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Digital path-following for a car-like robot ⋆

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Abstract: This paper deals with path following under digital control for a car-like robot via TFL. Assuming the existence of a continuous-time feedback, a multi-rate sampled-data control strategy is proposed and its effectiveness validated through simulations.

Keywords: Robotic systems, Novel control theory and techniques, Sampled-data nonlinear control

1. INTRODUCTION

Path following control deals with the confinement of the evolutions of a nonholonomic mechanical system to a prescribed geometric path. The assignment of time requirements on the way the system moves along the path represent an extra constraint which can be handled separately from the geometric requirement, allowing for better performances compared to trajectory tracking design (Aguiar et al. (2008)).
Several approaches solving path following problems are discussed in the continuous-time literature. Among others, let us mention the work relying on energy shaping through Immersion and Invariance by Yi et al. (2020), sliding mode path following by Dagci et al. (2003), smooth time-varying feedback and input scaling by De Luca et al. (2001) and, more relevant to this work, Transverse Feedback Linearization (TFL) by Banaszuk and Hauser (1995); Altafini (2002); Nielsen and Maggiore (2008).
Among these, the latter one aims to stabilize a general set which is made the zero dynamics sub-manifold associated with a suitable output function with well-defined relative degree. Accordingly, static (or possibly dynamic) feedback linearization is applied to stretch the trajectories onto the target set (Nielsen and Maggiore (2006)). In this sense, path following admits a solution via TFL when the path can be implicitly represented by the equations which specify the zero-dynamics sub-manifold (Akhtar et al. (2015)).
When approaching these problems in a digital context, one has to face well-known limitations due to the sampling process as, for instance, the loss of the relative degree and the rise of the unstable sampling zero-dynamics (see Aström et al. (1984); Monaco et al. (1986)) so directly affecting TFL in general. For those reasons, a first single rate solution preserving TFL in an approximate sense was proposed in Elobaid et al. (2020) providing, under suitable assumptions, a solution to the path following problem as well. For instance, when dealing with the car-like robot approximate TFL under sampling is preserved for the kinematic model under a preliminary continuous-time dynamic extension (Akhtar et al. (2015)). Beyond TFL, a multi-rate technique has been proposed in Di Giamberardino et al. (1996) to solve steering problems for mobile robots, when assuming a preliminary continuous-time feedback making the dynamics finitely discretizable. In the method we propose, both these demands are weakened.
In this work, following Monaco and Normand-Cyrot (1992), we propose a solution which circumvents the need for the dynamic extension, making use of a multi-rate sampled-data control strategy. Simulation results validate the proposed design approach in a comparative way with respect to the continuous-time solution and its direct implementation through Zero Order Holding (ZOH) referred to as emulation as well as the design approach previously recalled (Elobaid et al. (2020)). The proposed multi-rate solution provides efficient and improving results with respect to the preliminary ones.

The paper is organized as follows: Section II provides some background material and states the problem. The proposed control solution is developed in Section III in a constructive way. Simulations are discussed in Section IV. Concluding remarks end the manuscript.

Notations: $\mathbb{S}^1$ denotes the unit disk, i.e. $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| \leq 1\}$. $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers. Given a pair of matrices (A, B), $\text{col}(A, B) = \begin{pmatrix} A^\top & B^\top \end{pmatrix}^\top$ while $\text{diag}(A, B)$ is the block diagonal matrix with blocks A, B. The couple (A_n, B_n) with

$$A_n = \begin{pmatrix} \mathbf{0}_{(n-1)\times 1} & I_{n-1} \\ 0 & \mathbf{0}_{1\times(n-1)} \end{pmatrix}, B_n = \begin{pmatrix} \mathbf{0}_{(n-1)\times 1} \\ 1 \end{pmatrix}$$

is said to be in Brunovský's canonical form with I_{n-1} being the identity matrix of dimension $n-1$ and $\mathbf{0}_{m\times n}$ the zero matrix of dimension $m \times n$. Given a manifold M and a closed connected set $N \subset M$, N is said to be invariant under the dynamics $\dot{q} = f(q) + g(q)u$ if for all $q(0) \in N$

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and any control $u(\cdot)$, $q(t) \in N, \forall t$. N is *controlled* invariant if there exists a feedback $u^\star$ making N invariant for the closed loop system. The point-to-set distance is denoted $\|q_0\|_M = \inf\|q - q_0\|, q \in M$. L_f denotes the operator $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial q_i}$, $\mathrm{L}_f\mathrm{L}_g$ their composition. Given a real valued function $h(\cdot)$ on $\mathbb{R}^n$, $e^{\mathrm{L}_f}h(q)|_{q(k)}$ denotes the application of the Lie series operator e^{L_f} to the function $h(q)$ evaluated at the state $q(k)$. A continuous function $R(x, \delta)$ is of order $O(\delta^p)$ with $p \geq 1$ if, whenever it is defined, it can be written as $R(x, \delta) = \delta^{p-1}\tilde{R}(x, \delta)$ and there exists a function $\beta(\delta)$ of class κ_∞ and $\delta^\star > 0$ such that $\forall \delta \leq \delta^\star, |\tilde{R}(x, \delta)| \leq \beta(\delta)$.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 Path following for a car-like robot in continuous time

Consider the kinematic model of a car-like robot (Siciliano et al. (2010))

$$\begin{aligned} \dot{q} &= g_1(q)v + g_2(q)\omega \\ p &= (x\ y)^\top \end{aligned} \tag{1}$$

with (see Figure 1)

$$g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \dfrac{1}{\ell}\tan\phi \\ 0 \end{pmatrix}, \quad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$q = (q_1\ q_2\ q_3\ q_4)^\top = (x\ y\ \theta\ \phi)^\top \in \mathbb{R}^4$, $v \in \mathbb{R}$, the forward linear velocity, $\omega \in \mathbb{R}$, the angular velocity and ℓ the distance between the wheels. The position on the plane $p = h(q) := (q_1\ q_2)^\top \in \mathbb{R}^2$ is the output of the system.

The desired path is given by a regular parameterized curve $\varrho : \mathbb{D} \mapsto \mathbb{R}^2$ with no self-intersections. Following Nielsen and Maggiore (2004); Akhtar et al. (2015), we refer to for a precise characterization, let the path $\varrho(\mathbb{D})$ be an embedded sub-manifold of $\mathbb{R}^2$ of dimension 1; i.e., there exists a function $s(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ such that 0 is a regular value of s and $\varrho(\mathbb{D}) = \{w \in \mathbb{R}^2 \text{ s.t. } s(w) = 0\}$.

The path following problem for a car-like robot asks for the design of a feedback control law maneuvering the output of system (1) to approach and move along a given curve in a desired way. The problem is formally stated below.

Problem 2.1. Given a regular parameterized curve $\mathcal{C} = Im\{\varrho(\mathbb{D})\}$, find if possible, a smooth feedback law for system (1) such that for a set of some initial conditions $\mathcal{X}$, with $\mathcal{C} \subset \mathcal{X}$ the following holds true.

(1) *Invariance*: if $p(0) = h(q(0)) \in \mathcal{C}$ then $\forall t \geq 0$ $\|p(t)\|_\mathcal{C} = 0$.
(2) *Attractivity*: system (1) under feedback is s.t $\forall t \geq 0$, $\|p\|_\mathcal{C} \to 0$ as $t \to \infty$.
(3) *Motion on the curve*: system (1) traverses the curve $\mathcal{C}$ with a given desired velocity or acceleration profile $(\dot{\pi}_{ref}(t), \ddot{\pi}_{ref}(t))$.

It is known (Altafini (2002); Nielsen and Maggiore (2006)) that TFL is a natural tool to handle path following problems. To see this, denote the $n^\star = 3$ dimensional sub-manifold $\Gamma^\star \subseteq \{q \in \mathbb{R}^4 : q = (s \circ h)^{-1}(0)\}$ as the control invariant subset for (1); i.e., the set of all initial

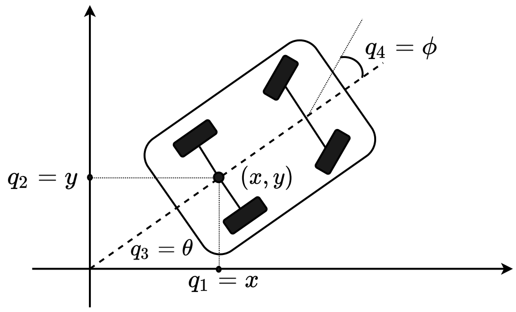

Fig. 1. Kinematics of a car-like robot

conditions $q_0 \in \mathbb{R}^4$ under which the car-like robot is forced to remain on the curve (i.e., $p(t) \in \mathcal{C}$ for all $t \geq 0$) under a suitably designed feedback. Consequently, $\Gamma^\star$ is referred to as the *path following sub-manifold* associated with the curve. With this in mind, Problem 2.1 is equivalent in the context of TFL to the problems of stabilizing $\Gamma^\star$ and zero dynamics assignment. The problem has been solved in Akhtar et al. (2015) making use of continuous-time TFL through dynamic control defining the function $\beta : \mathbb{R}^4 \to \mathbb{R}^2$

$$\beta(q) = (\alpha(q)\ \pi(q))^\top \tag{2}$$

with $\alpha(q) = (s \circ h)(q) \in \mathbb{R}$ the *transverse output function* (i.e., the curve implicit function) and $\pi(q) = \arctan(\frac{q_2}{q_1})$ is the *tangent output function.* Under the control dynamic extension

$$\begin{aligned} \dot{\eta}_1 &= \eta_2, & \dot{\eta}_2 &= u_1 \\ v &= \eta_1, & \omega &= u_2 \end{aligned} \tag{3}$$

the feedback system

$$\begin{aligned} \dot{\tilde{q}} &= \tilde{f}(\tilde{q}) + \tilde{g}_1(\tilde{q})u_1 + \tilde{g}_2(\tilde{q})u_2 \\ \tilde{f}(\tilde{q}) &= \left(\eta_1\cos\theta\ \ \eta_1\sin\theta\ \ \frac{\eta_1}{\ell}\tan\phi\ \ 0\ \ \eta_2\ \ 0\right)^\top \\ \tilde{g}_1(\tilde{q}) &= (0\ 0\ 0\ 0\ 0\ 1)^\top, \tilde{g}_2(\tilde{q}) = (0\ 0\ 0\ 1\ 0\ 0)^\top \end{aligned} \tag{4}$$

with $\tilde{q} = (q\ \eta_1\ \eta_2)^\top \in \mathbb{R}^{\tilde{n}}, \tilde{n} = 6$ possesses strong vector relative degree $r = (\tilde{n} - n^\star\ \ n^\star) = (3\ \ 3)$ with respect to the output function (2). Accordingly, the following result is recalled.

Theorem 2.1. (Akhtar et al. (2015)). Given a regular, three times differentiable curve $\mathcal{C}$ in the plane with $\Gamma^\star$ the path following sub-manifold associated with $\mathcal{C}$. Consider the dummy output function (2), then the Path-Following Problem 2.1 is solved by the feedback (3) with

$$u = \gamma(\tilde{q}, \nu) = A^{-1}(\tilde{q})(\nu - B(\tilde{q})) \tag{5}$$

with $u = (u_1, u_2)^\top$,

$$A(\tilde{q}) = \begin{pmatrix} L_{\tilde{g}_1}L_{\tilde{f}}^2\alpha(q) & L_{\tilde{g}_2}L_{\tilde{f}}^2\alpha(q) \\ L_{\tilde{g}_1}L_{\tilde{f}}^2\pi(q) & L_{\tilde{g}_2}L_{\tilde{f}}^2\pi(q) \end{pmatrix}, \ B(\tilde{q}) = \begin{pmatrix} L_{\tilde{f}}^3\alpha(q) \\ L_{\tilde{f}}^3\pi(q) \end{pmatrix}$$

and $\nu = (\nu_1,\ \nu_2)$ are respectively the transverse and tangential external stabilizing controls.

In the continuous-time case, dynamic extension is necessary for guaranteeing a well-defined relative degree $\tilde{n} - n^\star$ and, thus, TFL with respect to $\Gamma^\star$ via the input $\omega \in \mathbb{R}^n$.

Remark 2.1. Setting the feedback (5), and the coordinates change $\phi(\tilde{q}) = (\phi_1(\tilde{q})\ \phi_2(\tilde{q}))^\top$ with $\xi = \phi_1(\tilde{q}) =$

$(\alpha(\tilde{q})\ L_{\tilde{f}}\alpha(\tilde{q})\ L^2_{\tilde{f}}\alpha(\tilde{q}))^\top, z = \phi_2(\tilde{q}) = (\pi(\tilde{q})\ L_{\tilde{f}}\pi(\tilde{q})\ L^2_{\tilde{f}}\pi(\tilde{q}))^\top$ the extended system (4) takes the form

$$\dot{\xi} = A_3\xi + B_3\nu_1 \tag{6a}$$
$$\dot{z} = A_3 z + B_3\nu_2 \tag{6b}$$

where $A_3 \in \mathbb{R}^{3\times 3}, B_3 \in \mathbb{R}^3$. From the result above, $\xi \in \mathbb{R}^3$ is the *transverse* component to $\Gamma^\star$ while z describes the motion when the dynamics is restricted to $\Gamma^\star$ i.e. $\Gamma^\star = \{(\xi\ \ z)^\top : \xi = 0\}$. It is intuitively understood that $\Gamma^\star$ is precisely the zero dynamics sub-manifold of the system (4) with output $\alpha(\tilde{q})$.

Remark 2.2. Theorem 2.1 states that, under TFL with dynamics extension, (1) and (2) in Problem 2.1 are solved by $\nu_1 = -K\xi$ so to make (6a) asymptotically stable. Denoting $z = (z_1\ z_2\ z_3)^\top$, requirement (3) is solved setting

$$\nu_2 = -f_2(z_3 - \ddot{\pi}_{ref}(t)) - f_1(z_2 - \dot{\pi}_{ref}(t)) + \dddot{\pi}_{ref}(t)$$

with $f_1, f_2 > 0$ over the tangential dynamics (6b).

2.2 Path following for the car-like robot under sampling

In the sequel, we provide a solution to the path following problem for the car-like robot under sampled-data control and digital transverse feedback linearization as set in the following problem.

Problem 2.2. Design *a digital control* $(v(k)\ \ \omega(k)) = \gamma^{3\bar{\delta}}(q(k), \nu(k))$ with external inputs ν solving the Path Following Problem 2.1 for the car-like robot (1) at all sampling instants $t = k\delta$, $k \geq 0$ and $\delta > 0$ the sampling period.

In the present context, *digital control design* refers to design over the *sampled-data equivalent model* for which measures of the states are available at periodic sampling instants $t = k\delta, k \in \mathbb{Z}_{\geq 0}$, where δ is the sampling period, and controls kept constant over δ. A first solution to Problem 2.2 can be carried out over the extended model (4) with inputs (u_1, u_2) by directly applying the results in Elobaid et al. (2020). More in detail, let $u_i(t), i = 1, 2$ be constant over the sampling period, i.e. $u_i(t) = u_i(k\delta) = u_i(k)$ for $t \in [k\delta, (k+1)\delta[$. Then the sampled-data model equivalent to (4) takes he form

$$\tilde{q}(k+1) = \tilde{F}^\delta(\tilde{q}(k), u(k)) \tag{7}$$

with

$$\begin{aligned}\tilde{F}^\delta(\tilde{q}, u) =& e^{\delta(\mathrm{L}_{\tilde{f}} + u_1\mathrm{L}_{\tilde{g}_1} + u_2\mathrm{L}_{\tilde{g}_2})}\tilde{q}\\ =& \tilde{q} + \sum_{j\geq 1}\frac{\delta^j}{j!}(\mathrm{L}_{\tilde{f}} + u_1\mathrm{L}_{\tilde{g}_1} + u_2\mathrm{L}_{\tilde{g}_2})^j\tilde{q}\end{aligned}$$

with the function $\tilde{F}^\delta(\cdot, u)$ defined by its series expansion in powers of δ, Monaco and Normand-Cyrot (1997).

At this point, it is easily verified that the relative degree of the sampled-data model (7) with the output (2) falls to (1 1) for all $\delta > 0$ (Monaco and Normand-Cyrot (1987)). As a matter of facts, one gets that

$$\frac{\partial\alpha(\tilde{q}(k+1))}{\partial u_j(k)} = \frac{\delta^{\tilde{n}-n^\star}}{(\tilde{n}-n^\star)!}L_{\tilde{g}_j}L_{\tilde{f}}^{\tilde{n}-n^\star-1}\alpha(\tilde{q})\big|_{\tilde{q}(k)} + O(\delta^{\tilde{n}-n^\star+1})$$
$$\frac{\partial\pi(\tilde{q}(k+1))}{\partial u_j(k)} = \frac{\delta^{\tilde{n}^\star}}{n^\star!}L_{\tilde{g}_j}L_{\tilde{f}}^{\tilde{n}^\star-1}\pi(\tilde{q})\big|_{\tilde{q}(k)} + O(\delta^{\tilde{n}^\star+1})$$

for $n - n^\star = 3$ $j = 1, 2$ are non-zero (at least not simultaneously) by definition of the continuous-time relative degree. Thus, the path following sub-manifold $\Gamma^\star$ is no longer the zero dynamics sub-manifold for the sampled-data model (7) with output $\alpha(\tilde{q})$. In fact, the (typically unstable) zero dynamics of the sampled-data equivalent model evolves over a sub-manifold containing $\Gamma^\star \subset \mathbb{R}^3$ that is given by

$$\mathcal{Z}_{SD} = \{\tilde{q} : \alpha(\tilde{q}) = 0\} \subset \mathbb{R}^5$$
$$\mathcal{Z}_{SD} \supset \Gamma^\star = \{\tilde{q} : \alpha(\tilde{q}) = L_f\alpha(\tilde{q}) = L^2_{\tilde{f}}\alpha(\tilde{q}) = 0\}.$$

Accordingly, transverse feedback linearization is lost for the sampled-data model (7) with respect to the output function (2).

In Elobaid et al. (2020) TFL under sampling is achieved, up to a prescribed approximation order, by means of a δ-dependent dummy output computed from (2) as

$$\alpha^\delta(\tilde{q}) = \alpha(\tilde{q}) + \delta L_{\tilde{f}}\alpha(\tilde{q}) - \frac{\delta^2}{3}L^2_{\tilde{f}}\alpha(\tilde{q}) + O(\delta^3) \tag{8a}$$
$$\pi^\delta(q) = \pi(\tilde{q}) + \delta L_{\tilde{f}}\pi(\tilde{q}) - \frac{\delta^2}{3}L^2_{\tilde{f}}\pi(\tilde{q}) + O(\delta^3). \tag{8b}$$

The vector relative degree (3 3) is preserved under the dynamics (7) together with the zero dynamics sub-manifold $\Gamma^\star$ in $O(\delta^4)$, so that a digital solution can be computed to approximately solve the TFL problem under single-rate sampling. In fact, consider the coordinates change $q \mapsto (\xi^\delta\ \ z^\delta)$, with $\xi^\delta = T_3(\delta)\phi_1(\tilde{q})$, $z^\delta = T_3(\delta)\phi_2(\tilde{q})$, where $\phi_1(\cdot), \phi_2(\cdot)$ are as in Remark 2.1, and $T_3(\delta)$ as in Elobaid et al. (2020). This coordinates change, together with the piecewise continuous feedback for $t \in [k\delta, (k+1)\delta[$

$$\dot{\eta}(t) = A_2\eta(t) + B_2 u_1(k) \tag{9a}$$
$$u(k) = A^{-1}(\tilde{q}(k))(\nu^\delta(k) - B(\tilde{q}(k))) \tag{9b}$$

recovers the TFL normal form in an approximate sense, with (v, ω) piecewise continuous. The control $\nu^\delta = -K^\delta \mathrm{col}(\xi^\delta,\ z^\delta)$, $K^\delta : \sigma(I_3 + \delta(A_3 + B_3K^\delta)) \subset \mathbb{S}^1$ stabilizes the system to the path following sub-manifold and allows for solving Problem 2.2 in an approximate sense.

In the following we present an exact (and fully digital) solution for Problem 2.2 based on multi-rate sampling with no need of a preliminary continuous-time dynamic extension. Namely, we design a digital control $(v(k)\ \ \omega(k)) = \gamma^{3\bar{\delta}}(q(k), \nu(k))$ for solving path following based on the sampled-data equivalent model to the kinematic car-like robot in (1); i.e., one gets for $\omega(t) = \omega(k)$ and $v(t) = v(k)$ for $t \in [k\delta, (k+1)\delta[$ the sampled-data equivalent kinematics

$$q(k+1) = F^\delta(q(k), v(k), \omega(k)) \tag{10}$$

with

$$F^\delta(q, v, \omega) = e^{\delta(v\mathrm{L}_{g_1} + \omega\mathrm{L}_{g_2})}q = q + \sum_{j\geq 1}\frac{\delta^j}{j!}(v\mathrm{L}_{g_1} + \omega\mathrm{L}_{g_2})^j q.$$

3. EXACT PATH FOLLOWING FOR A CAR-LIKE ROBOT UNDER MULTI-RATE SAMPLING

As shown in the previous section, dynamic extension is used over system (1) to guarantee $v^2 L_{g_2}L^2_{g_1}\alpha(q) \neq 0$, that is relative degree 3 with respect to the dummy output component in (2) associated with $\Gamma^\star$ (i.e., $\alpha(q)$). However, the relative degree can also be guaranteed for the kinematic model (1) under multi-rate sampling without dynamic extension, as shown in Monaco and Normand-Cyrot (1992). To this end, we set in (10) a multi-rate of order $n - n^\star = 3$ over the input ω that is $\bar{\delta} = \frac{\delta}{3}$ and $\omega(t) = \omega_i(k)$ for $t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}[$ with $i = 1, 2, 3$ and $v(t) = v(k)$ for $t \in [k\delta, (k+1)\delta[$. Accordingly, the multi-rate model of (1) gets the form

$$q(k+1) = F_3^{\bar{\delta}}(q(k), v(k), \underline{\omega}(k)) \tag{11}$$

with

$$\begin{aligned} F_3^{\bar{\delta}}(q, v, \underline{\omega}) &= F^{\bar{\delta}}(\cdot, v, \omega_3) \circ F^{\bar{\delta}}(\cdot, v, \omega_2) \circ F^{\bar{\delta}}(q, v, \omega_1) \\ &= \sum_{j_1, j_2, j_3 \geq 0} \frac{\bar{\delta}^{j_1+j_2+j_3}}{j_1! j_2! j_3!} (vL_{g_1} + \omega_1 L_{g_2})^{j_1} \\ &\quad \circ (vL_{g_1} + \omega_2 L_{g_2})^{j_2} \circ (vL_{g_1} + \omega_3 L_{g_2})^{j_3} q. \end{aligned}$$

In this respect, with reference to Problem 2.2, we seek for a digital piecewise constant control that preserves TFL with respect to $\Gamma^\star$ for the multi-rate equivalent model (11) based on the dummy output (2).

The problem is set for (11) based on the augmented output vector (Monaco and Normand-Cyrot (1992))

$$H(q) = \left(\alpha(q) \ \dot{\alpha}(q) \ \ddot{\alpha}(q)\right)^\top \tag{12}$$

ensuring that the multi-rate model (11) possesses vector relative degree $(1, 1, 1)$ so that transverse feedback linearizability under sampling is guaranteed. Accordingly, requirements (1),(2) of Problem 2.1 at all sampling instants are satisfied by a digital feedback $\underline{\omega} = \underline{\omega}^{\bar{\delta}}(q)$ solution to the equality

$$H(F_3^{\bar{\delta}}(q, v, \underline{\omega})) = A^{3\bar{\delta}} H(q(k)) + B^{3\bar{\delta}} \nu_1(k) \tag{13}$$

with $A^{3\bar{\delta}} = e^{3\bar{\delta} A_3}$, $B^{3\bar{\delta}} = \int_0^{3\bar{\delta}} e^{\tau A_3} B_3 d\tau$, A_3, B_3 as in (6a) and ν_1 the external stabilizing *transverse* control. More in detail, the TFL feedback solution to (13) is the one making the $H(q) - \nu_1$ link in (11) linear. The following result asserts the existence of such feedback.

Proposition 3.1. Consider the kinematic model of the car-like robot (1), and a regular parameterized curve $\varrho : \mathbb{D} \mapsto \mathbb{R}^2$ under the hypotheses of Theorem 2.1. Then, requirement (1) of Problem 2.1 is guaranteed, at all $t = k\delta$ with $k \geq 0$, by the feedback $\omega = \underline{\omega}^{\bar{\delta}}(q, v, \nu_1)$ of the form

$$\underline{\omega}^{\bar{\delta}}(q, v, \nu_1) = \underline{\omega}_0(q, v, \nu_1) + \sum_{i > 0} \frac{\bar{\delta}^i}{(i+1)!} \underline{\omega}_i(q, v, \nu_1) \tag{14}$$

defined as the unique solution to (13). In addition, requirement (2) of Problem 2.1 is guaranteed, at all $t = k\delta$ with $k \geq 0$, setting

$$\nu_1(k) = -K^{\bar{\delta}} H(q(k)) \tag{15}$$

with $K^{\bar{\delta}}$ such that $\sigma(A^{3\bar{\delta}} - B^{3\bar{\delta}} K^{\bar{\delta}}) \subset \mathbb{S}^1$.

Proof: Rewriting (13) as a formal series equality in powers of $\bar{\delta}$ so getting

$$\begin{aligned} &\begin{pmatrix} e^{\bar{\delta}(v(k)L_{g_1} + \omega_1(k)\mathrm{L}_{g_2})} \ldots e^{\bar{\delta}(v(k)L_{g_1} + \omega_3(k)\mathrm{L}_{g_2})} \alpha(q) \\ e^{\bar{\delta}(v(k)L_{g_1} + \omega_1(k)\mathrm{L}_{g_2})} \ldots e^{\bar{\delta}(v(k)L_{g_1} + \omega_3(k)\mathrm{L}_{g_2})} \dot{\alpha}(q) \\ e^{\bar{\delta}(v(k)L_{g_1} + \omega_1(k)\mathrm{L}_{g_2})} \ldots e^{\bar{\delta}(v(k)L_{g_1} + \omega_3(k)\mathrm{L}_{g_2})} \ddot{\alpha}(q) \end{pmatrix} \\ &= A^{3\bar{\delta}} H(q(k)) + B^{3\bar{\delta}} \nu_1(k) \end{aligned} \tag{16}$$

The equations (16) rewrite as $S^{\bar{\delta}}(q, \underline{\omega}, v, \nu_1) = \mathbf{0}$ with

$$\begin{aligned} &S^{\bar{\delta}}(q, \underline{\omega}, v, \nu_1) \\ &= \left(\bar{\delta}^3 S_1^{\delta}(q, \underline{\omega}, v, \nu_1) \ \bar{\delta}^2 S_2^{\delta}(q, \underline{\omega}, v, \nu_1) \ \bar{\delta} S_3^{\delta}(q, \underline{\omega}, v, \nu_1)\right)^\top \end{aligned} \tag{17}$$

and

$$\begin{aligned} \bar{\delta}^{4-i} S_i^{\bar{\delta}} &= e^{\bar{\delta}(vL_{g_1} + \omega_1 L_{g_2})} \circ \ldots \circ e^{\bar{\delta}(vL_{g_1} + \omega_3 L_{g_2})} \alpha^{(i-1)}(q) \\ &\quad - \alpha^{(i-1)}(q) - \sum_{\ell=i}^{2} \frac{3^\ell \bar{\delta}^\ell}{\ell!} \alpha^{(\ell)}(q) - \frac{3^{4-i} \bar{\delta}^{4-i}}{(4-i)!} \nu_1 \end{aligned}$$

Accordingly, because $v(k)$ constant over the sampling interval, denoting $f(q) = g_1(q)v$, one looks for $\underline{\omega}$ satisfying (17), where each term can be written as $S_i(q, \underline{\omega}, v, \nu_1) = \sum_{j \geq 0} \bar{\delta}^j S_{i,j}(q, \underline{\omega}, v, \nu_1)$ with $i = 1, 2, 3$ and

$$S_{i,0}(q, \underline{\omega}, v, \nu_1) = \Delta_{4-i} \left(\mathrm{L}_{g_2} \mathrm{L}_f^2 \alpha(q) \underline{\omega} + \mathbf{1} \mathrm{L}_f^3 \alpha(q) - \mathbf{1} \nu_1(k)\right)$$

where $\mathbf{1} = (1 \ 1 \ 1)^\top$, $\Delta = \mathrm{col}(\Delta_3 \ \ \Delta_2 \ \ \Delta_1)$, and

$$\Delta_j = \frac{1}{j!} \left(j^{4-j} - (j-1)^{4-j}, \ (j-1)^{4-j} - (j-2)^{4-j}, \ 1\right)$$

with $j = 1, 2, 3$. Following Monaco and Normand-Cyrot (1997) , it results that the matrix

$$\frac{\partial}{\partial \underline{\omega}} S^{\bar{\delta}}(q, v, \underline{\omega})|_{\bar{\delta} \to 0} \to \Delta L_{g_2} L_f^2 \alpha(q)$$

is full rank because $L_{g_2} L_f^2 \alpha(q) \neq 0$ and Δ is invertible. Hence, by the Implicit Function Theorem, the existence of $\underline{\omega}^{\bar{\delta}}$ unique solution to (13) of the form (14) can be deduced (Mattioni et al., 2017, Proposition 4.1). Under the coordinates transformation $(\xi \ \bar{z})^\top := q \mapsto \phi(q) = (H(q) \ \pi(q))^\top$, the controlled dynamics reads

$$\begin{aligned} \xi(k+1) &= A^{3\bar{\delta}} \xi(k) + B^{3\bar{\delta}} \nu_1(k) \\ \bar{z}(k+1) &= \psi(\xi(k), \bar{z}(k), v(k), \nu_1(k)) \end{aligned}$$

where $\psi(\xi, \bar{z}, v, \nu_1) = \arctan \frac{q_2(k+1)}{q_1(k+1)}|_{(\xi \ \bar{z})^\top = \phi^{-1}(q)}$ bounded on the curve by definition and hence the result follows. $\triangleleft$

From the statement above, denoting $\dot{\pi}_{ref}(k) = \dot{\pi}_{ref}(k\delta)$ and $\ddot{\pi}_{ref}(k) = \ddot{\pi}_{ref}(k\delta)$, requirement (3) at all $t = k\delta$ is fulfilled by the discrete-time feedback

$$v(k+1) = v(k) + \delta a(k) + \frac{\delta^2}{2} \nu_2(k) \tag{18a}$$

$$a(k+1) = a(k) + \delta \nu_2(k) \tag{18b}$$

over the tangential component with $v(k), a(k)$ are the linear velocity and acceleration over the path respectively and

$$\begin{aligned} \nu_2(k) &= \left(\frac{1}{\delta^2}(a_0^\delta - a_1^\delta - 1) \ \ \frac{1}{2\delta}(a_0^\delta + a_1^\delta - 3)\right) e(k) \\ &\quad + \ddot{\pi}_{ref}(k+1) - \ddot{\pi}_{ref}(k) \end{aligned} \tag{19}$$

so to guarantee, asymptotically,

$$e(k) = \begin{pmatrix} v(k) - \dot{\pi}_{ref}(k) \\ a(k) - \ddot{\pi}_{ref}(k) \end{pmatrix} \to 0$$

for a_0^δ, a_1^δ given by

$$\begin{aligned} a_0^\delta &= e^{-\lambda_1 \delta} e^{-\lambda_2 \delta} \\ a_1^\delta &= -(e^{-\lambda_1 \delta} + e^{-\lambda_2 \delta}), \quad \lambda_1, \lambda_2 > 0. \end{aligned}$$

Those arguments, together with Proposition 3.1, constitute the proof of the result below.

Theorem 3.1. Consider the kinematic model of the car-like robot (1), and a regular parameterized curve $\varrho : \mathbb{D} \mapsto \mathbb{R}^2$ under the hypotheses of Theorem 2.1. Then, Problem 2.2 admits a solution under multi-rate control that is given by the feedback (v, ω) with: (i) $\omega = \underline{\omega}^{\bar{\delta}}(q, v, \nu_1)$ defined as in Proposition 3.1 with transverse control (15); (ii) v generated by the discrete dynamics (18) with tangential control (19).

Remark 3.1. It is important to stress the fact that, in the digital context, TFL is achieved working directly on the kinematic model of the car-like robot with no preliminary dynamic extension, contrarily to the continuous-time case.

The above discussion ascertains the intuitive expectation that whenever the path following problem is solvable in continuous time using TFL, a digital multi-rate feedback solution under sampling exists. As noted, the feedback component $\omega = \underline{\omega}^{\bar{\delta}}(q, v, \nu_1)$ soltion to (13) comes in the form of a series expansion in powers of $\bar{\delta}$. All terms of such an expansion (14) are computable through an iterative and constructive procedure solving, at each step, a linear equality in the corresponding unknown. For the first terms, one gets

$$\underline{\omega}_0(q, v, \nu_1) = \mathbf{1}\omega(q, v, \nu_1), \ \omega(q, v, \nu_1) = \frac{L_f^3\alpha(q) - \nu_1}{L_{g_2}L_f^2\alpha(q)} \tag{20a}$$

$$\underline{\omega}_1(q, v, \nu_1) = \begin{pmatrix} \frac{15}{4} & -\frac{15}{2} & \frac{87}{4} \end{pmatrix}^\top \dot{\omega}(q, v, \nu_1) \tag{20b}$$

when denoting $f(q) = g_1(q)v$ and $\dot{\omega}(q, v, \nu_1) = (vL_{g_1} + \omega(q, v, \nu_1)L_{g_2})\omega(q, v, \nu_1)$.

Remark 3.2. From the expression below, it is clear that the digital feedback $\omega = \underline{\omega}^{\bar{\delta}}(q, v, \nu_1)$ solution to (13) is an expansion around the continuous-time (static) TFL solution when no velocity and acceleration requirements on the path are enforced (i.e., when only (1) and (2) in Problem 2.1 are given).

Because a closed-form of (14) cannot be computed, only approximations can be implemented in practice as the truncation of the series at a finite order $\rho > 0$, i.e.,

$$\underline{\omega}^{\bar{\delta},[\rho]}(q, v, \nu_1) = \underline{\omega}_0(q, v, \nu_1) + \sum_{i=1}^{\rho} \frac{\bar{\delta}^i}{(i+1)!}\underline{\omega}_i(q, v, \nu_1). \tag{21}$$

Such approximate controllers (21) with (18)-(19) solve Problem 2.2 in a practical sense with trajectories of the closed-loop system converging to a neighbourhood of target set $\Gamma^\star$ in $O(\delta^{p+1})$, Mattioni et al. (2017).

4. SIMULATIONS

Consider the case where the car-like robot is required to follow a circle of radius $r = 2$. In this case; $\varrho : \mathbb{D} \mapsto \mathbb{R}^2$, $\lambda \mapsto (r \cos q_3 \ r \sin q_3)^\top$, satisfies the hypotheses of Theorem 2.1. Consequently, one has $\alpha(q) = q_1^2 + q_2^2 - r^2$. Suppose we are given a constant reference on the linear velocity $\dot{\pi}_{ref}(q) = 2$ with zero acceleration (i.e., $\ddot{\pi}_{ref}(q) = 0$). To follow the path, let $v(k)$ be as in (18)-(19), with $\lambda_i = 5, i = 1, 2$, and the first order approximate feedback be as in (21) with $\rho = 1$. In addition, for the transverse dynamics, let $\nu_1(k)$ be as in (15) with $K^{\bar{\delta}}$ placing the poles of $(A^{3\bar{\delta}} + B^{3\bar{\delta}}K^{\delta})$ in $(e^{-0.317\delta}, \ e^{(-1.34+1.16i)\delta)}, \ e^{(-1.34-1.16i)\delta)})$ for $\delta = 3\bar{\delta}$.

Simulation compare, for different initial conditions and increasing values of δ, the proposed multi-rate controller (21) with $\rho = 1$ (MR Sampling) with both the emulation-based (ZOH emulation) and the one proposed in Elobaid et al. (2020) (Approx Sampling), based on the continuous-time preliminary dynamic extension. For completeness, the results under continuous-time control are reported as well.

In Figures 2 and 3, the initial condition is fixed as $q_0 = (1 \ 1 \ \pi \ 0)^\top$ with increasing values of δ. The former one highlights that, albeit all controllers ensure convergence to the circle, the results under multi-rate control are slightly better compared to the other controllers. The latter one shows that as δ increases one notices deteriorated performances by both the approximate feedback in Elobaid et al. (2020) as well as emulation based control with higher peak values for the control effort. In all cases, the multi-rate control ensures satisfactory performances with improved control effort even with respect to the continuous time control law. In Figure 4, the initial condition is fixed as $q_0 = (3 \ 2 \ \pi \ 0)$ and $\delta = 0.3235s$. Whereas similar comments hold true in this case for the emulation and the approximate single-rate controllers (which strongly rely on the continuous-time design), for such initialization (outside the circle) both the aforementioned controllers are significantly sensible to variations of δ, with no guarantee for convergence to the path under emulation. In this case, the multi-rate solution provides still notable performances with a desired velocity profile and limited control effort.

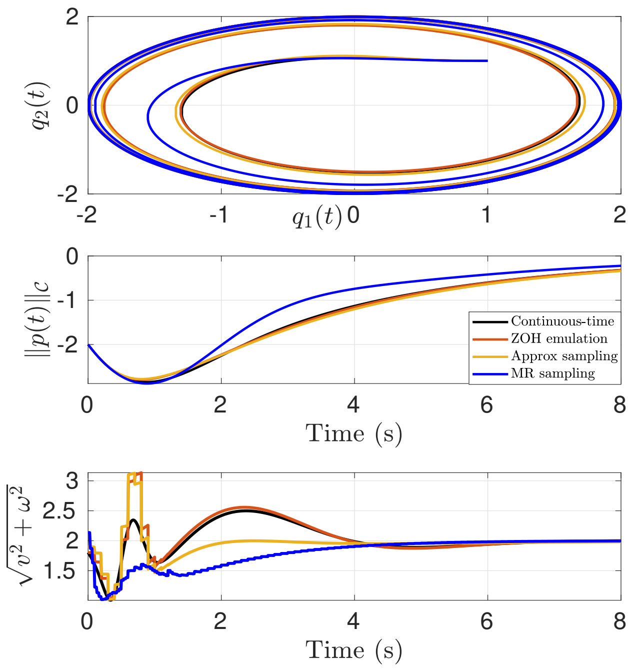

Fig. 2. From top to bottom: position on the plane, path-following error, control effort for $\delta = 0.1s$

5. CONCLUSIONS

Existence of a multi-rate digital feedback solution to the path following problem for a car-like robot via transverse feedback linearization was studied assuming a continuous time solution to the problem exists. In addition, this multi-rate solution was shown, through simulations, to provide better performances, both in terms of path-following position error and the required magnitude of input velocities to the robot. Perspectives concern the extension to the case of general nonlinear systems being transverse feedback linearizable in continuous time and the consequent application to further case studies.

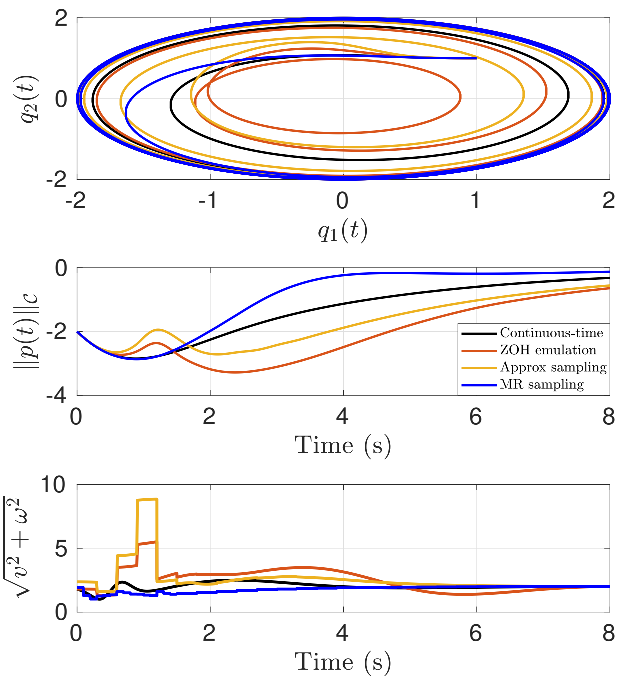

Fig. 3. From top to bottom: position on the plane, path-following error, control effort for $\delta = 0.3s$

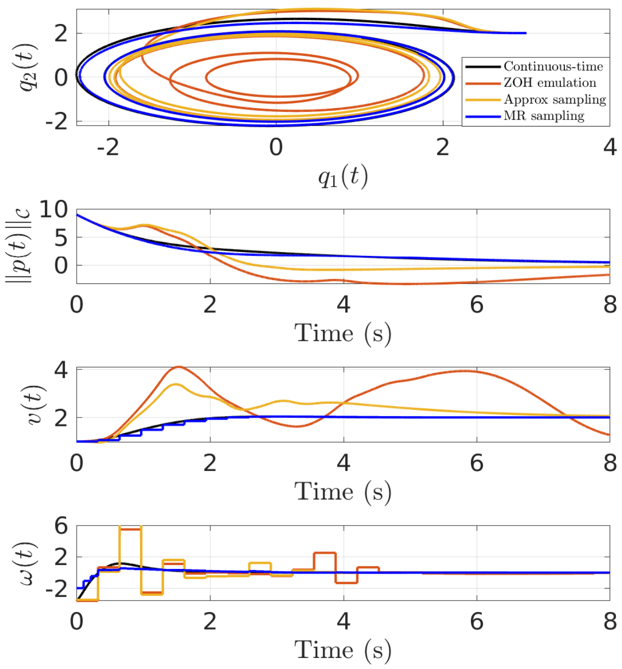

Fig. 4. From top to bottom: position on the plane, path-following error, linear velocity tracking, control effort for $\delta = 0.3235s$

REFERENCES

Aguiar, A.P., Hespanha, J.P., and Kokotović, P.V. (2008). Performance limitations in reference tracking and path following for nonlinear systems. *Automatica*, 44, 598–610.

Akhtar, A., Nielsen, C., and Waslander, S.L. (2015). Path following using dynamic transverse feedback linearization for car-like robots. *IEEE Transactions on Robotics*, 31, 269–279.

Altafini, C. (2002). Following a path of varying curvature as an output regulation problem. *IEEE Transactions on Automatic Control*, 47, 1551–1556.

Aström, K.J., Hagander, P., and Sternby, J. (1984). Zeros of sampled systems. *Automatica*, 20, 31–38.

Banaszuk, A. and Hauser, J. (1995). Feedback linearization of transverse dynamics for periodic orbits. *Systems & Control Letters*, 26, 95–105.

Dagci, O.H., Ogras, U.Y., and Ozguner, U. (2003). Path following controller design using sliding mode control theory. *American Control Conference*, 1, 903–908.

De Luca, A., Oriolo, G., and Vendittelli, M. (2001). Control of wheeled mobile robots: An experimental overview. *Ramsete*, 181–226.

Di Giamberardino, P., Grassini, F., Monaco, S., and Normand-Cyrot, D. (1996). Piecewise continuous control for a car-like robot: implementation and experimental results. *35th IEEE Conference on Decision and Control*, 3, 3564–3569.

Elobaid, M., Monaco, S., and Normand-Cyrot, D. (2020). Approximate transverse feedback linearization under digital control. *IEEE Control Systems Letters*.

Mattioni, M., Monaco, S., and Normand-Cyrot, D. (2017). Immersion and invariance stabilization of strict-feedback dynamics under sampling. *Automatica*, 76, 78–86.

Monaco, S. and Normand-Cyrot, D. (1992). An introduction to motion planning under multirate digital control. *31st IEEE Conference on Decision and Control*, 2, 1780–1785.

Monaco, S., Normand-Cyrot, D., and Stornelli, S. (1986). On the linearizing feedback in nonlinear sampled data control schemes. *25th IEEE Conference on Decision and Control*, 2056–2060.

Monaco, S. and Normand-Cyrot, D. (1997). On nonlinear digital control. In *Nonlinear control*, 127–155. Springer.

Monaco, S. and Normand-Cyrot, D. (1987). Minimum-phase nonlinear discrete-time systems and feedback stabilization. *26th IEEE Conference on Decision and Control*, 26, 979–986.

Nielsen, C. and Maggiore, M. (2004). Maneuver regulation via transverse feedback linearization: Theory and examples. *IFAC symposium on Nonlinear Control systems (NOLCOS)*, 37, 57–64.

Nielsen, C. and Maggiore, M. (2006). Output stabilization and maneuver regulation: A geometric approach. *Systems & Control Letters*, 55, 418–427.

Nielsen, C. and Maggiore, M. (2008). On local transverse feedback linearization. *SIAM Journal on Control and Optimization*, 47, 2227–2250.

Siciliano, B., Sciavicco, L., Villani, L., and Oriolo, G. (2010). *Robotics: modelling, planning and control.* Springer Science & Business Media.

Yi, B., Ortega, R., Manchester, I.R., and Siguerdidjane, H. (2020). Path following of a class of underactuated mechanical systems via immersion and invariance-based orbital stabilization. *International Journal of Robust and Nonlinear Control*, 30, 8521–8544.