



# A Coq Formalization of the Bochner Integral

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**RESEARCH  
REPORT**

**N° 9456**

February 2022

Project-Teams Toccata and  
Serena





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Research Report n° 9456 — February 2022 — 14 pages

**Abstract:** The Bochner integral is a generalization of the Lebesgue integral, for functions taking their values in a Banach space. Therefore, both its mathematical definition and its formalization in the Coq proof assistant are more challenging as we cannot rely on the properties of real numbers. Our contributions include an original formalization of simple functions, Bochner integrability defined by a dependent type, and the construction of the proof of the integrability of measurable functions under mild hypotheses (weak separability). Then, we define the Bochner integral and prove several theorems, including dominated convergence and the equivalence with an existing formalization of Lebesgue integral for nonnegative functions.

**Key-words:** Formal proof, Coq, Measure theory, Bochner integration

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## Une formalisation en Coq de l'intégrale de Bochner

**Résumé :** L'intégrale de Bochner est une généralisation de l'intégrale de Lebesgue pour des fonctions à valeurs dans un espace de Banach. Sa définition mathématique et sa formalisation dans l'assistant de preuve Coq en sont donc plus difficiles puisque l'on ne peut pas s'appuyer sur les propriétés des nombres réels. Nos contributions incluent une formalisation originale des fonctions simples, l'intégrabilité de Bochner définie par un type dépendant, et la construction de la preuve de l'intégrabilité de fonctions mesurables sous une hypothèse de séparabilité faible. Puis, nous définissons l'intégrale de Bochner et prouvons plusieurs théorèmes, dont la convergence dominée et l'équivalence avec une formalisation préexistante de l'intégrale de Lebesgue pour les fonctions mesurables positives.

**Mots-clés :** Preuve formelle, Coq, Théorie de la mesure, Intégrale de Bochner

## 1 Introduction

This work is devoted to the Coq formalization of the Bochner integral. Among a huge variety of integrals, e.g. see [7], the Bochner integral [3] is a generalization of the Lebesgue integral, for real-valued functions, to the case of functions taking their values in a Banach space, i.e. a complete normed vector space. Thus, it is perfectly suited for the study of partial differential equations involving time and space variables. For instance, given a real number  $T > 0$  and a regular enough space domain  $\Omega \subset \mathbb{R}^3$ , one might be interested in integrating functions mapping the time interval  $[0, T]$  to the Hilbert space  $L^2(\Omega)$  of functions  $\Omega \rightarrow \mathbb{R}$  that are square Lebesgue-integrable, which of course is also a Banach space. In a formal proof setting, it also allows us to have a single definition and set of theorems for integrating on either  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{R}^n$ .

The building of the Bochner integral follows a similar scheme to that of the Lebesgue integral: first consider simple functions, that only take a finite number of values, define their integral by summing terms of the form *measure of preimage*  $\times$  *value*, and then extend to the limit of simple functions. The main difference here is the absence of order in a normed vector space, which prevents the use of monotonicity and of the LUB property, as in  $\mathbb{R}$ , and thus prohibits infinite terms in the integral. Instead, it relies on completeness, and on the additional assumption of separability of the Banach space, or at least of the range of the integrand function. Note that some mathematical authors and the other formalizations prefer second countability, which is stronger than separability in general, but actually equivalent in the case of metric spaces (and Banach spaces are). Rather than the seminal paper by S. Bochner, or the monograph by J. Mikusiński [13], we chose to follow the modern presentation of the course in real analysis by G. Teschl [16].

The formalization is available at the following link:

[https://lipn.univ-paris13.fr/coq-num-analysis/tree/Bochner.1.0/Lebesgue/bochner\\_integral](https://lipn.univ-paris13.fr/coq-num-analysis/tree/Bochner.1.0/Lebesgue/bochner_integral)  
where the tag `Bochner.1.0` corresponds to the code of this article.

The paper is organized as follows. After drawing up the state of the art in Section 2, Section 3 presents the Coq formalization of the Lebesgue integral [6] we rely on, and Section 4 some preliminary topological results. Section 5 is dedicated to simple functions. Bochner integrability is addressed in Section 6, and the Bochner integral is defined in Section 7. Finally, Section 8 concludes and gives some perspectives.

## 2 State of the art

Measure theory and nonnegative Lebesgue integration has been formalized in formal proof assistants such as Mizar<sup>1</sup>, PVS [15], Isabelle/HOL [14], HOL4<sup>2</sup>, Lean [8, 10], and Coq<sup>3</sup>. We may cite [2, 9] in Mizar, dedicated libraries in PVS<sup>4</sup>, Isabelle/HOL<sup>5</sup>, and Lean<sup>6</sup>, [12] in HOL4, and dedicated libraries<sup>7</sup> and [6] in Coq.

There are few proof assistants that provide the Bochner integral, while the Riemann or Lebesgue integrals are more widespread. To the best of our knowledge, there are already two available formalizations.

First, Isabelle/HOL provides Bochner integrability and integral and the dominated convergence theorem [1]. Their goal is probability and the central limit theorem and this generic integral easily

<sup>1</sup><https://fm.mizar.org/>

<sup>2</sup><https://hol-theorem-prover.org/>

<sup>3</sup><https://coq.inria.fr/refman/>

<sup>4</sup>[https://shemesh.larc.nasa.gov/fm/ftp/larc/PVS-library/library/measure\\_integration.html](https://shemesh.larc.nasa.gov/fm/ftp/larc/PVS-library/library/measure_integration.html), <https://shemesh.larc.nasa.gov/fm/ftp/larc/PVS-library/library/lebesgue.html>

<sup>5</sup>[https://isabelle.in.tum.de/dist/library/HOL/HOL-Analysis/Measure\\_Space.html](https://isabelle.in.tum.de/dist/library/HOL/HOL-Analysis/Measure_Space.html), [https://isabelle.in.tum.de/dist/library/HOL/HOL-Analysis/Nonnegative\\_Lebesgue\\_Integration.html](https://isabelle.in.tum.de/dist/library/HOL/HOL-Analysis/Nonnegative_Lebesgue_Integration.html)

<sup>6</sup>[https://leanprover-community.github.io/mathlib\\_docs/measure\\_theory/measure/measure\\_space.html](https://leanprover-community.github.io/mathlib_docs/measure_theory/measure/measure_space.html), [https://leanprover-community.github.io/mathlib\\_docs/measure\\_theory/integral/lebesgue.html](https://leanprover-community.github.io/mathlib_docs/measure_theory/integral/lebesgue.html)

<sup>7</sup><https://github.com/math-comp/analysis/>

encompasses  $\mathbb{C}$  and  $\mathbb{R}^n$ . They assume a second-countable topology (while we add the weaker separability hypothesis at the only needed point). Their definitions are rather similar to ours except for Bochner integrability:  $f$  is Bochner-integrable if and only if  $f$  is measurable and its  $L^1$ -norm is finite (which is equivalent to saying that  $f$  is absolutely integrable). See our definition in Section 6.

Second and last, Lean provides Bochner integrability and integral, the dominated convergence theorem and the Fubini theorem [17]. They assume second-countable real Banach space. The main difference is that the quotient space  $L^1$  is defined and that the Bochner integral applies to equivalent classes of functions. Moreover, the definition of the integral is very different from ours: they extend to  $L^1$  the continuous linear map that is the integral on integrable simple functions.

As a conclusion on this state of the art, the proved theorems are similar, contrary to the definitions. A difference is that they require second-countability while we locally require weak separability. Another difference is the simple function definitions: they both rely on the fact that the image is a finite set while we use a dependent type. See our definition in Section 5.

The present formalization uses the real standard library of Coq, based on [11], and the Coquelicot library [4] extension. These libraries provide support for classical real numbers, which is consistent with the fact that the mathematics we are formalizing are based on classical logic, as most of the real analysis results do.

### 3 The Lebesgue integral in Coq

This work is based on several existing Coq libraries. We of course rely on the standard library, and in particular for the real numbers [11].

We also rely on Coquelicot [4], which is a conservative extension of the real numbers. We use several features of this library: the extended real numbers and their operations; the algebraic hierarchy in particular for the normed modules and Banach spaces; the underlying topology based on filters. We refer the reader to [4] for more details.

We have also taken inspiration from a recent work defining the Lebesgue integral for non-negative functions [6]<sup>8</sup> and require its latest version. Here are the important design choices of this library. The measurability of subsets of  $X$  is formalized as an inductive type parameterized by  $\text{gen} : (X \rightarrow \text{Prop}) \rightarrow \text{Prop}$ , that represents the corresponding generated  $\sigma$ -algebra. When  $X$  is a metric space, in Coquelicot  $X : \text{UniformSpace}$ , one generally uses the Borel  $\sigma$ -algebra that is generated by all the open subsets.

Simple functions are based on *lists*. More precisely, a function is a simple function when its image is included in a finite list of values. Then this list may be canonized (by removing unused values and duplicates and sorting the values) and this canonical list is used to compute the integral of a simple function (provided a given measure). This cannot be applied here as Banach space values cannot be sorted, contrary to real numbers.

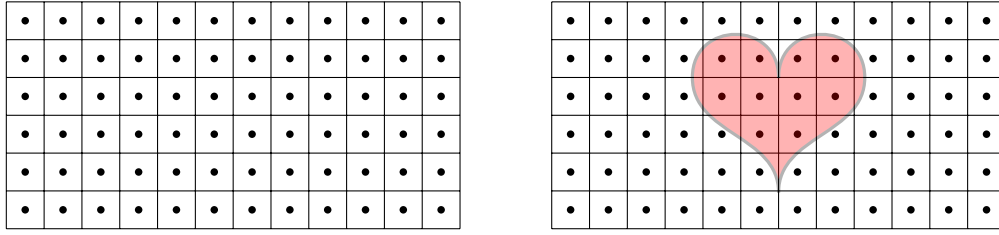
The integral for nonnegative measurable functions is then defined as in mathematics textbooks:

$$\int_{\mathcal{M}_+} f d\mu = \sup_{\substack{\psi \in \mathcal{SF}_+ \\ \psi \leq f}} \int_{\mathcal{SF}_+} \psi d\mu$$

with  $\mathcal{SF}_+$  being the set of nonnegative simple functions, and  $\mathcal{M}_+$  the set of nonnegative measurable functions. The integral of a nonnegative measurable function  $f$  is the supremum of the integral of the nonnegative measurable simple functions  $\psi$  less than or equal to  $f$  pointwise. Basic lemmas (such as monotony, scalar multiplication, addition) are provided in [6], as well the Beppo Levi (monotone convergence) theorem and Fatou's lemma.

For the sake of readability in the sequel, we do not always specify the scope in the Coq scripts.

<sup>8</sup><https://lipn.univ-paris13.fr/MILC/>



(a) The full space (in 2D) is separable with the given points.

(b) The subset  $Y$  is heart-shaped. For ensuring its separability, we may provide points inside  $Y$ . But it is easier to only ensure weak separability by considering the same points as on the left, that may or may not belong to  $Y$ .

Figure 1: Separability and weak separability: a figurative view.

## 4 Some topology in normed modules

We present here some preliminary needed results: a few lemmas above `Coquelicot` are given in Section 4.1 and separability is described in Section 4.2.

### 4.1 Additions to `Coquelicot`

In order to formalize the Bochner integral in `Coq`, one needs at first some topology in normed vector spaces, and especially in Banach spaces. In this development, we choose to use the existing formalization of filters and open subsets of `Coquelicot` [4]. The main notion required is the limit of sequences, which is straightforwardly given within `Coquelicot`. Given a type  $S : \text{UniformSpace}$ , we may denote

**Definition** `lim_seq`  $(u : \text{nat} \rightarrow S) := \text{lim} (\text{filtermap } u \text{ eventually})$ .

Starting from this definition, one can easily prove its equivalence with the more common textbook definition, that may also be more practical than filters in some cases.

**Lemma** `is_lim_seq_epsilon`  $\{A : \text{AbsRing}\} \{E : \text{NormedModule } A\} :$

$\forall u : \text{nat} \rightarrow E, \forall l : E, \text{is\_lim\_seq } u \ l \leftrightarrow$   
 $\forall \varepsilon, 0 < \varepsilon \rightarrow \exists N, \forall n, N \leq n \rightarrow \|\text{minus } (u \ n) \ l\| < \varepsilon$ .

Another useful lemma we may derive from this notion is the following, stating the (Borel) measurability of a pointwise limit of measurable functions in any vector space. (Actually, this is not as easy as in the real case, where one may use `LimSup` and `LimInf` to get a simple proof.)

**Lemma** `measurable_fun_lim_seq`  $\{X : \text{Set}\} \{\text{gen} : (X \rightarrow \text{Prop}) \rightarrow \text{Prop}\} :$

$\forall s : \text{nat} \rightarrow X \rightarrow E, (\forall n, \text{measurable\_fun } \text{gen } \text{open } (s \ n)) \rightarrow$   
 $\forall f : X \rightarrow E, (\forall x : X, \text{is\_lim\_seq } (\lambda n \Rightarrow s \ n \ x) (f \ x)) \rightarrow \text{measurable\_fun } \text{gen } \text{open } f$ .

Similarly, we define equivalent Cauchy sequences in a normed vector space, which may be easier to handle than Cauchy filters in practice:

**Definition** `NM_Cauchy_seq`  $\{A : \text{AbsRing}\} \{E : \text{NormedModule } A\} (u : \text{nat} \rightarrow E) : \text{Prop} :=$

$\forall \varepsilon, \varepsilon > 0 \rightarrow \exists n, \forall p \ q, p \geq n \rightarrow q \geq n \rightarrow \text{ball\_norm } (u \ p) \ \varepsilon \ (u \ q)$ .

### 4.2 Separability

Next, we need a formalization of separability in normed vector spaces. Let us remind the mathematical definition of this property.

**Definition 1** (separability). *A topological space  $(E, \tau)$  is said separable when it contains a countable dense subset, i.e. when there exists a sequence  $(u_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$  such that  $(U$  is any nonempty open subset)*

$$\forall U \in \tau, U \neq \emptyset \Rightarrow U \cap \{u_n \mid n \in \mathbb{N}\} \neq \emptyset.$$

It suffices in our case to define a more practical weaker version in which we do not require the countable part to dwell inside the separable one.

**Definition 2** (weak separability). *Let  $(E, \tau)$  be a topological space. A subset  $Y \subseteq E$  is said weakly separable in  $E$  when there exists a sequence  $(u_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$  such that*

$$\forall U \in \tau, U \cap Y \neq \emptyset \Rightarrow U \cap \{u_n \mid n \in \mathbb{N}\} \neq \emptyset.$$

For example, let  $E$  be  $\mathbb{R}$  equipped with the usual topology,  $Y$  be  $\mathbb{R} \setminus \mathbb{Q} \subseteq E$  and  $(u_n)_{n \in \mathbb{N}}$  be a sequence whose range is exactly  $\mathbb{Q}$ . Then  $Y$  (with the induced topology) and  $(u_n)_{n \in \mathbb{N}}$  does not satisfy the first definition since  $(u_n)_{n \in \mathbb{N}}$  is not a sequence in  $Y$ , but we may say from the second definition that  $Y$  is weakly separable in  $E$ , thus avoiding the building of a sequence of irrationals in  $Y$ . A more visual and figurative example is given in Figure 1.

For normed modules, the norm induces the topology, therefore we get the following characterization, easier to formalize.

**Lemma 3** (weak separability in normed vector spaces). *Let  $(E, \|\cdot\|)$  be a normed vector space. A subset  $Y \subseteq E$  is weakly separable in  $E$  if and only if there exists a sequence  $(u_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$  such that*

$$\forall y \in Y, \forall \varepsilon > 0, \exists n \in \mathbb{N}, \|y - u_n\| < \varepsilon.$$

Given `E:NormedModule R:AbsRing`, we define it in `Coq` as

```
Definition NM_seq_separable_weak (u : nat → E) (P : E → Prop) : Prop :=
  ∀ x : E, P x → ∀ ε : posreal, ∃ n, ball_norm x ε (u n).
```

Note that the sequence  $u$  is explicit in this definition.

For instance, `Coq` real numbers are weak separable.

```
Lemma NM_seq_separable_weakR :
```

```
  NM_seq_separable_weak (λ n ⇒ Q2R (bij_NQ n)) (λ _ : R_NormedModule ⇒ True).
```

The sequence  $u$  ranges over the rationals, relying on the bijection `bij_NQ` from  $\mathbb{N}$  onto  $\mathbb{Q}$ .

## 5 Formalizing simple functions

A first important step towards Bochner integrability and integral is the definition of simple functions on a Banach space. Even if the Lebesgue integral also needs simple functions [6], their formalization is not applicable in our case and we have provided an original definition described in Section 5.1, as well as the Bochner-integrability. The value of the integral is given in Section 5.2.

### 5.1 Definition and properties

Following [16], we start by formalizing simple functions. Then in Section 7, we define the Bochner integral as a limit of integrals of simple functions (as for the Lebesgue integral).

Let us consider a measurable space  $(X, \Sigma)$ , and  $E$  a normed vector space that is assumed to be equipped with its Borel  $\sigma$ -algebra (generated by all open subsets). In `Coq`, we have `X : Set`, the  $\sigma$ -algebra  $\Sigma$  is represented by some generator `gen : (X → Prop) → Prop` (see Section 3), `E : NormedModule A` with `A : AbsRing`, and its Borel  $\sigma$ -algebra is generated by the generic `open : (E → Prop) → Prop`. Then, the mathematical definition of (measurable) simple function is the following.

**Definition 4** (simple function). *A function  $f : X \rightarrow E$  is said simple when its range is finite and all the preimages are measurable.*

In [6], as explained in Section 3, the simple functions for the Lebesgue integral were defined by the existence of a list that collects the values taken by the function. This was chosen because by forcing the list to be sorted, and not to contain any duplicates or unnecessary value, one gets a canonical representation of a simple function. However, this is no longer possible with vector-valued functions, where no order can be used on the image space. But it is known that Definition 4 is equivalent to the two following characterizations.



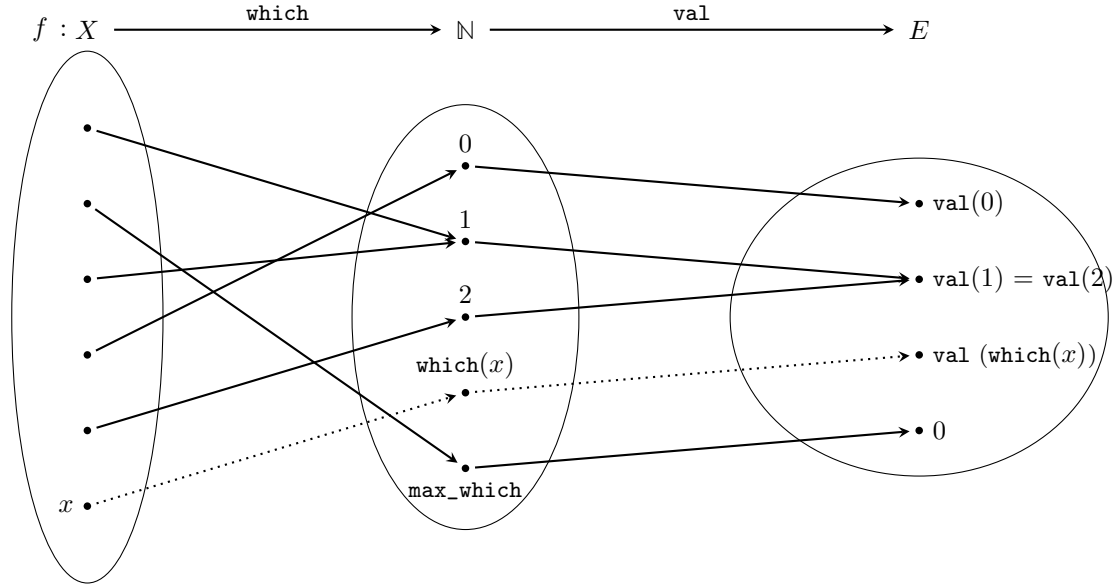


Figure 2: Coq representation of simple functions.

Given a cutting of the set  $X$  into  $\text{max\_which}+1$  pairwise disjoint measurable parts, the function  $\text{which}: X \rightarrow \text{nat}$  maps elements of each part to a distinct index, an integer in the range  $\llbracket 0, \text{max\_which} \rrbracket$ . Then, the function  $\text{val}: \text{nat} \rightarrow E$  maps each index of the previous range to some vector value in  $E$ . The greatest index  $\text{max\_which}$  is mapped to zero. The represented simple function  $f: X \rightarrow E$  is actually the composition of  $\text{which}$  and  $\text{val}$ . Note that several indices (here 1 and 2) may be mapped to the same vector value  $v$  (here  $\text{val } 1$  equals  $\text{val } 2$ ), meaning that the preimage  $f^{-1}(\{v\})$  is actually the (disjoint) union of the parts mapped to the indices. Thus, the representation is not unique. Of course, parts need not be convex, nor connected, and some of them may be empty, including the last one, associated with the value zero.

**Lemma 5** (characterization 1). *A function  $f: X \rightarrow E$  is simple if and only if it is a linear combination of characteristic functions of measurable subsets.*

**Lemma 6** (characterization 2). *A function  $f: X \rightarrow E$  is simple if and only if there exists a finite partition  $(A_i)_{i \in I}$  of  $X$  such that for all  $i \in I$ ,  $f$  is constant over  $A_i$ , and  $A_i$  is measurable.*

We tried to formalize both in Coq and the second proved to be much more efficient to handle. The chosen data structure takes the following form.

```
Record simpl_fun := mk_simpl_fun {
  which : X → nat;
  val : nat → E;
  max_which : nat;
  ax_val_max_which : val max_which = zero;
  ax_which_max_which : ∀ x : X, which x ≤ max_which;
  ax_measurable : ∀ n : nat, n ≤ max_which → measurable gen (λ x ⇒ which x = n);
}.
```

Such a record tells us that in order to build a simple function, there are three values that should be given to Coq: `which`, `val`, and `max_which`, see Figure 2, and several proofs. The function `which` corresponds to a cutting of the space  $X$ , or to an index in the finite set of preimages. The integer `max_which` is the maximal value allowed for `which` (it ensures the finiteness of the cutting). The function `val` provides the value corresponding to a given integer.

For instance, suppose we want to construct in Coq the simple function corresponding to  $f$  of type  $X \rightarrow E$  with the finite partition  $(A_i)_{i \in I}$  of Lemma 6. First of all, because  $I$  is finite, we can suppose that it is of the form  $\llbracket 0, n \rrbracket$  for some  $n \in \mathbb{N}$  (implicitly, here we also suppose that  $X$  is not empty), then:

- the number of parts minus one in our cutting (i.e.  $|I| - 1$  or  $n$  in the above description) is stored into `max_which`;
- because  $(A_i)_{i \in I}$  forms a pairwise disjoint cover of  $X$ , for each  $x \in X$ , there exists a unique  $i \in I$  such that  $x \in A_i$ . The function `which` associates this  $i \in \mathbb{N}$  with each  $x \in X$ . So inside Coq, this becomes `which : X → nat`.
- Finally, for every  $i \in I$ ,  $f$  takes over  $A_i$  a value  $v_i \in E$ . This is stored inside `val i`, for every  $i : \text{nat}$ . As we see below, the value of `val` for  $i > n$  does not matter in our formalization.

In addition to these three values, there is a need for properties that ensure such a structure correctly represents a simple function and behaves nicely.

- First, we need to ensure that `val` takes the value `zero` on `max_which`. The proof is stored into `ax_val_max_which`. This is not a mathematical consideration but a commodity in order to manipulate integrability of simple functions. This allows us to deal with the preimage of 0 separately from the others, and especially to allow  $A_n$  to be of infinite measure for integrable simple functions.
- Secondly, because `which` has its values in `nat` while mathematically, it should have it in  $\llbracket 0, \text{max\_which} \rrbracket$ , we must ensure that `which` does not exceed `max_which`. The proof is stored into `ax_which_max_which`.
- Finally, we have to ensure the measurability of the simple function, i.e. the measurability of all its preimages. The proof is stored into `ax_measurable`.

It is convenient to use the record defining a simple function as a function, so we also define the following coercion.

**Definition** `fun_sf (sf : simpl_fun) : X → E := λ x ⇒ sf.val (sf.which x).`

*(\* So we may write "sf x" for sf : simpl\_fun E gen, and x : X. \*)*

**Coercion** `fun_sf : simpl_fun >-> Funclass.`

For instance, for the indicator function of a (measurable) subset  $A$ , we would have `which` that returns 0 on  $A$  and 1 on  $\neg A$ ; `max_which` that is 1; and `val(n)` that is 1 when  $n = 0$  and 0 elsewhere. All assumptions hold. Therefore, for  $x \in A$ , we have `sf x = sf.val (sf.which x) = sf.val 0 = 1`, and for  $x \in \neg A$ , we have `sf x = sf.val (sf.which x) = sf.val 1 = 0`.

Note that the type `simpl_fun` actually carries more structure than just the definition of simple functions. As a consequence, the representation of a simple function  $f$  by an instance of the record is not unique. Indeed, the same value  $v \in E$  could be associated with several distinct indices, meaning that the actual preimage  $f^{-1}(\{v\})$  could be represented by several (pairwise disjoint) parts with distinct indices. And of course, this may occur for the value `zero`, already associated with the index `max_which`.

Nevertheless, we may recover usual properties about simple functions. An interesting one is their measurability as defined in [6].

**Lemma** `measurable_fun_sf : ∀ sf : simpl_fun E gen, measurable_fun gen open sf.`

We have also explicitly constructed an instance of `simpl_fun E gen` for the sum, opposite, subtraction, scalar product, norm or power of simple functions.

As an example, given two simple functions  $f, g : X \rightarrow E$  with respective decomposition  $(A_i)_{i \in \llbracket 0, n \rrbracket}$  and  $(B_j)_{j \in \llbracket 0, m \rrbracket}$ , we get a correct decomposition for  $f+g$  with  $(A_i \cap B_j)_{(i,j) \in \llbracket 0, n \rrbracket \times \llbracket 0, m \rrbracket}$ . To formalize this decomposition in Coq, we used an explicit bijection between  $\llbracket 0, n \rrbracket \times \llbracket 0, m \rrbracket$  and  $\llbracket 0, (n+1) \cdot (m+1) - 1 \rrbracket$ .

We use in the sequel the following notations:

`"sf + sg" := sf_plus sf sg.`

`"- sg" := sf_scal (opp one) sg.`

`"sf - sg" := sf_plus sf (sf_scal (opp one) sg).`

```

"a · sf" := sf_scal a sf.
"|| sf ||" := sf_norm sf.
"sf ^ p" := sf_power sf p.

```

Over these simple functions, one also needs to define the integrability property stating that preimages have a finite measure, except possibly for that of zero (corresponding at least to the index `max_which`). This allows in Section 5.2 to sum terms of the form *measure of preimage*  $\times$  *value*. For the sake of smoothness, we require all the parts of index smaller than `max_which` to have finite measure. It therefore prevents parts of infinite measure with `val n = zero` and `n < max_which`. This is allowed in mathematics but impractical in formal proofs and moreover, this case may be kept out (see below).

**Definition** `integrable_sf` (`sf : simpl_fun`) :=  
 $\forall n, n < \text{sf.max\_which} \rightarrow \text{is\_finite } (\mu (\lambda x \Rightarrow \text{sf.which } x = n))$ .

Indeed, this definition is not equivalent to the usual mathematical definition of integrability. Simple functions whose representation involves zero values for indices `n < max_which` are not recognized as integrable when the corresponding preimages have infinite measure. But, both definitions match when we ensure that the only part associated with the value zero is the last one (with index `max_which`). And this can be proved through the following result.

**Lemma** `sf_remove_zeros` (`sf : simpl_fun E gen`) :  
 $\{ \text{sf}' : \text{simpl\_fun } E \text{ gen} \mid (\forall x : X, \text{sf } x = \text{sf}' x) \wedge (\forall n, n < \text{sf'.max\_which} \rightarrow \text{sf'.val } n \neq \text{zero}) \}$ .

Here, we used a `sig` from Coq, that is a dependent type containing an instance of `simple_fun E gen` together with a proof of  $(\forall x : X, \text{sf } x = \text{sf}' x)$  and  $(\forall n, n < \text{sf'.max\_which} \rightarrow \text{sf'.val } n \neq \text{zero})$ . So we may use it to remove unwanted zero values from our structure representing a simple function.

The proof of this proposition is straightforward though tedious: browsing all the values in `val`, suppressing the redundant zeros and redefining `which` in order to have `which x = sf'.max_which` each time we have that `sf.val (sf.which x) = zero`.

Note also that the previous decomposition for the sum of simple functions maintains the integrability.

## 5.2 The Bochner integral for simple functions

Now that we have defined simple functions, we are able to define the integral of such functions.

Following Section 5.1, let us now consider a measure space  $(X, \Sigma, \mu)$  where  $\mu$  is a measure on the measurable space  $(X, \Sigma)$ , and  $E$  is now a normed vector space over  $\mathbb{R}$ . In Coq, we have now  `$\mu$  : measure gen` and  `$E$  : NormedModule R_AbsRing`. Then, we stick to the following mathematical definition.

**Definition 7** (Bochner integral of simple function). *Given an integrable simple function  $s$  of type  $X \rightarrow E$ , its Bochner integral (relatively to measure  $\mu$  on  $X$ ) is defined by*

$$\int s d\mu := \sum_{v \in E} \mu(f^{-1}\{v\}) \cdot v,$$

with the convention  $\infty \cdot 0_E := 0_E$ .

The former sum is finite according to the definition of integrable simple function, and this may be translated in our formalization by

**Definition** `BInt_sf` ( `$\mu$  : measure gen`) (`sf : simpl_fun _ gen`) : `E` :=  
`sum_n ( $\lambda n \Rightarrow \text{scal (real } (\mu (\text{nth\_carrier sf } n))) (\text{sf.val } n) (\text{sf.max\_which})$ )`.

where `nth_carrier` is the preimage defined by

**Definition** `nth_carrier` (`sf : simpl_fun`) (`n : nat`) : `(X  $\rightarrow$  Prop)` :=  `$\lambda x \Rightarrow \text{sf.which } x = n$` .

From this definition we may derive the usual properties of the integral such as linearity,

**Lemma** `BInt_sf_lin` {`sf sg : simpl_fun E gen`} (`a b : R`) :  
`integrable_sf  $\mu$  sf  $\rightarrow$  integrable_sf  $\mu$  sg  $\rightarrow$`   
`BInt_sf  $\mu$  (a · sf + b · sg) = a · (BInt_sf  $\mu$  sf) + b · (BInt_sf  $\mu$  sg).`

Note that  $\cdot$  is the scalar multiplication in the normed vector space. The mathematical proof implies some factorizations and finite sums inversion. This is basic linear algebra, but has proved slightly tedious. Here, the main difficulty is to handle the measure that takes values in  $\overline{\mathbb{R}}$ , and to manage separately:

- $\alpha \cdot v$  when  $v \in E$  and  $\alpha \in \mathbb{R}$  (i.e. with Coquelicot formalism, `is_finite`  $\alpha$ );
- $+\infty \cdot 0_E$ , which equals  $0_E$  by mathematical and Coquelicot conventions.

Another usual property is the triangle inequality.

**Lemma** `norm_Bint_sf_le` (`sf : simpl_fun E gen`) :  $\| \text{BInt\_sf } \mu \text{ sf} \| \leq \text{BInt\_sf } \mu \| \text{sf} \|$ .

This lemma reduces by definition to the usual triangle inequality for a finite sum.

As explained, our formalization of simple function is not canonical, so it must be proved that the value of `BInt_sf sf` only depends on the values taken by `sf` and not on the cutting we chose to represent this function. This is stated in the following extensionality lemma:

**Lemma** `BInt_sf_ext` {`sf sf' : simpl_fun E gen`} :  
`integrable_sf`  $\mu$  `sf`  $\rightarrow$  `integrable_sf`  $\mu$  `sf'`  $\rightarrow$   
 $(\forall x : X, \text{sf } x = \text{sf}' x) \rightarrow \text{BInt\_sf } \mu \text{ sf} = \text{BInt\_sf } \mu \text{ sf}'$ .

## 6 Bochner-integrable functions

Now we define the integrability of functions, by the means of an approximation by simple functions.

Following Section 5.2, let us still consider a measure space  $(X, \Sigma, \mu)$ , and  $E$  is now a *Banach space* over  $\mathbb{R}$ . In Coq, this becomes `E : CompleteNormedModule R_AbsRing`. As in textbooks, we consider  $f : X \rightarrow E$  as the pointwise limit of a sequence  $(s_n)_{n \in \mathbb{N}}$  of simple functions, and such that  $\int_{\mathcal{M}_+} \|f - s_n\| d\mu \xrightarrow{n \rightarrow \infty} 0$ , with  $\int_{\mathcal{M}_+}$  the Lebesgue integral over nonnegative measurable functions. In this case it may be proved that the sequence  $(\int s_n d\mu)_{n \in \mathbb{N}}$  is a Cauchy sequence, thus converges, thanks to the completeness of  $E$ , to a vector of  $E$  that we may define as the integral of  $f$ .

Therefore, we formally define Bochner-integrable functions as follows.

**Definition 8.** A function  $f : X \rightarrow E$  is said Bochner-integrable (with regard to  $\mu$ ) when there exists a sequence  $(s_n)_{n \in \mathbb{N}}$  of integrable simple functions such that

- $\forall x \in X, s_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ ;
- $\int_{\mathcal{M}_+} \|f - s_n\| d\mu \xrightarrow{n \rightarrow \infty} 0$ .

This becomes in Coq

```
Record Bif {f : X → E} := mk_Bif {
  seq : nat → simpl_fun E gen;
  ax_notempty : inhabited X;
  ax_int : ∀ n, integrable_sf μ (seq n);
  ax_lim_pw : ∀ x : X, is_lim_seq (λ n ⇒ seq n x) (f x);
  ax_lim_l1 : is_LimSup_seq' (λ n ⇒ LInt_p μ || f - seq n ||) 0
}.
```

A function is therefore Bochner-integrable when there exists such a record with the required values and proofs.

Once again, this record means that in order to prove that  $f$  is a Bochner-integrable function, we need to provide a sequence `seq : nat → simpl_fun E gen` of simple functions, and several properties corresponding to the mathematical requirements. In the above record, we used `is_LimSup_seq'` which is a generalization to  $\overline{\mathbb{R}}$ -valued sequences of `is_LimSup_seq` from Coquelicot that only takes reals.

The hypothesis `ax_notempty` is artificial. It is due to our will to be equivalent to the Lebesgue integral [6] that requires a nonempty set for preventing empty lists. A solution would be to convince the authors of [6] to switch to our simple functions.

We then prove several lemmas. First, a Bochner-integrable function is measurable as it is the pointwise limit of a sequence of measurable simple functions. Then, we also prove that  $\|f\|$  is integrable in the sense of Lebesgue integration of Section 3.

We also define approximating sequences of integrable simple functions for the sum, opposite, subtraction, scalar product and norm of a Bochner-integrable function. From these proofs and as before, we define useful notations:

```
"bf + bg" := Bif_plus bf bg.
"- bf" := Bif_scal (opp one) bf.
"bf - bg" := Bif_plus bf (Bif_scal (opp one) bg).
"a · bf" := Bif_scal a bf.
"|| bf ||" := Bif_norm bf.
```

But such a definition of integrability for vector-valued functions, though easy to use, does not make it really easy to prove that a given function is integrable so we look for equivalent properties. First of all, notice that if a function  $f : X \rightarrow E$  is the pointwise limit of simple functions, say  $(s_n)_{n \in \mathbb{N}}$ , then since every  $s_n$  has a finite image, the range of  $f$  must be weakly separable according to the previous definition. And since  $f$  must be integrable, we also know that  $\int_{\mathcal{M}_+} \|f\| d\mu$  is finite. Reciprocally, given those two properties, one may wonder if it is possible to prove that  $f$  is Bochner-integrable. The answer is yes, and we may even construct an explicit sequence of simple functions, which is useful for Coq to compute the value of the integral of  $f$ . To construct such a sequence, we followed [16]. Note that this requires some attention because it involves some careful splitting of the range of  $f$ .

**Lemma** `Bif_separable_range`  $\{f : X \rightarrow E\} \{u : \text{nat} \rightarrow E\}$  :

```
inhabited X → measurable_fun gen open f → NM_seq_separable_weak u (inRange f) →
is_finite (LInt_p μ (λ x : X ⇒ || f || x)) → Bif μ f.
```

The first consequence of this characterization is that any measurable function  $f : X \rightarrow \mathbb{R}$  such that  $\int_{\mathcal{M}_+} \|f\| d\mu < \infty$  is Bochner-integrable, because we already know that  $\mathbb{R}$  is (weakly) separable.

**Lemma** `R_Bif`  $\{f : X \rightarrow \mathbb{R}\text{-NormedModule}\}$  :

```
inhabited X → measurable_fun gen open f → is_finite (LInt_p μ (|| f ||)) → Bif μ f.
```

So here we recover exactly the definition of integrability for the Lebesgue integral, which makes both definitions compatible. But now, if we assume  $X$  separable and  $f$  continuous, one can prove that the range of  $f$  is separable. For example, we deduce that every continuous function  $f : \mathbb{R}^n \rightarrow E$  is Bochner-integrable.

For other cases where our function  $f$  seems too complicated to prove easily that its range is separable, we still have the ability to prove its Bochner integrability by using yet another equivalent property.

**Lemma 9.** *A function  $f : X \rightarrow E$  is Bochner-integrable if and only if*

- *it is the pointwise limit of a sequence of simple functions (without requiring any integrability);*
- $\int_{\mathcal{M}_+} \|f\| d\mu < \infty$ .

Functions which are pointwise limit of simple ones (i.e. which satisfies the first dot above) are said *strongly measurable*. This definition has been formalized inside the library too, and we have proved several useful properties about strongly measurable functions. The most striking example is that any pointwise limit of strongly measurable functions is still strongly measurable. Such a property is of great use to prove the dominated convergence theorem.

## 7 The Bochner integral

The definition of the Bochner integral is straightforward from the integrability definition. Let us still consider a measure space  $(X, \Sigma, \mu)$ , and  $E$  a Banach space over  $\mathbb{R}$ .

**Definition 10.** Let  $f : X \rightarrow E$ . Let  $(s_n)_{n \in \mathbb{N}}$  be a sequence of integrable simple functions such that

- $\forall x \in X, s_n(x) \xrightarrow[n \rightarrow \infty]{} f(x)$ ;
- $\int_{\mathcal{M}_+} \|f - s_n\| d\mu \xrightarrow[n \rightarrow \infty]{} 0$ .

Then the Bochner integral of  $f$  (relatively to measure  $\mu$  on  $X$ ) is defined by

$$\int f d\mu := \lim_{n \rightarrow \infty} \int s_n d\mu.$$

The Coq definition is quite short because the partial function `lim_seq` was used, so the convergence of the sequence do need to be checked while defining `BInt`. However, it was proved as an independent lemma, which is essential to be able to use the properties of `BInt` `bf` as a limit.

**Definition** `BInt {f : X → E} (bf : Bif μ f) := lim_seq (λ n ⇒ BInt_sf μ (seq bf n))`.

The first property to ensure is that this definition does neither depend on the chosen sequence  $(s_n)_{n \in \mathbb{N}}$ , nor on the integrability proof. This is stated as the following extensionality lemma.

**Lemma** `BInt_ext {f f' : X → E} :`

$$\forall (bf : Bif \mu f) (bf' : Bif \mu f'), (\forall x : X, f x = f' x) \rightarrow BInt bf = BInt bf'.$$

We then prove all the expected properties of the integral such as linearity or the triangular inequality, by taking the limit of the already proved properties over simple functions.

A larger proof is the equality of `BInt` and `LInt_p` for nonnegative real-valued integrable functions, as we had to prove the equivalence of the two formalizations (of simple functions, of integrability and of integrals). It makes our library compatible with the one about the Lebesgue integral.

The next lemmas were chosen to ease the main perspective of this work, that is the definition of Bochner spaces in Coq, which are a generalization of  $L^p$  spaces for the Lebesgue integral.

**Theorem 11.** A function  $f : X \rightarrow E$  is zero  $\mu$ -almost everywhere if and only if  $\int_{\mathcal{M}_+} \|f\| d\mu = 0$ .

**Theorem 12** (dominated convergence). Given a nonnegative integrable function  $g : X \rightarrow \mathbb{R}$ , and a pointwise convergent sequence  $(f_n)_{n \in \mathbb{N}}$  of Bochner-integrable functions such that  $\forall n \in \mathbb{N}$ , we have  $\|f_n\| \leq g$ , then  $f := (x \mapsto \lim_{n \rightarrow \infty} f_n(x))$  is Bochner-integrable, and

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

## 8 Conclusion and perspective

We have defined the Bochner integral with a constructive point of view for Bochner integrability. We have proved that a function is Bochner-integrable (with the constructive dependent type definition) provided it is the pointwise limit of simple functions and that its range is weakly separable. We have also proved that our definitions are consistent with those of a Coq formalization of the Lebesgue integral.

Our design choices are twofold. Mathematically, we have conscientiously followed Teschl [16] with a kind of *weak* separability instead of (regular) separability. Formally, we have simple functions with an index function and Bochner-integrable by a dependent type. We have succeeded in proving the common lemmas, from linearity to dominated convergence so this seems a good basis to build upon.

This opens the way to the formalization of Bochner spaces of strongly measurable functions for which the  $p$ -th power of the norm is Lebesgue integrable. They are a generalization of the usual  $L^p$  Lebesgue spaces, where functions equal almost everywhere are also identified. Such spaces are also Banach spaces for  $p \geq 1$ . For instance, given a regular enough space domain  $\Omega \subset \mathbb{R}^3$ , the

square-integrable functions  $\Omega \rightarrow \mathbb{R}$  form the Hilbert space  $L^2(\Omega)$  since  $\mathbb{R}$  is a Banach space on which Bochner integration applies. Moreover, given a real number  $T > 0$ , the square-integrable functions from  $[0, T]$  to  $L^2(\Omega)$  also form the Hilbert space  $L^2([0, T], L^2(\Omega))$ . And eventually, this could be used to apply the Lax–Milgram theorem [5] in the context of the resolution of some set of partial differential equations.

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Publisher  
Inria  
Domaine de Voluceau - Rocquencourt  
BP 105 - 78153 Le Chesnay Cedex  
[inria.fr](http://inria.fr)

ISSN 0249-6399