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## A Geometric Summation of the Geometric Series

Let  $k$  be a finite field of size  $q$ . The  $n$ -dimensional projective space over  $k$ ,  $\mathbb{P}^n(k)$ , is the set of orbits of  $k^{n+1} \setminus \{0\}$  under the action of  $k^*$  given by scalar multiplication. Therefore  $|\mathbb{P}^n(k)| = \frac{q^{n+1}-1}{q-1}$  since all orbits have the same size. Now, since the decomposition of  $\mathbb{P}^n(k)$  into a disjoint union of affine spaces  $\mathbb{P}^n(k) = A_0 \cup A_1 \cup \dots \cup A_n$  where

$$A_i := \{(x_0 : \dots : x_{i-1} : 1 : 0 : \dots : 0) \mid x_j \in k\}$$

is in bijection with the affine space  $k^i$ , then  $|\mathbb{P}^n(k)| = \sum_{i=0}^n q^i$ . Thus for all  $n$  and power of primes  $q$ ,

$$\sum_{i=0}^n q^i = \frac{q^{n+1} - 1}{q - 1}.$$

Since this identity of univariate rational functions over  $\mathbb{Q}$  is true for an infinite amount of numbers, it translates into an equality of rational functions. Hence

$$\sum_{i=0}^n X^i = \frac{X^{n+1} - 1}{X - 1}$$

with  $X$  indeterminate. Can this method be used to prove other identities by interpreting them geometrically using algebraic varieties over finite fields?

—Submitted by Josué Tonelli-Cueto, Inria Paris & IMJ-PRG