}{\|H^\top(x)\nabla V(x)\|} \tag{12}$$

for any $x \in \tilde{S}$, where $\tilde{S} = \{x \in \mathbb{R}^n : \ V(x) = 1\}$. Then conclusions of Theorem 3.1 remain true.

Proof of Corollary 3.1: The vector field f can be written in the following form :

$$f(x) = H(x)b(x), \quad \text{with}$$

$$H(x) = \mathbf{diag}\{H_0(x), \cdots, H_p(x)\}.$$

The assumptions of Corollary 3.1 imply that the homogeneous extension is defined as follows $F(x) = \bigcup_{s\in\mathbb{R}} \{H(x)\Lambda(s)b(\mathbf{d}(s)x)\}$ and the set $F(x)$ is bounded, $\sup_{h\in F(x)} \langle\nabla V(x), h\rangle < +\infty$ for every $x \in \mathbb{R}^n$.

Following the proof of Theorem 3.1 and using the condition (12), one deduces that the differential inclusion $\dot{x} \in F(x)$ is **GAS**, and there exist a positive definite radially unbounded $\mathbf{d}$-homogeneous Lyapunov function $V_{\mathbf{d}} \in C^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R}_+) \cap C(\mathbb{R}^n, \mathbb{R}_+)$ of degree 1 and positive definite $\mathbf{d}$-homogeneous function $W_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+$ of degree $1 + \nu$ such that (9) holds.

4 Finite-time and nearly fixed-time stability analysis using homogeneous extensions

In this section we derive finite-time stability and nearly fixed-time stability results for non-homogeneous systems by using the homogeneous extensions.

Corollary 4.1 *If the system* (1) *satisfy the assumptions of Theorem 3.1 (resp. Corollary 3.1) then the origin of the system* (1) *is globally*

- *uniformly finite-time stable for $\nu < 0$;*
- *nearly fixed-time stable for $\nu > 0$;*
- *exponentially stable for $\nu = 0$.*

Proof of Corollary 4.1: From Theorem 3.1 and Lemma 2.1 we derive

$$\sup_{h\in F(x)} \langle\nabla V_{\mathbf{d}}(x), h\rangle \leq -a_1 V_{\mathbf{d}}^{1+\nu}(x), \quad \forall x \neq 0,$$

where $a_1 = \min_{x\in\mathbb{R}^n : V_{\mathbf{d}}(x)=1} W_{\mathbf{d}}(x)$. The latter implies that for any strong solution ϕ_{x_0} of the differential inclusion (4), (5) we have

$$\frac{d}{dt}V_{\mathbf{d}}(\phi_{x_0}(t)) \overset{a.e.}{\leq} -a_1 V_{\mathbf{d}}^{\nu+1}(\phi_{x_0}(t)), \quad t > 0 : \phi_{x_0}(t) \neq 0.$$

Since V is differentiable on $\mathbb{R}^n\backslash\{0\}$ and ϕ_{x_0} is absolutely continuous then $t \to V_{\mathbf{d}}(\phi_{x_0}(t))$ is absolutely continuous as well. For $\nu < 0$ the latter inequality and the positive definiteness of $V_{\mathbf{d}}$ implies $\|\phi_{x_0}(t)\| = 0$ for $t \geq V_{\mathbf{d}}^{-\nu}(x_0)/(-\nu a_1)$

for any strong solution $\phi_{x_0}(t)$ of the differential inclusion (4), (5) and, in particular, for all solutions of the system (1).

For $\nu > 0$ we need to show that any neighborhood of the origin (1) is fixed-time time attractive. Indeed, since $V_{\mathbf{d}}$ is positive definite and continuous at 0 then for any neighborhood M of the origin there exists $\gamma > 0$ such that $\{x \in \mathbb{R}^n : V_{\mathbf{d}}(x) \le \gamma\} \subset M$. For $\nu > 0$ the above differential inequality implies

$$V_{\mathbf{d}}(\phi_{x_0}(t)) \le \gamma, \quad \forall t \ge \frac{1}{a_1 \nu \gamma^{\nu}}$$

independently of x_0.

Finally, for $\nu = 0$, one gets

$$\frac{d}{dt} V_{\mathbf{d}}(\phi_{x_0}(t)) \overset{a.e.}{\le} -a_1 V_{\mathbf{d}}(\phi_{x_0}(t)), \quad t > 0 : \phi_{x_0}(t) \ne 0.$$

This implies the exponential stability for the system (1). The proof is complete. ■

In the case of the scalar-valued function b the restriction (12) can be omitted.

Corollary 4.2 *If the vector field f of the system* (1) *satisfies the assumptions $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$, with $H : \mathbb{R}^n \to \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}$ and*

$$\inf_{x \in \mathbb{R}^n} |b(x)| > 0,$$

then conclusions of Theorem 3.1 remains true. In addition, the system (1) *is globally uniformly finite-time stable for $\nu < 0$ and globally nearly fixed-time stable for $\nu > 0$.*

Indeed, using the assumption $\mathcal{H}_3$ for $y \in K_{\gamma_1,\gamma_2}$ we derive

$$\langle \nabla V(y), H(y) b(\mathbf{d}(s)y)\rangle = \langle \nabla V(y), H(y) b(y)\rangle \frac{b(\mathbf{d}(s)y)}{b(y)}$$

$$\le \langle \nabla V(y), H(y) b(y)\rangle \underline{b}/\bar{b} \le -c,$$

where $\underline{b} := \inf_{x \in \mathbb{R}_n} |b(x)| \le \bar{b} := \sup_{x \in \mathbb{R}^n} |b(x)|$ and $c = -\underline{b}\,\bar{b}^{-1} \max_{y \in K_{\gamma_1,\gamma_2}} \langle \nabla V(y), f(y)\rangle$. All the remaining considerations literally repeat the proof of Theorem 3.1 and Corollary 4.1.

As a trivial example illustrating the presented corollary the system (7) can be considered. This system does not have a homogeneous approximation at zero, but it has a homogeneous extension that is globally uniformly finite-time stable.

5 A planar system example

In this section, we consider the class of planar systems studied in [25, 26], which has the following form:

$$\begin{cases} \dot{q} = v, \\ \dot{v} = f(q,v) + u(q,v), \end{cases} \tag{13}$$

where $q = (q_1, \cdots, q_n)^T \in \mathbb{R}^n$ is the vector of generalized coordinates, $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$ is the vector of generalized velocities while $f(q,v) = (f_1(q,v), \cdots, f_n(q,v))^T$ represents generalized forces and u is the vector of the inputs. For simplicity we consider $n = 2$ and

$$f_1(q,v) = -\beta_1 b_1(q,v) |v_1|^{\frac{1}{2}} (v_1),$$

$$f_2(q,v) = -\beta_2 b_2(q,v) |v_2|^{\frac{1}{2}} (v_2),$$

where $\beta_2 > 0, \beta_1 > 0$, (without loss of generality we choose $\beta_2 \ge \beta_1 > 0$) and continuous functions $b_i : \mathbb{R}^4 \backslash \{0\} \to \mathbb{R}$ satisfy the following conditions: $\exists \underline{b} > 0$ and $\exists \bar{b} > 0$ such that $\underline{b} \le b_i(q,v) \le \bar{b}$, $\forall (q,v) \in \mathbb{R}^4$, $i = 1, 2$. Under these conditions the system has only one equilibrium at the origin. The considered system

$$\begin{cases} \dot{q_1} &= v_1, \\ \dot{q_2} &= v_2, \\ \dot{v}_1 &= -\beta_1 b_1(q,v) |v_1|^{\frac{1}{2}} (v_1) + u_1(q,v), \\ \dot{v}_2 &= -\beta_2 b_2(q,v) |v_2|^{\frac{1}{2}} (v_2) + u_2(q,v), \end{cases} \tag{14}$$

can be represented as follows

$$\dot{z} = H(z)b(z),$$

with $z = (q, v)^\top$, $b(z) = (1, 1, b_1(z), b_2(z))^\top$ and

$$H(z) = \begin{pmatrix} v_1 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 \\ u_1(q,v) & 0 & -\beta_1 |v_1|^{\frac{1}{2}}(v_1) & 0 \\ 0 & u_2(q,v) & 0 & -\beta_2 |v_2|^{\frac{1}{2}}(v_2) \end{pmatrix}.$$

The function b is uniformly bounded and it is continuous on $\mathbb{R}^4 \setminus \{0\}$ due to imposed properties of b_1 and b_2. If we define $(u_1, u_2)^\top = \left(-\alpha_1 q_1^{\frac{1}{3}}, -\alpha_2 q_2^{\frac{1}{3}}\right)^\top$, $\alpha_2 \geq \alpha_1 > 0$ then the function $H : \mathbb{R}^4 \to \mathcal{M}_{4,4}$ is continuous and $\mathbf{d}$-homogeneous of the degree $\nu = -1$, provided that the dilation $\mathbf{d}$ is weighted homogeneous $\mathbf{d}(s) = \mathrm{diag}\{e^{3s}I, e^{2s}I\}$, $I \in \mathbb{R}^{2\times 2}$.

Consider the following Lyapunov function

$$V(z) = cU^{\frac{5}{4}}(z) + \alpha_1^{\frac{3}{4}} q_1 v_1 + \alpha_2^{\frac{3}{4}} q_2 v_2,$$

where $U(z) = \frac{3}{4}\sum_{i=1}^{2} \alpha_i |q_i|^{\frac{4}{3}} + \frac{1}{2}\|v\|^2$ and

$$c > \max\left\{ \frac{\alpha_2^{\frac{3}{4}}\left(1 + \beta_2 \frac{\bar{b}}{4}\right)}{\frac{5\underline{b}}{4}\beta_1 \left(\frac{1}{2}\right)^{\frac{1}{4}}}, \left(\frac{4}{5}\right)^{\frac{5}{4}} \right\}. \tag{15}$$

Applying Young's inequality produces

$$\left(\sum_{i=1}^{2} \alpha_i^{\frac{3}{4}} |q_i||v_i|\right)^{\frac{4}{5}} \leq \sum_{i=1}^{2} \left(\alpha_i^{\frac{3}{4}} |q_i|\right)^{\frac{4}{5}} |v_i|^{\frac{4}{5}} \leq \frac{3}{5}\sum_{i=1}^{2} \alpha_i |q_i|^{\frac{4}{3}} + \frac{2}{5}\|v\|^2.$$

Hence, V is positive definite for $c > (\frac{4}{5})^{\frac{5}{4}}$. From the definition of U, we obtain

$$\langle \nabla U(z), H(z)b(z)\rangle = -\beta_1 b_1(z)|v_1|^{\frac{3}{2}} - \beta_2 b_2(z)|v_2|^{\frac{3}{2}} \leq$$

$$-\beta_1 \underline{b}|v_1|^{\frac{3}{2}} - \beta_2 \underline{b}|v_2|^{\frac{3}{2}} \leq -\underline{b}\beta_1 \|v\|^{\frac{3}{2}},$$

$(U(z))^{\frac{1}{4}} \geq \left(\frac{1}{2}\right)^{\frac{1}{4}} \|v\|^{\frac{1}{2}}$ and again by applying Young's inequality, we have

$$\begin{aligned} \langle \nabla V(z), H(z)b(z)\rangle &= \tfrac{5c}{4}\langle \nabla U(z), H(z)b(z)\rangle U^{\frac{1}{4}}(z) + \alpha_1^{\frac{3}{4}} v_1^2 + \alpha_2^{\frac{3}{4}} v_2^2 \\ &\quad + \alpha_1^{\frac{3}{4}} q_1(f_1(z) + u_1(z)) + \alpha_2^{\frac{3}{4}} q_2(f_2(z) + u_2(z)) \\ &\leq -\left(\tfrac{1}{2}\right)^{\frac{1}{4}} \tfrac{5\underline{b}c}{4}\beta_1 \|v\|^2 + \alpha_2^{\frac{3}{4}}\beta_2 \tfrac{\bar{b}}{4}\|v\|^2 + \alpha_2^{\frac{3}{4}}\|v\|^2 \\ &\quad -\alpha_1^{\frac{7}{4}} \sum_{i=1}^{2} |q_i|^{\frac{4}{3}} + \alpha_2^{\frac{3}{4}}\beta_2 \tfrac{3\bar{b}}{4}\sum_{i=1}^{2} |q_i|^{\frac{4}{3}} \\ &= -k_1\|v\|^2 - k_2 \sum_{i=1}^{2} |q_i|^{\frac{4}{3}}, \end{aligned}$$

with $k_1 = \frac{5\underline{b}}{4}\beta_1 \left(\frac{1}{2}\right)^{\frac{1}{4}} c - \alpha_2^{\frac{3}{4}}\left(1 + \beta_2 \frac{\bar{b}}{4}\right)$, and $k_2 = \left(\alpha_1^{\frac{7}{4}} - \alpha_2^{\frac{3}{4}}\beta_2 \frac{3\bar{b}}{4}\right)$. By choosing $\alpha_1^{\frac{7}{4}} > \alpha_2^{\frac{3}{4}}\beta_2 \frac{3\bar{b}}{4}$ and c such that (15) holds,

we get $k_1, k_2 > 0$. This implies that the origin of the system (14) is globally asymptotically stable. According to Theorem 3.1 and Corollary 4.1 this system is finite-time stable provided that there exists $\beta \in (0,1)$ such that

$$\sup_{y \neq 0} \|b(y) - b(z)\| \leq \tfrac{-\beta \langle \nabla V(z), H(z)b(z)\rangle}{\|H^\top(z)\nabla V(z)\|}, \quad \forall z \in \tilde{S},$$

where $\tilde{S} = \{z \in \mathbb{R}^n : V(z) = 1\}$ (inequality (8) for this example). The latter inequality can be restricted to

$$\sqrt{2}(\bar{b} - \underline{b}) \leq -\tfrac{\beta \max_{z \in \tilde{S}} \langle \nabla V(z), H(z)b(z)\rangle}{\max_{z\in\tilde{S}} \|H^\top(z)\nabla V(z)\|}.$$

For $\bar{b}$ sufficiently close to $\underline{b}$ this condition holds and implies the finite-time stability of the system (14). Since

$$a_1\|z\|_{\mathbf{d}}^5 \leq V(z) \leq a_2\|z\|_{\mathbf{d}}^5,$$

with

$$\|z\|_{\mathbf{d}}^4 = \|v\|^2 + \sum_{i=1}^{2} |q_i|^{\frac{4}{3}},$$

$$a_1 = \left(c - (4/5)^{\frac{5}{4}}\right) \min\left\{(3\alpha_2/4)^{\frac{5}{4}}, (1/2)^{\frac{5}{4}}\right\}$$

and

$$a_2 = \left(c + (4/5)^{\frac{5}{4}}\right) \max\left\{(3\alpha_2/4)^{\frac{5}{4}}, (1/2)^{\frac{5}{4}}\right\}.$$

We obtain

$$-\max_{z\in\tilde{S}}\langle \nabla V(z), H(z)b(z)\rangle \geq \frac{\min\{k_1,k_2\}}{a_2^{\frac{4}{5}}} = 0.3985,$$

and

$$\begin{aligned}\max_{z\in\tilde{S}} \|H^\top(z)\nabla V(z)\| &\leq \max_{z\in\tilde{S}} \Big\{m_1\|z\|_{\mathbf{d}}^4 \\ &+m_2\frac{\|z\|_{\mathbf{d}}}{4} + m_2\|z\|_{\mathbf{d}}^5 + m_3\frac{\|z\|_{\mathbf{d}}^{\frac{9}{2}}}{4} + m_3\|z\|_{\mathbf{d}}^{\frac{1}{2}}\Big\} \\ &\leq \frac{m_1}{a_1^{\frac{4}{5}}} + m_2\left(\frac{1}{4a_1^{\frac{1}{5}}} + \frac{1}{a_1}\right) + m_3\left(\frac{1}{4a_1^{\frac{9}{10}}} + \frac{1}{a_1^{\frac{1}{10}}}\right) = 2.0057\end{aligned}$$

with

$$m_1 = \max\left\{\left(\alpha_2^{\frac{3}{4}} + \alpha_2^{\frac{5}{2}} + \alpha_2^{\frac{3}{2}}\beta_2^2/2\right)^{\frac{1}{2}}, \left(\alpha_2^{\frac{7}{2}} + \alpha_2^{\frac{5}{2}} + \alpha_2^{\frac{3}{2}}\beta_2^2/2\right)^{\frac{1}{2}}\right\},$$

$$m_2 = \beta_2 c \max\left\{\frac{1}{2}, \frac{3\alpha_2^2}{4}\right\}^{\frac{1}{4}},$$

$$m_3 = \beta_2\sqrt{c}\max\left\{\frac{1}{2}, \frac{3\alpha_2^2}{4}\right\}^{\frac{1}{8}},$$

$\alpha_1 = 1,\ \alpha_2 = 1,\ \beta_1 = 0.1,\ \beta_2 = 0.2,\ \underline{b} = 4,\ \bar{b} = 4.01$ and $c = 17.2776$. Then

$$\sqrt{2}(\bar{b} - \underline{b}) = 0.0141 \leq 0.0243\beta \leq -\frac{\beta\max_{z\in\tilde{S}}\langle\nabla V(z), H(z)b(z)\rangle}{\max_{z\in\tilde{S}}\|H^\top(z)\nabla V(z)\|}.$$

For $0.1987 < \beta = 0.7 < 1$, the functions $b_1(z) = 4.005 - 0.005\sin\left(\frac{1}{\|z\|}\right)$ and $b_2(z) = 4.005 - 0.005\cos\left(\frac{1}{\|z\|}\right)$ satisfy the condition (8). The simulation results for this case are depicted in Figure 1.

Remark 5.1 *To show an application of the obtained results, we can consider the system* (13) *with* $f_i(q,v) = -\beta_{i1}b_{i1}(q,v)|v_i|^{\frac{1}{2}}(v_i) - \beta_{i2}b_{i2}(q,v)|v_i|^{\frac{1}{3}}(v_i), \beta_{i2} \geq \beta_{i1} > 0,\ u_i = \alpha_i q^{\frac{1}{3}}, \alpha_2 \geq \alpha_1 > 0$ *for all* $i = 1,2$ *and* $b_i, i = 1,2$ *are chosen such that the assumption* $\mathcal{H}_3$ *holds. By using the same Lyapunov function* V*, we can prove the GAS property of* (13) *and then its FTS.*

Remark 5.2 *The system* (13) *with* $n \geq 2$ *with* $f_i(z) = -\beta_i b_i(z)|v_i|^{\frac{1}{2}}(v_i),\ \beta_n \geq \cdots \geq \beta_1 > 0,\ u_i = \alpha_i q^{\frac{1}{3}},\ \alpha_n \geq \cdots \geq \alpha_1 > 0$ *for all* $i = 1,\cdots,n$ *and* $b_i, i = 1,\cdots,n$ *can be studied similarly using the Lyapunov function* $V(z) = cU^{\frac{5}{4}}(z) + \sum_{i=1}^{n}\alpha_i^{\frac{3}{4}}q_iv_i$*, where* $U(z) = \frac{3}{4}\sum_{i=1}^{n}\alpha_i|q_i|^{\frac{4}{3}} + \frac{1}{2}\|v\|^2,\ z = (q,v) \in \mathbb{R}^n \times \mathbb{R}^n$ *and* $\alpha_i > 0,\ c > \frac{4}{5},\ \forall i = 1,\cdots,n.$

6 Conclusion

Finite-time stability and nearly fixed-time stability of a class of non-homogeneous systems is investigated using the homogeneous extension and Lyapunov function method. In particular, we presented some sufficient conditions (Corollaries 1, 2 and 3) to guarantee finite-time or nearly fixed-time stability of an asymptotically stable non-homogeneous ODE using homogeneity degree of a homogeneous extension of the system. The homogeneous extension could be useful for a homogeneity-based robustness analysis (see e.g. [17]) of dynamical systems which do not admit a homogeneous approximations. This can be considered as a possible direction for the future research.

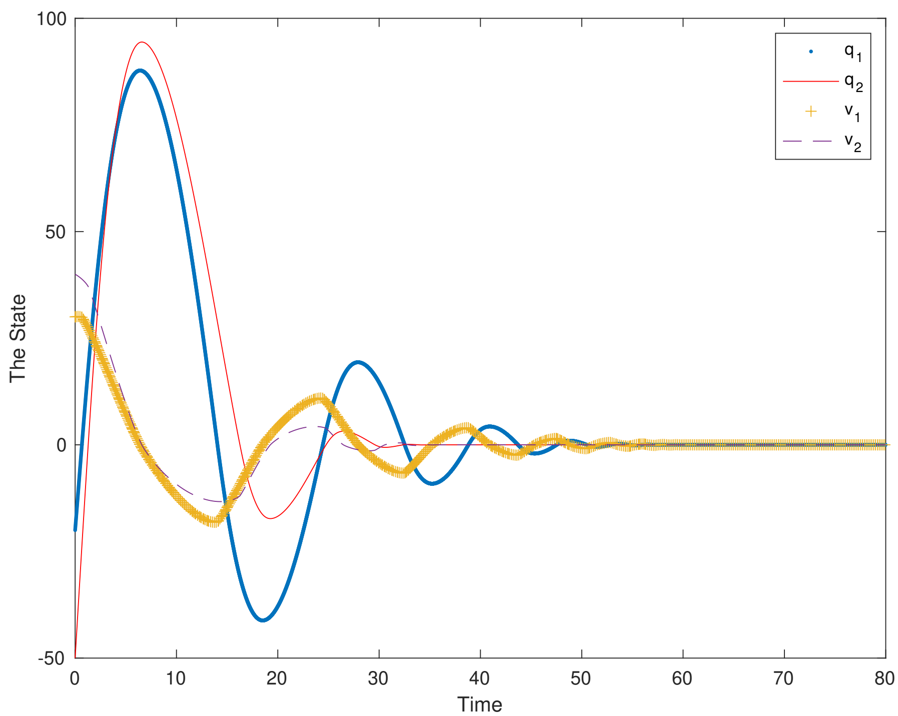

Figure 1: : The solutions of the system (14) with $(\alpha_1, \alpha_2) = (1, 1)$, $(\beta_1, \beta_2) = (0.1, 0.2)$, $b_1(z) = 4.005 - 0.005 \sin\left(\frac{1}{\|z\|}\right)$, $b_2(z) = 4.005 - 0.005 \cos\left(\frac{1}{\|z\|}\right)$ and the initial condition $(q_0, v_0) = (-20, -50, 30, 40)$.

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