



HAL
open science

Thin Layer Models for Geophysical Applications

Julien Diaz, Victor Péron

► **To cite this version:**

Julien Diaz, Victor Péron. Thin Layer Models for Geophysical Applications. Second Russian-French Conference on Computational Geophysics, Sep 2014, Novosibirsk, Russia. hal-03172030

HAL Id: hal-03172030

<https://inria.hal.science/hal-03172030v1>

Submitted on 17 Mar 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Thin Layer Models for Geophysical Applications

JULIEN DIAZ^{1,2}, VICTOR PÉRON^{2,1}

¹ INRIA Bordeaux Sud-Ouest, Team MAGIQUE 3D

² Université de Pau et des Pays de l'Adour

Second French-Russian Conference

Novosibirsk, September 25, 2014

Motivations

HPC-GA Project

- **High Performance Computing for Geophysical Applications**

FP7 Project : INRIA, BCAM, UFRGS, UNAM

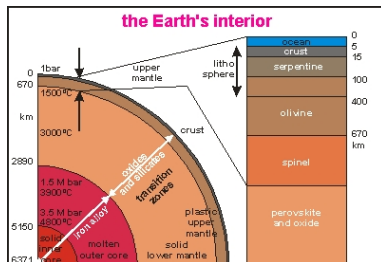
- Earthquakes
- Tsunami

- **Numerical Models (WP2)**

- Asymptotic method : To derive approximate models for the propagation of seismic waves and acoustic waves
- Elasto-acoustic coupling : To reproduce earthquakes

Motivations

Configurations of interest



- **Configuration A** : The medium of interest is a Land Area surrounded by a fluid zone
- **Configuration B** : The medium of interest is a Land Area surrounded by a fluid zone and a part of the atmosphere

Motivations

Configuration A

- **Configuration A:**

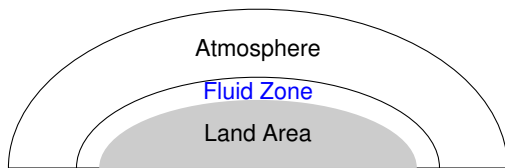


- The thickness of the **fluid zone** is negligible w.r.t. the wavelength
- **Sub-Configurations:**
 - 1 **Configuration A1** : A **thin layer** with a **uniform** thickness
 - 2 **Configuration A2** : A **thin layer** with a **variable** thickness

Motivations

Configuration B

- **Configuration B**



- **Difficulty**

Apply a FEM on a mesh with **thin cells** in the **Fluid Zone** and much larger outside

Approach : an Asymptotic Method

- 1 "Replace" the **thin layer** by an Equivalent Boundary Condition
- 2 Couple this condition with
 - the elastic equation (**Configuration A**)
 - elastic and acoustic equations (**Configuration B**)

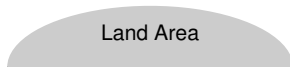


Figure: Configuration A
without the **thin layer**

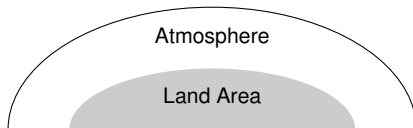


Figure: Configuration B
without the **thin layer**

- 3 Apply a (DG) Finite Element Method

Bibliography

- Derivation of Equivalent Conditions for **thin layer** Problems :



ENGQUIST-NÉDÉLEC, 93



ABBOUD-AMMARI, 96



BENDALI-LEMRABET, 96



LAFITTE, 99

...



PÉRON, 2014



BUREL, PHD THESIS, 2014



DE CASTRO, DIAZ, PÉRON, 2014

Outline

- 1 Energy Estimates (Conf. A1)
- 2 Equivalent Conditions (Conf. A1)
- 3 Numerical Results (Conf. A1)
- 4 Equivalent Conditions (Conf. A2 and B)

Outline

- 1 Energy Estimates (Conf. A1)**
- 2 Equivalent Conditions (Conf. A1)
- 3 Numerical Results (Conf. A1)
- 4 Equivalent Conditions (Conf. A2 and B)

A Model Problem

The Problem (\mathbf{P}_ε) set in a smooth domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

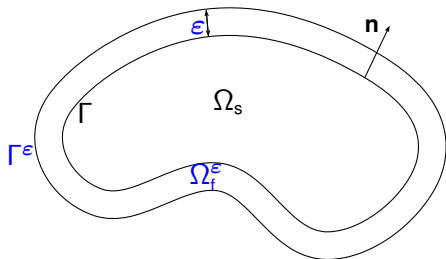
$$\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} \quad \text{in } \Omega_s$$

$$\Delta p + \left(\frac{\omega}{c_f}\right)^2 p = 0 \quad \text{in } \Omega_f^\varepsilon$$

$$\partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} \quad \text{on } \Gamma$$

$$\underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = -p \mathbf{n} \quad \text{on } \Gamma$$

$$p = 0 \quad \text{on } \Gamma^\varepsilon$$



Issue : Uniform Estimates for solutions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ of (\mathbf{P}_ε) as $\varepsilon \rightarrow 0$?

Framework

$$\text{Hooke's law : } \underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$$

Assumption

- (i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Framework

Hooke's law : $\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\epsilon}}(\mathbf{u})$

Assumption

- (i) $\underline{\underline{C}}(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klij}$
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ
- (iii) The tensor $\underline{\underline{C}}(x)$ is positive

Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem (\mathbf{P}_ε) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H_{0,\Gamma_\varepsilon}^1(\Omega_f^\varepsilon)$, and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C\|\mathbf{f}\|_{0,\Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$

Uniform Estimates

Theorem

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem (\mathbf{P}_ε) has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in H^1(\Omega_s) \times H_{0,\Gamma_\varepsilon}^1(\Omega_f^\varepsilon)$, and

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega_s} + \|p_\varepsilon\|_{1,\Omega_f^\varepsilon} \leq C \|\mathbf{f}\|_{0,\Omega_s}$$

Application : Convergence of an asymptotic expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ as $\varepsilon \rightarrow 0$
Key for the proof:

To introduce a "Scaled Problem" by means of the Scaling

$$S = \frac{s}{\varepsilon} \in (0, 1) \quad \text{where } s \in (0, \varepsilon)$$

Outline

- 1 Energy Estimates (Conf. A1)
- 2 Equivalent Conditions (Conf. A1)**
- 3 Numerical Results (Conf. A1)
- 4 Equivalent Conditions (Conf. A2 and B)

Methodology

- **Step 1** : Derive an Asymptotic Expansion for $(\mathbf{u}_\epsilon, p_\epsilon)$ when $\epsilon \rightarrow 0$

$$\begin{aligned}\mathbf{u}_\epsilon(x) &= \mathbf{u}_0(x) + \epsilon \mathbf{u}_1(x) + \epsilon^2 \mathbf{u}_2(x) + \dots \\ p_\epsilon(x) &= p_0(y_\alpha, \frac{s}{\epsilon}) + \epsilon p_1(y_\alpha, \frac{s}{\epsilon}) + \epsilon^2 p_2(y_\alpha, \frac{s}{\epsilon}) + \dots\end{aligned}$$

- **Step 2** : Equivalent Conditions of order $k + 1 \in \mathbb{N}$.
Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\epsilon} := \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \dots + \epsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\epsilon^{k+1})$$

- **Step 3** : Prove "Stability" & Convergence results for Equivalent models

Step 1 : Multiscale Expansion - First terms

1

$$p_0 = 0$$

2

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = 0 & \text{on } \Gamma \end{cases}$$

3

$$p_1(\cdot, S) = (S - 1) \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}, \quad S \in (0, 1)$$

4

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} & \text{on } \Gamma \end{cases}$$

5

$$p_2(\cdot, S) = -(S^2 - 1) \mathcal{H} \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} + (S - 1) \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

6

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} - \mathcal{H} \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} & \text{on } \Gamma \end{cases}$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\epsilon^N := \mathbf{u}_\epsilon - \sum_{n=0}^N \epsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\epsilon^N := p_\epsilon - \sum_{n=0}^N \epsilon^n p_n \quad \text{in } \Omega_f^\epsilon.$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{ll} \Delta r_\epsilon^N + \kappa^2 r_\epsilon^N & = f_\epsilon \quad \text{in } \Omega_f^\epsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{r}_\epsilon^N) + \omega^2 \rho \mathbf{r}_\epsilon^N & = 0 \quad \text{in } \Omega_s \\ \partial_{\mathbf{n}} r_\epsilon^N - \rho_f \omega^2 \mathbf{r}_\epsilon^N \cdot \mathbf{n} & = g_\epsilon \quad \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\epsilon^N) + r_\epsilon^N \mathbf{n} & = 0 \quad \text{on } \Gamma \\ r_\epsilon^N & = 0 \quad \text{on } \Gamma^\epsilon. \end{array} \right.$$

The RHS are explicit : $f_\epsilon = \mathcal{O}(\epsilon^{N-\frac{1}{2}})$ in Ω_f^ϵ , $g_\epsilon = \mathcal{O}(\epsilon^N)$ on Γ .

Optimal Estimates : $\|\mathbf{r}_\epsilon^N\|_{1,\Omega_s} + \sqrt{\epsilon} \|r_\epsilon^N\|_{1,\Omega_f^\epsilon} \leq C_N \epsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

For all $k \in \{0, 1, 2, 3\}$, we identify a simpler problem

$$(\mathbf{P}_{\epsilon}^k) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_{\epsilon}^k) + \omega^2 \rho \mathbf{u}_{\epsilon}^k = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_{\epsilon}^k) + \mathbf{B}_{k,\epsilon}(\mathbf{u}_{\epsilon}^k \cdot \mathbf{n}) \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

with $\mathbf{B}_{k,\epsilon}$ a surfacic differential operator.

Step 2 : Equivalent Conditions

1 Order 1 :

$$\mathbf{T}(\mathbf{u}_0) = 0$$

2 Order 2 :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - \epsilon \omega^2 \rho_f \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

3 Order 3 :

$$\mathbf{T}(\mathbf{u}_\epsilon^2) - \epsilon \omega^2 \rho_f (1 - \epsilon \mathcal{H}) \mathbf{u}_\epsilon^2 \cdot \mathbf{n} \mathbf{n} = 0$$

4 Order 4 :

$$\mathbf{T}(\mathbf{u}_\epsilon^3) - \epsilon \omega^2 \rho_f \left(1 - \epsilon \mathcal{H} + \frac{\epsilon^2}{3} [\Delta_\Gamma + \kappa^2 \mathbb{I} + 4\mathcal{H}^2 - \mathcal{K}] \right) \mathbf{u}_\epsilon^3 \cdot \mathbf{n} \mathbf{n} = 0$$

Step 3 : Stability and Convergence results

$$V_k = H^1(\Omega_s) \text{ when } k = 0, 1, 2$$

$$V_k = \{\mathbf{u} \in H^1(\Omega_s) \mid \mathbf{u} \cdot \mathbf{n}|_{\Gamma} \in H^1(\Gamma)\} \text{ when } k = 3$$

Proposition

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{P}_\varepsilon^k)$ with data $\mathbf{f} \in L^2(\Omega_s)$ has a unique solution $\mathbf{u}_\varepsilon^k \in V_k$ and

- Stability

$$\|\mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C \|\mathbf{f}\|_{0, \Omega_s}$$

- Convergence

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C\varepsilon^{k+1}$$

The case of a Fourier-Robin B.C.

Step 2 : Equivalent Conditions

When considering a Fourier-Robin boundary condition :

$$\partial_{\mathbf{n}} p_{\varepsilon} - i\kappa p_{\varepsilon} = 0 \quad \text{on } \Gamma^{\varepsilon}$$

- Order 1 :

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 \quad \text{on } \Gamma$$

- Order 2 :

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^1) - i\omega c_f \rho_f \left(1 + \varepsilon \left(-2\mathcal{H} + i\kappa^{-1} \Delta_{\Gamma}\right)\right) (\mathbf{u}_{\varepsilon}^1 \cdot \mathbf{n}) \mathbf{n} = 0$$

Outline

- 1 Energy Estimates (Conf. A1)
- 2 Equivalent Conditions (Conf. A1)
- 3 Numerical Results (Conf. A1)**
- 4 Equivalent Conditions (Conf. A2 and B)

Numerical simulations

We use a Discontinuous Galerkin Method (IPDGM).

- Computational domain : $\Omega_s = D(0; 0.01)$
- $\omega = 1.5 \times 10^6$
- Source on Γ ($\mathbf{f} = 0$) : $p_i(\mathbf{x}) = \exp(i\omega\mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- \mathbb{P}_3 -finite elements (Lagrange) in the Library [Hou10ni](#)
- Ω_s : Isotropic elastic material with Lamé coefficients :

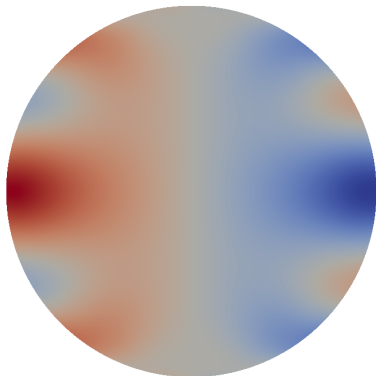
$$\mu = 26.32 \times 10^9 \quad \text{and} \quad \lambda = 51.08 \times 10^9 .$$

- $c = 1500 \text{ m.s}^{-1}$, $\rho_f = 1000 \text{ kg.m}^{-3}$, $\rho = 2700 \text{ kg.m}^{-3}$.

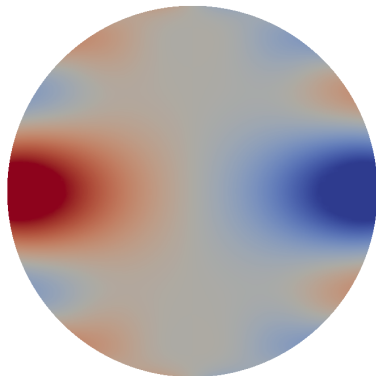
Numerical Simulations

Qualitative Comparisons

$$\varepsilon = 0.001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$
 \mathbf{u}_ε : analytical solution

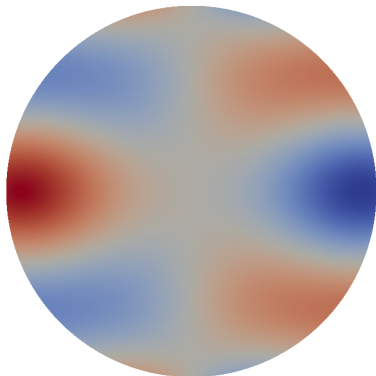


$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$
 \mathbf{u}_ε^2 (model of order 3) : IPDGM

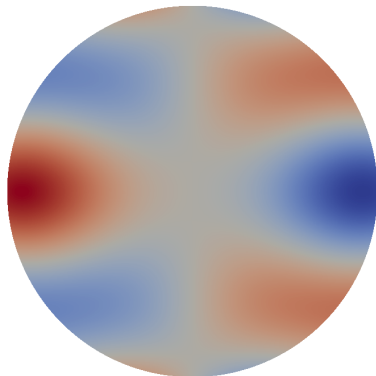
Numerical Simulations

Qualitative Comparisons

$$\varepsilon = 0.0001;$$



$|\operatorname{Re} \mathbf{u}_\varepsilon|$
 \mathbf{u}_ε : analytical solution



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$
 \mathbf{u}_ε^2 (model of order 3): IPDGM

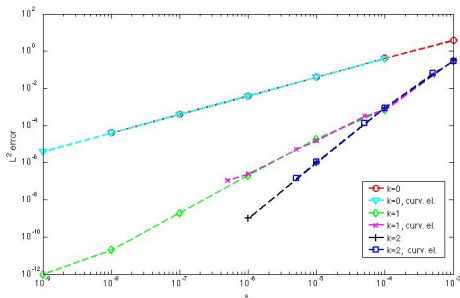
Convergence of the models

Dirichlet case

For $k \in \{0, 1, 2\}$, we plot the L^2 -error $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s}$ w.r.t. ϵ

\mathbf{u}_ϵ : analytical solution

\mathbf{u}_ϵ^k : analytical solution or numerical solution (IPDGM)



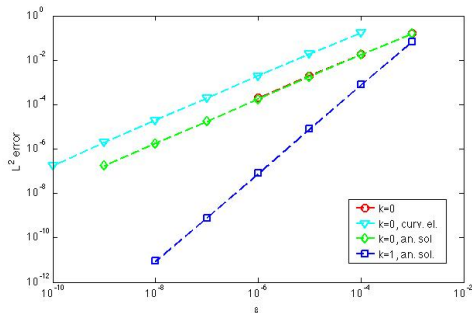
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2 -error $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0, \Omega_s}$ w.r.t. ε

\mathbf{u}_ε : analytical solution

\mathbf{u}_ε^k : analytical solution or numerical solution (IPDGM)



Outline

- 1 Energy Estimates (Conf. A1)
- 2 Equivalent Conditions (Conf. A1)
- 3 Numerical Results (Conf. A1)
- 4 Equivalent Conditions (Conf. A2 and B)**

Configuration A2

Framework in 2D

- Parameterization of the layer Ω_Γ^ϵ

$$\Omega_\Gamma^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

Configuration A2

Framework in 2D

- Parameterization of the layer Ω_f^ϵ

$$\Omega_f^\epsilon = \{ \mathbf{x}(t) + s f(t) \mathbf{n}(t) \in \mathbb{R}^2 \mid \mathbf{x}(t) \in \Gamma, \quad s \in (0, \epsilon) \}$$

t : arc-length on Γ

f : smooth and periodic function

$\mathbf{n}(t) = \mathbf{n}(\mathbf{x}(t))$: normal vector on Γ

- Euclidean metric of Ω_f^ϵ defined in the coordinates (t, s) :

$$\begin{pmatrix} (1 + s f(t) c(t))^2 + (s f'(t))^2 & s f(t) f'(t) \\ s f(t) f'(t) & f(t)^2 \end{pmatrix}$$

Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) = 0$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon f(t) \omega^2 \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

- $k = 2$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon f(t) \omega^2 \rho_f \left(1 - \frac{\varepsilon}{2} f(t) c(t) \right) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n} \mathbf{n} = 0$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) - i\omega c_f \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - i\omega c_f \rho_f \left(1 + \epsilon \left(-f(t)c(t) + i\kappa^{-1} [g(t)\partial_t + f(t)\partial_t^2] \right) \right) \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = 0$$

with

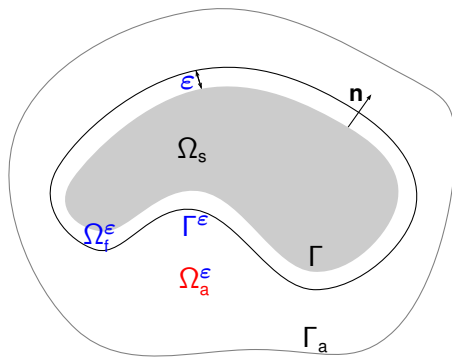
$$g(t) = (2 - f^2(t)) f'(t)$$

Configuration B

A Model Problem

(\mathbf{u}, p, p^a) satisfies

$$\left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} & \text{in } \Omega_s \\ \Delta p + \kappa^2 p = 0 & \text{in } \Omega_f^\epsilon \\ \Delta p^a + \kappa_a^2 p^a = 0 & \text{in } \Omega_a^\epsilon \\ \partial_n p = \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} & \text{on } \Gamma \\ \underline{\underline{\sigma}}(\mathbf{u}) \mathbf{n} = -p \mathbf{n} & \text{on } \Gamma \\ p^a = p & \text{on } \Gamma^\epsilon \\ c_a^2 \partial_n p^a = c_f^2 \partial_n p & \text{on } \Gamma^\epsilon \\ \text{ABC} & \text{on } \Gamma_a \end{array} \right.$$



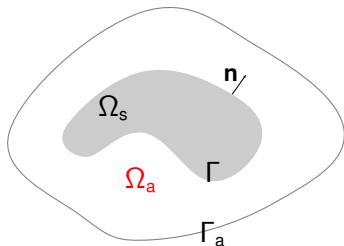
$$\kappa = \frac{\omega}{c_f}, \quad \kappa_a = \frac{\omega}{c_a}$$

Step 2 - Equivalent Conditions

Order 1

(\mathbf{u}_0, p_0) satisfies

$$\left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \Delta p_0^a + \kappa_a^2 p_0^a = 0 & \text{in } \Omega_a \\ \mathbf{T}(\mathbf{u}_0) = -p_0^a \mathbf{n} & \text{on } \Gamma \\ \partial_n p_0^a = \left(\frac{\alpha}{c_a}\right)^2 \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{on } \Gamma \\ \text{ABC} & \text{on } \Gamma_a \end{array} \right.$$



Step 2 - Equivalent Conditions

Order 2

$(\mathbf{u}_1, \mathbf{p}_1^a) := (\mathbf{u}_1^\varepsilon, \mathbf{p}_1^\varepsilon)$ satisfies

$$\left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = \mathbf{f} & \text{in } \Omega_s \\ \Delta \mathbf{p}_1^a + \kappa_a^2 \mathbf{p}_1^a = 0 & \text{in } \Omega_a \\ \mathbf{T}(\mathbf{u}_1) - \theta_{1,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} = -\mathbf{p}_1^a \mathbf{n} & \text{on } \Gamma \\ \partial_n \mathbf{p}_1^a + \varepsilon \left(\left(\frac{c_f}{c_a} \right)^2 - 1 \right) \Delta_\Gamma \mathbf{p}_1^a = \left(\frac{c_f}{c_a} \right)^2 \theta_{2,\varepsilon} \mathbf{u}_1 \cdot \mathbf{n} & \text{on } \Gamma \\ \text{ABC} & \text{on } \Gamma_a \end{array} \right.$$

$$\theta_{1,\varepsilon} = \varepsilon \rho_f \omega^2 \left[1 - \left(\frac{c_f}{c_a} \right)^2 \right]$$

$$\theta_{2,\varepsilon} = \rho_f \omega^2 - 2\varepsilon (1 - \rho_f \omega^2) \mathcal{H}$$

Prospect

- 1 To derive Equivalent Conditions adapted to high order ABCs
- 2 To implement Equivalent Conditions in configurations A2 and B
- 3 To implement Equivalent Conditions in 3D and in Time domain
- 4 To test these asymptotic models with real data

Thank you for your attention

Proof of Uniform Estimates

Variational Problem (VP_ϵ)

- $V_\epsilon = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f^\epsilon) \mid \gamma_0 p = 0 \text{ on } \Gamma^\epsilon\}$
- For all $\epsilon > 0$, (VP_ϵ) writes : Find $(\mathbf{u}_\epsilon, p_\epsilon) \in V_\epsilon$ such that

$$\forall (\mathbf{v}, q) \in V_\epsilon, \quad a_\epsilon((\mathbf{u}_\epsilon, p_\epsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V'_\epsilon, V_\epsilon}$$

$$a_\epsilon((\mathbf{u}, p), (\mathbf{v}, q)) := \int_{\Omega_f^\epsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, dx$$

$$+ \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}) \, dx + \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, ds$$

Formulation in a fixed domain

Scaled Problem (\mathfrak{P}_ε)

- $\Omega_f := \Gamma \times (0, 1)$
- $V = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f) \mid p(\cdot, 1) = 0 \text{ on } \Gamma\}$
- (\mathfrak{P}_ε) writes : Find $(\mathbf{u}, p) \in V$ such that $\forall (\mathbf{v}, q) \in V$

$$\varepsilon \int_0^1 \int_\Gamma [(\mathbb{I} + \varepsilon S\mathcal{R})^{-2} \nabla_\Gamma p \nabla_\Gamma \bar{q} + \varepsilon^{-2} \partial_S p \partial_S \bar{q} - \kappa^2 p \bar{q}] \det(\mathbb{I} + \varepsilon S\mathcal{R}) \, d\Omega_f$$

$$+ a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, q) \rangle_{V', V}$$

Scaled Problem (\mathfrak{P}_ε)

Uniform estimates

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, the problem (\mathfrak{P}_ε) with data $\mathfrak{F}_\varepsilon \in V'$ has a unique solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ which satisfies

$$\sqrt{\varepsilon} \|\nabla_{\Gamma} \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon}^{-1} \|\partial_s \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'} .$$

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, any solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ of problem (\mathfrak{P}_ε) with a data $\mathfrak{F}_\varepsilon \in V'$ satisfies

$$\|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'} .$$

Sketch of the proof of the Lemma

Assume : $\exists (\mathbf{u}_m, \mathbf{p}_m) \in V^{\mathbb{N}}$ satisfying $(\mathfrak{P}_{\varepsilon_m})$ with $\varepsilon_m \rightarrow 0$ and $\mathfrak{F}_m \in V'$ s.t.

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \rightarrow 0$$

We prove successively that :

1

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C$$

with $W = \{(\mathbf{u}, \mathbf{p}) \in H^1(\Omega_s) \times H^1(0, 1; L^2(\Gamma)) \mid \mathbf{p}(\cdot, 1) = 0 \text{ on } \Gamma\}$

2

$$\mathbf{u}_m \rightarrow \mathbf{u} \text{ in } L^2(\Omega_s) \quad \text{and} \quad \mathbf{p}_m \rightarrow \mathbf{p} = 0 \text{ in } L^2(\Omega_f)$$

3

$$\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} = 1$$

4 $\mathbf{u} = 0$ (using (SA)) : Contradiction.

Step 3 : Proof of Convergence

(i) Derive an expansion of \mathbf{u}_ε^k and show that :

$$\begin{aligned}\mathbf{u}_\varepsilon &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k, \\ \mathbf{u}_\varepsilon^k &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k.\end{aligned}$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1, \Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k, \varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma,$$

According to the Stability result, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1, \Omega_s} \leq C \varepsilon^{k+1}.$$