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A compact data structure for high dimensional Coxeter-Freudenthal-Kuhn triangulations

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1 — Abstract —

2 We consider a family of highly regular triangulations of \mathbb{R}^d that can be stored and queried
3 efficiently in high dimensions. This family consists of Freudenthal-Kuhn triangulations and their
4 images through affine mappings, among which are the celebrated Coxeter triangulations of type
5 \tilde{A}_d . Those triangulations have major advantages over grids in applications in high dimensions like
6 interpolation of functions and manifold sampling and meshing. We introduce an elegant and very
7 compact data structure to implicitly store the full facial structure of such triangulations. This
8 data structure allows to locate a point and to retrieve the faces or the cofaces of a simplex of any
9 dimension in an output sensitive way. The data structure has been implemented and experimental
10 results are presented.

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11 **1** Introduction

12 Subdivisions of Euclidean space are a major tool to efficiently answer geometric queries,
13 compute approximation of shapes or solve optimization problems. Among the most widely
14 used subdivision schemes are grids and triangulations. Both are subject to the curse of
15 dimensionality and their combinatorial complexity depends exponentially on the dimension
16 of the space. Triangulations are most flexible since their vertex set can be any set of points.
17 Differently, uniform grids depend only on the space but not on a given data set. The rigidity



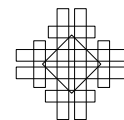
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18 of the grid structure has a major advantage: the grid, although of exponential size, need not
19 be represented explicitly and basic operations like locating a point or computing faces or
20 cofaces of a given cell in the grid can be done without storing an explicit representation of
21 the grid. In fact, the representation can be entirely implicit. This is clearly impossible with
22 general triangulations with arbitrary vertex sets.

23 The question of designing efficient data structure for triangulations and more general
24 simplicial complexes led to interesting developments recently. On one hand, one can take
25 advantage of the fact that special types of simplicial complexes allow compact representations.
26 Most notably, flag complexes (including the celebrated Vietoris-Rips complex) can be
27 represented by their 1-skeleton (or graph) and higher dimensional faces can be retrieved
28 by computing the cliques of the graph. One can also represent a simplicial complex by its
29 blockers, i.e. the simplices that do not belong to the complex but whose facets do [3].

30 On a different front, data structures have been proposed to efficiently store general
31 simplicial complexes such as the simplex tree [7] that uses a trie to store the faces of all
32 dimensions, or the Simplex Array List [6] that represents only the maximal faces, which allows
33 an exponential saving in storage since a simplex has exponential complexity. Nevertheless,
34 due to their generality and the fact that the represented complexes don't have any prespecified
35 symmetry, the data structures cannot compete with grids in terms of size and efficiency.

36 This paper explores middle ground and considers a special class of triangulations of \mathbb{R}^d
37 that have a high regularity. We propose a data structure for such triangulations that can be
38 as compact as for grids.

39 The class of triangulations we consider has different origins and names. The two founda-
40 tional works are due to Coxeter [14] and Freudenthal [18]. Coxeter triangulations derive from
41 geometric group theory, in particular affine Weyl groups, while Freudenthal triangulations
42 (also called Kuhn triangulations) are combinatorial in nature. Nevertheless, both triangula-
43 tions are the same up to a linear transformation, as remarked in [15] and fully proved in this
44 paper. This allows us to combine the nice geometric properties of Coxeter triangulations of
45 type \tilde{A}_d with the simple combinatorial definitions of the Freudenthal-Kuhn triangulation,
46 and its connection to permutahedra. Coxeter triangulations of type \tilde{A}_d are geometrically
47 attractive because each simplex is very well shaped (large volume compared to longest edge
48 length), and all d -simplices are identical up to reflections.

49 Although these triangulations do not depend on a given data set, they proved to be
50 very useful in to interpolate multivariate multivalued functions or to mesh geometric shapes
51 embedded in high dimensional spaces. Freudenthal-Kuhn triangulations have been known
52 in Applied Mathematics [1, 16, 25], and Coxeter triangulations have been used by Dobkin
53 et al. [15] to trace curves in high dimensions and are good candidates to trace manifolds
54 of any codimension [21, 4]. They have also been used in the context of Topological Data
55 Analysis [12].

56 In Section 2, we study these triangulations. This section recalls and extends to arbitrary
57 dimensions several results that were disseminated in many different places which are sometimes
58 difficult to access (see among others [18, 25, 16, 20, 27, 23, 15]).

59 Based on these results, we introduce in Section 3 a very compact data structure that
60 implicitly stores the full facial structure of such triangulations. The data structure allows to
61 locate a point in the triangulation and to retrieve the faces or the cofaces of a simplex of any
62 dimension in an output sensitive way.

63 The Data Structure has been implemented and fully tested. Section 4 reports on
64 experimental results and demonstrates that the data structure is highly practical and can
65 handle triangulations of high dimensional spaces. Using our data structure will allow to

66 extend the applicability of the methods based on such triangulations and to significantly
 67 improve their performance. It appears to be especially useful to trace low dimensional
 68 manifolds embedded in high dimensional spaces as encountered in statistics, dynamical
 69 systems, econometrics, or mechanics [10, 25, 22].

70 **2 Coxeter-Freudenthal-Kuhn triangulations**

71 Freudenthal-Kuhn triangulations are combinatorial structures that come from a specific
 72 triangulation of the d -cube. Their connections to permutahedra is at the heart of our data
 73 structure. Coxeter triangulations, we introduce in Section 2.3, have a different flavour and
 74 come with very nice geometric properties. Since both types of triangulations are the same
 75 up to an affine transformation, as first noted by Dobkin et al [15], they have the same
 76 combinatorial structure and our data structure will be able to handle both of them.

77 Although most ideas in this section were known previously, we give full proofs of the
 78 results that were not explicitly mentioned or not proved in full generality in the literature.

79 **2.1 Permutahedra**

80 We write $[i] = \{1, \dots, i\}$, and $[i, j] = \{i, \dots, j\}$.

81 **► Definition 1** (Permutahedron). *A d -permutahedron is a d -dimensional polytope, which is
 82 the convex hull \mathcal{P} of all points in \mathbb{R}^{d+1} , the coordinates of which are permutations of $[d + 1]$.
 83 Formally, this convex hull can be written as:*

84
$$\mathcal{P} = \text{conv}\{(\sigma(1), \dots, \sigma(d + 1)) \in \mathbb{R}^{d+1} \mid \sigma \in \mathfrak{S}_{d+1}\},$$

85 where \mathfrak{S}_{d+1} denotes the set of permutations of $[d + 1]$.

86 \mathcal{P} is at most d -dimensional since all its vertices lie on the hyperplane of equation

87
$$\sum_{i=1}^{d+1} x^i = \frac{d(d + 1)}{2}.$$

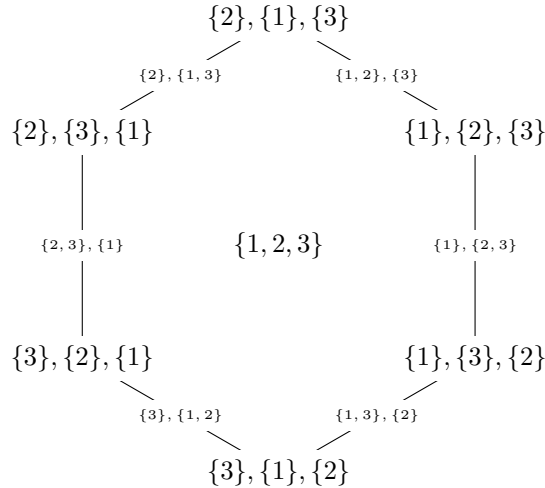
88 Moreover, it can be shown that there are $d + 1$ affinely independent vertices in \mathcal{P} , proving
 89 that \mathcal{P} is exactly d -dimensional (see for example [20, Lemma 3.4]). The facial structure of \mathcal{P}
 90 is best described in terms of ordered partitions [27]. Refer to Figure 1.

92 **► Definition 2** (Ordered partition). *Let T be a finite non-empty set, $|T|$ its cardinality, and
 93 $l \leq |T|$ a positive integer. An ordered partition of T in l parts is an ordered collection of l
 94 subsets $\omega = (\omega_1, \dots, \omega_l)$, such that $\omega_i \subseteq T$, and $\{\omega_1, \dots, \omega_l\}$ is a partition of T . The ω_i are
 95 called the parts. We write $OP_l[d]$ for the set of ordered partitions of $[d]$ with l parts, and just
 96 $OP[d]$ for the set of all ordered partitions of $[d]$.*

97 **► Definition 3** (Refinement). *Let ω , and ϖ be two ordered partitions of $[d + 1]$ in l , and p
 98 parts respectively, with $1 \leq l \leq p \leq d + 1$. We say that ϖ is a refinement of ω if there exist
 99 positive integers a_1, \dots, a_l , such that*

- 100 **■** $(\varpi_1, \dots, \varpi_{a_1})$ is an ordered partition of ω_1 in a_1 parts,
- 101 **■** $(\varpi_{a_1+1}, \dots, \varpi_{a_1+a_2})$ is an ordered partition of ω_2 in a_2 parts,
- 102 **■** \dots ,
- 103 **■** $(\varpi_{a_1+\dots+a_{l-1}+1}, \dots, \varpi_{a_1+\dots+a_l})$ is an ordered partition of ω_l in a_l parts.

104 We recall Theorem 3.6 of [20]:



91 **Figure 1** The 2-permutahedron and the ordered partitions associated to its faces.

105 **► Lemma 4** (Facial structure of the permutahedron). *The faces of a d -permutahedron are in*
 106 *bijection with the ordered partitions of $[d + 1]$. More precisely, the i -faces of \mathcal{P} correspond to*
 107 *ordered partitions of $[d + 1]$ into $l = d + 1 - i$ parts $(\omega_1, \dots, \omega_l)$. If σ , and τ are two faces of*
 108 *a d -permutahedron, σ is a subface of τ (denoted $\sigma \subseteq \tau$) if and only if the ordered partition*
 109 *associated to σ is a refinement of the ordered partition associated to τ .*

110 We also need the following result from [20, Corollary 3.15], and [23, Theorem 3].

111 **► Corollary 5.** *The number of $(d - i)$ -dimensional faces in a d -permutahedron is $(i + 1)! S(d +$
 112 $1, i + 1)$, where $S(\cdot, \cdot)$ is the Stirling number of the second kind. It is bounded by $2^{2(d+1) \log(i+1)}$.*

113 The following three corollaries seem to be new.

114 **► Corollary 6.** *The number $p_{0,i}$ of vertices of a i -face of a d -permutahedron is at most $(i + 1)!$,*
 115 *and at least $2^{\min(i, d-i)}$.*

116 The proof of Corollary 6 is based on:

117 **► Lemma 7** (Lemma 3.11 of [20]). *The face of a permutahedron corresponding to an ordered*
 118 *partition $\omega = (\omega_1, \dots, \omega_{l+1})$ is combinatorially*

119
$$\mathcal{P}(|\omega_1|) \times \dots \times \mathcal{P}(|\omega_{l+1}|),$$

120 *where $|\omega_p|$ denotes the length of the p -th part of the ordered partition, and $\mathcal{P}(n)$ the permuta-*
 121 *hedron of dimension $n - 1$.*

122 **Proof of Corollary 6.** Write $l = d - i$. Since the number of vertices of the product of two
 123 polytopes is the product of the vertices, and a $(n - 1)$ -dimensional permutahedron has $n!$
 124 vertices, we see that the total number of vertices of a i -face of a d -dimensional permutahedron
 125 corresponding to an ordered partition $\omega = (\omega_1, \dots, \omega_{l+1})$ is

126
$$\prod_{p=1}^{l+1} (|\omega_p|!).$$

127 Let $1 \leq j < k \leq d$ be integers such that $j + k = d + 1$. By definition $j!k! < (j - 1)!(k + 1)!$,
 128 and thus $j!k! \leq 1d!$. Generalizing this, we see that the product of the $|\omega_p|!$ is maximal when
 129 all parts are singletons except the biggest part which has $d + 1 - l$ elements. Therefore

$$130 \quad \prod_{p=1}^{l+1} (|\omega_p|!) \leq (d - l + 1)!.$$

131 Using the inverse argument, the lower bound is obtained when each part in the ordered
 132 partition are as small as possible that is all parts have almost equal size. In this case,
 133 $|\omega_p| \geq \lfloor \frac{d+1}{l+1} \rfloor$, so that

$$134 \quad \prod_{p=1}^{l+1} (|\omega_p|!) \geq \left(\left\lfloor \frac{d+1}{l+1} \right\rfloor ! \right)^{l+1}$$

135 More accurately, let r' be the remainder of $d + 1$ after division by $l + 1$, that is $r' = d + 1$
 136 $\text{mod } l + 1$ then:

$$137 \quad \prod_{p=1}^{l+1} (|\omega_p|!) \geq \left(\left\lfloor \frac{d+1}{l+1} \right\rfloor ! \right)^{l-r'+1} \left(\left(\left\lfloor \frac{d+1}{l+1} \right\rfloor + 1 \right) ! \right)^{r'}$$

138 We now distinguish two cases:

139 ■ If $\lfloor \frac{d+1}{l+1} \rfloor \geq 2$, and thus $\frac{d+1}{2} \geq l + 1$, then we see that

$$140 \quad \prod_{p=1}^{l+1} (|\omega_p|!) \geq 2^{l+1}.$$

141 ■ If $\lfloor \frac{d+1}{l+1} \rfloor = 1$, we have that $\frac{d+1}{2} < l + 1 \leq d + 1$, and thus $r' = d + 1 \text{ mod } l + 1 = d - l$.

142 Hence,

$$143 \quad \prod_{p=1}^{l+1} (|\omega_p|!) \geq 2^{d-l}.$$

144 Because $l + 1 = d - l$, or $2l + 1 = d$ is precisely the point where you go from the first to
 145 the second case we see that

$$146 \quad \prod_{p=1}^{l+1} (|\omega_p|!) \geq 2^{\min\{l+1, d-l\}}$$

147 ◀

148 ► **Corollary 8.** *The number of facets of an i -face σ of a d -permutahedron is at most 2^{i+1} .*

149 **Proof.** Write $l = d - i$. We first recall a set of $d > 2$ objects can be subdivided in two
 150 non-empty ordered subsets A and B in $2^d - 2$ ways. This is not hard to see. Because we
 151 pick for each element if it will be put in A or B there are 2^d possibilities. Excluding that A
 152 or B is empty gives $2^d - 2$. Let $\omega = (\omega_1, \dots, \omega_l)$ again be an ordered partition. To find a
 153 refinement of ω in $l + 1$ parts, we need to first pick a $1 \leq p \leq l$, such that $|\omega_p| > 1$, and then
 154 we need to break ω_p up into two (ordered) parts, for which there are $2^{|\omega_p|} - 2$ possibilities
 155 as we have seen above. This means that if $I = \{p \mid 1 \leq p \leq l, |\omega_p| > 1\}$, the number of
 156 refinements is

$$157 \quad \sum_{p \in I} 2^{|\omega_p|} - 2.$$

158 Let now $1 \leq s < t \leq d$ be integers such that $s + t = d + 1$. Then $2^s + 2^t < 2^{s-1} + 2^{t+1}$.
 159 Generalizing this, we see that the sum of the $2^{|\omega_p|} - 2$ is maximal when all $|\omega_p| = 1$ except
 160 the biggest part which has $d - l + 1 = i + 1$ elements. ◀

161 ▶ **Corollary 9.** *Let $p_{i,j}$ denote the number of i -faces of a j -face of the d -permutahedron. We*
 162 *have*

$$163 \quad p_{i,j} \leq \frac{1}{2^{\min(i,d-i)}} \binom{j}{i} (j+1)!$$

164

165 Corollary 9 generalizes the previous two corollaries. For $i = 0$, the bound in Corollary 9
 166 is the same as the upper bound in Corollary 6. For $i = j - 1$, the bound is comparable but
 167 weaker than the bound in Corollary 8.

168 **Proof of Corollary 9.** Let σ be a j -face of the d -permutahedron. Write $F_{i,\sigma}$ for the set of
 169 i -faces of σ , and c_v for the number of i -cofaces of a vertex v of σ , i.e. the number of simplices
 170 of $F_{i,\sigma}$ that contain v . For $\tau \in F_{i,\sigma}$, we write p_τ for the number of vertices of τ . By double
 171 counting the incidences between vertices, and i -faces inside σ , we have

$$172 \quad \sum_{\tau \in F_{i,\sigma}} p_\tau = \sum_{v \in F_{0,\sigma}} c_v.$$

173 Now observe that the d -permutahedron is a simple polytope (this follows from the fact
 174 that its dual is simplicial since it is a star in the FK-triangulation). The faces of simple
 175 polytopes are also simple polytopes which implies that the vertices of a j -face are incident to
 176 $\binom{j}{i}$ faces of dimension i [8, Lemma 7.1.14]. Moreover, $|F_{0,\sigma}| \leq (j+1)!$ by Corollary 6. Hence

$$177 \quad \sum_{v \in F_{0,\sigma}} c_v = \binom{j}{i} |F_{0,\sigma}| \leq \binom{j}{i} (j+1)!.$$

178 In addition, by Corollary 6, we have

$$179 \quad \sum_{\tau \in F_{i,\sigma}} p_\tau \geq 2^{\min(i,d-i)} |F_{i,\sigma}|$$

180 The inequality follows since σ is any j -face of the d -permutahedron. ◀

181 2.2 Freudenthal-Kuhn triangulation

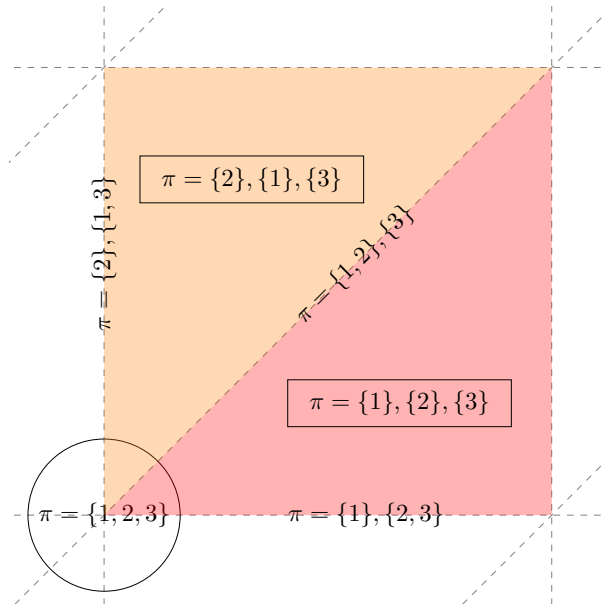
182 The Freudenthal-Kuhn (FK for short) triangulation is obtained from the d -grid, i.e. the
 183 unit cubical tessellation of \mathbb{R}^d that consists of copies of the unit d -cube along the integer
 184 lattice \mathbb{Z}^d . By triangulating each d -cube in the grid in an appropriate way to be described
 185 now, we obtain the FK-triangulation of \mathbb{R}^d . The results and definitions below were known
 186 to Freudenthal [18], Todd [25], or Eaves [16], mainly for top dimensional simplices and in
 187 different guises. We combine these results and extend to simplices of arbitrary (co)dimension,
 188 where necessary.

189 ▶ **Definition 10.** *Let $x \in \mathbb{R}^d$, and write $z^i = x^i - \lfloor x^i \rfloor$. We denote by e_1, \dots, e_d the basis*
 190 *vectors and introduce, for reasons that will be clear later, the extra vector*

$$191 \quad e_{d+1} = - \sum_{i=1}^d e_i.$$

192 We introduce the convention that $z^{d+1} = 0$. We associate to x the ordered partition $\omega =$
 193 $(\omega_1, \dots, \omega_{l+1})$ of $[d + 1]$ where the ω_i are obtained by sorting the z^i in decreasing order.
 194 Specifically, with $\omega_i = \{\omega_i(1), \dots, \omega_i(m_i)\}$, we have (see Figure 2):

$$195 \quad 1 > z^{\omega_1(1)} = \dots = z^{\omega_1(m_1)} > \dots > z^{\omega_l(1)} = \dots = z^{\omega_l(m_l)} \\
 196 \quad > z^{\omega_{l+1}(1)} = \dots = z^{\omega_{l+1}(m_{l+1})} = 0. \tag{1}$$



197 **Figure 2** The ordered partitions associated to the faces of the FK-triangulation of \mathbb{R}^2 that have
 198 the same minimal vertex v_0 (circled).

199 **Lemma 11.** Suppose that $\omega = (\omega_1, \dots, \omega_{l+1})$ is an ordered partition of $[d + 1]$ such that
 200 $d + 1 \in \omega_{l+1}$, and let $\sigma = \{v_0, \dots, v_l\}$ be the l -simplex whose vertices are the points

$$201 \quad v_0 = ([x^1], \dots, [x^d]), \quad v_i = v_{i-1} + \sum_{j \in \omega_i} e_j \quad i = 1, \dots, l. \tag{2}$$

202 Then x is a point in the relative interior of σ if and only if $z^i = x^i - [x^i]$, $i = 1, \dots, d + 1$
 203 (with, as above, $z^{d+1} = 0$), satisfy (1).

204 **Proof.** Because the whole problem is translation invariant, we assume that $v_0 = 0$ without
 205 loss of generality, so that the expressions are shorter. Using barycentric coordinates, $z \in \sigma$
 206 can be written as

$$207 \quad z = \sum_{i=0}^l \lambda_i v_i = \sum_{i=0}^l \lambda_i \sum_{k=1}^i \sum_{j \in \omega_k} e_j \\
 208 \quad = \lambda_l \left(\sum_{k \in \omega_l} e_k \right) + (\lambda_l + \lambda_{l-1}) \left(\sum_{k \in \omega_{l-1}} e_k \right) + \dots + (\lambda_l + \dots + \lambda_1) \left(\sum_{k \in \omega_1} e_k \right). \tag{3}$$

209 Here the $\lambda_i > 0$, $i \in [0, l]$, $\sum_{i=0}^l \lambda_i = 1$, are the barycentric coordinates of z in σ . We have

$$\begin{aligned}
 210 \quad & \alpha_{\omega_l(1)} = \cdots = \alpha_{\omega_l(m_l)} = \lambda_l \\
 211 \quad & \quad \quad \quad \vdots \\
 212 \quad & \alpha_{\omega_1(1)} = \cdots = \alpha_{\omega_1(m_1)} = \lambda_l + \cdots + \lambda_1
 \end{aligned} \tag{4}$$

213 From (3), we see that $\alpha_{\omega_i(j)}$ is the $\omega_i(j)$ th coordinate of z , denoted by $z^{\omega_i(j)}$, while all
 214 coordinates $z^{\omega_{i+1}(1)}, \dots, z^{\omega_{i+1}(m_{i+1})}$ are zero, because $e_{\omega_{i+1}(i)}$ does not occur in (3), for all i .
 215 Moreover, because $\lambda_l + \cdots + \lambda_i > \lambda_l + \cdots + \lambda_{i-1}$, we see that (1) is satisfied.

216 Conversely, given a point z such that its coordinates satisfy (1), we can read of its
 217 barycentric coordinates with respect to the v_i , as defined by (2), from (4). ◀

218 ▶ **Theorem 12.** *The equivalence classes of the points of \mathbb{R}^d with a same ordered partition*
 219 *are the simplices of a triangulation of \mathbb{R}^d called the FK-triangulation (see Figure 2).*

220 **Proof.** Lemma 11 implies that:

- 221 ■ Any face of a simplex in the FK-triangulation also lies in the FK-triangulation.
- 222 ■ The intersection of two simplices in the FK-triangulation also lie in the FK-triangulation.
- 223 ■ For any point $x \in \mathbb{R}^d$, there is a unique simplex σ such that x lies in the relative interior
 224 of σ . Indeed, x has uniquely defined barycentric coordinates with respect to the vertices
 225 of σ , and thus is mapped to a unique point in σ .

226 Hence the partition we have defined is a well-defined triangulation of \mathbb{R}^d . ◀

227 ▶ **Remark 13.** *We note that, by construction, v_0 in Lemma 11 is the smallest vertex of σ*
 228 *in the lexicographical order. Lemma 11 also implies an observation of Freudenthal [18]: all*
 229 *d -simplices in the FK-triangulation can be described by monotone paths along the edges of*
 230 *the cube from vertex $(0, \dots, 0) + v_0$ to vertex $(1, \dots, 1) + v_0$. Conversely, any monotone path*
 231 *along the edges of the cubes from $(0, \dots, 0) + v_0$ to $(1, \dots, 1) + v_0$ gives a simplex in the*
 232 *FK-triangulation.*

233 2.3 CFX-triangulations

234 Freudenthal-Kuhn triangulations are closely related to Coxeter triangulations of type \tilde{A}_d [11]
 235 and both are arrangements of hyperplanes as demonstrated now.

236 Let E be a finite set of vectors of \mathbb{R}^d , and consider the set of hyperplanes $H_E = \{x \in$
 237 $\mathbb{R}^d \mid \langle x, u \rangle = k, u \in E, k \in \mathbb{Z}\}$. In generalization of the theory of Coxeter triangulations, we
 238 call the set E the set of roots for historic reasons, as mentioned below.

239 These hyperplanes partition \mathbb{R}^d in a cell complex called the *arrangement* of the hyperplanes.
 240 We denote it by \mathcal{H}_E .

241 ▶ **Lemma 14.** *The Freudenthal-Kuhn triangulation is the hyperplane arrangement $\mathcal{H}_{E_{FK}}$*
 242 *associated to the set of vectors $E_{FK} = \{e_1, \dots, e_d\} \cup \{u_{i,j} = e_j - e_i \mid 1 \leq i < j \leq d\}$.*

243 **Proof.** Thanks to Lemma 11, we know that if $x \in \sigma$, where σ is a simplex of dimension less
 244 than d , there is at least one equality in (1) on top of $z^{d+1} = 0$. That is $x^i - \tilde{v}_0^i = x^j - \tilde{v}_0^j$ or
 245 $x^i - \tilde{v}_0^i = 0$ for some $i, j \neq d+1$. Note that $\tilde{v}_0^i, \tilde{v}_0^j \in \mathbb{Z}$. The converse direction of Lemma 11
 246 gives that if $x^i - \tilde{v}_0^i = x^j - \tilde{v}_0^j$ or $x^i - \tilde{v}_0^i = 0$ for some $i, j \neq d+1$ for $x \in \mathbb{R}^d$, then there is a
 247 simplex σ of dimension strictly less than d in the FK-triangulation such that $x \in \sigma$. ◀

248 Observe that the norms of the vectors in E_{FK} are either 1 or $\sqrt{2}$. By definition, this
 249 implies that the distance between the two hyperplanes $\langle x, u \rangle = k$, and $\langle x, u \rangle = k + 1$, where
 250 $u \in E_{FK}$, is either 1 or $1/\sqrt{2}$.

251 Let H be the hyperplane of \mathbb{R}^{d+1} of equation $\langle x, \mathbf{1} \rangle = 0$ where $\mathbf{1}$ is the vector of \mathbb{R}^{d+1}
 252 whose coordinates are all 1. We now define a linear map μ from \mathbb{R}^d to H by showing how
 253 it acts on the basis of \mathbb{R}^d : $\mu(e_i) = r_{1,i} = \sum_{j=1}^i s_j$, where $s_i = e_i - e_{i+1}$, $i = 1, \dots, d$. The
 254 vectors s_j are called simple roots and play an important role in algebra. We refer to [11] for
 255 more information.

256 ► **Lemma 15.** μ maps E_{FK} bijectively onto the set E_C defined as

$$257 \quad E_C = \left\{ r_{i,j} = \sum_{l=i}^j s_l = e_i - e_{j+1} \mid 1 \leq i \leq j \leq d \right\}$$

258 **Proof.** The vector $\mu(e_i) = r_{1,i}$ lies in E_C , by definition. For $u_{i,j} \in E_{FK}$, with $i < j$, we see
 259 that

$$260 \quad \begin{aligned} \mu(u_{i,j}) &= \mu(e_j - e_i) = \mu(e_j) - \mu(e_i) = r_{1,j} - r_{1,i} \\ 261 \quad &= \sum_{l=1}^j s_l - \sum_{l=1}^i s_l = \sum_{l=i+1}^j s_l = r_{i+1,j}. \end{aligned}$$

262 Hence $\mu(u_{i,j})$ lies in E_C . By reading the previous calculation backwards, we see that μ^{-1}
 263 maps each $r \in E_C$ to a vector in E_{FK} . ◀

264 Observe that all vectors in E_C have length $\sqrt{2}$. By definition, this implies that the
 265 distance between the two hyperplanes $\langle x, u \rangle = k$, and $\langle x, u \rangle = k + 1$, where $u \in E_C$, is $1/\sqrt{2}$.

266 The image by μ of the Freudenthal-Kuhn triangulation is a triangulation of \mathbb{R}^d which is
 267 the arrangement \mathcal{H}_{E_C} associated to the set of vectors E_C . This triangulation is called the
 268 *Coxeter triangulation* of type \tilde{A}_d of \mathbb{R}^d . By definition, it has the same combinatorial structure
 269 as the FK-triangulation. In addition, it has remarkable geometric properties [15, 11]. First,
 270 it is a non-degenerate Delaunay triangulation, and its dual complex is a Voronoi diagram.
 271 Moreover, its simplices have an exceptionally large thickness (the ratio of the smallest altitude
 272 of a simplex over its diameter or longest edge length).

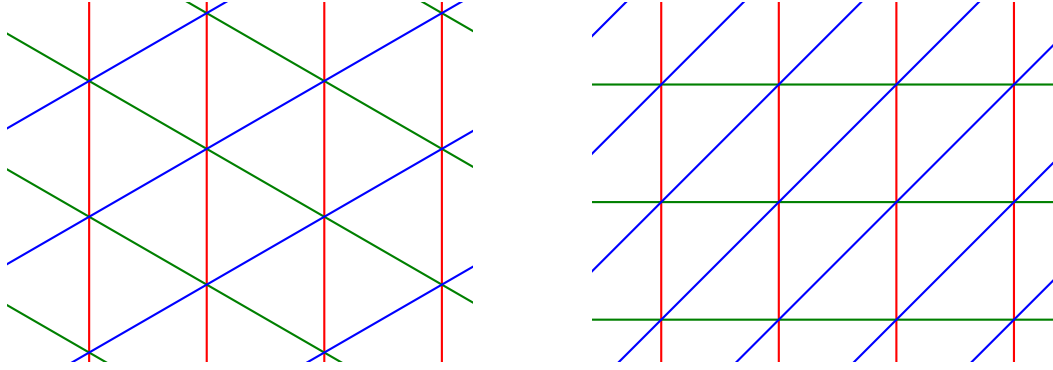
273 We will call any triangulation of \mathbb{R}^d that is the image of a Freudenthal-Kuhn triangulation
 274 under a non-degenerate affine map a Coxeter-Freudenthal-Kuhn triangulation, or CFK-
 275 triangulation for short. This includes the Coxeter triangulation of type \tilde{A}_d (as embedded in
 276 \mathbb{R}^d).

278 3 Data Structure

279 We introduce our data structure in this section. We first consider the case of FK-triangulations
 280 in Sections 3.1 and 3.2. The extension to CFK-triangulations in Section 3.3 is straightforward,
 281 since all these triangulations have the same combinatorial structure.

282 3.1 Permutahedral representation of FK-triangulations

283 **Cycles and the permutahedron.** In Remark 13 we have seen that simplices can be
 284 described by monotone paths (increasing coordinates) along the the edges of the cube. As



277 **Figure 3** The Coxeter and Freudenthal-Kuhn triangulations in the plane.

285 observed by Eaves [16], these monotone paths can be made into a cycle using the extra vector
 286 $e_{d+1} = -\sum_{i=1}^d e_i$ because by construction

287
$$v_0 = v_l + \sum_{i \in \omega_{l+1}} e_i,$$

288 with ω as in Definition 10. Because it is a cycle, we can take any vertex of the cycle as a
 289 starting point, which means that v_0 no longer has a special role as a starting point of a
 290 monotone edge walk. A cycle can now be represented by an ordered partition of $[d+1]$, for
 291 which it is not longer necessary that $d+1$ lies in ω_{l+1} , and an (arbitrary) starting point. We
 292 now formalize these general cyclical paths:

293 **Definition 16** (Permutahedral representation). *Let $(v_0, \omega) \in \mathbb{Z}^d \times OP_{l+1}[d]$. To this pair*
 294 *we associate a simplex $\sigma^\omega = \{v_0 = v_0^\omega, v_1^\omega, \dots, v_l^\omega\}$ with*

295
$$v_i^\omega = v_{i-1}^\omega + \sum_{i \in \omega_i} e_i \quad i = 1, \dots, l. \tag{5}$$

296 *We say that (v_0, ω) is the permutahedral representation of the simplex σ^ω . If $d+1 \in \omega_{l+1}$,*
 297 *we say that (v_0, ω) is the canonical permutahedral representation of σ^ω . In this case, σ^ω is*
 298 *a simplex in the FK-triangulation in the cube of which v_0 is the minimal vertex with respect*
 299 *to the lexicographical order, as we have seen above.*

300 In Lemma 19, and Proposition 20, we will see that, more generally, $\{(v_0, \omega) \mid \omega \in OP[d+1]\}$
 301 is the star of v_0 in the FK-triangulation, where we identify simplices with their permutahedral
 302 representations.

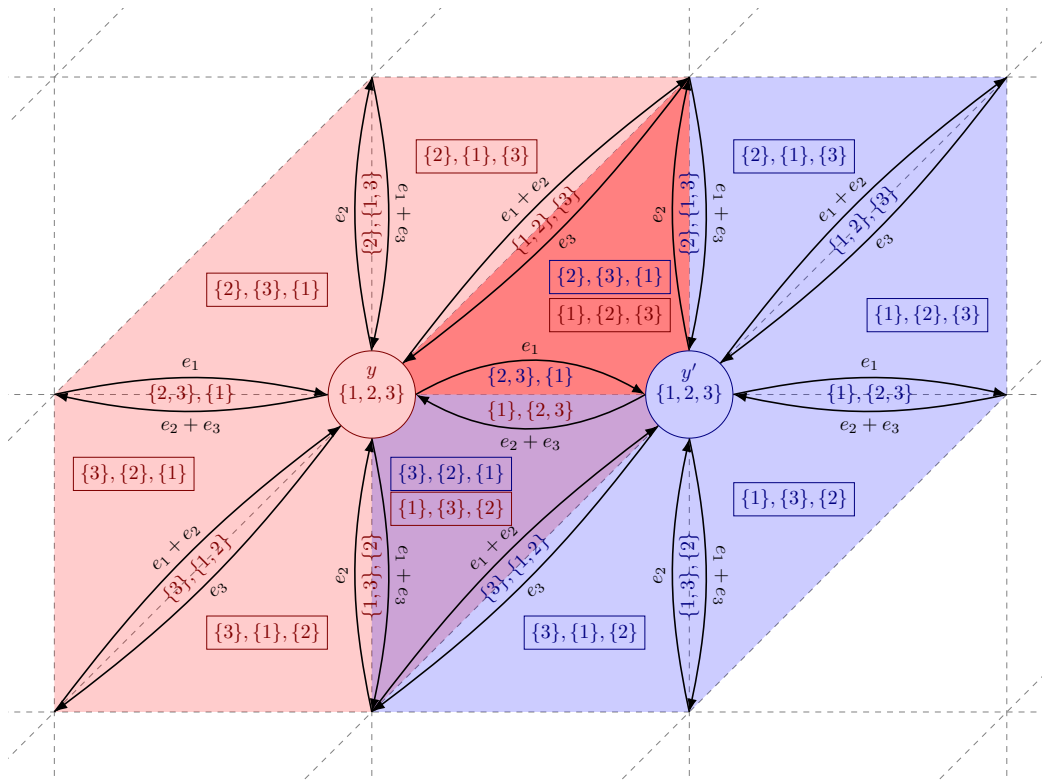
303 **Definition 17** (Cyclic shifts). *Let (v_0, ω) be a permutahedral representation. We define the*
 304 *cyclic shift of (v_0, ω) of length k to the left as (v'_0, ω') , where*

305
$$v'_0 = v_0 + \sum_{j=1}^k \sum_{i \in \omega_j} e_i \quad \omega'_j = \omega_{(j+k-1) \bmod (l+1)+1}. \tag{6}$$

306 *Here we use the convention that the sum from 1 to 0 is empty. We write $(v'_0, \omega') = (v_0, \omega) \oplus k$.*

308 **Lemma 18.** *The cyclic shift $(v'_0, \omega') = (v_0, \omega) \oplus k$ defines the same simplex as (v_0, ω) .*

309 **Proof.** Follows by inserting (6) in (5). ◀



307 ■ **Figure 4** The permutahedral representation of the simplices in the stars of vertices y and y' .

310 We now prove that the all permutahedral representations for a fixed v_0 , form the star of
 311 v_0 . This is a crucial property that will be used to efficiently compute faces, and cofaces, and
 312 traverse the triangulation.

313 ► **Lemma 19.** *The set $\{(v_0, \omega) \mid \omega \in OP[d + 1]\}$, where $OP[d + 1]$ is the set of all ordered
 314 partitions of $[d + 1]$, gives all the simplices in the star of v_0 in FK-triangulation.*

315 **Proof.** Let (v_0, ω) , with $\omega \in OP_{l+1}[d + 1]$, be such that $d + 1 \in \omega_k$. Let $(v'_0, \omega') =$
 316 $(v_0, \omega) \oplus (l - k + 1)$. By Definition 17, and Lemma 18, (v_0, ω) , and (v'_0, ω') represent the same
 317 simplex. Moreover $d + 1 \in \omega'_{l+1}$, that is (v'_0, ω') is a canonical permutahedral representation.
 318 This implies that (v'_0, ω') lies in the FK-triangulation by Lemma 11, and Theorem 12.

319 Conversely, suppose that (v'_0, ω') is the canonical permutahedral representation of a
 320 simplex in the star of v_0 , that is there is some k such that $v'_k = v_0$, with v'_k as in (2). Then
 321 $(v_0, \omega) = (v'_0, \omega') \oplus k$ is also a permutahedral representation of the same simplex. ◀

322 **Faces.** From (5) it is clear that merging two consecutive parts in the ordered partition
 323 $\omega = (\omega_1, \dots, \omega_{l+1})$ corresponds to removing a vertex from the simplex, that is taking a facet.
 324 Here we stress that we allow to merge ω_1 , and ω_{l+1} , but in that case we have to change
 325 the base point of the cycle to $v_0 + \sum_{l \in \omega_1} e_l$ to obtain the canonical representation. For
 326 example, when looking at the two dimensional example in Figure 4, we see that the edges
 327 that contain y in the red triangle with permutahedral representation $(y, (\{1\}, \{2\}, \{3\}))$ are
 328 $(y, (\{1, 2\}, \{3\}))$, and $(y, (\{1\}, \{2, 3\}))$. The third edge of the red triangle is $(y', (\{2\}, \{1, 3\}))$.

329 Generally, given an ordered partition ω in $l + 1$ parts all $(l - j)$ -faces can be found by merging
 330 j consecutive parts in ω (for example merging ω_1 with ω_2 , and ω_3 with ω_4), where we allow
 331 ω_{l+1} to merge with ω_1 , but in this case we again need to change the base point to obtain the
 332 canonical representation.

333 Because the combinatorial structure of the faces is compatible with the permutahedron,
 334 Lemma 19 immediately gives the following result. We recall that two complexes are dual if
 335 there is a bijection between the faces that inverses the inclusion relationships (see for example
 336 [8, Section 11.3]).

337 ► **Proposition 20.** *The star of v_0 is dual to a permutahedron (combinatorially).*

338 This proposition explains the nomenclature permutahedral representation.

339 This is equivalent to the following more geometric result:

340 ► **Proposition 21.** *The Voronoi cell of a Coxeter triangulation of type \tilde{A}_d is a permutahedron.*

341 This can be found in [13, Chapter 21, Section 3.F], see also [12]. We note that Coxeter
 342 triangulations of type \tilde{A}_d are combinatorially equivalent to FK-triangulations as discussed
 343 below in Section 2.3. In the appendix we give a new proof that is more direct than the one
 344 in [13].

345 **Duality.** We can associate to a FK-triangulation \mathcal{T} its dual complex \mathcal{T}^* . Since \mathcal{T} is a
 346 simplicial complex, \mathcal{T}^* is a simple complex, that is a cell complex whose faces are all simple
 347 polytopes [8]. By Proposition 20, each d -dimensional cell of \mathcal{T}^* is a d -permutahedron.

348 **3.2 Basic operations on FK-triangulations**

349 **Point location.** Given a point $x \in \mathbb{R}^d$ Lemma 11 tells us how to find the canonical
 350 permutahedral representation of the simplex in which x is contained. The complexity of
 351 point location is dominated by the sorting of the $z^i = x^i - \lfloor x^i \rfloor$, which takes $O(d \log d)$ time,
 352 and requires $O(d)$ space.

353 **Face computation.** Let σ be an l -simplex whose canonical permutahedral representation
 354 is (v_0, ω) , where ω is an ordered partition of $[d + 1]$ into $l + 1$ parts. The computation of all
 355 k -faces of σ goes as follows. We use Ehrlich's subset generation algorithm [17] to compute
 356 all the subsets of $k + 1$ elements from $\{v_0, \dots, v_l\}$. Let $\tau = \{v_{m_0}, \dots, v_{m_k}\}$ be such a subset.
 357 τ is a k -face of σ . We then compute the canonical permutahedral representation of all those
 358 k -faces τ .

359 We first sort the m_i so that $m_0 < \dots < m_k$ using counting sort. Then, the canonical
 360 permutahedral representation (\tilde{v}'_0, ω') of τ is found by merging consecutive parts of ω so as
 361 to obtain $k + 1$ parts as follows :

$$362 \quad v'_0 = v_{m_0} = v_0 + \sum_{j \in \omega_1} e_j + \dots + \sum_{j \in \omega_{m_0-1}} e_j$$

$$363 \quad \omega'_i = \omega_{m_{i-1}} \cup \dots \cup \omega_{m_i-1} \quad \text{for } i \in \{1, \dots, k\}$$

$$364 \quad \omega'_{k+1} = (\omega_1 \cup \dots \cup \omega_{m_0-1}) \cup (\omega_{m_k} \cup \dots \cup \omega_{l+1}).$$

365 The complexity of computing all subsets of $k + 1$ vertices of σ using Ehrlich's algorithm
 366 takes time $O(k + s)$ where $s = \binom{l+1}{k+1}$ is the number of subsets. Computing, for each such
 367 k -simplex its permutahedral representation takes $O(d)$ time.

368 ► **Lemma 22.** *Let σ be an l -simplex in the FK-triangulation of \mathbb{R}^d given by its canonical
 369 permutahedral representation. Computing the canonical permutahedral representations of*

370 all its k -faces can be done in time $O(ds)$, where $s = \binom{l+1}{k+1}$ is the number of k -faces of an l
 371 simplex. The space complexity of the algorithm is $O(l)$.

372 **Coface computation.** Computing the faces of a simplex σ consists in coarsifying its
 373 ordered partition. The computation of cofaces is the reverse. Here we refine the ordered
 374 partition. Specifically, if σ is a k -simplex represented by its canonical permutahedral repre-
 375 sentation (v_0, ω) , and we want to compute its l -cofaces, we need to compute all refinements
 376 of ω into $l + 1$ parts.

377 More precisely, we need to subdivide each ω_i in $a_i \leq |\omega_i|$ subparts so that $\sum_{i=1}^{k+1} a_i = l + 1$.
 378 This can be done in time proportional to the number $k + 1$ of the generated subparts. We
 379 then need to consider all the permutations of these subparts since we are interested in ordered
 380 partitions. Using known algorithms by Walsh [26], and Ruskey and Savage [24], we can
 381 compute all the ordered partitions associated to the l -cofaces of σ in time proportional to
 382 the number of such cofaces. We thus obtain all the permutahedral representations (v_0, ω') of
 383 all the l -cofaces of σ .

384 It is important to notice that all cofaces of σ have v_0 as a vertex. However v_0 is not
 385 necessarily the minimal vertex of some of the computed cofaces. We thus have to identify the
 386 minimal vertex of each computed coface, and use cyclic shifts (as in Lemma 19) to obtain
 387 the canonical permutahedral representation of the coface. The next Lemma follows. The
 388 bound on the number s of cofaces follows, by duality, from Corollary 9.

389 **► Lemma 23.** *Let σ be a k -simplex in the FK-triangulation of \mathbb{R}^d given by its permutahedral
 390 representation. Computing the permutahedral representations of all its l -cofaces can be done
 391 in time $O(ds)$, where*

$$392 \quad s = p_{d-l, d-k} \leq \frac{1}{2^{\min(l, d-l)}} \binom{d-k}{d-l} (d-k+1)!$$

393 *is the number of l -cofaces of a k -simplex in the FK-triangulation. The space complexity of
 394 the algorithm is $O(d)$.*

395 3.3 Data structure for CFK-triangulations

396 We store a CFK-triangulation as follows. The combinatorial structure of the triangulation is
 397 given through the canonical permutahedral representation of its simplices, and the algorithms
 398 from Section 2.2. The geometry of the triangulation is specified by the affine transformation
 399 that maps the FK-triangulation of \mathbb{R}^d to the CFK-triangulation. The affine transformation
 400 is given by a $d \times d$ matrix Λ , and a d -vector b . For the FK-triangulation, Λ is the identity
 401 matrix, and $b = 0$; therefore no storage is required. For the Coxeter triangulation of type \tilde{A}_d ,
 402 Λ is sparse as indicated in Section 2.3.

403 4 Experimental results

404 The data structure and the basic operations have been implemented in C++ and are currently
 405 under review to be integrated in the GUDHI library. We report on the execution time of the
 406 face and coface generation algorithms for the FK-triangulations.

??:14 **A compact data structure for high dimensional Coxeter-Freudenthal-Kuhn triangulations**

407 In Tables 1-4, we consider an ambient space of moderate dimension $d = 30$ and compute
 408 the higher dimensional faces of various high dimensional simplices, of dimensions ranging
 409 from 22 to 30.

410 Each entry in Table 1 corresponds to the total time in milliseconds of computing all the
 411 k -dimensional faces of a set of l -dimensional simplices in \mathbb{R}^{30} . The l -dimensional simplices
 412 are picked at random in the triangulation and the results are averaged over 1 000 simplices.
 413 Note that the time 11 904.7ms is the time of computing all 5 852 925 faces of dimension 22 of
 414 a simplex of dimension 30.

415 Table 2 shows the same running times per computed face. As we can see, except for the
 416 case $l = k$, the running time per computed face is around $2\mu s$.

Face dimension k	22	23	24	25	26	27	28	29	
Simplex dimension l	22	0.006							
	23	0.042	0.006						
	24	0.503	0.05	0.008					
	25	4.88	0.645	0.058	0.008				
	26	33.76	5.697	0.697	0.062	0.008			
	27	162.114	35.108	6.824	0.758	0.064	0.008		
	28	885.293	190.441	40.856	6.906	0.739	0.058	0.006	
	29	3420.99	973.455	246.88	49.896	6.657	0.735	0.058	0.006
	30	11904.7	4175.92	1247.97	275.776	50.932	7.348	0.778	0.058

426 ■ **Table 1** Total running time of the face generation algorithm (in milliseconds).

Face dimension k	22	23	24	25	26	27	28	29	
Simplex dimension l	22	0.006							
	23	0.0018	0.006						
	24	0.0017	0.002	0.008					
	25	0.0019	0.002	0.0022	0.008				
	26	0.0019	0.0019	0.002	0.0023	0.008			
	27	0.0016	0.0017	0.0021	0.002	0.0023	0.006		
	28	0.0019	0.0016	0.0017	0.0019	0.0018	0.002	0.006	
	29	0.0017	0.0016	0.0017	0.0018	0.0016	0.0017	0.0019	0.006
	30	0.0015	0.0016	0.0017	0.0016	0.0016	0.0016	0.0017	0.0019

436 ■ **Table 2** Running time of the face generation algorithm *per computed face* (in milliseconds).

437 In Tables 3 and 4, we present analogous tables for the coface computation algorithm.
 438 Similarly, the running time per computed coface in Table 4 is around $2\mu s$ with the exception
 439 of when k is close to l .

Coface dimension l		23	24	25	26	27	28	29	30
Simplex dimension k	22	0.11	1.274	9.577	43.848	86.699	96.407	59.935	15.487
	23	0.043	0.114	0.729	3.499	9.337	13.523	10.058	3.049
	24		0.047	0.1	0.381	1.183	2.132	1.871	0.653
	25			0.046	0.097	0.23	0.423	0.426	0.193
	26				0.047	0.076	0.128	0.15	0.093
	27					0.049	0.069	0.081	0.063
	28						0.047	0.061	0.054
	29							0.05	0.053
	30								0.05

449 ■ **Table 3** Total running time of the *coface* generation algorithm (in milliseconds).

Coface dimension l		23	24	25	26	27	28	29	30
Simplex dimension k	22	0.002	0.0013	0.0013	0.0015	0.0016	0.0016	0.0016	0.0017
	23	0.042	0.003	0.0017	0.0016	0.0016	0.0016	0.0016	0.0017
	24		0.045	0.004	0.0019	0.0017	0.0017	0.0017	0.0018
	25			0.045	0.0053	0.0025	0.002	0.0019	0.0022
	26				0.047	0.0073	0.0035	0.0028	0.0036
	27					0.048	0.0103	0.0058	0.0068
	28						0.048	0.0145	0.0133
	29							0.05	0.026
	30								0.05

459 ■ **Table 4** Running time of the *coface* generation algorithm *per computed face* (in milliseconds).

460 The next results are motivated by the problem of tracing a manifold of low dimension m
 461 embedded in \mathbb{R}^d for high d . The crucial operations in this context consist in computing the
 462 facets and cofacets of simplices of codimension m in a triangulation of \mathbb{R}^d [4].

463 In Table 5, we present the execution time of the *facet* generation algorithm applied to
 464 simplices of low codimension m , ranging from 1 to 7, in FK-triangulations of high dimensions
 465 d (up to $d = 400$). In Table 6, we present the execution time of the *cofacet* generation
 466 algorithm under the same circumstances.

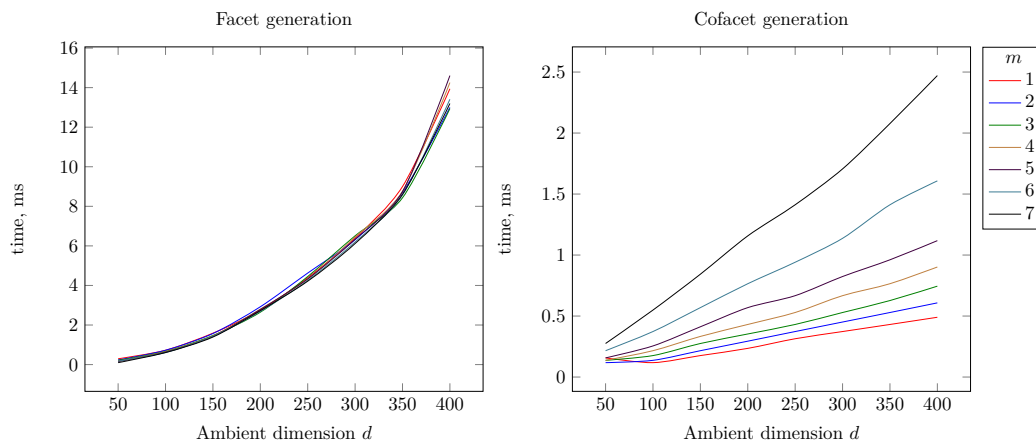
Ambient dimension d		50	100	150	200	250	300	350	400
Face codimension m	1	0.166	0.612	1.438	2.862	5.376	8.69	12.184	15.924
	2	0.166	0.643	1.417	2.858	5.607	8.375	11.806	16.261
	3	0.168	0.607	1.395	2.888	5.866	8.232	12.008	16.527
	4	0.162	0.589	1.373	2.864	5.491	8.447	11.936	16.08
	5	0.154	0.587	1.349	2.76	5.77	8.371	11.814	15.88
	6	0.148	0.579	1.321	2.737	5.735	8.351	12.038	15.798
	7	0.136	0.575	1.313	2.553	5.701	8.313	12.11	15.754

474 ■ **Table 5** Average running times in milliseconds of the *facet* generation algorithm.

Ambient dimension d		50	100	150	200	250	300	350	400
Simplex codimension m	1	0.068	0.134	0.228	0.281	0.423	0.605	0.611	0.848
	2	0.082	0.17	0.267	0.341	0.483	0.723	0.731	0.966
	3	0.098	0.194	0.303	0.401	0.525	0.733	0.866	1.124
	4	0.112	0.226	0.351	0.467	0.665	0.806	0.974	1.295
	5	0.132	0.265	0.423	0.545	0.966	0.928	1.128	1.477
	6	0.162	0.329	0.515	0.713	0.948	1.124	1.361	1.76
	7	0.2	0.415	0.651	0.878	1.166	1.421	1.784	2.283

482 ■ **Table 6** Average running times in milliseconds of the *cofacet* generation algorithm.

483 A graphical display of the results of Tables 5 and 6 is shown in Figure 5.



484 ■ **Figure 5** Graphical display of the results of Tables 5 and 6.

5 Conclusion

We have introduced a data structure to represent any triangulation in a family of highly regular triangulations of \mathbb{R}^d , which we called the CFK-triangulations. This family includes the celebrated Freudenthal-Kuhn triangulation and Coxeter triangulations of type \tilde{A}_d . The representation is implicit and extremely compact, which makes the usage of these triangulations almost as easy and efficient as that of Cartesian grids. This is advantageous in high dimensions, because a cube has 2^d vertices while a simplex has only $d + 1$ vertices. We refer to our companion paper [5], where we describe how this data structure can be used to trace isomanifolds of \mathbb{R}^d of any intrinsic dimension, in time that is polynomial in the ambient dimension.

In this paper, we have only considered uniform triangulations where each simplex has (almost) the same diameter. A natural and rather easy extension of the present work would be to generalize the data structure so as to provide different resolutions in different regions.

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564 **A Proof**

565 **Proof of Proposition 21.** We start by recalling a number of results. Let $P = \{(x^i) \in \mathbb{R}^{d+1} \mid$
566 $\sum_i x^i = 0\}$ and consider the d -simplex with vertices u_k in P .

567
$$u_0 = \left(0^{\{d+1\}}\right) \quad u_k = \left(\left(-\frac{d+1-k}{d+1}\right)^{\{k\}}, \left(\frac{k}{d+1}\right)^{\{d+1-k\}}\right), \quad k \in [d],$$

568 where $x^{\{k\}}$ denotes k consecutive coordinates x . This simplex is a simplex in the Coxeter
569 triangulation, as defined in Section 2.3. In [11] we have seen that the circumcentre of this
570 simplex is

571
$$c = \left(-\frac{d-2i}{2(d+1)}\right),$$

572 with $i \in \{0, \dots, d\}$. The circumcentre of a Delaunay simplex is a Voronoi vertex. We recall
573 that

- 574 ■ All simplices in the star of 0 in the Coxeter triangulation are found by consecutive
575 reflection of a single simplex (in this star) in the hyperplanes of \mathcal{H}_{EC} that go through
576 0, that is the hyperplanes with normals $r_{j,k} = e_j - e_k$, with $j \neq k$. See for example
577 [9, 19, 11]. We also call these reflections the action of the Weyl group.

578 ■ The reflection $R_{j,k}$ in a plane that goes through the origin with normal $r_{j,k}$ is given by

579
$$R_{j,k}(v) = v - 2 \frac{v \cdot r_{j,k}}{r_{j,k} \cdot r_{j,k}} r_{j,k} = v - (v \cdot r_{j,k}) r_{j,k}.$$

580 We find that

581
$$R_{j,k}(c)^i = (c - (c \cdot r_{j,k}) r_{j,k})^i = -\frac{d-2i}{2(d+1)} - \frac{2j-2k}{2(d+1)} (\delta_{ij} - \delta_{ik}),$$

582 which permutes the j th and k th coordinate of c . Here we used the upper index i to denote
 583 the i th coordinate. Using the cycle notation for the permutation group, see for example [2,
 584 Chapter 6], this coincides the 2-cycle $(j\ k)$. Let now

585
$$c_\pi = \left(-\frac{d-2\pi_i}{2(d+1)} \right),$$

586 with $\{\pi_i\}$ some permutation of $\{0, \dots, d\}$. We find that

587
$$R_{j,k}(c_\pi)^i = (c_\pi - (c_\pi \cdot r_{j,k}) r_{j,k})^i = -\frac{d-2\pi_i}{2(d+1)} - \frac{2\pi_j-2\pi_k}{2(d+1)} (\delta_{ij} - \delta_{ik}),$$

588 which again permutes the j th and k th coordinate. Now recall that all permutations are
 589 generated by 2-cycles, see for example [2, Theorem 6.1]. This implies that, for any permutation
 590 π , we can find c_π from c by the action of the Weyl group. This also means that we have
 591 explicitly described the Voronoi cell of 0 in the Coxeter triangulation of type \tilde{A}_d as a
 592 permutahedron. Because of symmetry, this now holds for any Voronoi cell. ◀