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► **To cite this version:**

Weiqiong Cao, Hongsong Shi, Hua Chen, Wei Wei, Jiazhe Chen. Lattice-Based Weak Curve Fault Attack on ECDSA. 36th IFIP International Conference on ICT Systems Security and Privacy Protection (SEC), Jun 2021, Oslo, Norway. pp.146-161, 10.1007/978-3-030-78120-0_10 . hal-03746039

HAL Id: hal-03746039

<https://inria.hal.science/hal-03746039v1>

Submitted on 4 Aug 2022

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Lattice-based weak curve fault attack on ECDSA

Weiqiong Cao^{1,2}, Hongsong Shi¹(✉), Hua Chen², Wei Wei¹, Jiazhe Chen¹

¹ China Information Technology Security Evaluation Center. Building 1, yard 8, Shangdi West Road, Haidian District, Beijing 100085, China.

² Trusted Computing and Information Assurance Laboratory, Institute of Software, Chinese Academy of Sciences, South Fourth Street 4#, ZhongGuanCun, Beijing 100190, China.

Email: caoweqion@163.com, hssshi@163.com

Abstract. ECDSA algorithm is usually used in ICT system to ensure the authenticity of communication. But the weaknesses in various implementations of ECDSA may make its security deviate from theoretical guarantee. This paper proposes a new lattice-based weak curve fault attack on ECDSA. Since the ECDLP is not required to be computationally practical on the whole group of $\langle G \rangle$ (G is the specified base point of ECDSA), our approach extends the existing attacks along this line. In detail, the proposed attack assumes a segment of consecutive bits of the curve parameter a in the Weierstrass equation of ECDSA can be disturbed randomly by fault injection and thus is changed into a' . An analysis about the density of smooth numbers demonstrates the faulty parameter a' can be used for our attack with high probability. Then we show a' can be recovered by a dedicated quadratic residue distinguisher. Some reduced information about the nonce used in ECDSA signature generation can be obtained by solving the instances of ECDLP on the new curve about a' . With the help of these information, we can construct a new model of lattice to recover the private key with the lattice basis reduction techniques. Further, we show the same strategy can defeat the nonce masking countermeasure if the random mask is not too long, and makes the commonly employed countermeasures ineffective. To our knowledge, the problem remains untraceable to the existing weak curve fault attacks. Thus the proposed approach can find more applications than the existing ones. This is demonstrated by the experimental analysis.

Keywords: ECDSA, Weak Curve, Fault Attack, Lattice Attack

1 Introduction

1.1 Existing work on fault attacks

Elliptic curve digital signature algorithm (ECDSA) has found its extensive use in practice. It is mainly used for data authentication in network communication protocols (e.g., TLS protocol), financial IC cards and various embedded cryptographic devices. Over the last decades, side channel attack (SCA) and fault

attack (FA) have been found to be exploitable on different implementations of ECDSA. As in the SCAs [3,1], obtaining useful side channel information is generally the initial step of various attacks. For example, in [11,20,21], an adversary, when collecting enough side channel information about the nonce or some intermediates, can construct specific instances of shortest vector problem (SVP) or closest vector problem (CVP) in lattice, and then employ the lattice basis reduction methods to recover the signature key.

In this paper, we are interested in fault attacks. The structure of FA is similar to SCA on ECDSA. Firstly, FA manages to obtain some valid bit-leaked information about the nonce in ECDSA, and then translates the obtained information into a lattice so as to recover the private key by lattice basis reduction algorithms. The difference of FA from SCA is that, the leakage information is actively induced by fault injection approaches, such as laser injection, electromagnetic injection or voltage glitch interference. The induced signal can perturb the execution flow of the signature generation, which makes some instructions skipped or intermediates faulty, and then makes the target produce faulty signatures. A dozen of fault attacks on ECDSA have been proposed since very early of this century. Here we only review those being related to lattice analysis.

In PKC 2005 [18], Nacache *et al.* introduced a lattice-based fault attack on DSA. In their approach, some least significant bits of nonce are set to be 0 by inducing voltage glitch, and then the private key in DSA is recovered by solving an instance of CVP in lattice. Schmidt *et al.* in FDTC 2009 [24] introduced a new differential fault model. If a point addition or doubling operation during the scalar multiplication (ECSM) in signature generation is skipped by fault injection, some bits of the nonce can be obtained by differential analysis. Nguyen *et al.* [22] summarized this kind of fault attacks and called them lattice-based fault attacks. Cao *et al.* in ICISC 2015 [4] also introduced a random fault model targeting the y -coordinate of intermediate point during the calculation of ECSM, which can tolerate more random faulty bits. In [14], Kim *et al.* showed that the fault on modulus p can also be applied to do FA on ECDSA. The attack assumes fault injection can flip every bit of the modulus p , and then obtain a weak curve on which solving elliptic curve discrete logarithm problem (ECDLP) is computationally practical. The solution reveals some leakage information about nonce k , by which two faulty signatures are enough for constructing a lattice to recover the private key. However, the approach requires a strong fault model that only one bit (or a few bits) of p is flipped and the faulty modulus p' is known to the adversary. Moreover, it requires all the prime factors p_i of p' and the orders n_i of subgroups $\mathbb{Z}/p_i^{e_i}$ (where $p' = \prod_{i=1}^u p_i^{e_i}$, $p_i < p_j$ for $1 \leq i < j \leq u$ and $e_i \in \mathbb{N}$) to be relatively small, such that the time complexity $O(\sqrt{n_u})$ of solving ECDLP in this case is practical. In addition, in order to mount lattice attack, the product $n' (= \prod_{i=1}^u n_i)$ of all the orders n_i should satisfy $n' \geq n^{1/2}$, that is, the bit length of n' should be greater than half of the key length of ECDSA, which restrains the applicability of the fault attack.

1.2 Our approach

In this paper, we propose a lattice-based weak curve fault attack on ECDSA. Here an elliptic curve is defined as weak curve if ECDLP in a *subgroup* of the point group $\langle G \rangle$ is computationally solvable in practice, where G is the base point of ECDSA algorithm. (See Definition 3 for detail.) Note the definition does not require ECDLP in the whole group of $\langle G \rangle$ being computationally solvable, which is the main difference of our approach from the existing attacks.

In more detail, we consider a continuous segment of the curve parameter a can be randomly disturbed by fault injection. The faulty value of a , called a' hereafter, is not required to be known, but can be guessed by a specific quadratic residual distinguisher, see the Algorithm ALG-GUESS-PARA in Section 3.2. Then if the induced curve is weak such that the ECDLP in some subgroup of $\langle G \rangle$ can be solved practically, we can obtain enough reduced information about the nonce. Formally, the weakness is characterized by a factor d of the order n' of G in the weak curve. And the reduced information is expressed in the form of modulo d , see the Algorithm ALG-OBTAIN-NONCEINFO in Section 3.2. Then this type of information can be employed to construct a new model of lattice for key recovery (see Section 3.3). The dimension N of the lattice is tightly related to the module d . In short, the bigger d is, the smaller N would be.

In addition, for ECDSA with random scalar masking, the proposed approach is still practical without any additional masked bits leakage. For example, if $k' = k + \lambda n$, where k is the real scalar and λ is a 64-bit random number, the approach can succeed by selecting a bigger modulus d (see Section 3.4).

Our study on the density of smooth numbers shows the probability of existing some big prime factors in n' is much greater than that of all the prime factors of n' being small (see Section 3.5). It thus indicates the proposed approach can find more applications than the existing weak curve fault attacks. This is demonstrated by the experimental analysis in Section 4.

2 Preliminaries

In this paper, we consider the elliptic curves on prime field \mathbb{F}_p , where p is a prime. Generally, the Weierstrass equation of the elliptic curves on \mathbb{F}_p is given by

$$E(a, b) : y^2 = x^3 + ax + b \pmod{p},$$

where parameters $a, b \in \mathbb{F}_p$ satisfy $4a^3 + 27b^2 \neq 0$. The group of rational points on the elliptic curve $E(a, b)$ is defined by

$$\mathbf{G} = \{(x, y) | y^2 = x^3 + ax + b \pmod{p}, x, y \in \mathbb{F}_p\} \cup \{\mathcal{O}\},$$

where \mathcal{O} is the infinite point. Let G be an element in \mathbf{G} with order n (which is usually a prime), $\langle G \rangle$ be the additive subgroup of \mathbf{G} generated by G (where \mathcal{O} is the identity element of $\langle G \rangle$). For any $P = (x, y) \in \langle G \rangle$, its inverse element $-P \in \langle G \rangle$ is $(x, -y)$. For any integer $k \in \mathbb{Z}_n$, the calculation of $kG = G + G + \dots + G$

(k times) is called ECSM in $E(a, b)$, and can be calculated using point doubling and addition operations [10].

Point Addition

If $P = (x_1, y_1) \in \langle G \rangle$, $Q = (x_2, y_2) \in \langle G \rangle$, and $P \neq \pm Q$, then $(x_3, y_3) = P+Q$ satisfies

$$x_3 = \lambda^2 - x_2 - x_1, \text{ where } \lambda = \frac{y_2 - y_1}{x_2 - x_1}.$$

$$y_3 = \lambda(x_1 - x_3) - y_1$$

Point doubling

If $P = (x_1, y_1) \in \langle G \rangle$ and $P \neq -P$, then $(x_3, y_3) = 2P$ satisfies

$$x_3 = \lambda^2 - 2x_1, \text{ where } \lambda = \frac{3x_1^2 + a}{2y_1}.$$

$$y_3 = \lambda(x_1 - x_3) - y_1$$

An important notice is that the parameter b is not involved in the calculation of point doubling and addition. The order of $E(a, b)$, denoted by $\#E(a, b)$ can be calculated using SEA algorithm [9].

2.1 ECDSA digital signature algorithm

The ECDSA signature generation algorithm is described in Algorithm 1 with some less important details being abstracted away. The randomly generated nonce k is involved in the calculations of ECSM kG (step 3) and s (step 6), which are exactly the targets of our attacks.

Algorithm 1 Signature generation of ECDSA

Input: The definition of a specific elliptic curve $E(a, b)$, a base point G of the curve with order n , private key $d_A \in \mathbb{Z}_n$, message m .

Output: Signature pair (r, s) .

- 1: $e = H(m)$, where H is a cryptographic hash function;
 - 2: Generate k randomly from \mathbb{Z}_n ;
 - 3: $Q(x_1, y_1) = kG$;
 - 4: $r = x_1 \bmod n$;
 - 5: **if** $r = 0$ **then** goto step 2;
 - 6: $s = k^{-1}(e + d_A r) \bmod n$;
 - 7: **if** $s = 0$ **then** goto step 2;
 - 8: **return** (r, s)
-

2.2 Smoothness of weak curve order

The following definitions are required to better describe our approach. For all of them, let n be the order of point G in $E(a, b)$.

Definition 1. Denote the prime factorization of n by $n = \prod_{i=1}^u q_i^{e_i}$, where $q_i \in \mathbb{N}$ is a prime factor of n , $e_i \in \mathbb{N}$ denotes the degree of q_i in the factorization and $q_i < q_j$ for $1 \leq i < j \leq u$. For $y \in \mathbb{N}$, if the biggest prime factor q_u meets $q_u \leq y$, then n is called y -smooth.

Definition 2. *The elliptic curve discrete logarithm problem (ECDLP) in $E(a, b)$ is defined as: given $G \in \mathbf{G}$ with order n and an element $Q \in \langle G \rangle$, compute the value $k \in \mathbb{Z}_n$ such that $Q = kG$.*

To our knowledge, the best known generic algorithm [28,23] in classical computer for solving ECDLP in arbitrary elliptic curves needs $O(\sqrt{q_u})$ group operations in computation. We call an ECDLP instance practically solvable if its solving complexity is not bigger than a predefined constant x . In this paper, we set $\text{PRAC_COMP} = 2^{64}$ group operations by considering currently achievable computing power of classical computers, which can be redefined to adapt to the future development of computing technology.

Definition 3. *With respect to the group $\langle G \rangle$, we call n practically solvable s -smooth if the ECDLP on $\langle G \rangle$ is practically solvable; we call n partially solvable smooth if there exists a factor d of n such that the ECDLP on $\langle (n/d)G \rangle$ (with order d) is practically solvable; Finally, we call n practically unsolvable smooth if n does not belong to the above two cases.*

In this paper, $E(a, b)$ is defined as weak curve if the order n of the chosen base point G of $E(a, b)$ is partially solvable smooth or practically solvable smooth.

2.3 Existing fault attacks on weak curves

In this section, we introduce an existing fault attack on the weak curves with solvable smooth order [14,2], which can also be used to the weak curves with partially solvable smooth order.

It is assumed that the y -coordinate of G is disturbed by a fault injection process, i.e., $G = (x_G, y_G)$ is changed into $G' = (x_G, y_{G'})$ with $y_G \neq y_{G'}$. Then with overwhelming probability, the faulty G' is not on the original curve $E(a, b)$ (the only exception being $y_G = -y_{G'}$). Note since parameter b is not involved in the calculation of scalar multiplication, G' can be viewed on a new curve $E(a, b')$, and then $Q' = kG'$ is calculated on the curve $E(a, b')$, where $b' = y_{Q'}^2 - x_{Q'}^3 - ax_{Q'} = y_{G'}^2 - x_G^3 - ax_G$.

Assume the induced curve has a solvable smooth order $n' = \prod_{i=1}^u q_i^{e_i}$ with respect to the group $\langle G' \rangle$. Then given Q' , G' and n' , the following approach can be used to compute the scalar k . Firstly, the reduced value $k \bmod q_i$ can be obtained by solving the ECDLP instance $\frac{n'}{q_i} Q' = k \frac{n'}{q_i} G' (i = 1, \dots, u)$ with Pollard-rho algorithm [28]. Next, the reduced value $k_i = k \bmod q_i^{e_i} (i = 1, \dots, u)$ can be obtained by Pohlig-Hellman algorithm [23]. Finally, the modulo- n' reduced value $t = k \bmod n'$ can be obtained by Chinese remainder theorem (CRT). Hence, $k = t + \varepsilon n'$, where $\varepsilon \in \{0, \dots, \lfloor n/n' \rfloor\}$. Enumerate all the possible values of ε to calculate the corresponding k , when $Q = kG$, k is just the correct one that we are looking for.

The above approach shows that n' must be solvable smooth so as to solve the ECDLP instances on $\langle G' \rangle$ and $\lfloor n/n' \rfloor \leq \text{PRAC_COMP}$. Otherwise, the approach cannot be applied in practice.

2.4 Lattice basis reduction

Lattice analysis is a key technique to our approach. We thus give some fundamental background about lattice attacks.

Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_N\} \subseteq \mathbb{R}^m$ be a series of N linearly independent vectors. The lattice generated by B is defined as $\mathcal{L}(B) = \left\{ \sum_{i=1}^N x_i \mathbf{b}_i : x_i \in \mathbb{Z} \right\}$, where B serves as a basis for the lattice $\mathcal{L}(B)$, and we call the integers N and m its rank and dimension respectively. If $m = N$, \mathcal{L} is called a N -dimensional full rank lattice. The shortest vector problem (SVP) and closest vector problem (CVP) are two computational complexity problems crucial to lattice-based cryptography. We give them below.

Definition 4. [17]

- (1) **Shortest Vector Problem:** Given a basis of a lattice \mathcal{L} , find a lattice vector $\mathbf{v} \neq \mathbf{0}$ such that $\|\mathbf{v}\| \leq \|\mathbf{u}\|$ for any nonzero vector $\mathbf{u} \in \mathcal{L}$.
- (2) **Closest Vector Problem:** Given a basis of a lattice \mathcal{L} and a target vector $\mathbf{t} \in \mathbb{R}^m$, find a lattice vector $\mathbf{v} \in \mathcal{L}$ closest to the target \mathbf{t} , which means $\text{dist}(\mathbf{v}, \mathbf{t}) \leq \text{dist}(\mathbf{u}, \mathbf{t})$ for any vector $\mathbf{u} \in \mathcal{L}$, where $\text{dist}(\cdot, \cdot)$ denotes the Euclid norm of two points.

For a N -dimensional approximate SVP, there exist some polynomial-time basis reduction algorithms which can output short lattice vectors when the approximate factor is large enough. Among those algorithms, LLL algorithm [15] is the most typical one, and BKZ-algorithm [6] has been the most practical algorithm for lattice basis reduction based on a series of optimizing technique [25,26].

For random lattices with dimension N , Gaussian heuristic gives a probable estimation on the length of shortest lattice vector in the sense of average as in [19], from which Gaussian expected the shortest length of a N -dimensional lattice \mathcal{L} could be defined to be

$$\sigma(\mathcal{L}) = \sqrt{\frac{N}{2\pi e}} \text{vol}(\mathcal{L})^{1/N},$$

where vol denotes the volume or determinant of \mathcal{L} .

Generally, the actual shortest lattice vector is much easier to be found as the increment of the gap between the shortest length and the Gaussian heuristic. If it is significantly shorter than $\sigma(\mathcal{L})$, it can be located in polynomial time by using LLL and related algorithms. Heuristically, assuming the lattice \mathcal{L} behaves like a random lattice, if there exists a lattice vector whose distance from the target is much shorter than $\sigma(\mathcal{L})$, this lattice vector is expected to be the closest vector to the target. Accordingly, this special CVP instance usually could be solved by Babai algorithm or embedding-based SVP.

3 Lattice-based weak curve attack

In this section, we present our lattice-based weak curve attack on ECDSA. The attack consists of two steps: 1) Obtain the reduced information of the nonce

by weak curve fault attack; 2) Construct an instance of CVP by virtue of the reduced information, and resolve it to recover the private key.

3.1 Fault model

The fault attacks we consider in this paper aim at modifying the curve parameter a by inducing fault to the corresponding physical storage cells (RAM, EEPROM or CPU register for example). Further, we mainly consider a type of random fault, in which a continuous l -bit segment of a is modified randomly by fault injection and the starting bit location of fault is also randomly picked. The fault can be permanent or transient. A permanent fault means the value corresponding to the parameter is definitely changed, and fixed on the faulty value. A transient fault means the parameter keeps the faulty value unless the original value is explicitly restored. The length l of faulty bits is usually valued from $\{1, 8, 16, 24, 32\}$ considering the byte-based cell structure of storage.

Assume, before running the signature generation (Algorithm 1), an adversary induces a permanent (or transient) fault to the parameter a of elliptic curve $E(a, b)$. Denote the modified parameter by a' and suppose it is different from a in a continuous l -bit segment. Therefore, the base point $G = (x_G, y_G)$ will be on a new curve $E(a', b') : y^2 = x^3 + a'x + b' \pmod{p}$, and $b' = y_G^2 - x_G^3 - a'x_G \pmod{p}$. Correspondingly, the ECSM $Q = kG$ of step 3 in Algorithm 1 will be actually computed on the new curve $E(a', b')$ (since parameter b is not involved in the calculation of ECSM). Let the new order of G on $E(a', b')$ be n' . Finally, the faulty signatures (r', s') are output to the adversary.

Note that our single fault model has the following limitations: 1) parameter a must be involved in the calculation of ECSM (except that a is sometimes substituted with $p - 3$ when $a = p - 3$ for the sake of resource optimization); 2) There is no point validation checking whether the input or output point is on the original elliptic curve during the calculation of ECSM. Otherwise, our attack will not work.

3.2 Proposed fault attack on weak curves

Suppose the signatures on the weak curve can be retrieved after the signature generation procedure. We will run the following two algorithms sequentially to obtain some reduced information about the nonce k . Then, in Section 3.3 we will use the obtained information to construct lattices and recover the private key.

Algorithm ALG-GUESS-PARA: Guess and determine a' and x_Q

Step 1-1. If assuming the fault injection step induced a randomly located and randomly valued continuous l -bit segment errors to a , then a' could be characterized by $a' = a \oplus \beta 2^\sigma$, where $\beta \in \mathbb{Z}_{2^l}$ is an unknown l -bit random integer, l_a is the bit length of a and σ is an unknown random integer in interval $[0, l_a - l]$. Let \mathbf{T} be the set for storing all the possible values of a' . Hence, the initial number of the possible values for a' is $|\mathbf{T}| = (l_a - l + 1)2^l$.

Step 1-2. Run the signature generation procedure to obtain a faulty signature pair (r', s') . Then deduce the x -coordinate $x_{Q'} \bmod p$ of the faulty point Q' from r' . We separate two cases to consider the deduction.

- If $p < n$ then $x_{Q'} \bmod p = r' \bmod n$;
- If $p > n$, we have the deduction as follows. Note that $hn = \#E(a, b)$ (where $\#E(a, b)$ is the number of point and h is the cofactor on $E(a, b)$). By Hasse theorem [10], we have $p + 1 - 2\sqrt{p} \leq hn \leq p + 1 + 2\sqrt{p}$. Hence, $x_{Q'} \bmod p = r' + \lambda n < p$, and the integer λ is valued not greater than h (h is usually 1 or 2 in a standard curve) when $p > 2^6$, i.e., $x_{Q'}$ has at most $h + 1$ values depending on the concrete values of p and n .

Step 1-3. Sieve \mathbf{T} to find valid a' . For each possible $a' \in \mathbf{T}$, calculate $b' = y_G^2 - x_G^3 - a'x_G \bmod p$, and for each value $x_{Q'}$ derived in step 2, compute

$$Y = x_{Q'}^3 + a'x_{Q'} + b' \bmod p.$$

If Y is a quadratic residue modulo p , keep a' in \mathbf{T} . Otherwise, eliminate it from \mathbf{T} .

Step 1-4. If $|\mathbf{T}|$ is greater than 1, go to **Step 1-2**; otherwise, regard the only value in \mathbf{T} as the faulty parameter a' . Then run the SEA algorithm [9] to compute the order n' of G on $E(a', b')$, and factorize it using some algorithms in subexponential time (such as Pollard $p - 1$ or number field sieve algorithms). Note when $n' \leq 512$, the factorization step is practical which is enough to factorize n in current standard ECC. If the factorization result shows that n' is partially solvable smooth, we get a valid a' ; otherwise repeat Algorithm ALG-GUESS-PARA to induce a new curve until getting a valid a' with partially solvable smooth n' . (Note we don't require n' to be *solvable smooth*.) The condition can be satisfied after a number of trials considering the density of smooth numbers (See Section 3.5 for detail). So in the following we assume there exists a factor d of n' such that d is practically solvable smooth with respect to $E(a', b')$ (see the definitions in Section 2.2). Finally, we end the algorithm.

The above ‘‘quadratic residue’’ distinguisher can eliminate about a half of the invalid values in each invocation. Hence, the total time complexity is about $O((l_a - l + 1)2^{l+1})$.

Algorithm ALG-OBTAIN-NONCEINFO: Obtain reduced value of nonce k

Based the derived valid a' and the selected small factor d of n' , run the following steps to collect as much as useful reduced information about the nonce.

Step 2-1. Run the signature generation procedure with different message to get a signature (r', s') , and based on the derived (a', b') compute possible values $\{x_{Q'}\}$ and Y as above. Since there is only one correct value for $x_{Q'}$, to remove the erroneous computed values, we discard the signature (r', s') if more than one Y derived from it are quadratic residue modulo p , and re-generate a new signature (r', s') until the condition is satisfied. When the correct $x_{Q'}$ on curve $E(a', b')$ is calculated, we could obtain two possible points $(x_{Q'}, \pm\sqrt{Y})$ of Q' .

Step 2-2. Without loss of generality, assume $Q'_1 = (x_{Q'}, \sqrt{Y}) = \mu G$ for $\mu \in \mathbb{Z}_n$. Let $\ell = n'/d$, we can construct the instance of ECDLP

$$\ell Q'_1 = \mu(\ell G)$$

in the d -ordered subgroup $\langle \ell G \rangle$ and solve it to obtain $\mu \bmod d$. Specifically, if Q'_1 is the correct choice of Q' , we have $k = \mu \bmod d$; otherwise $-Q'_1$ is the correct choice of Q' , and we have $k = (d - \mu) \bmod d$.

Step 2-3. Repeat the above two steps N times to obtain N groups of valid signature result $\{r_i, s_i\}_{i=1}^N$ and N groups of reduced information about the nonces $\{k_i\}_{i=1}^N$, where each reduced information is denoted by

$$k_i = c_i + \lambda_i d$$

$$\text{for } i = 1, \dots, N, \text{ where } c_i = \begin{cases} \mu_i, & \text{for } y_{Q'} = \sqrt{Y} \\ d - \mu_i, & \text{for } y_{Q'} = -\sqrt{Y} \end{cases} \text{ and } 0 < \lambda_i < n/d.$$

In the above Algorithm ALG-OBTAIN-NONCEINFO, solving the ECDLP in the d -ordered subgroup consumes the main computation time, and its time complexity $O(\sqrt{q_{max}})$ (where q_{max} is the biggest prime factor of d) is less than PRAC_COMP since d is selected to be practically solvable (see Section 2.3). Hence, Algorithm ALG-OBTAIN-NONCEINFO is practically feasible.

3.3 Proposed lattice-based ECDSA key recovery algorithm

We show how to use the signature result $\{r_i, s_i\}_{i=1}^N$ and the reduced information about $\{k_i\}_{i=1}^N$ to construct a new model of lattice and then recover the private key d_A .

For each $i \in \{1, \dots, N\}$, we first assume the correct value of c_i is identified. Substituting the corresponding k_i into s_i , we have

$$s_i(c_i + \lambda_i d) = e_i + r_i d_A, \quad (1)$$

where e_i is the hash value of message m_i and $0 < \lambda_i < n/d$. The identification of correct value of c_i is discussed at the end of this subsection.

The equation (1) can be transformed as

$$\lambda_i = A_i d_A - (B_i - n/(2d)) \bmod n, \quad (2)$$

where $A_i = s_i^{-1} d^{-1} r_i \bmod n$ and $B_i = d^{-1} (c_i - s_i^{-1} e_i) \bmod n + n/(2d)$. Since $0 < \lambda_i < n/d$, there exists a $h_i \in \mathbb{Z}$ such that

$$|A_i d_A + h_i n - B_i| < n/(2d) \quad (i = 1, \dots, N). \quad (3)$$

We can construct a lattice \mathcal{L} by the above inequalities (3), and the row vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_{N+1}\}$ of the matrix

$$\mathbf{M} = \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & n & 0 \\ A_1 & A_2 & \cdots & A_N & 1/(2d) \end{bmatrix}$$

construct a basis of \mathcal{L} .

Let the target vector $\mathbf{t} = (B_1, \dots, B_N, 0) \in \mathbb{R}^{N+1}$. There exists a lattice vector $\mathbf{v} = \mathbf{x}\mathbf{M} = (h_1n + A_1d_A, \dots, h_Nn + A_Nd_A, d_A/(2d))$ with the coordinate vector $\mathbf{x} = (h_1, \dots, h_N, d_A) \in \mathbb{Z}^{N+1}$. From inequalities (3), we have

$$\|\mathbf{v} - \mathbf{t}\| < \sqrt{N+1}n/(2d). \quad (4)$$

As introduced in [22], we also assume heuristically \mathcal{L} behaves like a random lattice. According to Section 2.4, if $\|\mathbf{v} - \mathbf{t}\|$ is much less than $\sigma(\mathcal{L}) (= \sqrt{\frac{N+1}{2\pi e}} \text{vol}(\mathcal{L})^{\frac{1}{N+1}})$, we expect \mathbf{v} to be the closest vector from \mathbf{t} in \mathcal{L} , where $\text{vol}(\mathcal{L}) = \det(\mathbf{M}) = n^N/(2d)$ ($\det(\mathbf{M})$ is the determinant of \mathbf{M}). Hence, it is required

$$\|\mathbf{v} - \mathbf{t}\| < \sqrt{N+1}n/(2d) \ll \sqrt{\frac{N+1}{2\pi e}} (n^N/(2d))^{\frac{1}{N+1}}. \quad (5)$$

Let $f = \lceil \log n \rceil$ and $l_d = \lceil \log d \rceil$. If $N > \frac{f + \log \sqrt{2\pi e}}{l_d + 1 - \log \sqrt{2\pi e}}$ and $l_d > \log \sqrt{2\pi e} - 1$, heuristically the inequality (4) can be viewed as a special instance of CVP in lattice \mathcal{L} . Consequently, the vector \mathbf{v} can be determined by solving the instance of CVP to reveal the private key d_A . If $d_A G$ is equal to the public key P_A , the attack is successful.

In addition, the inequality (3) is equivalent to

$$|A_i d_A + h_i n - B_i| < n/2^{l_d} \quad (i = 1, \dots, N), \quad (6)$$

which is a hidden number problem(HNP)[20,21]. By the same way, it can be transformed into a CVP to recover d_A .

It is assumed above that all the values of $\{c_i\}_{i=1}^N$ are correctly guessed before the lattice attack. The reality is that there are two solutions for each c_i after the Algorithm ALG-OBTAIN-NONCEINFO, and it is not sure which one is the correct. Hence, 2^N kinds of combinations about the possible c_i would be generated and only one combination is correct. Although it is computationally difficult to make it certain in general, from the above analysis, we know the required number N of faulty signatures for lattice attack depends on the size of factor d . Or in order to ensure the practicality of the lattice attack, d should not be too small. In fact, the larger d is, the smaller N would be. For example, d is generally recommended to satisfy $d \geq 2^8$ such that $N \approx 45$ for 256-bit ECDSA. So when N is not too big, it is still possible to enumerate all the possible c_i in practice. Specifically, the worse-case time complexity for the enumerating lattice attack in this case is $2^N T$, where T represents the time required for each running of the lattice attack.

3.4 Attack on ECDSA with scalar masking

Generally, scalar masking is one of the most common countermeasures for ECDSA to resist SCA. For example, nonce k_i during signature generation is masked as k'_i with a random number α_i , i.e., $k'_i = k_i + \alpha_i n$ ($i = 1, \dots, N$). This countermeasure also could block the existing lattice attacks [11,20,21], since it is required

to obtain all the bit leakage information of $\{\alpha_i\}_{i=1}^N$. By comparison, our attack is affected much less, specifically as follows.

With the masked nonce k'_i , we have

$$Q = k'_i G \text{ and } s_i = k_i^{-1}(e_i + r_i d_A) = k'_i{}^{-1}(e_i + r_i d_A) \pmod n.$$

Accordingly, the reduced information derived by weak curve fault attack meets $k'_i = c_i + \lambda_i d$, where l_{α_i} denotes the bit length of α_i and $\lambda_i < 2^{f+l_{\alpha_i}-l_d}$. Substitute k'_i into s_i and mount lattice attack. If $l_d > \log \sqrt{2\pi e} + l_{\alpha_i} - 1$ and $N > \frac{f+\log \sqrt{2\pi e}}{l_d-l_{\alpha_i}+1-\log \sqrt{2\pi e}}$, the private key d_A can be recovered by constructing an instance of CVP. There is no bit leakage of α_i required in our lattice attack except a bigger d . Moreover, the required d can be obtained with high proportion in experiments. For example, if $l_{\alpha_i} = 32$, the experimental success rate of fault injection is still up to 80% since l_d is recommended as 40 (see Section 4). Obviously, our lattice attack is more practical on ECDSA with scalar masking.

3.5 The density of smooth numbers

When comparing with existing weak curve fault attacks in [14,2,7] (which require n' must be solvable smooth so as to solve ECDLP instances on $\langle G' \rangle$ and $\lfloor n/n' \rfloor \leq \text{PRAC_COMP}$), our attack puts weaker condition on the process of fault injection. Specifically, the order of the induced weak curve in the proposed method is only required to be partially solvable smooth in the proposed attack, which is weaker than the existing attacks. The following analysis on the smoothness of a random number demonstrates that the weaker condition improves the applicability of proposed attack significantly.

Let z be an integer with prime factorization $z = \prod_{i=1}^u p_i^{e_i}$. We say z is y -smooth if $\max_{1 \leq i \leq u} \{p_i\} \leq y$, as mentioned in Section 2.2. We denote by $\psi(x, y)$ the number of integers $z \leq x$ such that z is y -smooth. In [8], a result on the bound of $\psi(x, y)$ shows smooth numbers with suitable x, y are relatively common to meet. Specifically, let ϵ be an arbitrary positive constant, then for $x \geq 10$ and $y \geq (\ln x)^{1+\epsilon}$, we have

$$\psi(x, y)/x = e^{-(1+o(1))u \ln u} \quad \text{as } x \rightarrow \infty,$$

where $u = \ln x / \ln y$ and e is the natural number. Note for a fixed x the density of smooth numbers (i.e., $\psi(x, y)/x$) is an increasing function with respect to the bound y of factors. For instance, we can roughly get $\psi(2^{256}, 2^{247})/2^{256} = 0.963$ and $\psi(2^{256}, 2^{128})/2^{256} = 0.25$ (where $o(1)$ is set to be 0 in the approximation). It means smooth (integer) numbers in the scope of $[1, x]$ with at least one large factor could be much more frequent than those with only smaller factors. Note the action of n' is not uniformly random, since the injected fault is only considered to impact very limited bits of the parameter a of the curve. However, as in [8], we make a heuristic assumption that the probability of sampling n' in this method is subject to the density given by $\psi(x, y)/x$. In this view, Figure

1 shows the probability (Y-axis) of generating a point in 256-bit ECDSA curve with y -smooth order n' (X-axis). If considering practicality the 2^{128} -smooth n' (with probability 25% in Figure 1) is required by existing weak curve attacks. Our attack extends the possibility of fault attacks on ECDSA with 256-bit private key. In more detail, considering the case that n' is 2^{247} -smooth (with probability 96.3%), though solving ECDLP in the case is currently unpractical, if collecting enough reduced information about the nonce with respect to a small factor d of n' (which is close to 2^9), our attack is still practical. This is supported by the experiments in Section 4.

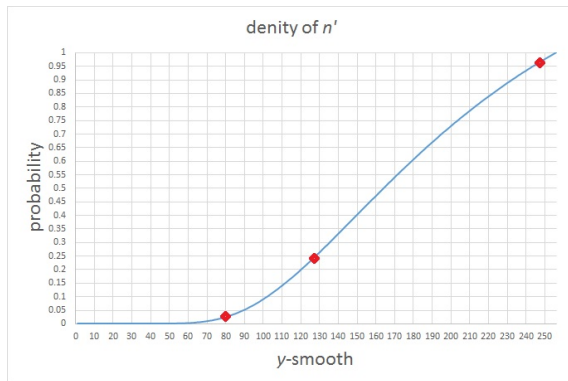


Fig. 1. The density for y -smooth n'

In conclusion, since with high probability the order n' has at least a big prime factor in factorization (determined by PRAC_COMP), the existing weak curve attacks (which require n' to be practically solvable smooth) may not be efficient in practice. In comparison, our attack can survive in this case with high probability, since we only need n' to be partially solvable smooth. The price is that our new constructed lattice model needs more faulty signatures (in number N), which is much greater than the one in [14]. But this is affordable in practice, since we can usually ask the target to generate enough faulty signatures once the fault has been induced, especially when it is permanent.

4 Experimental analysis

In this section, we do the experiments to validate the applicability of the proposed attack. The emphasis is on checking Algorithm ALG-OBTAIN-NONCEINFO and the lattice-based key recovery algorithm. Therefore, the fault injection process is not conducted in experiments. For more information on engineering aspects of fault injection, we refer to the reference [13]. The experiments were conducted on a computer with 3.4GHz 8-core CPU, 8G memory and Windows7

OS. The order n' (derived by the faulty a') is calculated by SEA algorithm implemented in Miracl library [16], and the constructed CVP instances are solved by BKZ algorithm [6] implemented in NTL library [27] with block size 10.

Two types of 256-bit curves over prime field \mathbb{F}_p are targeted in experiments, which are NIST P-256 [5] (hereafter called P-256) and the curve recommended in SM2 digital signature algorithm (hereafter called SM2-curve which still can be employed as the curve of ECDSA) [12] respectively. Then for each curve, two types of bit perturbation experiments are simulated, including the single-bit flipped fault and 16-bit random fault. The single-bit flipped fault is to flip a bit-by-bit. Then there are 256 cases in total. The 16-bit random fault is to randomly pick a starting bit location, characterized by a random integer $\sigma \in [0, 240]$, and then XOR the continuous 16-bit segment (identified by σ) of parameter a with a 16-bit random number $\beta \in \{0, 1\}^{16}$, such that $a' = a \oplus (\beta 2^\sigma)$. The experiments are also done for 256 times. As a whole, four types of different experiments are conducted, and each is done 256 times.

We then use the obtained a' to compute n' for each experiment. Figure 2 shows the proportion γ (Y-axis) of partially solvable smooth n' when its factors d satisfies $l_d \geq X$ (X-axis), and Figure 3 shows the proportion γ (Y-axis) of the unsolvable smooth n' available for our attack when d satisfies $l_d \geq X$ (X-axis) in each type of experiment. (Note each experiment type includes 256 concrete experiments.) From Figure 2, we can find that the proportion of partially solvable smooth orders accounts for 94.9% at least when $l_d \geq 8$, which is far greater than the proportion (35%) of solvable smooth orders required in the previous weak curve attacks. Moreover, as shown in Figure 3, even under the case that n' is unsolvable smooth (accounting for about 65% proportion in each 256 experiments), there still are 92.5% unsolvable smooth n' available for our attack. Obviously, most of the weak curves derived from 256 fault simulations can be applied to our attack, which increases the success rate of fault injection sharply. In addition, from Figures 2 and 3, no matter whether the curve is P-256 or SM2-curve, and the fault type is single-bit flipped fault or 16-bit random fault, all the proportions of the four types of experiments are roughly similar with the density mentioned in Section 3.1. Hence, our attack is effective for most of ECC signatures based on prime field.

Finally, based on a faulty a' in SM2-curve, select d with different bit length, and carry on the corresponding lattice attacks. As shown in Table 1, N is the number of signatures required to achieve 100% success rate of lattice attack, T is the required time of each lattice attack in seconds and O is the maximum time complexity of the attack including the enumerating of c_i (see Section 3.3). For example, when $l_d = 16$, the require time for each lattice attack is about 0.255 seconds, the required number N is 19 and hence the total maximum time require in the whole attack is about 0.255×2^{19} seconds. From Table 1, even under the worst case $l_d = 8$, the complexity of enumerating the correct c_i in lattice attack is computationally feasible since $2^{45} < \text{PRAC_COMP}$. In addition, to speed up enumerating, the case $l_d \geq 16$ with time complexity $2^{19}T$ is generally selected in experiments, whose proportion of partially solvable smooth n' is also up to 92.6%

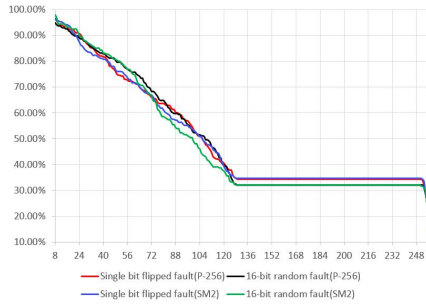


Fig. 2. The proportion γ of partially solvable smooth n' when $l_d > X$

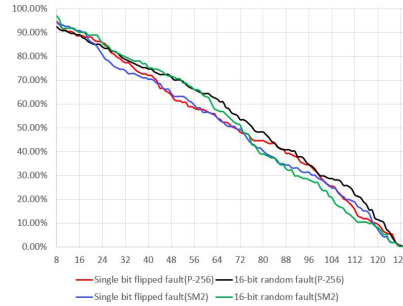


Fig. 3. The proportion γ of unsolvable smooth n' available for our attack when $l_d > X$

at least (see Figure 2). The results show that the success rate of fault injection is significantly high when ensuring the successful lattice attacks. To sum up, our attack is feasible and effective towards most of ECDSA-like algorithms based on Weierstrass equations of prime finite fields, and increases the success rate of fault injection dramatically.

Table 1. The number of faulty signatures and complexity for lattice attack

Items	$l_d = 8$	$l_d = 9$	$l_d = 16$	$l_d = 32$	$l_d = 64$
N	45	40	19	9	5
$T(s)$	≈ 5.788	≈ 3.675	≈ 0.255	≈ 0.021	≈ 0.005
O	$2^{45}T$	$2^{40}T$	$2^{19}T$	2^9T	2^5T

5 Conclusion

We propose a new lattice-based weak curve fault attack on ECDSA which combines the advantages of weak curve fault attack and lattice attack. The order n' of the weak curve generated by faulty a' is not required to be solvable smooth, and the reduced information of nonces is obtained by solving the ECDLP constructed in a small subgroup, by which a new model of lattice attack is constructed to recover the private key. The experiments showed the success rate of fault injection that there exists a solvable smooth factor d of n' satisfying $l_d \geq 8$ can be as high as 94.9%. In addition, for ECDSA with w -bit scalar masking, our attack still works with high success rate of fault injection by selecting an appropriate d satisfying $l_d - w > \log \sqrt{2\pi e} - 1$.

Acknowledgments.

This work is supported by the National Key Research and Development Program of China (No.U1936209), the National Cryptography Development Fund of China (No.MMJJ20170214, MMJJ20170211) and the National Natural Science Foundation of China (No.61802439).

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