High-Dimensional Private Empirical Risk Minimization by Greedy Coordinate Descent

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Abstract

In this paper, we study differentially private empirical risk minimization (DP-ERM). It has been shown that the worst-case utility of DP-ERM reduces polynomially as the dimension increases. This is a major obstacle to privately learning large machine learning models. In high dimension, it is common for some model's parameters to carry more information than others. To exploit this, we propose a differentially private greedy coordinate descent (DP-GCD) algorithm. At each iteration, DP-GCD privately performs a coordinate-wise gradient step along the gradients' (approximately) greatest entry. We show theoretically that DP-GCD can achieve a logarithmic dependence on the dimension for a wide range of problems by naturally exploiting their structural properties (such as quasi-sparse solutions). We illustrate this behavior numerically, both on synthetic and real datasets.

1 INTRODUCTION

Machine Learning (ML) crucially relies on data, which can be sensitive or confidential. Unfortunately, trained models are prone to leaking information about specific training points (Shokri et al., 2017). A standard approach for training models while provably controlling the amount of leakage is to solve an empirical risk minimization (ERM) problem under differential privacy (DP) constraints (Chaudhuri et al., 2011). In this work, we consider the generic problem formulation:

$$w^* \in \operatorname*{arg\,min}_{w \in \mathbb{R}^p} \left\{ f(w) = \frac{1}{n} \sum_{i=1}^{n} \ell(w; d_i) \right\} \ , \qquad (1)$$

where $D = (d_1, \dots, d_n)$ is a dataset of n samples drawn from a universe $\mathcal{X}$, and $\ell(\cdot, d) : \mathbb{R}^p \to \mathbb{R}$ is a loss function which is convex and smooth for all $d \in D$.

The DP constraint in DP-ERM induces a trade-off between the precision of the solution (utility) and privacy. Bassily et al. (2014) proved lower bounds on utility under a fixed DP budget. These lower bounds scale polynomially with the dimension p. Since machine learning models are often high-dimensional (*e.g.,* $n \approx p$ or even $n \ll p$), this is a massive drawback for the use of DP-ERM.

To go beyond this negative result, one has to leverage the fact that high-dimensional problems often exhibit some *structure*. In particular, some parameters are typically more significant than others: it is notably (but not only) the case when models are sparse, which is often desired in high dimension (Tibshirani, 1996). Private learning algorithms could thus be designed to exploit this by focusing on the most significant parameters of the problem. Several works have tried to exploit such high-dimensional problems' structure to reduce the dependence on the dimension, *e.g.,* from polynomial to logarithmic. Talwar et al. (2015), Bassily et al. (2021), and Asi et al. (2021) proposed a DP Frank-Wolfe algorithm (DP-FW) that exploits the solution's sparsity. However, their algorithm only works on ℓ_1-constrained DP-ERM, restricting its range of application. For sparse linear regression, Kifer et al. (2012) proposed to first identify some support and then solve the DP-ERM problem on the restricted support. Unfortunately, their approach requires implicit knowledge of the solution's sparsity. Finally, Kairouz et al. (2021) and Zhou et al. (2021) used public data to estimate lower-dimensional subspaces, where the gradient can be computed at a reduced privacy cost. A key limitation is that such public data set, from the same domain as the private data, is typically not available in many learning scenarios involving sensitive data.

Proceedings of the 26th International Conference on Artificial Intelligence and Statistics (AISTATS) 2023, Valencia, Spain. PMLR: Volume 206.

In this work, we propose a private algorithm that does not have these pitfalls: the differentially private greedy coordinate descent algorithm (DP-GCD). At each iteration, DP-GCD privately determines the gradient's greatest coordinate, and performs a gradient step in its direction. It focuses on the most useful parameters, avoiding wasting privacy budget on updating non-significant ones. Our algorithm works on any smooth, unconstrained DP-ERM problem. We also propose a proximal version to tackle non-smooth regularizers. Crucially, DP-GCD is adaptive to the sparsity of the solution, and is able to ignore small (but non-zero) parameters, improving utility even on non-sparse problems.

Formally, we show that DP-GCD reduces the dependence on the dimension from $\sqrt{p}$ or p to $\log(p)$ for a wide range of unconstrained problems. This is the first algorithm to obtain such gains without relying on ℓ_1 or ℓ_0 constraints. In fact, DP-GCD's utility naturally depends on ℓ_1-norm quantities (*i.e.,* distance from initialization to optimal or strong-convexity parameter) and spans two different regimes. When these ℓ_1-norm quantities are $O(1)$ as assumed in DP-FW, DP-GCD attains $O(\log(p)/n^{2/3}\epsilon^{2/3})$ and $O(\log(p)/n^2\epsilon^2)$ utility on convex and strongly-convex problems respectively, outperforming existing DP-FW algorithms without solving a constrained problem. In the second regime, when the ℓ_2-norm counterpart of the above quantities are $O(1)$ as assumed for DP-SGD and its variants, we show that DP-GCD adapts to the problem's underlying structure. Specifically, it is able to *interpolate between logarithmic and polynomial dependence on the dimension*. In addition to these general utility results, we prove that for strongly convex problems with quasi-sparse solutions (including but not limited to sparse problems), DP-GCD converges to a good approximate solution in few iterations. This improves utility in the ℓ_2-norm setting, replacing the polynomial dependence on the ambient space's dimension by the quasi-sparsity level of the solution. We evaluate both our algorithms numerically on real and synthetic datasets, validating our theoretical observations.

Our contributions can be summarized as follows:

1. We propose differentially private greedy coordinate descent (DP-GCD), a method that performs updates along the (approximately) greatest entry of the gradient. We formally establish its privacy guarantees, and derive high probability utility upper bounds.

2. We prove that DP-GCD exploits structural properties of the problem (*e.g.,* quasi-sparse solutions) to improve utility. Importantly, DP-GCD does not require prior knowledge of this structure to exploit it.

3. We empirically validate our theoretical results on a variety of synthetic and real datasets, showing that DP-GCD outperforms existing private algorithms on high-dimensional problems with quasi-sparse solutions.

The rest of the paper is organized as follows. First, we discuss related work in more details in Section 2, and present the relevant mathematical background in Section 3. Section 4 then introduces DP-GCD, and formally analyzes its privacy and utility. We validate our theoretical results numerically in Section 5. Finally, we conclude and discuss the limitations of our results in Section 6.

2 RELATED WORK

DP-ML in Euclidean geometry. Most of the work on differentially private empirical risk minimization (DP-ERM) and differentially private stochastic convex optimization (DP-SCO)[1] consider problem quantities (*e.g.,* bounds on the domain and regularity assumptions) expressed in ℓ_2 norm. In this Euclidean setting, Bassily et al. (2014) analyzed the theoretical properties of DP-SGD for DP-ERM, and derived matching utility lower bounds. Faster algorithms based on SVRG (Johnson and Zhang, 2013; Xiao and Zhang, 2014) were designed by Wang et al. (2017). Wu et al. (2017) studied a variant of DP-SGD with output perturbation, that is efficient when only few passes on the data are possible. For DP-SCO, Bassily et al. (2019) used algorithmic stability arguments (following work from Hardt et al., 2016; Bassily et al., 2020) to show that in some regimes, the population risk is the same as in non-private SCO. Feldman et al. (2020) and Wang et al. (2022) then developed efficient (linear-time) algorithm to solve this problem. In all of the above work, *the utility upper bounds scale polynomially in p*, which is not suitable in high dimension. In contrast, our approach provably achieves logarithmic dependence on the dimension for some problems.

DP-ML in high dimension. Several approaches have been explored to reduce the dependence on the dimension. One option is to consider ℓ_1-constrained problems. For DP-ERM, Talwar et al. (2015) and Talwar et al. (2016) used a differentially private Frank-Wolfe algorithm (DP-FW) (Frank and Wolfe, 1956; Jaggi, 2013) to achieve utility that scales logarithmically with the dimension. Asi et al. (2021) and Bassily et al. (2021) proposed stochastic DP-FW algorithms, extending the above results to DP-SCO. For more general domains (*e.g.,* polytopes), Kasiviswanathan and Jin (2016) randomly project the data on a smaller-dimensional space, and lift the result back onto the original space. The dependence in the dimension is encoded by the Gaussian width of the domain, again leading to $O(\log p)$ error for the ℓ_1 ball or the simplex. Wang et al. (2017) derived a faster mirror descent algorithm for DP-ERM whose utility also depends on the Gaussian width of the domain. Our approach matches the $O(\log p)$ dependence of the above methods when key quantities are bounded in ℓ_1 norm, but can also achieve such gains for more general problems,

[1] See (Dwork et al., 2015; Bassily et al., 2016; Jung et al., 2021) for techniques to convert DP-ERM results to DP-SCO.

e.g., when the problem has a quasi-sparse solution. Kifer et al. (2012) previously leveraged the solution sparsity for the specific problem of sparse linear regression: they first identify some support, and then solve DP-ERM on this restricted support. Similarly, Wang and Gu (2019) and Hu et al. (2022) proposed hard thresholding-based algorithms for DP-ERM and DP-SCO under sparsity (ℓ_0 norm) constraints. Both approaches achieve an error of $O(\log p)$ but rely either on prior knowledge on the solution's sparsity, or on the tuning of an additional hyperparameter. In contrast, our approach automatically adapts to the sparsity and works also when solutions are only quasi-sparse. Finally, Kairouz et al. (2021) and Zhou et al. (2021) estimate lower-dimensional gradient subspaces using public data. This reduces noise addition, but in practice, public data is only rarely available.

Coordinate descent. CD algorithms have a long history in optimization (see Wright, 2015; Shi et al., 2017, for detailed reviews on CD). Most approaches have focused on randomized or cyclic choices of coordinates (Tseng, 2001; Nesterov, 2012), with proximal and parallel variants (Richtárik and Takáč, 2014; Fercoq and Richtárik, 2014; Hanzely et al., 2018), sometimes applied to the dual problem (Shalev-Shwartz and Zhang, 2013). In this work, our focus is on greedy coordinate descent methods, which update the coordinate with greatest gradient entry (Luo and Tseng, 1992; Tseng and Yun, 2009; Dhillon et al., 2011). Nutini et al. (2015) showed improved convergence rates for smooth, strongly-convex functions, by measuring strong convexity in the ℓ_1-norm. Our work builds upon these results to design and analyze the first *private* greedy CD approach. Although techniques such as fast nearest-neighbor schemes have been proposed to compute the (approximate) greedy update more efficiently (Dhillon et al., 2011; Nutini et al., 2015; Karimireddy et al., 2019), greedy CD methods are often slower (in wall-clock time) than their randomized or cyclic counterparts (Massias et al., 2017). However, in the private setting we consider, the main focus is not on computing time but on achieving the best privacy-utility trade-off. This gives a distinct advantage to greedy CD, as it provides a way to perform the (approximately) most useful coordinate update under a given privacy budget instead of wasting budget on updating random (potentially useless) coordinates. The analysis of proximal extensions of greedy CD for composite problems with non-smooth parts is known to be challenging even in the non-private setting. Karimireddy et al. (2019) proved convergence rates only for ℓ_1- and box-regularized problems, using a modified greedy CD algorithm. In this work, we propose and empirically evaluate a proximal extension of our DP-GCD algorithm with formal privacy guarantees, but leave its utility analysis for future work; see the discussion in Section 6.

Private coordinate descent. Differentially Private Coordinate Descent (DP-CD) was recently studied by Mangold et al. (2022), who analyzed its utility and derived corresponding lower bounds. They showed that DP-CD can exploit coordinate-wise regularity assumptions to use larger step-sizes, outperforming DP-SGD when gradient coordinates are imbalanced. Our DP-GCD also shares this property. Damaskinos et al. (2021) proposed a dual coordinate descent algorithm for generalized linear models. Private CD has also been used by Bellet et al. (2018) in a decentralized setting. All these works use random selection, which fails to exploit key problem's properties such as quasi-sparsity, and thus suffer a polynomial dependence on the dimension p. In contrast, our private greedy selection rule focuses on the most useful coordinates, thereby reducing the dependence on p to only logarithmic in such settings.

3 PRELIMINARIES

In this section, we introduce important technical notions that will be used throughout the paper.

Norms. We start by defining two conjugate norms that will allow to keep track of coordinate-wise quantities. Let $M = \mathrm{diag}(M_1, \dots, M_p)$ with $M_1, \dots, M_p > 0$, and

$$\|w\|_{M,1} = \sum_{j=1}^{p} M_j^{\frac{1}{2}} |w_j| \ , \ \|w\|_{M^{-1},\infty} = \max_{j \in [p]} M_j^{-\frac{1}{2}} |w_j| \ .$$

When M is the identity matrix I, $\|\cdot\|_{M,1}$ is the standard ℓ_1-norm and $\|\cdot\|_{M^{-1},\infty}$ is the ℓ_∞-norm. We also define the Euclidean dot product $\langle u, v \rangle = \sum_{j=1}^{p} u_i v_i$ and corresponding norms $\|\cdot\|_{M,2} = \langle \cdot, M \cdot \rangle^{\frac{1}{2}}$ and $\|\cdot\|_{M^{-1},2} = \langle \cdot, M^{-1} \cdot \rangle^{\frac{1}{2}}$. Similarly, we recover the standard ℓ_2-norm when $M = I$.

Regularity assumptions. We recall classical regularity assumptions along with ones specific to the coordinate-wise setting. We denote by ∇f the gradient of a differentiable function f, and by $\nabla_j f$ its j-th coordinate. We denote by e_j the j-th vector of $\mathbb{R}^p$'s standard basis.

(Strong)-convexity. For $q \in \{1, 2\}$, a differentiable function $f : \mathbb{R}^p \to \mathbb{R}$ is $\mu_{M,q}$-strongly-convex w.r.t. the norm $\|\cdot\|_{M,q}$ if for all $v, w \in \mathbb{R}^p$, $f(w) \geq f(v) + \langle \nabla f(v), w - v \rangle + \frac{\mu_{M,q}}{2} \|w - v\|_{M,q}^2$. The case $M_{1,q} = \cdots = M_{p,q} = 1$ recovers standard $\mu_{I,q}$-strong convexity w.r.t. the ℓ_q-norm. When $\mu_{M,q} = 0$, the function is just said to be *convex*.

Component Lipschitzness. A function $f : \mathbb{R}^p \to \mathbb{R}$ is L-component-Lipschitz for $L = (L_1, \dots, L_p)$ with $L_1, \dots, L_p > 0$ if for $w \in \mathbb{R}^p$, $t \in \mathbb{R}$ and $j \in [p]$, $|f(w + te_j) - f(w)| \leq L_j |t|$. For $q \in \{1, 2\}$, f is Λ_q-Lipschitz w.r.t. $\|\cdot\|_q$ if for $v, w \in \mathbb{R}^p$, $|f(v) - f(w)| \leq \Lambda_q \|v - w\|_q$.

Component smoothness. A differentiable function $f : \mathbb{R}^p \to \mathbb{R}$ is M-component-smooth for $M_1, \dots, M_p > 0$ if for $v, w \in \mathbb{R}^p$, $f(w) \leq f(v) + \langle \nabla f(v), w - v \rangle + \frac{1}{2}\|w - v\|^2_{M,2}$. When $M_1 = \cdots = M_p = \beta$, f is said to be β-smooth.

Component-wise regularity assumptions are not restrictive: for $q \in \{1, 2\}$, Λ_q-Lipschitzness w.r.t. $\|\cdot\|_q$ implies $(\Lambda_q, \dots, \Lambda_q)$-component-Lipschitzness and β-smoothness implies $(\beta, \dots, \beta)$-component-smoothness. Yet, the actual component-wise constants of a function can be much lower than what can be deduced from their global counterparts. In the following of this paper, we will use $M_{\min} = \min_{j \in [p]} M_j$, $M_{\max} = \max_{j \in [p]} M_j$, and their Lipschitz counterparts $L_{\min}$ and $L_{\max}$.

Differential privacy (DP). Let $\mathcal{D}$ be a set of datasets and $\mathcal{F}$ a set of possible outcomes. Two datasets $D, D' \in \mathcal{D}$ are said *neighboring* (denoted by $D \sim D'$) if they differ on at most one element.

Definition 3.1 (Differential Privacy, Dwork 2006)**.** A randomized algorithm $\mathcal{A} : \mathcal{D} \to \mathcal{F}$ is (ϵ, δ)-differentially private if, for all neighboring datasets $D, D' \in \mathcal{D}$ and all $S \subseteq \mathcal{F}$ in the range of $\mathcal{A}$:

$$\Pr[\mathcal{A}(D) \in S] \leq e^{\epsilon} \cdot \Pr[\mathcal{A}(D') \in S] + \delta \ .$$

In this paper, we consider the classic central model of DP, where a trusted curator has access to the raw dataset and releases a model trained on this dataset[2].

A common principle for releasing a private estimate of a function $h : \mathcal{D} \to \mathbb{R}^p$ is to perturb it with noise. To ensure privacy, the noise is scaled with the sensitivity $\Delta_q(h) = \sup_{D \sim D'} \|h(D) - h(D')\|_q$ of h, with $q = 1$ for Laplace, and $q = 2$ for Gaussian mechanism. In coordinate descent methods, we release coordinate-wise gradients. The j-th coordinate of a loss function's gradient $\nabla_j \ell : \mathbb{R}^p \to \mathbb{R}$ has sensitivity $\Delta_1(\nabla_j f) = \Delta_2(\nabla_j f)$ ($\nabla_j f$ is a scalar). For L-component-Lipschitz losses, these sensitivities are upper bounded by $2L_j$ (Mangold et al., 2022).

In our algorithm, we will also need to compute the index of the gradient's maximal entry privately. To this end, we use the report-noisy-argmax mechanism (Dwork and Roth, 2013). This mechanism perturbs each entry of a vector with Laplace noise, calibrated to its *coordinate-wise* sensitivities, and releases the index of a maximal entry of this noisy vector. Revealing only this index allows to greatly reduce the noise, in comparison to releasing the full gradient. This will be the cornerstone of our greedy algorithm.

[2]In fact, our privacy guarantees hold even if all intermediate iterates are released (not just the final model).

4 PRIVATE GREEDY CD

In this section, we present our main contribution: the differentially private greedy coordinate descent algorithm (DP-GCD). As described in Section 4.1, DP-GCD updates only one coordinate per iteration, which is selected greedily as the (approximately) largest entry of the gradient so as to maximize the improvement in utility at each iteration. We establish privacy (Section 4.2) and utility (Section 4.3) guarantees for DP-GCD. We further show in Section 4.4 that DP-GCD enjoys improved utility for high-dimensional problems with a *quasi-sparse* solution (*i.e.,* with a fraction of the parameters dominating the others). We then provide a proximal extension of DP-GCD to non-smooth problems (Section 4.5) and conclude with a discussion of DP-GCD's computational complexity in Section 4.6.

4.1 The Algorithm

At each iteration, DP-GCD (Algorithm 1) updates the parameter with the greatest gradient value (rescaled by the inverse square root of the coordinate-wise smoothness constant). This corresponds to the Gauss-Southwell-Lipschitz rule (Nutini et al., 2015). To guarantee privacy, this selection is done using the report-noisy-max mechanism (Dwork and Roth, 2013) with noise scales λ'_j along j-th entry ($j \in [p]$). DP-GCD then performs a gradient step with step size $\gamma_j > 0$ along this direction. The gradient is privatized using the Laplace mechanism (Dwork and Roth, 2013) with scale λ_j.

Remark 4.1 (Sparsity of iterates)**.** When initialized at $w^0 = 0$, DP-GCD generates sparse iterates. Since it chooses its updates greedily, this gives a screening ability to the algorithm (Fang et al., 2020). We discuss the implications of this property in Section 4.4, where we show that DP-GCD's utility is improved when the problem's solution is (quasi-)sparse.

4.2 Privacy Guarantees

The privacy guarantees of DP-GCD depends on the noise scales λ_j and λ'_j. In Theorem 4.2, we describe how to set these values so as to ensure that DP-GCD is (ϵ, δ)-differentially private.

Theorem 4.2. *Let $\epsilon, \delta \in (0, 1]$. Algorithm 1 with $\lambda_j = \lambda'_j = \frac{8L_j}{n\epsilon}\sqrt{T \log(1/\delta)}$ is (ϵ, δ)-DP.*

Sketch of Proof. (Detailed proof in Appendix A) Let $\epsilon' = \epsilon / \sqrt{16T \log(1/\delta)}$. At an iteration t, data is accessed twice. First, to compute the index j_t of the coordinate to update. It is obtained as the index of the largest noisy entry of f's gradient, with noise $\mathrm{Lap}(\lambda'_j)$. By the report-noisy-argmax mechanism, j_t is ϵ'-DP. Second, to compute the gradient's j_t's entry, which is released with noise $\mathrm{Lap}(\lambda_j)$.The

Algorithm 1 DP-GCD: Differentially Private Greedy Coordinate Descent

Input: initial $w^0 \in \mathbb{R}^p$, iteration count $T > 0, \forall j \in [p]$, noise scales λ_j, λ'_j, step sizes $\gamma_j > 0$.
for $t = 0$ to $T-1$ **do**
  $j_t = \arg\max_{j' \in [p]} \frac{|\nabla_{j'} f(w^t) + \chi^t_{j'}|}{\sqrt{M_{j'}}}$, with $\chi^t_{j'} \sim \mathrm{Lap}(\lambda'_{j'})$. ▷ Choose j_t using report-noisy-max.
  $w^{t+1} = w^t - \gamma_{j_t}(\nabla_{j_t} f(w^t) + \eta^t_{j_t}) e_{j_t}$, with $\eta^t_{j_t} \sim \mathrm{Lap}(\lambda_{j_t})$. ▷ Update the chosen coordinate.
return w^T.

Laplace mechanism ensures that this computation is also ϵ'-DP. Algorithm 1 is thus the $2T$-fold composition of ϵ'-DP mechanisms, and the result follows from DP's advanced composition theorem (Dwork and Roth, 2013). □

Remark 4.3. The assumption $\epsilon \in (0, 1]$ is only used to give a closed-form expression for the noise scales λ, λ's. In practice, we tune them numerically to obtain the desired value of $\epsilon > 0$ by the advanced composition theorem (see eq. (2) in Appendix A), removing the assumption $\epsilon \leq 1$.

Computing the greedy update requires injecting Laplace noise that scales with the coordinate-wise Lipschitz constants $L_1, \dots, L_p$ of the loss. These constants are typically smaller than their global counterpart. This allows DP-GCD to inject less noise on smaller-scaled coordinates.

4.3 Utility Guarantees

We now prove utility upper bounds for DP-GCD. We show that in favorable settings (see discussion below), DP-GCD decreases the dependence on the dimension from polynomial to logarithmic. Theorem 4.4 gives asymptotic utility upper bounds, where $\widetilde{O}$ ignores non-significant logarithmic terms. Complete non-asymptotic results can be found in Appendix B.

Theorem 4.4. *Let $\epsilon, \delta \in (0, 1]$. Assume $\ell(\cdot; d)$ is a convex and L-component-Lipschitz loss function for all $d \in \mathcal{X}$, and f is M-component-smooth. Define $\mathcal{W}^*$ the set of minimizers of f, and f^* the minimum of f. Let $w_{priv} \in \mathbb{R}^p$ be the output of Algorithm 1 with step sizes $\gamma_j = 1/M_j$, and noise scales $\lambda_1, \dots, \lambda_p$, $\lambda'_1, \dots, \lambda'_p$ set as in Theorem 4.2 (with T chosen below) to ensure (ϵ, δ)-DP. Then, the following holds for any $\zeta \in (0, 1]$:*

1. *When f is convex, we define the quantity $R_{M,1} = \max_{w \in \mathbb{R}^p} \max_{w^* \in \mathcal{W}^*} \{\|w - w^*\|_{M,1} \,|\, f(w) \leq f(w^0)\}$. Assume the initial optimality gap is $f(w^0) - f^* \geq 16 L_{\max} \sqrt{T \log(1/\delta) \log(2Tp/\zeta)} / M_{\min} n \epsilon$, and set $T = O(n^{2/3} \epsilon^{2/3} R_{M,1}^{2/3} M_{\min}^{1/3} / L_{\max}^{2/3} \log(1/\delta)^{1/3})$. Then with probability at least $1 - \zeta$,*

$$f(w_{priv}) - f^* = \widetilde{O}\left(\frac{R_{M,1}^{4/3} L_{\max}^{2/3} \log(1/\delta) \log(p/\zeta)}{n^{2/3} \epsilon^{2/3} M_{\min}^{1/3}} \right) .$$

2. *When f is $\mu_{M,1}$-strongly convex w.r.t. $\|\cdot\|_{M,1}$, set $T = O\left(\frac{1}{\mu_{M,1}} \log\left(\frac{M_{\min} \mu_{M,1} n \epsilon (f(w^0) - f(w^*))}{L_{\max} \log(1/\delta) \log(2p/\zeta)} \right) \right)$. Then with probability at least $1 - \zeta$,*

$$f(w_{priv}) - f^* = \widetilde{O}\left(\frac{L_{\max}^2 \log(1/\delta) \log(2p/\mu_M \zeta)}{M_{\min} \mu_{M,1}^2 n^2 \epsilon^2} \right) .$$

Sketch of Proof. (Detailed proof in Appendix B). We start by proving a noisy "descent lemma". Since f is smooth, we have $f(w^{t+1}) \leq f(w^t) - \frac{1}{2M_j} \nabla_j f(w^t)^2 + \frac{1}{2M_j} (\eta^t_j)^2$. The greedy selection of j gives that $-\frac{1}{M_j} (\nabla_j f(w^t) + \chi_j)^2 \leq -\|\nabla f(w^t) + \chi\|^2_{M^{-1},\infty}$. We then use the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$, and convexity arguments to prove the lemma. When f is convex, we have

$$f(w^{t+1}) - f(w^*) \leq f(w^t) - f(w^*) - \frac{(f(w^t) - f(w^*))^2}{8 \|w^t - w^*\|^2_{M,1}} + \frac{|\eta^t_j|^2}{2M_j} + \frac{|\chi^t_j|^2}{2M_j} + \frac{|\chi^t_{j^*}|^2}{4M_{j^*}} .$$

There, we observe that, at each iteration, either (i) w^t is far enough from the optimum, and the value of the objective decreases with high probability, either (ii) w^t is close to the optimum, then all future iterates remain in a ball whose radius depends on the scale of the noise. We prove this key property rigorously in Appendix B.3.2.

When f is $\mu_{M,1}$-strongly-convex w.r.t. $\|\cdot\|_{M,1}$, we obtain

$$f(w^{t+1}) - f(w^*) \leq \left(1 - \frac{\mu_{M,1}}{4}\right)(f(w^t) - f(w^*)) + \frac{|\eta^t_j|^2}{2M_j} + \frac{|\chi^t_j|^2}{2M_j} + \frac{|\chi^t_{j^*}|^2}{4M_{j^*}} ,$$

and the result follows by induction. In both settings, we use Chernoff bounds to obtain a high-probability result. □

Remark 4.5. The lower bound on $f(w^0) - f^*$ in Theorem 4.4 is a standard assumption in the analysis of inexact coordinate descent methods: it ensures that sufficient decrease is possible despite the noise. A similar assumption is made by Tappenden et al. (2016), see Theorem 5.1 therein.

Discussion of the utility bounds. One of the key properties of DP-GCD is that its utility is dictated by ℓ_1-norm quantities ($R_{M,1}$ and $\mu_{M,1}$). Remarkably, this arises without enforcing any ℓ_1 constraint in the problem, which is in stark contrast with private Frank-Wolfe algorithms (DP-FW) that require such constraints (Talwar et al., 2015; Asi

et al., 2021; Bassily et al., 2021). To better grasp the implications of this, we discuss our results in two regimes considered in previous work (see Section 2): (i) when these ℓ_1-norm quantities are bounded (similarly to DP-FW algorithms), and (ii) when their ℓ_2-norm counterparts are bounded (similarly to DP-SGD-style algorithms).

Bounded in ℓ_1-norm. When $R_{M,1}$ and $\mu_{M,1}$ are $O(1)$, as assumed in prior work on DP-FW (Talwar et al., 2015; Asi et al., 2021; Bassily et al., 2021), DP-GCD's dependence on the dimension is *logarithmic*. For convex objectives, its utility is $O(\log(p)/n^{2/3}\epsilon^{2/3})$, matching that of DP-FW and known lower bounds (Talwar et al., 2015). For strongly-convex problems, DP-GCD is the first algorithm to achieve a $O(\log(p)/n^2\epsilon^2)$ utility. Indeed, the only competing result in this setting, due to Asi et al. (2021), obtains a worse utility of $O(\log(p)^{4/3}/n^{4/3}\epsilon^{4/3})$ by using an impractical reduction of DP-FW to the convex case. DP-GCD outperforms this prior result without suffering the extra complexity due to the reduction.

Bounded in ℓ_2-norm. Consider $R_{M,2}$ and $\mu_{M,2}$, the ℓ_2-norm counterparts of $R_{M,1}$ and $\mu_{M,1}$. Assume that $R_{M,2}$ and $\mu_{M,2}$ are both $O(1)$, as considered in DP-SGD and its variants (Bassily et al., 2014; Wang et al., 2017). We compare these quantities using the following inequalities (see Stich et al., 2017; Nutini et al., 2015):

$$R_{M,2}^2 \leq R_{M,1}^2 \leq pR_{M,2}^2 \ , \quad \tfrac{1}{p}\mu_{M,2} \leq \mu_{M,1} \leq \mu_{M,2} \ .$$

In the best case of these inequalities, the $O(\log p)$ utility bounds of the bounded ℓ_1 norm regime are preserved in the bounded ℓ_2 scenario. In the worst case, the utility of DP-GCD becomes $\widetilde{O}(p^{2/3}/n^{2/3}\epsilon^{2/3})$ and $\widetilde{O}(p^2/n^2\epsilon^2)$ for convex and strongly-convex objectives respectively. These worst-case results match DP-FW's utility in the convex setting (see *e.g.,* Asi et al. (2021)), but they do not match DP-SGD's utility. However, this sheds light on an interesting phenomenon: DP-GCD *interpolates between ℓ_1- and ℓ_2-norm regimes*. Indeed, it lies somewhere between the two extreme cases we just described, depending on how the ℓ_1- and ℓ_2-norm constants compare. Most interestingly, it does so without *a priori* knowledge of the problem or explicit constraint on the domain. Whether there exists an algorithm that yields optimal utility in all regimes is an interesting open question.

Coordinate-wise regularity. Due to its use of coordinate-wise step sizes, DP-GCD can adapt to coordinate-wise imbalance of the objective in the same way as its randomized counterpart, DP-CD, where coordinates are chosen uniformly at random (Mangold et al., 2022). This adaptivity notably appears in Theorem 4.4 through the measurement of $R_{M,1}$ and $\mu_{M,1}$ relatively to the scaled norm $\|\cdot\|_{M,1}$ (as defined in Section 3). We refer to (Mangold et al., 2022) for detailed discussion of these quantities and the associated gains compared to full gradient methods like DP-SGD.

4.4 Better Utility on Quasi-Sparse Problems

In addition to the general utility results presented above, we now exhibit a specific setting where DP-GCD performs especially well, namely strongly-convex problems whose solutions are dominated by a few parameters. We call such vectors quasi-sparse.

Definition 4.6 ((α,τ)-quasi-sparsity)**.** A vector $w \in \mathbb{R}^p$ is (α,τ)-quasi-sparse if it has at most τ entries superior to α (in modulus). When $\alpha = 0$, the vector is called τ-sparse.

Note that any vector in $\mathbb{R}^p$ is $(0,p)$-quasi-sparse, and for any τ there exists $\alpha > 0$ such that the vector is (α,τ)-quasi-sparse. In fact, α and τ are linked, and $\tau(\alpha)$ can be seen as a function of α. Of course, quasi-sparsity will only yield meaningful improvements when α and τ are small simultaneously.

We now state the main result of this section, which shows that DP-GCD (initialized with $w^0 = 0$) converges to a good approximate solution in few iterations for problems with quasi-sparse solutions.

Theorem 4.7 (Proof in Appendix B.4.3)**.** *Consider f satisfying the hypotheses of Theorem 4.4, with Algorithm 1 initialized at $w^0 = 0$. We denote its output w^T, and assume that its iterates remain s-sparse for some $s \leq p$. Assume that f is $\mu_{M,2}$-strongly-convex w.r.t. $\|\cdot\|_{M,2}$, and that the (unique) solution of problem (1) is (α,τ)-quasi-sparse for some $\alpha,\tau \geq 0$. Let $0 \leq T \leq p-\tau$ and $\zeta \in [0,1]$. Then with probability at least $1-\zeta$:*

$$f(w^T) - f^* \leq \prod_{t=1}^{T}\Big(1 - \frac{\mu_{M,2}}{4(\tau+\min(t,s))}\Big)(f(w^0)-f^*) + \widetilde{O}\Big((T+\tau)(p-\tau)\alpha^2 + \frac{L_{\max}^2 T(T+\tau)}{M_{\min}\mu_{M,2}n^2\epsilon^2}\Big) \ .$$

Setting $T = \frac{s+\tau}{\mu_{M,2}}\log((f(w^0)-f^*)M_{\min}\mu_{M,2}n^2\epsilon^2/L^2)$, and assuming $\alpha^2 = O\left(L_{\max}^2(s+\tau)/M_{\min}\mu_{M,2}^2 pn^2\epsilon^2\right)$, we obtain that with probability at least $1-\zeta$,

$$f(w^T) - f^* = \widetilde{O}\left(\frac{L_{\max}^2}{M_{\min}}\frac{(s+\tau)^2\log(2p/\zeta)}{\mu_{M,2}n^2\epsilon^2}\right) \ .$$

Here, strong convexity is measured in ℓ_2 norm but the dependence on the dimension is reduced from p, the ambient space dimension, to $(s+\tau)^2$, the *effective dimension of the space where the optimization actually takes place*. For high-dimensional sparse problems, the latter is typically much smaller and yields a large improvement in utility. Note that it is not necessary for the solution to be perfectly sparse: it suffices that most of its mass is concentrated in a fraction of the coordinates. Notably, when $\alpha^2 = O(L_{\max}^2 T/M_{\min}\mu_{M,2}pn^2\epsilon^2)$, the lack of sparsity is

smaller than the noise, and does not affect the rate. It generalizes the results by Fang et al. (2020) for non-private and sparse settings, that we recover when $\alpha = 0$ and $\epsilon \to +\infty$.

In practice, the assumption over the iterates' sparsity is often met with $s \ll p$. In the non-private setting, greedy coordinate descent is known to focus on coordinates that are non-zero in the solution (Massias et al., 2017): this keeps iterates' sparsity close to the one of the solution. Furthermore, due to privacy constraints, DP-GCD will often run for $T \ll p$ iterations. This is especially true in high-dimensional problems, where the amount of noise required to guarantee privacy does not allow many iterations (*cf.* experiments in Section 5).

4.5 Proximal DP-GCD

In Section 4.4, we proved that DP-GCD's utility is improved when problem's solution is (quasi-)sparse. This motivates us to consider problems with sparsity-inducing regularization, such as the ℓ_1 norm of w (Tibshirani, 1996). To tackle such non-smooth terms, we propose a proximal version of DP-GCD (for which the same privacy guarantees hold), building upon the multiple greedy rules that have been proposed for the nonsmooth setting (see *e.g.,* Tseng and Yun, 2009; Nutini et al., 2015). We describe this extension in Appendix C, and study it numerically in Section 5.

4.6 Computational Cost

Each iteration of DP-GCD requires computing a full gradient, but only uses one of its coordinates. In non-private optimization, one would generally be better off performing the full update to avoid wasting computation. This is not the case when gradients are private. Indeed, using the full gradient requires privatizing p coordinates, even when only a few of them may be needed. Conversely, the report noisy max mechanism (Dwork and Roth, 2013) allows to select these entries *without paying the full privacy cost of dimension*. Hence, the greedy updates of DP-GCD reduce the noise needed at the cost of more computation.

In practice, the higher computational cost of each iteration may not always translate in a significantly larger cost overall: as shown by our theoretical results, DP-GCD is able to exploit the *quasi-sparsity* of the solution to progress fast and only a handful of iterations may be needed to reach a good private solution. In contrast, most updates of classic private optimization algorithms (like DP-SGD) may not be worth doing, and lead to unnecessary injection of noise. We illustrate this phenomenon numerically in Section 5.

5 EXPERIMENTS

In this section, we evaluate the practical performance of DP-GCD on linear models using the logistic and squared loss with ℓ_1 and ℓ_2 regularization. We compare DP-GCD to two competitors: differentially private stochastic gradient descent (DP-SGD) with batch size 1 (Bassily et al., 2014; Abadi et al., 2016), and differentially private randomized coordinate descent (DP-CD) (Mangold et al., 2022). The code is available online[3] and in the supplementary.

Datasets. The first two datasets, coined `log1` and `log2`, are synthetic. We generate a design matrix $X \in \mathbb{R}^{1,000\times 100}$ with unit-variance, normally-distributed columns. Labels are computed as $y = Xw^{(true)} + \varepsilon$, where ε is normally-distributed noise and $w^{(true)}$ is drawn from a log-normal distribution of parameters $\mu = 0$ and $\sigma = 1$ or 2 respectively. This makes $w^{(true)}$ quasi-sparse. The `square` dataset is generated similarly, with $X \in \mathbb{R}^{1,000\times 1,000}$ and $w^{(true)}$ having only 10 non-zero values. The `california` dataset can be downloaded from `scikit-learn` (Pedregosa et al., 2011) while `mtp`, `madelon` and `dorothea` are available in the `OpenML` repository (Vanschoren et al., 2014); see summary in Table 1. We discuss the levels of (quasi)-sparsity of each problem's solution in Appendix D.

Algorithmic setup. *(Privacy.)* For each algorithm, the tightest noise scales are computed numerically to guarantee a suitable privacy level of $(1, 1/n^2)$-DP, where n is the number of records in the dataset. For DP-CD and DP-SGD, we privatize the gradients with the Gaussian mechanism (Dwork and Roth, 2013), and account for privacy tightly using Rényi differential privacy (RDP) (Mironov, 2017). For DP-SGD, we use RDP amplification for the subsampled Gaussian mechanism (Mironov et al., 2019).

(Hyperparameters.) For DP-SGD, we use constant step sizes and standard gradient clipping (Abadi et al., 2016). For DP-GCD and DP-CD, we set the step sizes to $\eta_j = \frac{\gamma}{M_j}$, and adapt the coordinate-wise clipping thresholds from one hyperparameter, as proposed by Mangold et al. (2022). For each algorithm, we thus tune two hyperparameters: one step-size and one clipping threshold; see also Appendix D.

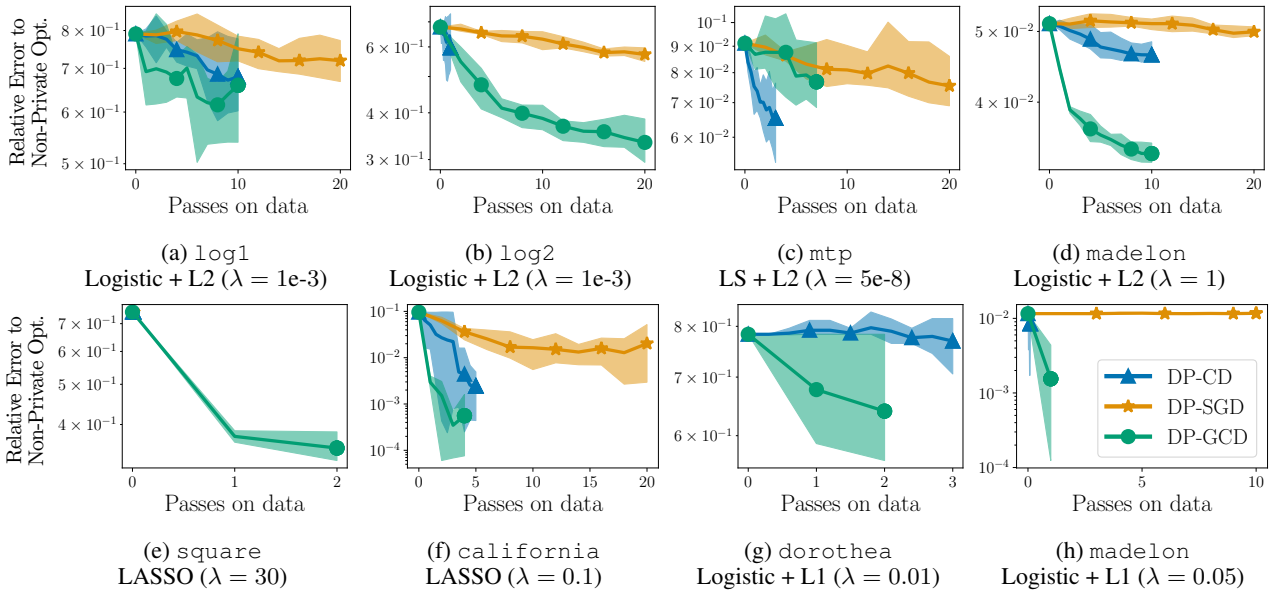
(Plots.) In all experiments, we plot the relative error to the *non-private* optimal objective value for the best set of hyperparameters (averaged over 5 runs), as a function of the number of passes on the data. Each pass corresponds to p iterations of DP-CD, n iterations of DP-SGD and 1 iteration of DP-GCD. This guarantees the same amount of computation for each algorithm, for each x-axis tick.

DP-GCD exploits problem structure. In the higher-dimensional datasets `square` and `dorothea`, where $p \geq n$, DP-GCD is the only algorithm that manages to do multiple iterations and to decrease the objective value (see Figures 1e and 1g). In both problems, solutions are sparse due

[3] https://gitlab.inria.fr/pmangold1/greedy-coordinate-descent

Table 1: Number of records and features in each dataset.

	log1, log2	square	mtp	dorothea	california	madelon
Records	1,000	1,000	4,450	800	20,640	2,600
Features	100	1,000	202	88,119	8	501



(a) log1 Logistic + L2 (λ = 1e-3) (b) log2 Logistic + L2 (λ = 1e-3) (c) mtp LS + L2 (λ = 5e-8) (d) madelon Logistic + L2 (λ = 1)

(e) square LASSO (λ = 30) (f) california LASSO (λ = 0.1) (g) dorothea Logistic + L1 (λ = 0.01) (h) madelon Logistic + L1 (λ = 0.05)

Figure 1: Relative error (min/mean/max over 5 runs) to non-private optimal for DP-GCD (our approach) versus DP-CD and DP-SGD. On the x-axis, 1 tick represents a full access to the data: p iterations of DP-CD, n iterations of DP-SGD and 1 iteration of DP-GCD. Number of iterations, clipping thresholds and step sizes are tuned simultaneously for each algorithm.

to the ℓ_1 regularization. This shows that DP-GCD's greedy selection of updates can exploit this property to find relevant non-zero coefficients (see Table 3 in Appendix D), even when this selection is noisy. The lower-dimensional datasets log1, log2 and madelon (where $p < n$) are still too high dimensional (relatively to n) for DP-SGD and DP-CD to make significant progress. In contrast, DP-GCD exploits the fact that solutions are quasi-sparse to find good approximate solutions quickly (see Figures 1a, 1b, 1d, 1e, 1g and 1h). On the low-dimensional dataset california, DP-GCD is roughly on par with DP-SGD and DP-CD (see Figure 1f). This is due to the additional noise term introduced by the greedy selection rule: in such setting, the lower number of iterations does not compensate for this as much as in higher-dimensional problems. A similar phenomenon arise in mtp (Figure 1c), whose solution is not imbalanced enough for DP-GCD to be superior to its competitors.

Computational complexity. As discussed in Section 4.6, one iteration of DP-GCD requires a full pass on the data. This is as costly as p iterations of DP-CD or n iterations of DP-SGD. Nonetheless, on many problems, DP-GCD requires just as many passes on the data as DP-CD and DP-SGD (Figures 1a and 1c to 1f). When more computation is required, it also provides significantly better solutions than DP-CD and DP-SGD (Figure 1b). This is in line with our theoretical results from Section 4.4.

6 CONCLUSION AND DISCUSSION

We proposed DP-GCD, a greedy coordinate descent algorithm for DP-ERM. In favorable settings, DP-GCD achieves utility guarantees of $O(\log(p)/n^{2/3}\epsilon^{2/3})$ and $O(\log(p)/n^2\epsilon^2)$ for convex and strongly-convex objectives. It is the first algorithm to achieve such rates without solving an ℓ_1-constrained problem. Instead, we show that DP-GCD depends on ℓ_1-norm quantities and automatically adapts to the structure of the problem. Specifically, DP-GCD interpolates between logarithmic and polynomial dependence on the dimension, depending on the problem. Thus, DP-GCD constitutes a step towards the design of an algorithm that adjusts to the appropriate ℓ_p structure of a problem (see Bassily et al., 2021; Asi et al., 2021).

We also showed that DP-GCD adapts to the quasi-sparsity of the problem, without requiring *a priori* knowledge about it. In such problems, it converges to a good approximate solution in few iterations. This improves utility, and reduces the polynomial dependence on the dimension to a polyno-

mial dependence on the (much smaller) quasi-sparsity level of the solution.

We also proposed and evaluated a proximal variant of DP-GCD, allowing non-smooth, sparsity-inducing regularization. While it is not covered by our utility guarantees, we note that the only existing analysis of such variants in the non-private setting is the one of Karimireddy et al. (2019) for ℓ_1 and box constraints. Their proof relies on an alternation between `good` (that provably progress) and `bad` steps (that do not increase the objective), which does not transfer to the private setting. Extending such results to DP-ERM is an exciting direction for future work.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous reviewers who provided useful feedback on previous versions of this work, which helped to improve the paper.

This work was supported by the Inria Exploratory Action FLAMED and by the French National Research Agency (ANR) through grant ANR-20-CE23-0015 (Project PRIDE), ANR-20-CHIA-0001-01 (Chaire IA CaMeLOt) and ANR 22-PECY-0002 IPOP (Interdisciplinary Project on Privacy) project of the Cybersecurity PEPR.

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A PROOF OF PRIVACY

Theorem 4.2. *Let $\epsilon, \delta \in (0, 1]$. Algorithm 1 with $\lambda_j = \lambda'_j = \frac{8L_j}{n\epsilon}\sqrt{T\log(1/\delta)}$ is (ϵ, δ)-DP.*

Proof. In each iteration of Algorithm 1, the data is accessed twice: once to choose the coordinate and once to compute the private gradient. In total, data is thus queried $2T$ times.

Let $\lambda_j = \lambda'_j = \frac{2L_j}{n\epsilon'}$. For $j \in [p]$, the gradient's j-th entry has sensitivity $2L_j$. Thus, by the report noisy max mechanism (Dwork and Roth, 2013), the greedy choice of j is ϵ'-DP. By the Laplace mechanism (Dwork and Roth, 2013), computing the corresponding gradient coordinate is also ϵ'-DP.

The advanced composition theorem for differential privacy thus ensures that the $2T$-fold composition of these mechanisms is (ϵ, δ)-DP for $\delta > 0$ and

$$\epsilon = \sqrt{4T\log(1/\delta)}\epsilon' + 2T\epsilon'(\exp(\epsilon') - 1) \ , \tag{2}$$

where we recall that $\epsilon' = \frac{2L_j}{n\lambda_j} = \frac{2L_j}{n\lambda'_j}$ for all $j \in [p]$. When $\epsilon \leq 1$, we can give a simpler expression (see Corollary 3.21 of Dwork and Roth, 2013): with $\epsilon' = \epsilon/4\sqrt{T\log(1/\delta)}$, Algorithm 1 is (ϵ, δ)-DP for $\lambda_j = \lambda'_j = 8L_j\sqrt{T\log(1/\delta)}/n\epsilon$. □

B PROOF OF UTILITY

In this section, we prove Theorem 4.4 and Theorem 4.7, giving utility upper bounds for DP-GCD. We obtain these high-probability results through a careful examination of the properties of DP-GCD's iterates, and obtain high-probability results by using concentration inequalities (see Appendix B.1).

In Appendix B.2, we prove a general descent lemma, which implies that iterates of DP-GCD converge (with high probability) to a neighborhood of the optimum. This property is proven rigorously in Appendix B.3.2, and we give the utility results for general convex functions in Appendix B.3.3. Under the additional assumption that the objective is strongly convex, we prove better utility bounds in Appendix B.4. These bounds follow from a key lemma (see Appendix B.4.1), which implies linear convergence to a neighborhood of the optimum. We then use this result in two settings, obtaining two different rates: first in a general setting (in Appendix B.4.2), then under the additional assumption that the problem's solution is quasi-sparse (in Appendix B.4.3).

B.1 Concentration Lemma

To prove high-probability utility results, we first bound (in Lemma B.1) the probability for a sum of squared Laplacian variables to exceed a given threshold.

Lemma B.1. *Let $K > 0$ and $\lambda_1, \dots, \lambda_K > 0$. Define $X_k \sim \mathrm{Lap}(\lambda_k)$ and $\lambda_{\max} = \max_{k\in[K]} \lambda_k$. For any $\beta > 0$, it holds that*

$$\Pr\left[\sum_{k=1}^{K} X_k^2 \geq \beta\right] \leq 2^K \exp\left(-\frac{\sqrt{\beta}}{2\lambda_{\max}}\right) \ . \tag{3}$$

Proof. We first remark that $(\sum_{k=1}^K |X_k|)^2 = \sum_{k=1}^K \sum_{k'=1}^K |X_k||X_{k'}| \geq \sum_{k=1}^K X_k^2$. Therefore

$$\Pr\left[\sum_{k=1}^{K} X_k^2 \geq a^2\right] \leq \Pr\left[\Big(\sum_{k=1}^{K} |X_k|\Big)^2 \geq a^2\right] = \Pr\left[\Big(\sum_{k=1}^{K} |X_k|\Big) \geq a\right] \ . \tag{4}$$

Chernoff's inequality now gives, for any $\gamma > 0$,

$$\Pr\left[\sum_{k=1}^{K} |X_k| \geq a\right] \leq \exp(-\gamma a)\mathbb{E}\left[\exp(\gamma \sum_{k=1}^{K} |X_k|)\right] \ . \tag{5}$$

By the properties of the exponential and the X_k's independence, we can rewrite the inequality as

$$\Pr\left[\sum_{k=1}^{K} |X_k| \geq a\right] \leq \exp(-\gamma a)\mathbb{E}\left[\prod_{k=1}^{K} \exp\Big(\gamma|X_k|\Big)\right] = \exp(-\gamma a)\prod_{k=1}^{K} \mathbb{E}\Big[\exp\Big(\gamma|X_k|\Big)\Big] \ . \tag{6}$$

We can now compute the expectation of $\exp(\gamma|X_k|)$ for $k \in [K]$,

$$\mathbb{E}\Big[\exp\Big(\gamma|X_k|\Big)\Big] = \frac{1}{2\lambda_k}\int_{-\infty}^{+\infty}\exp(\gamma|x|)\exp(-\frac{|x|}{\lambda_k})dx = \frac{1}{\lambda_k}\int_0^{+\infty}\exp\Big((\gamma-\frac{1}{\lambda_k})x\Big)dx \ . \tag{7}$$

We choose $\gamma = 1/2\lambda_{\max}$, such that $\gamma \leq 1/2\lambda_k$ for all $k \in [K]$ and obtain

$$\mathbb{E}\Big[\exp\Big(\gamma|X_k|\Big)\Big] = \frac{1}{\lambda_k}\frac{1}{\frac{1}{\lambda_k}-\gamma} = \frac{1}{1-\gamma\lambda_k} \leq 2 \ . \tag{8}$$

Plugging everything together, we have proved that

$$\Pr\left[\sum_{k=1}^K X_k^2 \geq a^2\right] \leq \Pr\left[\sum_{k=1}^K |X_k| \geq a\right] \leq 2^K \exp(-\frac{a}{2\lambda_{\max}}) \ , \tag{9}$$

and taking $a = \sqrt{\beta}$ gives the result. □

B.2 Descent Lemma

We now prove a noisy descent lemma for DP-GCD (Lemma B.2). This lemma bounds the suboptimality $f(w^{t+1}) - f(w^*)$ at time $t+1$ as a function of the suboptimality $f(w^t) - f(w^*)$ at time t, of the gradient's largest entry and of the noise. At this point, we remark that when the gradient is large enough, it is very probable that $\frac{1}{8}\|\nabla f(w^t)\|^2_{M^{-1},\infty} \geq \frac{1}{2M_j}|\eta_j^t|^2 + \frac{1}{2M_j}|\chi_j^t|^2 + \frac{1}{4M_{j^*}}|\chi_{j^*}^t|^2$: this implies that the value of the objective function decreases with high probability, even under the presence of noise. This observation will be crucial for proving utility for general convex functions.

Lemma B.2. *Let $t \geq 0$ and $w^t, w^{t+1} \in \mathbb{R}^p$ two consecutive iterates of Algorithm 1, with $\gamma_j = 1/M_j$ and λ_j, λ'_j chosen as in Theorem 4.2 to ensure ϵ, δ-DP. We denote by $j \in [p]$ the coordinate chosen at this step t, and by $j^* = \arg\max_{j\in[p]} |\nabla_j f(w^t)|/\sqrt{M_j}$ the coordinate that would have been chosen without noise. The following inequality holds*

$$f(w^{t+1}) - f(w^*) \leq f(w^t) - f(w^*) - \frac{1}{8}\|\nabla f(w^t)\|^2_{M^{-1},\infty} + \frac{1}{2M_j}|\eta_j^t|^2 + \frac{1}{2M_j}|\chi_j^t|^2 + \frac{1}{4M_{j^*}}|\chi_{j^*}^t|^2 \ . \tag{10}$$

Proof. The smoothness of f gives a first inequality

$$\begin{aligned}
f(w^{t+1}) &\leq f(w^t) + \langle \nabla f(w^t), w^{t+1} - w^t\rangle + \frac{1}{2}\|w^{t+1} - w^t\|^2_M & (11)\\
&= f(w^t) - \frac{1}{M_j}\nabla_j f(w^t)(\nabla_j f(w^t) + \eta_j^t) + \frac{1}{2M_j}(\nabla_j f(w^t) + \eta_j^t)^2 & (12)\\
&= f(w^t) - \frac{1}{M_j}\nabla_j f(w^t)^2 - \frac{1}{M_j}\nabla_j f(w^t)\eta_j^t + \frac{1}{2M_j}(\nabla_j f(w^t))^2 \\
&\quad + \frac{1}{M_j}\nabla_j f(w^t)\eta_j^t + \frac{1}{2M_j}(\eta_j^t)^2 & (13)\\
&= f(w^t) - \frac{1}{2M_j}\nabla_j f(w^t)^2 + \frac{1}{2M_j}(\eta_j^t)^2 \ . & (14)
\end{aligned}$$

We will make the noisy gradient appear, so as to use the noisy greedy rule. To do so, we remark that the classical inequality $(a+b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$ implies that $-a^2 \leq -\frac{1}{2}(a+b)^2 + b^2$. Applied with $a = \nabla_j f(w^t)/\sqrt{M_j}$ and $b = \chi_j^t/\sqrt{M_j}$, this results in

$$-\frac{1}{2M_j}\nabla_j f(w^t)^2 \leq -\frac{1}{4M_j}(\nabla_j f(w^t) + \chi_j^t)^2 + \frac{1}{2M_j}(\chi_j^t)^2 \ . \tag{15}$$

And, by the noisy greedy rule, $\frac{1}{\sqrt{M_{j^*}}}|\nabla_{j^*} f(w^t) + \chi_{j^*}^t| \leq \frac{1}{\sqrt{M_j}}|\nabla_j f(w^t) + \chi_j^t|$. We replace in (15) and use the inequality $-a^2 \leq -\frac{1}{2}(a+b)^2 + b^2$ with $a = (\nabla_{j^*} f(w^t) + \chi_{j^*})/\sqrt{M_{j^*}}$ and $b = -\chi_{j^*}/\sqrt{M_{j^*}}$ to obtain

$$\begin{aligned}
-\frac{1}{2M_j}\nabla_j f(w^t)^2 &\leq -\frac{1}{4M_{j^*}}(\nabla_{j^*} f(w^t) + \chi_{j^*}^t)^2 + \frac{1}{2M_j}(\chi_j^t)^2 & (16)\\
&\leq -\frac{1}{8M_{j^*}}(\nabla_{j^*} f(w^t))^2 + \frac{1}{4M_{j^*}}(\chi_{j^*}^t)^2 + \frac{1}{2M_j}(\chi_j^t)^2 \ . & (17)
\end{aligned}$$

The result follows from (14) and $\frac{1}{M_{j^*}}(\nabla_{j^*} f(w^t))^2 = \|\nabla f(w^t)\|^2_{M^{-1},\infty}$. □

B.3 Utility for General Convex Functions

In this section, we derive an upper bound on the utility of DP-GCD for convex objective functions. First, we use convexity of f to upper bound the decrease described in Lemma B.2. This gives Lemma B.3 in Appendix B.3.1, where the suboptimality gap $f(w^{t+1}) - f(w^*)$ at time $t+1$ is upper bound by a function of the suboptimality gap $f(w^t) - f(w^*)$ at time t and the noise injected in step t. The novelty of our analysis lies in Lemma B.4, where examine the decrease of the objective. Specifically, we show that either (i) $f(w^t)$ is far from its minimum, and the suboptimality gap decreases with high probability, either (ii) $f(w^t)$ is close to its minimum, then all future iterates of DP-GCD will remain in a ball whose radius is determined by the variance of the noise. This observation is essential for proving the utility results stated in Section 4.3.

B.3.1 Descent Lemma for Convex Functions

Lemma B.3. *Under the hypotheses of Lemma B.2, for a convex objective function f, we have*

$$f(w^{t+1}) - f(w^*) \leq f(w^t) - f(w^*) - \frac{(f(w^t) - f(w^*))^2}{8\|w^t - w^*\|_{M,1}^2} + \frac{1}{2M_j}|\eta_j^t|^2 + \frac{1}{2M_j}|\chi_j^t|^2 + \frac{1}{4M_{j^*}}|\chi_{j^*}^t|^2 \ . \tag{18}$$

Proof. Since f is convex, it holds that

$$f(w^*) \geq f(w^t) + \langle \nabla f(w^t), w^* - w^t \rangle \ . \tag{19}$$

After reorganizing the terms, we can upper bound them using Hölder's inequality

$$f(w^t) - f(w^*) \leq \langle \nabla f(w^t), w^t - w^* \rangle \tag{20}$$

$$\leq \|\nabla f(w^t)\|_{M^{-1},\infty} \|w^t - w^*\|_{M,1} \ , \tag{21}$$

where the second inequality holds since $\|\cdot\|_{M,1}$ and $\|\cdot\|_{M^{-1},\infty}$ are conjugate norms. We now divide (21) by $\|w^t - w^*\|_{M,1}$, square it and reorganize to get $-\|\nabla f(w^t)\|_{M^{-1},\infty}^2 \leq -\frac{(f(w^t)-f(w^*))^2}{\|w^t - w^*\|_{M,1}^2}$. Replacing in Lemma B.2 gives the result. □

B.3.2 Key Lemma on the Behavior of DP-GCD's Iterates

Now that we have an inequality in the form of Lemma B.3, we prove that iterates of DP-GCD converge to a vicinity of the optimum. In the general lemma below, think of ξ_t as $f(w^t) - f(w^*)$ and of β as the variance of the term. This result will be combined with Lemma B.1 to obtain high-probability bounds.

Lemma B.4. *Let $\{c_t\}_{t\geq 0}$ and $\{\xi_t\}_{t\geq 0}$ be two sequences of positive values that satisfy, for all $t \geq 0$,*

$$\xi_{t+1} \leq \xi_t - \frac{\xi_t^2}{c_t} + \beta, \tag{22}$$

such that if $\xi_t \leq \xi_0$ then $c_t \leq c_0$. Assume that $\beta \leq c_0$ and $\xi_0 \geq 2\sqrt{\beta c_0}$. Then:

1. *For all $t > 0$, $c_t \leq c_0$, and there exists $t^* > 0$ such that $\xi_{t+1} \leq \xi_t$ if $t < t^*$ and $\xi_t \leq 2\sqrt{\beta c_0}$ if $t \geq t^*$.*
2. *For all $t \geq 1$, $\xi_t \leq \frac{c_0}{t} + 2\sqrt{\beta c_0}$.*

Proof. 1. Assume that for $t \geq 0$, $\sqrt{\beta c_0} \leq \xi_t \leq \xi_0$. Then,

$$\xi_{t+1} \leq \xi_t - \frac{\xi_t^2}{c_t} + \beta \leq \xi_t - \frac{\sqrt{\beta c_0}^2}{c_0} + \beta = \xi_t \ , \tag{23}$$

where the second inequality comes from $\xi_t \geq \sqrt{\beta c_0}$ and $\xi_t \leq \xi_0$ (which implies $c_t \leq c_0$). We now define the following value t^*, which defines the point of rupture between two regimes for ξ_t:

$$t^* = \min \left\{ t \geq 0 \,\middle|\, \xi_t \leq \sqrt{\beta c_0} \right\} \ . \tag{24}$$

Let $t < t^*$, assume that $\xi_t \leq \xi_0$, then (23) holds, that is $\xi_{t+1} \leq \xi_t \leq \xi_0$. By induction, it follows that for all $t < t^*$, $\xi_{t+1} \leq \xi_t \leq \xi_0$ and $c_t \leq c_0$.

Remark now that $\xi_{t^*} \leq \sqrt{\beta c_0}$, we prove by induction that ξ_t stays under $2\sqrt{\beta c_0}$ for $t \geq t^*$. Assume that for $t \geq t^*$, $\xi_t \leq 2\sqrt{\beta c_0}$. Then, there are two possibilities. If $\xi_t \leq \sqrt{\beta c_0}$, then

$$\xi_{t+1} \leq \xi_t - \frac{\xi_t^2}{c_t} + \beta \leq \sqrt{\beta c_0} + \beta \leq 2\sqrt{\beta c_0} \ , \tag{25}$$

and $\xi_{t+1} \leq 2\sqrt{\beta c_0}$. Otherwise, $\sqrt{\beta c_0} \leq \xi_t \leq 2\sqrt{\beta c_0} \leq \xi_0$ and (23) holds, which gives $\xi_{t+1} \leq \xi_t \leq 2\sqrt{\beta c_0}$. We proved that for $t \geq t^*$, $\xi_t \leq 2\sqrt{\beta c_0}$, which concludes the proof of the first part of the lemma.

2. We start by proving this statement for $0 < t < t^* - 1$. Define $\omega = \frac{2u}{c_0}$ and $u = \sqrt{\beta c_0}$. The assumption on ξ_t implies, by the first part of the lemma, $\xi_{t+1} \leq \xi_t - \frac{\xi_t^2}{c_t} + \beta \leq \xi_t - \frac{\xi_t^2}{c_0} + \beta$, which can be rewritten

$$\xi_{t+1} - u \leq (1-\omega)(\xi_t - u) - \frac{(\xi_t - u)^2}{c_0} \ , \tag{26}$$

since $(1-\omega)(\xi_t - u) - \frac{(\xi_t - u)^2}{c_0} = \xi_t - \omega\xi_t - u + \omega u - \frac{\xi_t^2}{c_0} - \frac{2\xi_t u}{c_0} - \frac{u^2}{c_0} = \xi_t - \frac{\xi_t^2}{c_0} - u + \omega u - \frac{u^2}{c_0}$, and $\omega u - \frac{u^2}{c_0} = \frac{u^2}{c_0} = \beta$. Since $t < t^* - 1$, $\xi_{t+1} - u > 0$ and $\xi_t - u > 0$, we can thus divide (26) by $(\xi_{t+1} - u)(\xi_t - u)$ to obtain

$$\frac{1}{\xi_t - u} \leq \frac{1-\omega}{\xi_{t+1} - u} - \frac{\xi_t - u}{(\xi_{t+1} - u)c_0} \leq \frac{1-\omega}{\xi_{t+1} - u} - \frac{1}{c_0} \leq \frac{1}{\xi_{t+1} - u} - \frac{1}{c_0} \ , \tag{27}$$

where the second inequality comes from $\xi_{t+1} - u \leq \xi_t - u$ from the first part of the lemma. By applying this inequality recursively and taking the inverse of the result, we obtain the desired resuld $\xi_t \leq \frac{c_0}{t} + \sqrt{\beta c_0} \leq \frac{c_0}{t} + 2\sqrt{\beta c_0}$ for all $0 < t < t^*$.

For $t \geq t^*$, we have already proved that $\xi_t \leq 2\sqrt{\beta c_0} \leq \frac{c_0}{t} + 2\sqrt{\beta c_0}$, which concludes our proof. □

B.3.3 Convex Utility Result

Theorem 4.4. *(Convex Case) Let $\epsilon, \delta \in (0,1]$. Assume $\ell(\cdot; d)$ is a convex and L-component-Lipschitz loss function for all $d \in \mathcal{X}$, and f is M-component-smooth. Define $\mathcal{W}^*$ the set of minimizers of f, and f^* the minimum of f. Let $w_{priv} \in \mathbb{R}^p$ be the output of Algorithm 1 with step sizes $\gamma_j = 1/M_j$, and noise scales $\lambda_1, \dots, \lambda_p, \lambda'_1, \dots, \lambda'_p$ set as in Theorem 4.2 (with T chosen below) to ensure (ϵ, δ)-DP. Then, the following holds for $\zeta \in (0,1]$:*

$$f(w_{priv}) - f(w^*) \leq \frac{8R_M^2}{T} + \sqrt{32 R_M^2 \beta} \ , \tag{28}$$

where $\beta = \frac{2\lambda_{\max}^2}{M_{\min}} \log(\frac{8Tp}{\zeta})^2$, *and* $R_M = \max_{w \in \mathbb{R}^p} \min_{w^* \in \mathcal{W}^*} \{\|w - w^*\|_{M,1} \mid f(w) \leq f(w^0)\}$. *If we set* $T = \left(\frac{n^2\epsilon^2 R_M^2 M_{\min}}{2^7 L_{\max}^2 \log(1/\delta)}\right)^{1/3}$, *then with probability at least* $1 - \zeta$,

$$f(w^T) - f(w^0) = \widetilde{O}\Big(\frac{R_M^{4/3} L_{\max}^{2/3} \log(p/\zeta)}{M_{\min}^{1/3} n^{2/3} \epsilon^{2/3}}\Big) \ . \tag{29}$$

Proof. Let $\xi_t = f(w^t) - f(w^*)$. We upper bound the following probability by the union bound, and the fact that for $t \geq 0$, the events E_j^t : "coordinate j is updated at step t" for $j \in [p]$ partition the probability space:

$$\Pr\left[\exists t, \xi_{t+1} \geq \xi_t - \frac{\xi_t^2}{8\|w^t - w^*\|_{M,1}^2} + \beta\right] \leq \sum_{t=0}^{T-1} \Pr\left[\xi_{t+1} \geq \xi_t - \frac{\xi_t^2}{8\|w^t - w^*\|_{M,1}^2} + \beta\right] \tag{30}$$

$$= \sum_{t=0}^{T-1} \sum_{j=1}^{p} \Pr\left[\xi_{t+1} \geq \xi_t - \frac{\xi_t^2}{8\|w^t - w^*\|_{M,1}^2} + \beta \ \wedge \ E_j^t\right] \ . \tag{31}$$

Lemma B.3 gives $\xi_{t+1} \leq \xi_t - \frac{\xi_t^2}{8\|w^t - w^*\|_{M,1}^2} + \frac{1}{2M_j}|\eta_j^t|^2 + \frac{1}{2M_j}|\chi_j^t|^2 + \frac{1}{4M_{j^*}}|\chi_{j^*}^t|^2$. We thus have the following upper bound:

$$\Pr\left[\exists t, \xi_{t+1} \geq \xi_t - \frac{1}{8\|w^t - w^*\|_{M,1}^2}\xi_t^2 + \beta\right] \leq \sum_{t=0}^{T-1}\sum_{j=1}^{p} \Pr\left[\frac{|\eta_j^t|^2}{2M_j} + \frac{|\chi_j^t|^2}{2M_j} + \frac{|\chi_{j^*}^t|^2}{4M_{j^*}} \geq \beta\right] \tag{32}$$

$$\leq \sum_{t=0}^{T-1}\sum_{j=1}^{p} \Pr\left[|\eta_j^t|^2 + |\chi_j^t|^2 + |\chi_{j^*}^t|^2 \geq 2M_{\min}\beta\right] \ . \tag{33}$$

By Lemma B.1 with $X_1 = \eta_j^t \sim \mathrm{Lap}(\lambda_j)$, $X_2 = \chi_j^t \sim \mathrm{Lap}(\lambda'_j)$ and $X_3 = \chi_{j^*}^t \sim \mathrm{Lap}(\lambda'_{j^*})$, it holds that

$$\Pr\left[|\eta_j^t|^2 + |\chi_j^t|^2 + |\chi_{j^*}^t|^2 \geq 2M_{\min}\beta\right] \leq 8\exp\left(-\frac{\sqrt{2M_{\min}\beta}}{2\lambda_{\max}}\right) = \frac{\zeta}{Tp} \ , \tag{34}$$

where the last equality comes from $\beta = \frac{2\lambda_{\max}^2}{M_{\min}}\log(\frac{8Tp}{\zeta})^2$. We have proved that

$$\Pr\left[\exists t, \xi_{t+1} \geq \xi_t - \frac{\xi_t^2}{8\|w^t - w^*\|_{M,1}^2} + \beta\right] \leq \sum_{t=0}^{T-1}\sum_{j=1}^{p}\frac{\zeta}{Tp} = \zeta \ . \tag{35}$$

We now use our Lemma B.4, with $\xi_t = f(w^t) - f(w^*)$; $c_0 = 8R_M^2$ and $c_t = 8\|w^t - w^*\|_{M,1}^2$ for $t > 0$; and $\beta = \frac{2\lambda_{\max}^2}{M_{\min}}\log(\frac{8Tp}{\zeta})^2$. These values satisfies the assumptions of Lemma B.4 since, by the definition of R_M, it holds that $c_t \leq c_0$ whenever $\xi_t \leq \xi_0$ (*i.e.,* $f(w^t) - f(w^*) \leq f(w^0) - f(w^*)$). Additionally, $f(w^0) - f(w^*) \geq \sqrt{32R_M^2\beta}$, therefore $f(w^0) - f(w^*) \geq 2\sqrt{\beta c_0}$, and $\beta \leq c_0$.

We obtain the result, with probability at least $1 - \zeta$:

$$f(w^t) - f(w^0) \leq \frac{c_0}{t} + 2\sqrt{\beta c_0} = \frac{8R_M^2}{t} + \frac{64R_M L_{\max}\log(8Tp/\zeta)\sqrt{T\log(1/\delta)}}{\sqrt{M_{\min}}n\epsilon} \ . \tag{36}$$

For $T = \frac{R_M^{2/3}M_{\min}^{1/3}n^{2/3}\epsilon^{2/3}}{4L_{\max}^{2/3}\log(1/\delta)^{1/3}}$, we obtain that, with probability at least $1 - \zeta$,

$$f(w^t) - f(w^0) \leq \frac{64R_M^{4/3}L_{\max}^{2/3}\log(1/\delta)^{1/3}}{M_{\min}^{1/3}n^{2/3}\epsilon^{2/3}}\log\left(\frac{pR_M^{2/3}M_{\min}^{1/3}n^{2/3}\epsilon^{2/3}}{4\zeta L_{\max}^{2/3}\log(1/\delta)^{1/3}}\right) \ , \tag{37}$$

which is the result of the theorem. □

B.4 Utility for Strongly-Convex Functions

B.4.1 A Key Inequality for Strongly-Convex Functions

We now prove a link between f's largest gradient entry and the suboptimality gap, under the assumption that there exists a unique minimizer w^* of f that is (α, τ)-quasi-sparse. Note that this assumption is not restrictive in general as any vector in $\mathbb{R}^p$ is $(0, p)$-quasi-sparse, and for any τ there exists $\alpha > 0$ such that the vector is (α, τ)-quasi-sparse. We will denote by $\mathcal{W}_{\tau,\alpha} \subseteq \mathbb{R}^p$ the set of (α, τ)-quasi-sparse vectors of $\mathbb{R}^p$:

$$\mathcal{W}_{\tau,\alpha} = \{w \in \mathbb{R}^p \mid |\{j \in [p] \mid |w_j| \geq \alpha\}| \leq \tau\} \ . \tag{38}$$

When $\alpha = 0$, we simply write $\mathcal{W}_\tau = \mathcal{W}_{\tau,0}$, that is the set of τ-sparse vectors. We also define the associated thresholding operator π_α, that puts to 0 the coordinates that are smaller than α, "projecting" vectors from $\mathcal{W}_{\tau,\alpha}$ to $\mathcal{W}_\tau$, *i.e.,* for $w \in \mathbb{R}^p$,

$$\pi_\alpha(w) = \begin{cases} 0 & \text{if } |w_j| \leq \alpha \ , \\ w_j & \text{otherwise} \ . \end{cases} \tag{39}$$

Importantly, restricting a function to τ-sparse vectors changes its strong-convexity parameter. Let $\tau \geq 0$ and $q \in \{1, 2\}$, we say a function is $\mu^{(\tau)}_{M,q}$-strongly-convex when restricted to τ-sparse vectors if for all τ-sparse vectors $v, w \in \mathcal{W}_\tau$,

$$f(w) \geq f(v) + \langle \nabla f(v), w - v \rangle + \frac{\mu^{(\tau)}_{M,q}}{2} \|w - v\|^2_{M,q} \ . \tag{40}$$

Remark that when $\tau \geq p$, we recover the usual strong-convexity parameters. The parameters w.r.t. ℓ_1- and ℓ_2-norms can be compared using the following inequality (Fang et al., 2020), for all $\tau \geq 0$,

$$\frac{1}{\tau}\mu^{(\tau)}_{M,2} \leq \mu^{(\tau)}_{M,1} \leq \mu^{(\tau)}_{M,2} \ . \tag{41}$$

We are ready to prove Lemma B.5.

Lemma B.5. *Let $f : \mathbb{R}^p \to \mathbb{R}$ be a function that is M-component-smooth, and $\mu^{(\tau)}_{M,1}$-strongly-convex w.r.t. $\|\cdot\|_{M,1}$ when restricted to τ-sparse vectors, for $\tau \geq 0$. Assume that the unique minimizer w^* of f is (τ, α)-quasi-sparse, for $\alpha, \tau \geq 0$. Let $w^t \in \mathbb{R}^p$ be a t-sparse vector for some $t \geq 0$. Then we have*

$$-\frac{1}{2}\|\nabla f(w^t)\|_{M^{-1},\infty} \leq -\mu^{(t+\tau)}_{M,1}(f(w^t) - f(w^*)) + \frac{1}{2} M_{\max} \mu^{(t+\tau)}_{M,1} (p - \tau)\alpha^2 \ . \tag{42}$$

Proof. Let $w^t \in \mathbb{R}^p$ be a t-sparse vector. Remark that w^* is (α, τ)-quasi-sparse, meaning that $\pi_\alpha(w^*)$ is τ-sparse. The union of w^t and $\pi_\alpha(w^*)$'s supports ($\mathrm{supp}(w^t)$ and $\mathrm{supp}(\pi_\alpha(w^*))$) thus satisfies $|\mathrm{supp}(w) \cup \mathrm{supp}(\pi_\alpha(w^*))| \leq t + \tau$. As the function f is $\mu^{(t+\tau)}_{M,1}$-strongly-convex with respect to $\|\cdot\|_{M,1}$ and $t + \tau$ sparse vector,

$$f(\pi_\alpha(w)) \geq f(w^t) + \langle \nabla f(w^t), \pi_\alpha(w) - w^t \rangle + \frac{\mu^{(t+\tau)}_{M,1}}{2} \|\pi_\alpha(w) - w^t\|^2_{M,1} \ . \tag{43}$$

Since $\pi_\alpha : \mathcal{W}_{\tau,\alpha} \to \mathcal{W}_{\tau,0}$ is surjective, minimizing this equation for $w \in \mathcal{W}_{\tau,\alpha}$ on both sides gives

$$\inf_{w \in \mathcal{W}_\tau} f(w) \geq f(w^t) - \sup_{w \in \mathcal{W}_{\tau,\alpha}} \left\{ \langle -\nabla f(w^t), w^t - \pi_\alpha(w) \rangle - \frac{\mu^{(t+\tau)}_{M,1}}{2} \|\pi_\alpha(w) - w^t\|^2_{M,1} \right\} \tag{44}$$

$$\geq f(w^t) - \sup_{w \in \mathbb{R}^p} \left\{ \langle -\nabla f(w^t), w^t - w \rangle - \frac{\mu^{(t+\tau)}_{M,1}}{2} \|w - w^t\|^2_{M,1} \right\} \ . \tag{45}$$

The second term corresponds to the conjugate of the function $\frac{1}{2}\|\cdot\|^2_{M,1}$, that is $\frac{1}{2}\|\cdot\|^2_{M^{-1},\infty}$ (Boyd and Vandenberghe, 2004). This gives

$$\inf_{w \in \mathcal{W}_\tau} f(w) \geq f(w^t) - \left(\frac{\mu^{(t+\tau)}_{M,1}}{2} \|\cdot\|^2_1 \right)^* (-\nabla f(w')) \tag{46}$$

$$= f(w^t) - \frac{1}{2\mu^{(t+\tau)}_{M,1}} \|\nabla f(w')\|^2_{M^{-1},\infty} \ . \tag{47}$$

Finally, w^* is the minimizer of f (which is convex), thus $\nabla f(w^*) = 0$. The smoothness of f gives, for any $w \in \mathbb{R}^p$, $f(w) \leq f(w^*) + \frac{1}{2}\|w - w^*\|^2_{M,2}$. Hence

$$\inf_{w \in \mathcal{W}_\tau} f(w) \leq f(w^*) + \inf_{w \in \mathcal{W}_\tau} \frac{1}{2}\|w - w^*\|^2_{M,2}. \leq f(w^*) + \frac{1}{2}\|\pi_\alpha(w^*) - w^*\|^2_{M,2} \ , \tag{48}$$

where the second inequality comes from $\pi_\alpha(w^*) \in \mathcal{W}_\tau$, since $w^* \in \mathcal{W}_{\tau,\alpha}$. It remains to observe that $\|\pi_\alpha(w^*) - w^*\|^2_{M,2} \leq M_{\max}(p - \tau)\alpha^2$ to get the result. □

Corollary B.6. *For τ-sparse vectors, we have $\alpha = 0$ and thus $(p - \tau)\alpha = 0$. Lemma B.5 can thus be simplified as*

$$-\frac{1}{2}\|\nabla f(w^t)\|^2_{M^{-1},\infty} \leq -\mu^{(t+\tau)}_{M,1}(f(w^t) - f(w^*)) \ . \tag{49}$$

When vectors are not sparse ($\tau = p$), we recover the inequality $-\frac{1}{2}\|\nabla f(w^t)\|^2_{M^{-1},\infty} \leq -\mu_{M,1}(f(w^t) - f(w^*))$.

B.4.2 General Strongly-Convex Utility Result

Theorem 4.4. *(Strongly-Convex Case) Let $\epsilon, \delta \in (0,1]$. Assume $\ell(\cdot; d)$ is a $\mu_{M,1}$-strongly-convex w.r.t. $\|\cdot\|_{M,1}$ and L-component-Lipschitz loss function for all $d \in \mathcal{X}$, and f is M-component-smooth. Let $\mathcal{W}^*$ be the set of minimizers of f, and f^* the minimum of f. Let $w_{priv} \in \mathbb{R}^p$ be the output of Algorithm 1 with step sizes $\gamma_j = 1/M_j$, and noise scales $\lambda_1, \dots, \lambda_p, \lambda'_1, \dots, \lambda'_p$ set as in Theorem 4.2 (with T chosen below) to ensure (ϵ, δ)-DP. Then, the following holds for $\zeta \in (0,1]$:*

$$f(w^T) - f(w^*) \leq (1 - \frac{\mu_{M,1}}{2})^T (f(w^0) - f(w^*)) + \frac{64TL_{\max}^2 \log(1/\delta)}{M_{\min}\mu_{M,1} n^2 \epsilon^2} \log(\frac{2Tp}{\zeta}) \ . \tag{50}$$

If we set $T = \frac{2}{\mu_{M,1}} \log(\frac{M_{\min}\mu_{M,1} n^2 \epsilon^2 (f(w^0) - f(w^*))}{32 L_{\max}^2 \log(1/\delta)})$, then with probability at least $1-\zeta$,

$$f(w^T) - f(w^*) = \widetilde{O}\Big(\frac{L_{\max}^2 \log(p/\zeta)}{M_{\min}\mu_{M,1}^2 n^2 \epsilon^2}\Big) \ . \tag{51}$$

Proof. When f is $\mu_{M,1}$-strongly-convex w.r.t. the norm $\|\cdot\|_{M,1}$, Corollary B.6 with $\tau = p$ and $\alpha = 0$ (which holds for any vector) yields

$$-\frac{1}{2}\|\nabla f(w^t)\|_{M^{-1},\infty}^2 \leq -\mu_{M,1}(f(w^t) - f(w^*)) \ . \tag{52}$$

We replace this in Lemma B.2 to obtain

$$f(w^{t+1}) - f(w^*) \leq (1 - \frac{\mu_{M,1}}{4})(f(w^t) - f(w^*)) + \frac{1}{2M_j}|\eta_j^t|^2 + \frac{1}{2M_j}|\chi_j^t|^2 + \frac{1}{4M_{j^*}}|\chi_{j^*}^t|^2 \ . \tag{53}$$

Analogously to the proof of Theorem 4.4, we define $\xi_t = f(w^t) - f(w^*)$ for all $0 \leq t \leq T$, and show that $\Pr\left[\exists t, \xi_{t+1} \geq (1 - \frac{\mu_{M,1}}{4})\xi_t + \beta\right] \leq \zeta/Tp$, with $\beta = \frac{2\lambda_{\max}^2}{M_{\min}} \log(\frac{8Tp}{\zeta})^2$. This yields that, with probability at least $1-\zeta$,

$$f(w^T) - f(w^*) \leq (1 - \frac{\mu_{M,1}}{4})^T (f(w^0) - f(w^*)) + \sum_{t=0}^{T-1} (1 - \frac{\mu_{M,1}}{4})^{T-t} \beta \tag{54}$$

$$\leq (1 - \frac{\mu_{M,1}}{4})^T (f(w^0) - f(w^*)) + \frac{4}{\mu_{M,1}} \frac{32TL_{\max}^2 \log(1/\delta)}{M_{\min} n^2 \epsilon^2} \log\Big(\frac{8Tp}{\zeta}\Big)^2 \ , \tag{55}$$

With $T = \frac{4}{\mu_{M,1}} \log\left(\frac{\mu_{M,1} M_{\min} n^2 \epsilon^2 (f(w^0) - f(w^*))}{128 L_{\max}^2 \log(1/\delta) \log(8p/\zeta)}\right)$ we have, with probability at least $1-\zeta$,

$$f(w^T) - f(w^*) \leq \frac{128 L_{\max}^2 \log(1/\delta) \log(8p/\zeta)^2}{\mu_{M,1} M_{\min} n^2 \epsilon^2} + \frac{512 L_{\max}^2 \log(1/\delta) \log(8Tp/\zeta)^2}{\mu_{M,1}^2 M_{\min} n^2 \epsilon^2} \log\left(\frac{\mu_{M,1} M_{\min} n^2 \epsilon^2 (f(w^0) - f(w^*))}{128 L_{\max}^2 \log(1/\delta) \log(8p/\zeta)^2}\right) \ , \tag{56}$$

which is the desired result. □

B.4.3 Better Utility for Quasi-Sparse Solutions

Theorem 4.7. *Consider f satisfying the hypotheses of Theorem 4.4, with Algorithm 1 initialized at $w^0 = 0$. We denote its output w^T, and assume that its iterates remain s-sparse for some $s \leq p$. Assume that, for all $\tau' \geq 0$, f is $\mu_{M,1}^{(\tau')}$-strongly-convex w.r.t. $\|\cdot\|_{M,1}$ for τ'-sparse vectors and $\mu_{M,2}$-strongly-convex w.r.t. $\|\cdot\|_{M,2}$, and that the (unique) solution of problem (1) is (α, τ)-quasi-sparse for some $\alpha, \tau \geq 0$. Let $T \geq 0$, $\zeta \in [0,1]$, and $\beta = \frac{2\lambda_{\max}^2}{M_{\min}} \log(TP/\zeta)^2$. Then for all $t \leq T$ we have that, with probability at least $1-\zeta$:*

$$f(w^T) - f(w^*) \leq \Big(1 - \frac{\mu_{M,1}^{(\min(s,T)+\tau)}}{4}\Big)^T (f(w^0) - f(w^*)) + \frac{4(\min(s,T)+\tau)\beta}{\mu_{M,2}} + \frac{\min(s,T)+\tau}{8}(p-\tau)\alpha^2 \tag{57}$$

$$\leq \Big(1 - \frac{\mu_{M,2}}{4(\min(s,T)+\tau)}\Big)^T (f(w^0) - f(w^*)) + \frac{4(\min(s,T)+\tau)\beta}{\mu_{M,2}} + \frac{\min(s,T)+\tau}{8}(p-\tau)\alpha^2 \ . \tag{58}$$

Setting $T = \frac{s+\tau}{\mu_{M,2}} \log((f(w^0) - f^*) M_{\min} \mu_{M,2} n^2 \epsilon^2 / L^2)$*, and assuming* $\alpha^2 = O\left(L_{\max}^2 (s+\tau) / M_{\min} \mu_{M,2}^2 p n^2 \epsilon^2\right)$*, we obtain that with probability at least* $1-\zeta$*,*

$$f(w^T) - f^* = \widetilde{O}\left(\frac{L_{\max}^2}{M_{\min}} \frac{(s+\tau)^2 \log(2p/\zeta)}{\mu_{M,2} n^2 \epsilon^2}\right) \ . \tag{59}$$

Proof. First, we remark that at each iteration, we change only one coordinate. Therefore, after t iterations, the iterate w^t is at most t-sparse. Since all iterates are also s-sparse, it is $\min(s,t)$-sparse. Additionally, we assumed that w^* is (τ, α)-almost-sparse. Therefore, Lemma B.5 yields

$$-\frac{1}{2}\|\nabla f(w^t)\|_{M^{-1},\infty} \leq -\mu_{M,1}^{(\min(s,t)+\tau)}(f(w^t) - f(w^*)) + \frac{\mu_{M,1}^{(\min(s,t)+\tau)}}{2}(p-\tau)\alpha^2 \ , \tag{60}$$

and Lemma B.2 becomes

$$\begin{aligned} f(w^{t+1}) - f(w^*) \leq (1 - \frac{\mu_{M,1}^{(\min(s,t)+\tau)}}{4})(f(w^t) - f(w^*)) + \frac{\mu_{M,1}^{(\min(s,t)+\tau)}}{8}(p-\tau)\alpha^2 \\ + \frac{1}{2M_j}|\eta_j^t|^2 + \frac{1}{2M_j}|\chi_j^t|^2 + \frac{1}{4M_{j^*}}|\chi_{j^*}^t|^2 \ . \end{aligned} \tag{61}$$

Then by Chernoff's equality, we obtain (similarly to the proof of Theorem 4.4 for the convex case) that with probability at least $1-\zeta$, for $T \geq 0$,

$$\begin{aligned} f(w^T) - f(w^*) \leq \prod_{t=0}^{T}\Big(1 - \frac{\mu_{M,1}^{(\min(s,t)+\tau)}}{4}\Big)(f(w^0) - f(w^*)) \\ + \sum_{t=0}^{T-1}\prod_{k=T-t}^{T}\Big(1 - \frac{\mu_{M,1}^{(\min(s,k)+\tau)}}{4}\Big)\Big(\beta + \frac{\mu_{M,1}^{(\min(s,t)+\tau)}}{8}(p-\tau)\alpha^2\Big) \ . \end{aligned} \tag{62}$$

Since for $k \in [T]$, $\mu_{M,1}^{\min(s,k)+\tau} \geq \mu_{M,1}^{\min(s,T)+\tau}$, we can further upper bound $\mu_{M,1}^{(\min(s,t)+\tau)} \leq \mu_{M,1}^{(\tau)}$, and $1 - \frac{\mu_{M,1}^{(\min(s,t)+\tau)}}{4} \leq 1 - \frac{\mu_{M,1}^{(\min(s,T)+\tau)}}{4}$ and

$$\sum_{t=0}^{T-1}\prod_{k=T-t}^{T}\Big(1 - \frac{\mu_{M,1}^{(\min(s,k)+\tau)}}{4}\Big) \leq \sum_{t=0}^{T-1}\Big(1 - \frac{\mu_{M,1}^{(\min(s,T)+\tau)}}{4}\Big)^t \leq \frac{4}{\mu_{M,1}^{(\min(s,T)+\tau)}} \ , \tag{63}$$

which allows to simplify the above expression to

$$f(w^T) - f(w^*) \leq \Big(1 - \frac{\mu_{M,1}^{(\min(s,T)+\tau)}}{4}\Big)^T (f(w^0) - f(w^*)) + \frac{4}{\mu_{M,1}^{(\min(s,T)+\tau)}}\Big(\beta + \frac{\mu_{M,1}^{(\tau)}}{8}(p-\tau)\alpha^2\Big) \tag{64}$$

$$\leq \Big(1 - \frac{\mu_{M,2}}{4(\min(s,T)+\tau)}\Big)^T (f(w^0) - f(w^*)) + \frac{4(\min(s,T)+\tau)}{\mu_{M,2}}\Big(\beta + \frac{\mu_{M,2}}{8}(p-\tau)\alpha^2\Big) \tag{65}$$

$$\leq \Big(1 - \frac{\mu_{M,2}}{4(\min(s,T)+\tau)}\Big)^T (f(w^0) - f(w^*)) + \frac{4(\min(s,T)+\tau)\beta}{\mu_{M,2}} + \frac{\min(s,T)+\tau}{8}(p-\tau)\alpha^2 \ , \tag{66}$$

where the second inequality follows from $\mu_{M,1}^{(\min(s,T)+\tau)} \geq \frac{\mu_{M,2}^{(\min(s,T)+\tau)}}{\min(s,T)+\tau} \geq \frac{\mu_{M,2}}{\min(s,T)+\tau}$ and $\mu_{M,1}^{(\tau)} \leq \mu_{M,2}$. We have proven inequalities (57) and (58) of the theorem.

When $\alpha^2 = O\left(L_{\max}^2 (s+\tau) / M_{\min} \mu_{M,2}^2 p n^2 \epsilon^2\right)$, the two additive terms of (66) are $O((s+\tau)\beta/\mu_{M,2})$. Since $\min(s,T) + \tau \leq s + \tau$, we choose $T = \frac{s+\tau}{\mu_{M,2}} \log((f(w^0) - f^*) M_{\min} \mu_{M,2} n^2 \epsilon^2 / L^2)$ to balance all the terms and obtain the result. □

C GREEDY COORDINATE DESCENT FOR COMPOSITE PROBLEMS

Consider the problem of privately approximating

$$w^* \in \arg\min_{w \in \mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} \ell(w; d_i) + \psi(w), \tag{67}$$

where $D = (d_1, \dots, d_n)$ is a dataset of n samples drawn from a universe $\mathcal{X}$, $\ell : \mathbb{R}^p \times \mathcal{X} \to \mathbb{R}$ is a loss function which is convex and smooth in w, and $\psi : \mathbb{R}^p \to \mathbb{R}$ is a convex regularizer which is separable (*i.e.,* $\psi(w) = \sum_{j=1}^p \psi_j(w_j)$) and typically nonsmooth (*e.g.,* ℓ_1-norm).

Algorithm 2 DP-GCD (Proximal Version): Private Proximal Greedy CD

Input: initial $w^0 \in \mathbb{R}^p$, iteration count $T > 0, \forall j \in [p]$, noise scales λ_j, λ'_j, step sizes $\gamma_j > 0$.
for $t = 0$ to $T - 1$ **do**
  Select j_t by the noisy `GS-s`, `GS-r` or `GS-q` rule.
  $w^{t+1} = w^t + (\operatorname{prox} \gamma_j \psi_j(w^t - \gamma_j(\nabla_j f(w^t) + \eta^t_{j_t})) - w^t_j)e_j$, $\eta^t_{j_t} \sim \operatorname{Lap}(\lambda_{j_t})$.
return w^T.

We propose a proximal greedy algorithm to solve (67), see Algorithm 2. The proximal operator is the following (we refer to Parikh and Boyd, 2014, for a detailed discussion on proximal operator and related algorithms):

$$\operatorname{prox}_{\gamma\psi}(v) = \arg\min_{x \in \mathbb{R}^p} \{\frac{1}{2}\|v - x\|_2^2 + \gamma\psi(x)\} \ . \tag{68}$$

The same privacy guarantees as for the smooth DP-GCD algorithm hold since, privacy-wise, the proximal step is a post-processing step. We also adapt the greedy selection rule to incorporate the non-smooth term. We can use one of the following three rules

$$j_t = \arg\max_{j \in [p]} \min_{\xi_j \in \partial\psi_j(w_j)} \frac{1}{\sqrt{M_j}} |\nabla_j f(w^t) + \eta^t_j + \xi_j| \ , \tag{GS-s}$$

$$j_t = \arg\max_{j \in [p]} \sqrt{M_j} |\operatorname{prox}_{\frac{1}{M_j}\psi_j}(w^t_j - \frac{1}{M_j}(\nabla_j f(w^t) + \eta^t_j) - w^t_j| \ , \tag{GS-r}$$

$$j_t = \arg\max_{j \in [p]} \min_{\alpha \in \mathbb{R}} \nabla_j f(w^t)\alpha + \frac{M_j}{2}\alpha^2 + \psi_j(w^t_j + \alpha) - \psi_j(w^t_j) \ . \tag{GS-q}$$

These rules are commonly considered in the non-private GCD literature (see *e.g.,* Tseng and Yun, 2009; Shi et al., 2017; Karimireddy et al., 2019), except for the noise η^t_j and the rescaling in the `GS-s` and `GS-r` rules.

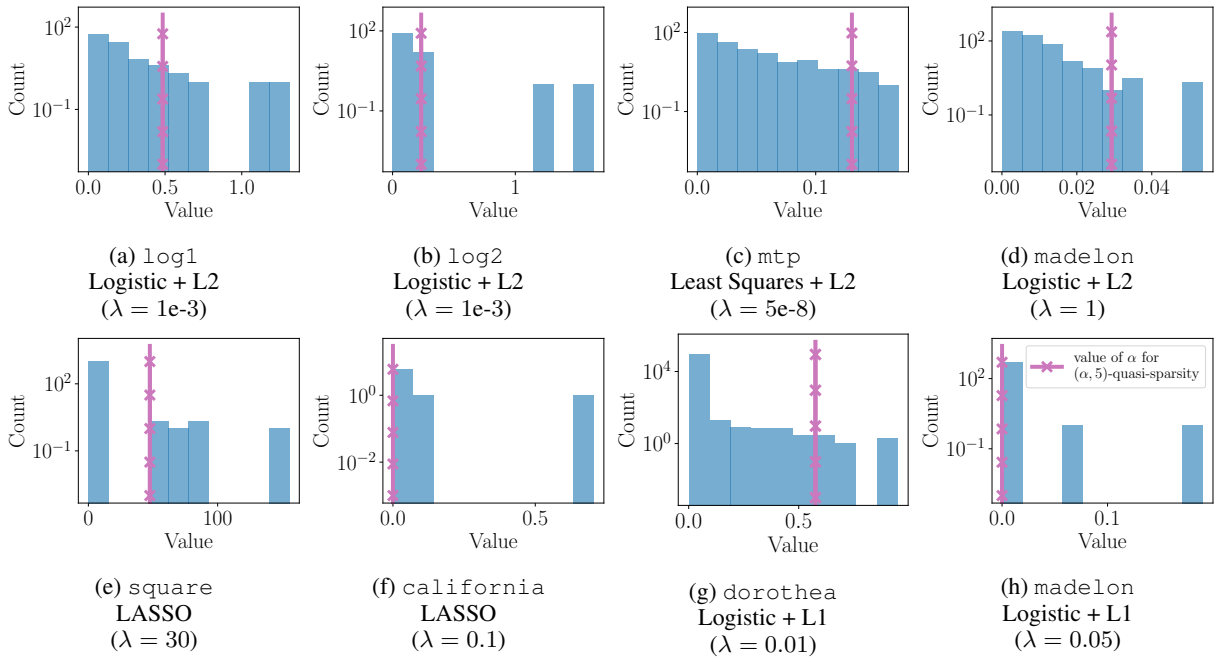
D EXPERIMENTAL DETAILS

In this section, we provide more information about the experiments, such as details on implementation, datasets and the hyperparameter grid we use for each algorithm. We then give the full results on our L1-regularized, non-smooth, problems, with the three greedy rules (as opposed to Section 5 where we only plotted results for the `GS-r` rule). Finally, we provide runtime plots.

Code and setup. The algorithms are implemented in C++ for efficiency, together with a Python wrapper for simple use. It is provided as supplementary. Experiments are run on a computer with a Intel (R) Xeon(R) Silver 4114 CPU @ 2.20GHz and 64GB of RAM, and took about 10 hours in total to run (this includes all hyperparameter tuning).

Datasets. The datasets we use are described in Table 1. In Figure 2, we plot the histograms of the absolute value of each problem solution's parameters. The purple line indicates the value of α that ensures that the parameters of the solution are $(\alpha, 5)$-quasi-sparse. Note the logarithmic scale on the y-axis. On the `log1`, `log2`, `madelon`, `square`, `california` and `dorothea` datasets, the solutions are very imbalanced. In these problems, a very limited number of parameters stand out, and DP-GCD is able to exploit this property. This illustrates the results from Section 4.4, since DP-GCD can exploit this structure even in quasi-sparse problems, where α is non zero. Conversely, the `mtp` solution is more balanced: the structural properties of this dataset are not strong enough for DP-GCD to outperform its competitors.

Hyperparameters. On all datasets, we use the same hyperparameter grid. For each algorithm, we choose between roughly the same number of hyperparameters. The number of passes on data represents p iterations of DP-CD, n iterations of DP-SGD, and 1 iteration of DP-GCD. The complete grid is described in Table 2, and the chosen hyperparameters for each problem and algorithm are given in Table 4.



(a) `log1` Logistic + L2 (λ = 1e-3)
(b) `log2` Logistic + L2 (λ = 1e-3)
(c) `mtp` Least Squares + L2 (λ = 5e-8)
(d) `madelon` Logistic + L2 ($\lambda = 1$)
(e) `square` LASSO ($\lambda = 30$)
(f) `california` LASSO ($\lambda = 0.1$)
(g) `dorothea` Logistic + L1 ($\lambda = 0.01$)
(h) `madelon` Logistic + L1 ($\lambda = 0.05$)

Figure 2: Histograms of the absolute value of each problem solution's parameters. Purple line indicates the α for which the plotted vector is $(\alpha, 5)$-quasi-sparse. Y-axis is logarithmic.

Recovery of the support. In Table 3, we report the number of coordinates that are correctly/incorrectly identified as non-zero on ℓ_1 regularized problems. Contrary to DP-SGD and DP-CD, DP-GCD never incorrectly identifies a coordinate as non-zero. Additionally, the suboptimality gap is lower for DP-GCD: its updates thus lead to better solutions.

Additional experiments on proximal DP-GCD. In Figure 3, we show the results of the proximal DP-GCD algorithm, after tuning the hyperparameters with the grid described above for each of the `GS-s`, `GS-r` and `GS-q` rules.

The three rules seem to behave qualitatively the same on `square`, `dorothea` and `madelon`, our three high-dimensional non-smooth problems. There, most coordinates are chosen about one time. Thus, as described by Karimireddy et al. (2019), all the steps are "`good`" steps (along their terminology): and on such good steps, the three rules coincide. On the lower-dimensional dataset `california`, coordinates can be chosen more than one time, and "`bad`" steps are likely to happen. On these steps, the three rules differ.

Table 2: Hyperparameter grid used in our experiments.

Algorithm	Parameter	Values
DP-CD	Passes on data	`[0.001, 0.01, 0.1, 1, 2, 3, 5, 10, 20]`
	Step sizes	`np.logspace(-2, 1, 10)`
	Clipping threshold	`np.logspace(-4, 6, 50)`
DP-SGD	Passes on data	`[0.001, 0.01, 0.1, 1, 2, 3, 5, 10, 20]`
	Step sizes	`np.logspace(-6, 0, 10)`
	Clipping threshold	`np.logspace(-4, 6, 50)`
DP-GCD	Passes on data	`[1, 2, 4, 7, 10, 15, 20]`
	Step sizes	`np.logspace(-2, 1, 10)`
	Clipping threshold	`np.logspace(-4, 6, 50)`

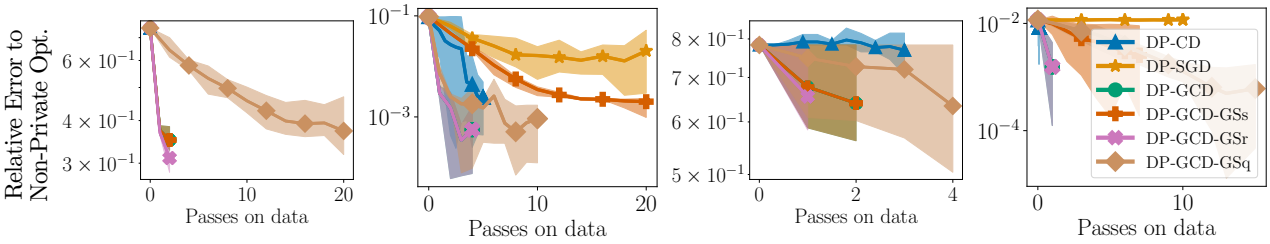
Table 3: Coordinates correctly/incorrectly identified as non-zeros by each algorithm, and relative suboptimality gap $(f(w^{priv}) - f^*)/f^*$ (averaged over 5 runs).

	`square`	`california`	`dorothea`	`madelon`
$\|w^*\|_0$	7	3	72	3
DP-CD	0 / 0 (0.75)	3 / 2 (0.0024)	1 / 1 (0.77)	0 / 0 (0.0085)
DP-SGD	0 / 3 (0.75)	3 / 5 (0.020)	0 / 0 (0.78)	0 / 0 (0.012)
DP-GCD	2 / 0 (0.35)	2 / 0 (0.00056)	1 / 0 (0.64)	1 / 0 (0.0015)

Table 4: Selected hyperparameters for every dataset and algorithm.

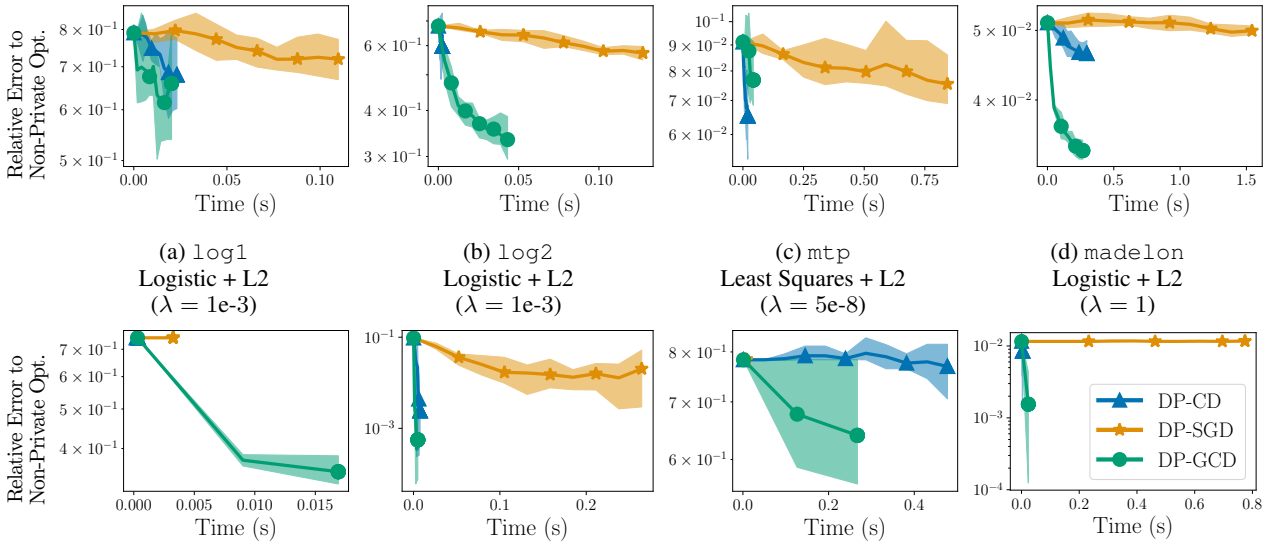
Dataset	Loss	Algorithm	Passes on data	Clipping threshold	Step size
california	LeastSquares + L1	DP-CD	5.0	2.02e+01	1.00e+00
square	LeastSquares + L1	DP-CD	0.01	9.10e+03	1.00e+01
mtp	LeastSquares + L2	DP-CD	3.0	2.02e+01	2.15e-02
madelon	Logistic + L1	DP-CD	0.1	7.91e+00	2.15e+00
log1	Logistic + L2	DP-CD	10.0	1.84e-01	1.00e+00
log2	Logistic + L2	DP-CD	1.0	7.54e-01	2.15e+00
madelon	Logistic + L2	DP-CD	10.0	1.21e+00	1.00e-01
dorothea	Logistic + L1	DP-CD	3.0	4.50e-02	4.64e+00
california	LeastSquares + L1	DP-SGD	20.0	1.26e+01	2.15e-05
square	LeastSquares + L1	DP-SGD	0.01	4.94e+00	1.00e-04
mtp	LeastSquares + L2	DP-SGD	20.0	1.26e+01	2.15e-05
madelon	Logistic + L1	DP-SGD	10.0	6.87e-03	1.00e+00
log1	Logistic + L2	DP-SGD	20.0	1.84e-01	4.64e-04
log2	Logistic + L2	DP-SGD	20.0	1.84e-01	4.64e-04
madelon	Logistic + L2	DP-SGD	20.0	1.84e-01	1.00e-04
dorothea	Logistic + L1	DP-SGD	0.001	1.00e-04	1.00e-06
california	LeastSquares + L1	DP-GCD	4	5.18e+01	1.00e+00
square	LeastSquares + L1	DP-GCD	2	1.46e+04	2.15e+00
mtp	LeastSquares + L2	DP-GCD	7	2.02e+01	4.64e-01
madelon	Logistic + L1	DP-GCD	1	7.91e+00	2.15e+00
log1	Logistic + L2	DP-GCD	10	3.09e+00	2.15e+00
log2	Logistic + L2	DP-GCD	20	1.93e+00	4.64e-01
madelon	Logistic + L2	DP-GCD	10	7.91e+00	1.00e+00
dorothea	Logistic + L1	DP-GCD	2	1.26e+01	2.15e+00

Runtime. Finally, we report the runtime of DP-GCD, in comparison with DP-CD and DP-SGD in Figure 4, that is the counterpart of Figure 1, except with runtime on the x-axis. These results confirm the fact that DP-GCD can be efficient, although its iterations are expensive to compute. Indeed, in imbalanced problems, the small number of iterations of DP-GCD enables it to run faster than DP-SGD, and in roughly the same time as DP-CD, while improving utility.



(a) `sparse` LASSO ($\lambda = 30$) (b) `california` LASSO ($\lambda = 0.1$) (c) `dorothea` Logistic + L1 ($\lambda = 0.01$) (d) `madelon` Logistic + L1 ($\lambda = 0.05$)

Figure 3: Relative error to non-private optimal for DP-CD, proximal DP-GCD (with `GS-r`, `GS-s` and `GS-q` rules) and DP-SGD on different problems. On the x-axis, 1 tick represents a full access to the data: p iterations of DP-CD, n iterations of DP-SGD and 1 iteration of DP-GCD. Number of iterations, clipping thresholds and step sizes are tuned simultaneously for each algorithm. We report min/mean/max values over 5 runs.



(a) `log1` Logistic + L2 ($\lambda =$ 1e-3) (b) `log2` Logistic + L2 ($\lambda =$ 1e-3) (c) `mtp` Least Squares + L2 ($\lambda =$ 5e-8) (d) `madelon` Logistic + L2 ($\lambda = 1$)

(e) `sparse` LASSO ($\lambda = 30$) (f) `california` LASSO ($\lambda = 0.1$) (g) `dorothea` Logistic + L1 ($\lambda = 0.01$) (h) `madelon` Logistic + L1 ($\lambda = 0.05$)

Figure 4: Relative error to non-private optimal for DP-CD, DP-GCD and DP-SGD on different problems, as a function of running time. Number of iterations, clipping thresholds and step sizes are tuned simultaneously for each algorithm. We report min/mean/max values over 5 runs.