



**HAL**  
open science

# PML applied to spacetime Trefftz-DG numerical formulation for the acoustic wave equation

Vinduja Vasanthan, H el ene Barucq, Henri Calandra, Julien Diaz

## ► To cite this version:

Vinduja Vasanthan, H el ene Barucq, Henri Calandra, Julien Diaz. PML applied to spacetime Trefftz-DG numerical formulation for the acoustic wave equation. Icosahom 2020 - International Conference on Spectral and High-Order Methods, Jul 2021, Vienna, Austria. hal-03533342

**HAL Id: hal-03533342**

**<https://inria.hal.science/hal-03533342v1>**

Submitted on 19 Jan 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destin ee au d ep ot et  a la diffusion de documents scientifiques de niveau recherche, publi es ou non,  emanant des  tablissements d'enseignement et de recherche franais ou  trangers, des laboratoires publics ou priv es.

*Inria*



Depth Imaging Partnership  
**DIP**



ICOSAHOM 2020 – JULY 12-16, 2021

# PML applied to spacetime Trefftz-DG numerical formulation for the acoustic wave equation

V. Vasanthan<sup>1,\*</sup>, H. Barucq<sup>1</sup>, H. Calandra<sup>2</sup>, J. Diaz<sup>1</sup>

1. Project-Team Makutu, Inria, E2S UPPA, CNRS, France,  
firstname.lastname@inria.fr

2. TotalEnergies, France, firstname.lastname@totalenergies.com

- Aim of our work ?
  - ↪ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↪ gain precision,
  - ↪ reduce computational time.
- What kind of problems ?
  - ↪ acoustic,
  - ↪ elastodynamic,
  - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↪ polynomial basis functions,
  - ↪ Green's functions.

# Acoustic Wave equation

First order acoustic wave equation :

$$\left\{ \begin{array}{ll} \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, p(\cdot, 0) = p_0 & \text{in } \mathcal{D}, \\ \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad \begin{array}{l} \Omega = \mathcal{D} \times \mathcal{I} \\ \mathcal{I} = [0, T] \end{array}$$

with

$$\mathbf{x} = (x, y),$$

$$p = p(\mathbf{x}, t), \text{ pressure,}$$

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (u, v), \text{ velocity,}$$

$$f = f(\mathbf{x}, t), \text{ source term,}$$

$$c = c(\mathbf{x}), \text{ wave speed,}$$

$$\rho = \rho(\mathbf{x}), \text{ density.}$$

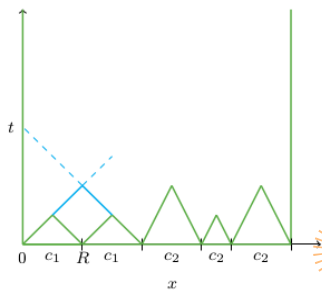
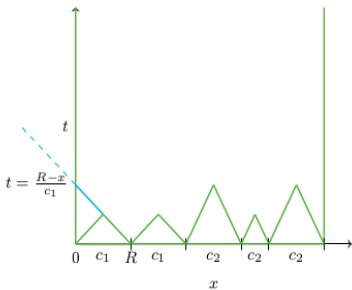
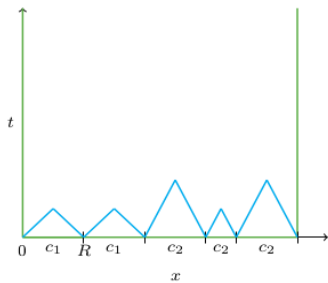
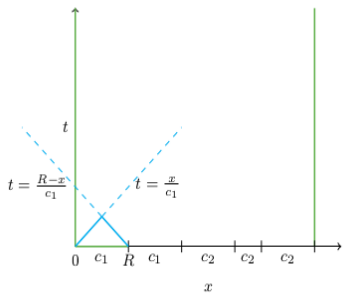
- Basis functions are local solutions of the initial PDEs
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
- Physical characteristics of the solution are included in the discrete solution
  
- Basis functions are discontinuous  $\rightarrow$  Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
  
- Implicit scheme

How to reduce computational costs ?

- computations on time slabs
- approximate inverse matrix
- solve element-by-element if mesh respects constraints
  - ↪ Tent-Pitcher algorithm
    - introduced for hyperbolic problems in spacetime domain,
    - consists in building a causal mesh, which respects the wave propagation speed,
    - no approximation of the inverse matrix,
    - conducive to parallel computing.



# Tent-pitching process 1d+t



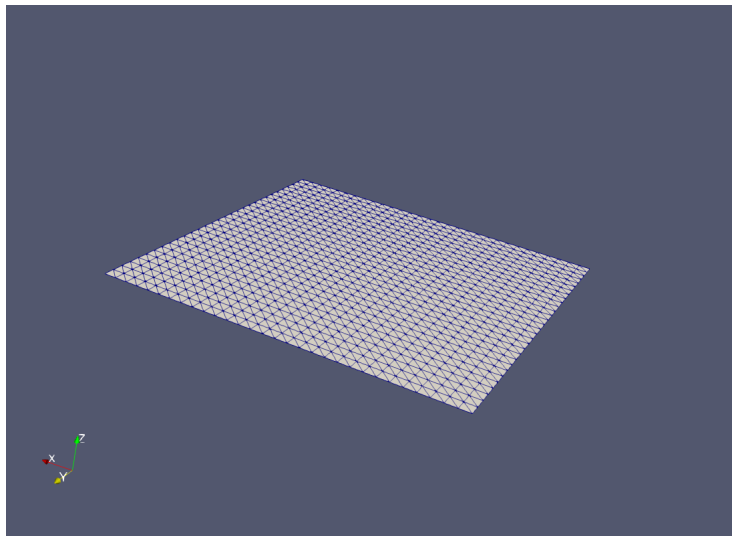


Figure: Initial spatial mesh



# Tent-pitching process $2d+t$

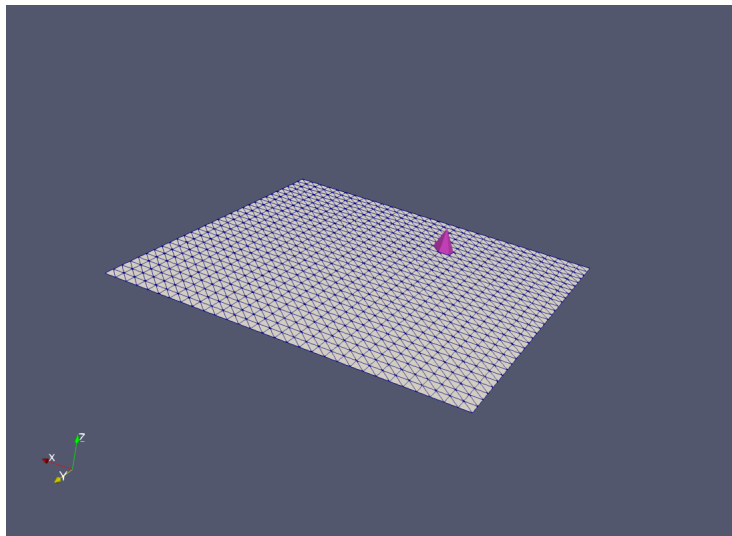


Figure: 1<sup>st</sup> iteration

# Tent-pitching process 2d+t

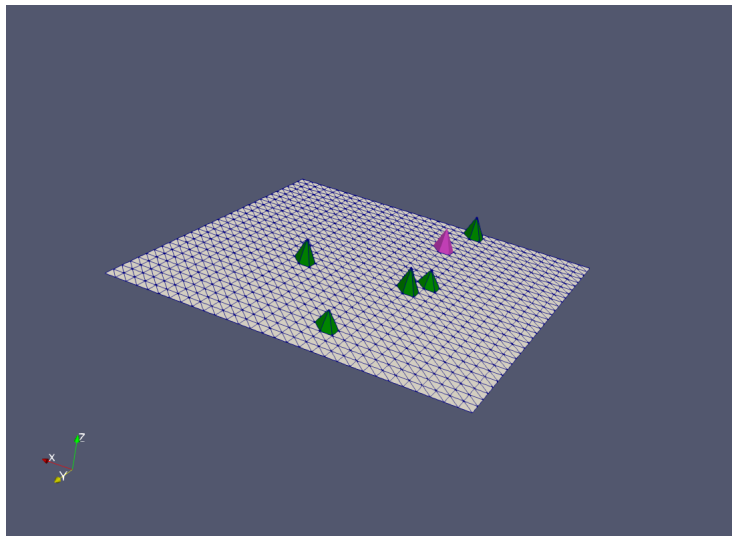


Figure: 5<sup>th</sup> iteration

# Tent-pitching process 2d+t

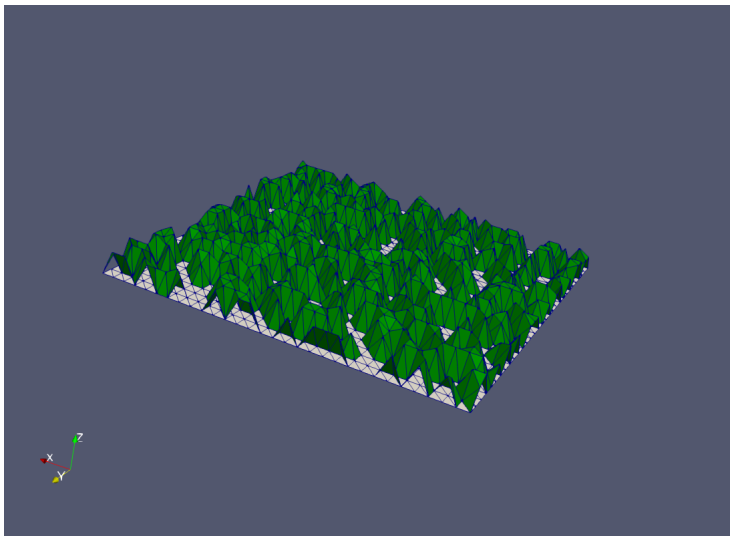


Figure: 450<sup>th</sup> iteration



# Tent-pitching process 2d+t

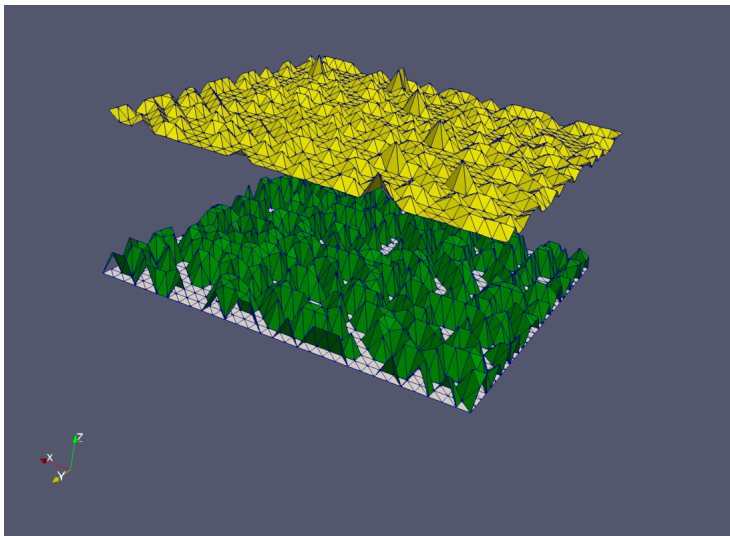


Figure: 6698<sup>th</sup> iteration

- Neumann boundary condition

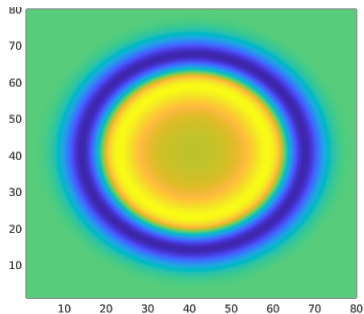


Figure: Pressure at time  $t=0.308$  with Trefftz-DG+Tent-Pitching

- polynomial basis functions

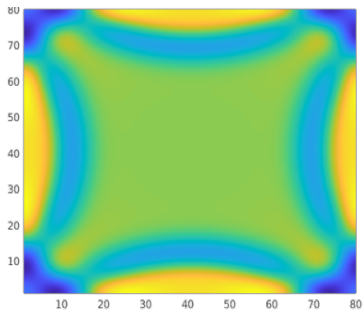
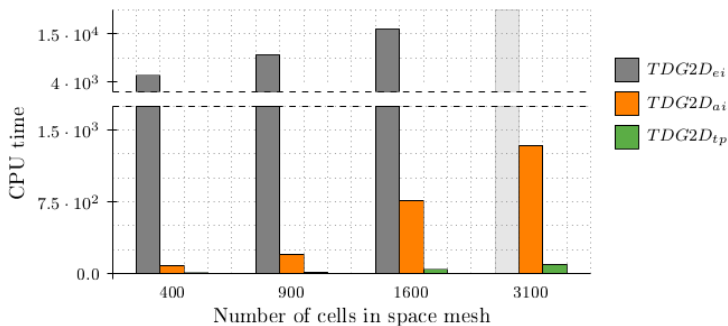


Figure: Pressure at time  $t=0.616$  with Trefftz-DG+Tent-Pitching



$$CPU_{exact} = 20 \times CPU_{approx}. \quad CPU_{approx} = 10 \times CPU_{tent-pitch}.$$

Thesis of E. Shishenina [2018]

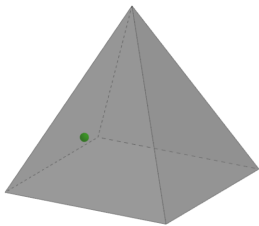
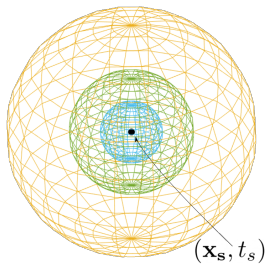
State of the art :

- not much work on absorbing boundary conditions with polynomials on spacetime Trefftz,
- no Perfectly Matched Layers for spacetime Trefftz.

Green's functions :

- are exact local solutions,
- have interesting behaviour at infinity,
- can be computed analytically inside the PML.

# Source Points





Notations:  $\mathcal{X} = \mathbf{x} - \mathbf{x}_s$ ,  $r = \sqrt{\mathcal{X} \cdot \mathcal{X}}$   
 $\mathcal{X}_1 = x - x_s$ ,  $\mathcal{X}_2 = y - y_s$ ,  $\mathcal{T} = t - t_s$

Cagniard-De Hoop method :

$$G_p(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{1}{2\pi \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}},$$

$$G_u(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_1}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s),$$

$$G_v(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{T} \mathcal{X}_2}{\rho r^2} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s).$$

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \times w \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \times z \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \times q \end{array} \right.$$

□  $w, z, q$ : test variables

$$\begin{aligned} & \int_{\mathcal{F}_h^{\text{out}}} \left( \frac{1}{c^2 \rho} p_h q + \rho u_h w + \rho v_h z \right) n_t + (u_h q + p_h w) n_x + (v_h q + p_h z) n_y \\ & + \int_{\mathcal{F}_h^{\text{in}}} (0.5 - \alpha) (\rho u_h w + \rho v_h z) n_t + (0.5 - \beta) \left( \frac{1}{c^2 \rho} p_h q \right) n_t \\ & + 0.5 \int_{\mathcal{F}_h^{\text{in}}} q (u_h n_x + v_h n_y) + p_h (w n_x + z n_y) \\ & = \int_{\mathcal{F}_h^{\text{in}}} - (0.5 + \alpha) (\rho u^0 w + \rho v^0 z) n_t - (0.5 + \beta) \left( \frac{1}{c^2 \rho} p^0 q \right) n_t \\ & - 0.5 \int_{\mathcal{F}_h^{\text{in}}} p^0 (w n_x + z n_y) + q (u^0 n_x + v^0 n_y) \end{aligned}$$

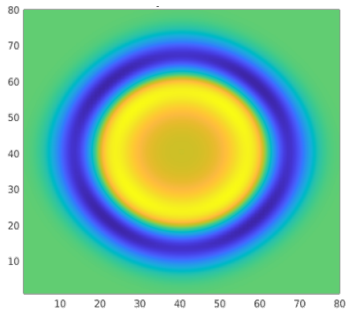


Figure: Pressure at time  $t=0.308$   
with Green's functions

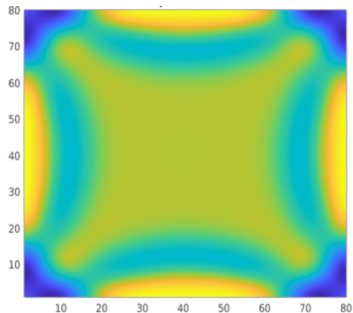
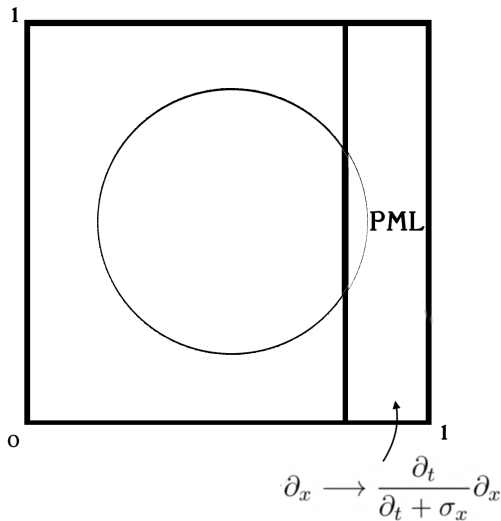


Figure: Pressure at time  $t=0.616$   
with Green's functions

# Perfectly Matched Layers



$$\rho \frac{\partial u}{\partial t} + \sigma_x u + \frac{\partial p}{\partial x} = 0,$$

$$\rho \left( \frac{\partial}{\partial t} + \sigma_x \right) u + \frac{\partial p}{\partial x} = 0,$$

$$\rho \left( \frac{\partial}{\partial t} + \sigma_x \right) \frac{\partial u}{\partial t} + \frac{\partial}{\partial t} \frac{\partial p}{\partial x} = 0,$$

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \frac{\sigma_x}{\frac{\partial}{\partial t} + \sigma_x} \frac{\partial p}{\partial x} = 0,$$

□  $\sigma_x$ : absorbing coefficient in the  $x$ -direction

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - p_1 = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - u_1 = 0, \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) p_1 = \sigma_x \frac{\partial p}{\partial x}, \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) u_1 = \sigma_x \frac{\partial u}{\partial x}. \end{array} \right.$$

□  $u_1, p_1$ : auxiliary variables

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - p_1 = 0, \times w \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \times z \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - u_1 = 0, \times q \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) p_1 = \sigma_x \frac{\partial p}{\partial x}, \times w_1 \\ \left( \frac{\partial}{\partial t} + \sigma_x \right) u_1 = \sigma_x \frac{\partial u}{\partial x}, \times q_1 \end{array} \right.$$

- $w, z, q$ : test variables
- $w_1, q_1$ : auxiliary test variables

# PML - Variational Formulation

$$\begin{aligned}
 & \int_{\mathcal{F}_h^{\text{out}}} \left( \frac{1}{c^2 \rho} p_h q + \rho u_h w + \rho v_h z \right) n_t + (u_h q + p_h w) n_x + (v_h q + p_h z) n_y \\
 & \quad + (u_1 q_1 + p_1 w_1) n_t - \sigma_x (u q_1 + p w_1) n_x \\
 & + \int_{\mathcal{F}_h^{\text{in}}} (0.5 - \alpha) (\rho u_h w + \rho v_h z + u_1 q_1) n_t + (0.5 - \beta) \left( \frac{1}{c^2 \rho} p_h q + p_1 w_1 \right) n_t \\
 & + 0.5 \int_{\mathcal{F}_h^{\text{in}}} q (u_h n_x + v_h n_y) + p_h (w n_x + z n_y) - \sigma_x (u q_1 + p w_1) n_x \\
 & = \int_{\mathcal{F}_h^{\text{in}}} - (0.5 + \alpha) (\rho u^0 w + \rho v^0 z + u_1^0 q_1) n_t - (0.5 + \beta) \left( \frac{1}{c^2 \rho} p^0 q + p_1^0 w_1 \right) n_t \\
 & - 0.5 \int_{\mathcal{F}_h^{\text{in}}} p^0 (w n_x + z n_y) + q (u^0 n_x + v^0 n_y) - \sigma_x (u^0 q_1 + p^0 w_1) n_x
 \end{aligned}$$



# PML - Basis functions

- $G_p^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \exp^{-A} \cos(B) G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)$
- $G_u^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left( \mathcal{T} \mathcal{X}_1 \cos(B) + h \mathcal{X}_2 \sin(B) \right)$
- $G_v^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma_x) = \frac{\exp^{-A} G_p(\mathbf{x}, t, \mathbf{x}_s, t_s)}{\rho r^2} \left( \mathcal{T} \mathcal{X}_2 \cos(B) + h \mathcal{X}_1 \sin(B) \right)$

with

$$A = A(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_1}{r^2} \Sigma(x, x_s) \mathcal{T},$$

$$B = B(\mathbf{x}, t, \mathbf{x}_s, t_s) = \frac{\mathcal{X}_2}{r^2} \Sigma(x, x_s) h,$$

$$\Sigma(x, x_s) = \int_0^{\mathcal{X}_1} \sigma(\mathbf{x}, \mathbf{x}_s) d\mathcal{X},$$

$$h = h(\mathbf{x}, t, \mathbf{x}_s, t_s) = \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}.$$

$$\left\{ \begin{array}{l} \rho \frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} - \sigma_x \frac{\partial q_1}{\partial x} = 0, \\ \rho \frac{\partial z}{\partial t} + \frac{\partial q}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial q}{\partial t} + \frac{\partial w}{\partial x} + \frac{\partial z}{\partial y} - \sigma_x \frac{\partial w_1}{\partial x} = 0, \\ \left( \frac{\partial}{\partial t} - \sigma_x \right) q_1 = -q, \\ \left( \frac{\partial}{\partial t} - \sigma_x \right) w_1 = -w. \end{array} \right.$$

- $G_q^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = G_p^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$
- $G_w^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = G_u^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$
- $G_z^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, \sigma) = G_v^{pml}(\mathbf{x}, t, \mathbf{x}_s, t_s, -\sigma)$

# PML - Results

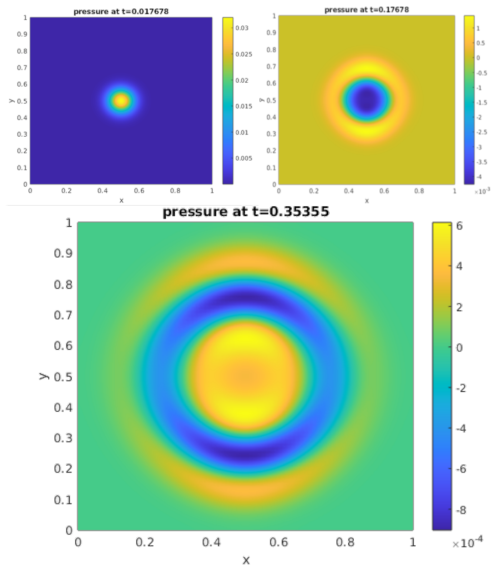


Figure: Pressure at different time steps with PML

- Coupling PML with computational domain
- Parallelization
- Extend to unstructured meshes
- Extend 2D to 3D

# References

Bérenger.

A perfectly matched layer for the absorption of electromagnetic waves.

*Journal of Computational Physics*, 114(2):185 – 200, 1994.

Björn Engquist and Andrew Majda.

Absorbing boundary conditions for numerical simulation of waves.

*Proceedings of the National Academy of Sciences*, 74(5):1765–1766, 1977.

I. Herrera.

Trefftz method: A general theory.

*Eq.*, 16:561–580, 11 2000.

Ralf Hiptmair, Andrea Moiola, and Ilaria Perugia.

A survey of trefftz methods for the helmholtz equation.

06 2015.

V.D. Kupradze and M.A. Aleksidze.

An approximate method of solving certain boundary-value problems.

*Soobšč. Akad. Nauk Gruzin. SSR*, 30:529–536, 01 1963.

Rudolf Mathon and R.L. Johnston.

The approximate solution of elliptic boundary-value problems by fundamental solutions.

*SIAM Journal on Numerical Analysis*, 14, 09 1977.

Ilaria Perugia, Joachim Schöberl, Paul Stocker, and Christoph Wintersteiger.

Tent pitching and trefftz-dg method for the acoustic wave equation.

*Computers & Mathematics with Applications*, 2020.

Elvira Shishenina.

*Space-Time Discretization of Elasto-Acoustic Wave Equation in Polynomial Trefftz-DG Bases.*

PhD thesis, 2018.

Erich Trefftz.

Ein gegenstück zum ritzschen verfahren.

In *Proc. 2nd Int. Cong. Appl. Mech. Zurich*, pages 131–137, 1926.

Alper Üngör and Alla Sheffer.

Tent-pitcher: A meshing algorithm for space-time discontinuous galerkin methods.

In *In proc. 9th int'l. meshing roundtable*, pages 111–122, 01 2000.

The background of the slide features a series of wavy, concentric orange lines that create a sense of depth and movement, resembling a stylized fingerprint or a topographical map. The lines are thin and closely spaced, filling the entire frame.

Thank you for your attention !