On Galerkin Method for Homogeneous Infinite-Dimensional Systems

Andrey Polyakov

On Galerkin Method for Homogeneous Infinite-Dimensional Systems

Andrey Polyakov*

Abstract. The paper proposes Galerkin-like projection method which preserves dilation symmetries of linear and nonlinear (possibly unbounded) operators. The method is developed for approximation of generalized homogeneous evolution equations in Hilbert spaces. It is shown that the obtained reduced-order model preserve stability and convergence properties (such as finite-time and fixed-time stability) of the original system. The proposed method is compared on simulations with the classical Galerkin method for the Burgers equation.

Key word. approximation of distributed parameter systems; semigroup and operator theory.

1. Introduction. Symmetries are specific features of many mathematical models, which play an important role in analysis and control design. The standard (conventional) homogeneity, studied by Leonhard Euler, is a symmetry with respect to the uniform dilation $x \to \lambda x$, where $\lambda \in \mathbb{R}$ and x is an element of a vector space. For example, a function $x \to f(x)$ is standard homogeneous if $f(\lambda x) = \lambda^{\nu} f(x)$, $\forall \lambda > 0, \forall x$, where ν is a real number called the homogeneity degree. It seems the generalized homogeneity (a symmetry with respect to a non-uniform dilation) was first studied by Vladimir Zubov in [33]. The generalized homogeneity was shown to be useful for analysis of nonlinear finite-dimensional dynamical systems [14], [16], [3] as well as for control design and estimation problems [12], [22], [19], [20]. Homogeneity specifies a convergence rate of an asymptotically stable system (see e.g. [21]) and allows simple methods of robustness analysis of nonlinear control systems to be developed [1], [2]. Elements of the homogeneous control theory of infinite dimensional systems are presented in [29], [27]. It is well-known (see [11], [10], [25]) that the generalized homogeneity is the specific feature of linear differential operators and nonlinear models of mathematical physics like Saint-Venant, Navier–Stokes, Burgers and KdV equations. The dilation symmetry of homogeneous operators is preserved in mathematical objects induced by these operators. For example, the well-known Euler's homogeneous function theorem implies that derivatives of homogeneous functions are homogeneous as well. The similar conclusions can be made for Frechét differentiable operators in Hilbert spaces as well as for solutions of generalized homogeneous evolution equations in Banach spaces [25]. Therefore, preserving the dilation symmetry is important for systems analysis and design of high performance control and estimation algorithms. Many methods of control engineering are based on reduced order models obtained by a conventional approximation techniques which do not preserve the dilation symmetries of the original nonlinear system in the general case.

Methods of geometric numerical integration are known to preserve geometric properties of the system of differential equations allowing better global numerical results to be obtained (see e.g. [13], [31]). Symmetries persevering numerical algorithms are surveyed in [5]. They are essentially based on the approach proposed by Vladimir Dorodnitsyn in 1989, [7], which applies Lie group methods to a finite-difference equations. The mentioned algorithms allow

*Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France (andrey.polyakov@inria.fr).

some Lie symmetries of partial differential equations (PDEs) to be preserved in their finite-difference approximations. Recently [4], the similar schemes were adapted to finite-element methods for ordinary differential equations (ODEs). To the best of the author's knowledge, homogeneity preserving Galerkin-type methods for evolution equations in Hilbert spaces have never been presented before.

Numerical schemes preserving properties induced by homogeneity (e.g. convergence rates) of control systems in $\mathbb{R}^n$ are also designed in [28]. Being developed in the context of control systems theory, the corresponding schemes do not use explicitly the moving frames and the invariantization procedure known for the symmetry preserving integration algorithms [5]. The original model was simply transformed to an equivalent ODE allowing the conventional discretization algorithms to be applied in such a way that all important features of homogeneous systems are preserved. The present paper develops a Galerkin-type method in the similar way. The obtained reduced-order ODE model preserves the dilation symmetry and the homogeneity degree of the original infinite-dimensional evolution equation. The numerical experiments for the Burgers equation confirm a large improvement of the estimation precision of the homogeneous Galerkin projection in comparison with the classical Galerkin technique. It is also shown that the proposed scheme allows some properties important for control systems design and analysis to be preserved as well, e.g. Lyapunov/asymptotic stability and convergence rates (such as finite-time or fixed-time convergence). Together with [28] the proposed method provides a unified framework for a consistent discrete-time approximation of homogeneous control systems as well as digital implementation of homogeneous controllers for distributed parameter systems.

Notation. $\mathbb{R}$ is the field of real numbers; $\mathbb{R}_+ = [0,+\infty)$; $\|\cdot\|_{\mathbb{B}}$ denotes a norm in a real Banach space $\mathbb{B}$; $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ denotes an inner product in a real Hilbert space $\mathbb{H}$; $\mathbf{0}$ is the zero element of a Banach space; I (resp. O) denotes the identity (resp. the zero) operator; $\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)$ denotes the space of linear bounded operators $\mathbb{B}_1\to\mathbb{B}_2$ with the norm $\|A\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} = \sup_{\|x\|_{\mathbb{B}_1}=1}\|Ax\|_{\mathbb{B}_2}$, where $A\in\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)$; if $\mathbb{B}_1=\mathbb{B}_2=\mathbb{B}$ we write shortly $\|A\|_{\mathbb{B}}$; $S_{\mathbb{B}}$ is the unit sphere in $\mathbb{B}$; for $r>1$ the set $K_{\mathbb{B}}(r)\subset\mathbb{B}$ is defined as follows $K_{\mathbb{B}}(r) := \{x\in\mathbb{B} : 1/r<\|x\|_{\mathbb{B}}<r\}$; $\mathcal{D}(A)$ denotes the domain of an operator A; $C([t_1,t_2],\mathbb{B})$ is the space of continuous functions $x:[t_1,t_2]\to\mathbb{B}$ with the uniform norm $\|x\|_C = \max_{t\in[t_1,t_2]}\|x(t)\|$ with $-\infty<t_1<t_2<+\infty$; $C_c^\infty(\Omega,\mathbb{R}^m)$ is a set of infinitely smooth functions $\mathbb{R}^n\to\mathbb{R}^m$ with compact supports in an open connected set $\Omega\subset\mathbb{R}^n$ with a sufficiently smooth boundary (or $\Omega=\mathbb{R}^n$); $L^2(\Omega,\mathbb{R}^m)$ is the Lebesgue space with $\|u\|^2_{L^2(\Omega,\mathbb{R}^m)} = \langle u,u\rangle_{L^2(\Omega,\mathbb{R}^m)}$ and the inner product $\langle u,v\rangle_{L^2(\Omega,\mathbb{R}^m)} = \int_\Omega u\cdot v$, where $u,v\in L^2(\Omega,\mathbb{R}^m)$; $H^p(\Omega,\mathbb{R}^m)$ is the Sobolev space with $\|u\|^2_{H^p(\Omega,\mathbb{R}^m)} = \langle u,u\rangle_{H^p(\Omega,\mathbb{R}^m)}$ and the inner product $\langle u,v\rangle_{H^p(\Omega,\mathbb{R}^m)} = \sum_{i=0}^p\langle\nabla^i u,\nabla^i v\rangle_{L_2(\Omega,\mathbb{R}^m)}$ for $u,v\in H^p(\Omega,\mathbb{R}^m)$; H_0^p is a completion of C_c^∞ in the norm of H^p; ℓ^2 is the Hilbert spaces of quadratically summable sequences of real numbers with the inner product $\langle x,y\rangle = \sum_i x_i y_i$, $x=(x_1,x_2,...)^\top$, $y=(y_1,y_2,...)^\top\in\ell^2$; $L^1((t_1,t_2),\mathbb{B})$ is the space of Bochner integrable functions $(t_1,t_2)\to\mathbb{B}$, where $-\infty\le t_1<t_2\le+\infty$; the symbol $\stackrel{a.e}{=}$ (resp. $\stackrel{a.e}{\in}$) means that an identity (resp. inclusion) holds almost everywhere; $\dot{x}(t) := \lim_{h\to 0}\frac{x(t+h)-x(t)}{h}$ is a time derivative of the function $x:\mathbb{R}\to\mathbb{B}$; $\lambda_{\max}(P)$ (resp. $\lambda_{\min}(P)$) denotes the minimal (resp. maximal) eigenvalue of the symmetric matrix $P=P^\top\in\mathbb{R}^{n\times n}$.

2. Homogeneous Systems.

2.1. Dilations in Banach/Hilbert spaces and their generators. In this paper we deal only with linear dilations [25]. For more details about theory of dilations in finite-dimensional and infinite-dimensional spaces we refer the reader to [15], [17].

Definition 2.1. *A mapping $\mathfrak{d}:\mathbb{R}\to\mathcal{L}(\mathbb{B},\mathbb{B})$ is said to be a group of linear dilations (or, simply, dilation) in a Banach space $\mathbb{B}$ if*

1) (Group property) $\mathfrak{d}(0)=I$, $\mathfrak{d}(t+s)=\mathfrak{d}(t)\mathfrak{d}(s)$, $t,s\in\mathbb{R}$;

2) (Limit property) $\lim\limits_{s\to-\infty}\|\mathfrak{d}(s)u\|_{\mathbb{B}}=0$ and $\lim\limits_{s\to+\infty}\|\mathfrak{d}(s)u\|_{\mathbb{B}}=+\infty$ for any $u\neq\mathbf{0}$.

Obviously, $\mathfrak{d}$ is one-parameter group of linear bounded invertible operators and $\mathfrak{d}(-s)=(\mathfrak{d}(s))^{-1}, \forall s\in\mathbb{R}$. The limit property specifies groups being dilations in $\mathbb{B}$.

Example 1. *Let us consider the one-parameter group of linear invertible operators in the Lebesgue space $L^p(\mathbb{R}^n,\mathbb{R}^m)$ and in the Sobolev space $H^p(\mathbb{R}^n,\mathbb{R}^m)$ given by*

$$(\mathfrak{d}(s)x)(z)=e^{\alpha s}x(e^{\beta s}z),\quad s\in\mathbb{R},\quad x\in L^p(\mathbb{R}^n,\mathbb{R}^m),\quad z\in\mathbb{R}^n, \tag{2.1}$$

where $\alpha,\beta\in\mathbb{R}$ are constant parameters. Since

$$\|\mathfrak{d}(s)x\|_{L^p}=e^{(\alpha-\beta n/p)s}\|x\|_{L^p}$$

then $\mathfrak{d}$ is a dilation in $L^p(\mathbb{R}^n,\mathbb{R}^m)$ provided that $\alpha-\beta n/p>0$. On the other hand, since

$$\|\mathfrak{d}(s)x\|^2_{H^p}=\sum_{i=0}^{p}\|\nabla^i\mathfrak{d}(s)x\|^2_{L_2}=\sum_{i=0}^{p}\|e^{\beta is}\mathfrak{d}(s)\nabla^i x\|^2_{L_2}=\sum_{i=0}^{p}e^{2\alpha-\beta n+2i\beta}\|\nabla^i x\|^2_{L_2},$$

then $\mathfrak{d}$ is a dilation in $H^p(\mathbb{R}^n,\mathbb{R}^m)$ provided that $\alpha>\beta(0.5n/2-i), i=0,1,...,p$.

Definition 2.2. *A dilation $\mathfrak{d}$ is strongly (uniformly) continuous if $s\to\mathfrak{d}(s)u$ (resp. $s\to\mathfrak{d}(s)$) is continuous in $\mathbb{B}$ (resp. in $L(\mathbb{B},\mathbb{B})$) for any $u\in\mathbb{B}$.*

Any continuous linear dilation in $\mathbb{R}^n$ is given by $\mathfrak{d}(s)=e^{sG_{\mathfrak{d}}}=\sum_{i=0}^{\infty}\frac{s^iG_{\mathfrak{d}}^i}{i!}$, where $G_{\mathfrak{d}}\in\mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix. Nonlinear dilations in $\mathbb{R}^n$ are studied in [18], [17], [32].

Example 2. *The dilation $\mathfrak{d}$ given by (2.1) is strongly continuous in $H^p(\mathbb{R}^n,\mathbb{R}^m)$. Indeed, considering $p=0$ and $x_\infty\in C^\infty(\mathbb{R}^n,\mathbb{R}^m)$ we derive*

$$\langle\mathfrak{d}(s)x_\infty-x_\infty,x_\infty\rangle_{L^2}=\int_{B_{\mathbb{R}^n}(c)}\Big(e^{\alpha s}x_\infty(e^{\beta s}z)-x_\infty(z)\Big)\cdot x_\infty(z)dz,\quad |s|\leq s_0$$

for some finite $c\geq 0$ (dependent of x_∞ and $s_0>0$). Since $x_\infty\in C^\infty$ is a uniformly continuous function then $\|x_\infty(e^{\beta s}z)-x_\infty(z)\|_{\mathbb{R}^m}\leq\sigma(\|e^{\beta s}z-z\|_{\mathbb{R}^n})\leq\sigma(c(e^{\beta s}-1))$ for all $z\in B_{\mathbb{R}^n}(c)$ and some $\sigma\in\mathcal{K}^\infty$. Hence $\langle\mathfrak{d}(s)x_\infty-x_\infty,x_\infty\rangle_{L_2(\mathbb{R}^n,\mathbb{R}^m)}\to 0$ as $s\to 0$ and

$$\|\mathfrak{d}(s)x_\infty-x_\infty\|^2_{L^2}=\|\mathfrak{d}(s)x_\infty\|^2_{L^2}-\|x_\infty\|^2_{L^2}-2\langle\mathfrak{d}(s)x_\infty-x_\infty,x_\infty\rangle_{L^2}$$

tends to zero as $s\to 0$. Taking into account that C^∞ is dense in H^p, the continuity of $s\to\mathfrak{d}(s)x$ can be shown for any $x\in H^p(\mathbb{R}^n,\mathbb{R}^m)$.

Being a strongly continuous group of linear bounded operators, the linear dilation always has an infinitesimal generator [23] that is a closed densely defined linear operator $G_{\mathfrak{d}} : \mathcal{D}(G_{\mathfrak{d}}) \subset \mathbb{B} \to \mathbb{B}$ given by $G_{\mathfrak{d}} u = \lim_{s\to 0} \frac{\mathfrak{d}(s)u-u}{s}$, $u \in \mathcal{D}(G_{\mathfrak{d}})$.

Example 3. *The generator $G_{\mathfrak{d}}$ of the group $\mathfrak{d}$ considered in Example 1 is given by (see [25])*

$$(G_{\mathfrak{d}}x)(z) = \alpha x(z) + \beta(z\cdot\nabla)x(z), \quad z \in \mathbb{R}^n, \quad x \in \mathcal{D}(G_{\mathfrak{d}}) \subset H^p(\mathbb{R}^n,\mathbb{R}^m), \tag{2.2}$$

where $z\cdot\nabla = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + ... + z_n\frac{\partial}{\partial z_n}$, the domain $\mathcal{D}(G_{\mathfrak{d}})$ is a completion of C_c^∞ with respect to the norm $\|x\|_{H^p} + \|G_{\mathfrak{d}}x\|_{H^p}$. All derivatives are understood in the weak sense.

Definition 2.3. *A set $\mathcal{D} \subseteq \mathbb{B}$ is a $\mathfrak{d}$-homogeneous cone in $\mathbb{B}$ if $\mathfrak{d}(s)u \in \mathcal{D}, \forall u \in \mathcal{D}, \forall s \in \mathbb{R}$.*

Notice that $\mathcal{D}$ becomes the conventional positive cone in $\mathbb{B}$ provided that $\mathfrak{d}$ is the uniform dilation $\mathfrak{d}(s) = e^s I, s \in \mathbb{R}$. Homogeneous cones are domains of unbounded homogeneous operators in $\mathbb{B}$ (see below).

3. Monotone dilations. For further constructions it is also important to know a relation between a dilation and the norm topology of the vector space.

Definition 3.1. *A dilation $\mathfrak{d}$ is strictly monotone in $\mathbb{B}$ if $\exists \beta > 0 : \|\mathfrak{d}(s)\|_{\mathbb{B}} \le e^{\beta s}, \forall s \le 0$.*

Monotonicity of the dilation implies the monotonicity of the function $s \to \|\mathfrak{d}(s)u\|_{\mathbb{B}}$ for any $u \in \mathbb{B}$. Any linear dilation in $\mathbb{R}^n$ is strictly monotone under a proper selection of the weighted Euclidean norm [24].

Example 4. *The dilation (2.1) is strictly monotone in $L^p(\mathbb{R}^n,\mathbb{R}^m)$ if $\alpha - \beta n/p > 0$ and strictly monotone in $H^p(\mathbb{R}^n,\mathbb{R}^m)$ provided that $\alpha > \beta(0.5n/2 - i), i = 0,1,...,p$ (see Example 1). For more details, about linear dilations in function spaces we refer the reader to [25, Chapter 6], where in particular the generalized dilations like $(\mathfrak{d}(s)x)(z) = e^{G_\alpha s}x(e^{G_\beta \beta s}z)$, $G_\alpha \in \mathbb{R}^{n\times n}, G_\beta \in \mathbb{R}^{m\times m}$ are studied.*

Definition 3.2. *A vector $z_0 \in \mathbb{B} : \|z_0\|_{\mathbb{B}} = 1$ is said to be a homogeneous projection of the vector $z \in \mathbb{B}$ if is there exists $s_0 \in \mathbb{R}$ such that $z_0 = \mathfrak{d}(s_0)z$, where $\mathfrak{d}$ is a dilation in $\mathbb{B}$.*

Monotonicity of the dilation $\mathfrak{d}$ guarantees the uniqueness of the homogeneous projection for any vector $z \in \mathbb{B}\backslash\{\mathbf{0}\}$. For dilations in Hilbert spaces the following simple criterion of strict monotonicity can be provided in terms of the generator.

Proposition 3.3. *[25] A strongly continuous dilation group $\mathfrak{d}$ in a real Hilbert space $\mathbb{H}$ is strictly monotone if and only if there exists $\gamma > 0$ and a $\mathfrak{d}$-homogeneous cone $\mathcal{D}$ dense in $\mathcal{D}(G_{\mathfrak{d}})$ such that*

$$\langle G_{\mathfrak{d}}z, z\rangle_{\mathbb{H}} \ge \gamma\|z\|_{\mathbb{H}}^2 \quad \textit{for any} \quad z \in \mathcal{D}, \tag{3.1}$$

where $G_{\mathfrak{d}}$ is the generator of $\mathfrak{d}$.

Example 5. *For the generator $G_{\mathfrak{d}}$ of the dilation $\mathfrak{d}$ given by (2.1) we have*

$$\langle G_{\mathfrak{d}}z, z\rangle_{L_2} = \left(\alpha - \frac{\beta n}{2}\right)\langle z, z\rangle_{L_2}, \quad \forall z \in \mathcal{D}(G_{\mathfrak{d}}).$$

3.1. The canonical homogeneous norm. The dilation introduces an alternative norm topology in $\mathbb{B}$ using the so-called canonical homogeneous norm.

Definition 3.4 ([27]). *The functional* $\|\cdot\|_{\mathfrak{d}}:\mathbb{B}\to\mathbb{R}_+$ *given by*

$$\|u\|_{\mathfrak{d},\mathbb{B}}=e^{s_u},\quad \textit{where } s_u\in\mathbb{R}:\|\mathfrak{d}(-s_u)u\|_{\mathbb{B}}=1,\quad u\neq\mathbf{0} \tag{3.2}$$

is called the canonical homogeneous norm in $\mathbb{B}$, *where* $\mathfrak{d}$ *is a strictly monotone dilation in* $\mathbb{B}$.

Obviously, $\|\mathfrak{d}(s)u\|_{\mathfrak{d},\mathbb{B}}=e^s\|u\|_{\mathfrak{d},\mathbb{B}}$ and $\|u\|_{\mathfrak{d},\mathbb{B}}=\|-u\|_{\mathfrak{d},\mathbb{B}}$ for $\forall u\in\mathbb{B}$ and $\forall s\in\mathbb{R}$.Notice that $\|\cdot\|_{\mathfrak{d},\mathbb{B}}=\|\cdot\|_{\mathbb{B}}$ provided that $\mathfrak{d}$ is the uniform (standard) dilation $\mathfrak{d}(s)=e^sI$, $s\in\mathbb{R}$. The functional $\|\cdot\|_{\mathfrak{d},\mathbb{B}}$ is not even a semi norm. However, in control literature (see [16], [12], [1]), the name "norm" is frequently utilized for positive definite homogeneous functionals like $\|\cdot\|_{\mathfrak{d},\mathbb{B}}$. We follow this tradition in the paper.

Theorem 3.5 ([25], Lemmas 7.1, 7.2). *If* $\mathfrak{d}$ *is a strongly continuous strictly monotone dilation then* $\|\cdot\|_{\mathfrak{d}}$ *is single-valued, positive definite, locally Lipschitz continuous on* $\mathbb{B}\backslash\{\mathbf{0}\}$ *and there exist* $\omega\geq\gamma>0$, $C\geq1:\frac{1}{C}\|u\|_{\mathfrak{d}}^{\omega}\leq\|u\|_{\mathbb{B}}\leq\|u\|_{\mathfrak{d}}^{\gamma}$, $u\in B_{\mathfrak{d}}(1)$ *and* $\|u\|_{\mathfrak{d}}^{\gamma}\leq\|u\|_{\mathbb{B}}\leq C\|u\|_{\mathfrak{d}}^{\omega}$, $u\in\mathbb{B}\backslash B_{\mathfrak{d}}(1)$. *Moreover, there exist* $\underline{\sigma},\overline{\sigma}\in\mathcal{K}^{\infty}:\underline{\sigma}(\|u\|_{\mathbb{B}})\leq\|u\|_{\mathfrak{d}}\leq\overline{\sigma}(\|u\|_{\mathbb{B}})$, $\forall u\in\mathbb{B}$.

In [25,Theorem 7.1] it is also proven that $\|\cdot\|_{\mathfrak{d}}$ is a norm in a Banach space $\tilde{\mathbb{B}}$ homeomorphic to $\mathbb{B}$. This justifies the name "norm" for the functional $\|\cdot\|_{\mathfrak{d}}$.

Example 6. *Since the strongly continuous strictly monotone dilation* (2.1) *in* $H^p(\mathbb{R}^n,\mathbb{R}^m)$ *satisfies*

$$\|\mathfrak{d}(s)x\|^2_{H^p}=\sum_{i=0}^{p}e^{s(\alpha-\beta n/2+i\beta)}\|\nabla^i x\|^2_{L^2},\quad x\in H^p(\mathbb{R}^n,\mathbb{R}^m)$$

then the canonical homogeneous norm in H^p *is defined as* $\|x\|_{\mathfrak{d},H^p}=1/V$, *where* $V>0$ *is a unique real positive root of the following fractional polynomial equation*

$$1=\sum_{i=0}^{p}V^{\alpha-\beta n/2+i\beta}a_i,$$

where $a_i=\|\nabla^i x\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^m)}$, $x\neq\mathbf{0}$, $\alpha>\beta n/2-i\beta, i=0,1...,p$. *Since the right-hand side of the latter equation is continuously differentiable in* $a_i,V\in(0,+\infty)$ *then, by the implicit function theorem, the function* $(a_0,...,a_p)\to V$ *defined implicitly by this equation is continuously differentiable in* $a_i\in(0,+\infty)$ *as well. In other words, the canonical homogeneous norm* $\|x\|_{\mathfrak{d},H^p}$ *is a continuously differentiable function of* $\|\nabla^i x\|_{L^2}, i=0,1,...,p$ *for* $x\in H^p(\mathbb{R}^n,\mathbb{R}^m)\backslash\{\mathbf{0}\}$.

The canonical homogeneous can be utilized as a Lyapunov function candidate for some homogeneous systems. The differentiability properties of the homogeneous norm are important for the corresponding analysis. In the case of an abstract Hilbert space $\mathbb{H}$, the canonical homogeneous norm is Frechét differentiable at least on the domain of the generator $G_{\mathfrak{d}}$.

Lemma 3.6 ([25], Lemma 7.4). *Let* $\mathfrak{d}$ *be a strongly continuous strictly monotone dilation group in a Hilbert space* $\mathbb{H}$ *then the homogeneous norm* $\|\cdot\|_{\mathfrak{d}}$ *is differentiable on* $\mathcal{D}(G_{\mathfrak{d}})\backslash\{\mathbf{0}\}$ *and the Fréchet derivative of* $\|\cdot\|_{\mathfrak{d},\mathbb{H}}$ *at* $u\in\mathcal{D}(G_{\mathfrak{d}})\backslash\{\mathbf{0}\}$ *is given by*

$$(D\|u\|_{\mathfrak{d},\mathbb{H}})(\cdot)=\frac{\langle\mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})\,\cdot\,,\ \mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})u\rangle}{\langle G_{\mathfrak{d}}\mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})u,\mathfrak{d}(-\ln\|u\|_{\mathfrak{d},\mathbb{H}})u\rangle}\|u\|_{\mathfrak{d},\mathbb{H}}. \tag{3.3}$$

Example 7. *Let us consider again the strongly continuous strictly monotone dilation* (2.1) *in* $H^p(\mathbb{R}^n,\mathbb{R}^m)$ *with* $\alpha>\beta n/2-i\beta, i=0,1,...,p$. *From Example* 6 *we conclude the Fréchet differentiability of* $\|\cdot\|_{\mathfrak{d},H^p}$ *on* $H^p(\mathbb{R}^n,\mathbb{R}^m)\backslash\{\mathbf{0}\}$. *Notice that for* $v\in H^1(\mathbb{R}^n,\mathbb{R}^m)$ *using integration by parts we derive*

$$\langle G_{\mathfrak{d}}v,v\rangle_{H^1}=\langle \alpha v+\beta(z\cdot\nabla)v,v\rangle_{L_2}+\langle\nabla\left(\alpha v+(\beta z\cdot\nabla)v\right),\nabla v\rangle_{L_2}=$$
$$(\alpha-\beta n/2)\left\langle v,v\right\rangle_{L_2}+(\alpha+\beta(1-n/2))\left\langle\nabla v,\nabla v\right\rangle_{L_2}$$

and the Fréchet derivative of $\|\cdot\|_{\mathfrak{d},H^1}$ *at the point* $x\in H^1(\mathbb{R}^n,\mathbb{R}^m)$ *can be computed as follows*

$$\left(D\|x\|_{\mathfrak{d},H^1}\right)(q)=\frac{\left\langle \mathfrak{d}(-\ln\|x\|_{\mathfrak{d},H^1})q,\,v\right\rangle_{H^1}}{(\alpha-\beta n/2)\langle v,v\rangle_{L_2}+(\alpha+\beta(1-n/2))\langle\nabla v,\nabla v\rangle_{L_2}}\|x\|_{\mathfrak{d},H^1},\ q\in H^1(\mathbb{R}^n,\mathbb{R}^m), \tag{3.4}$$

where $v:=\mathfrak{d}\left(-\ln\|x\|_{\mathfrak{d},H^1}\right)x$.

3.2. Homogeneous operators and functionals. Homogeneous functionals and operators on a Banach space $\mathbb{B}$ (see [29]) are defined similarly to homogeneous functions and vector fields (see e.g. [17], [12]) taking into account their possible unboundedness.

Definition 3.7. *An operator* $f:\mathcal{D}(f)\subset\mathbb{B}\to\mathbb{B}$ *(a functional* $h:\mathcal{D}(h)\subset\mathbb{B}\to\mathbb{R}$*) is said to be* $\mathfrak{d}$*-homogeneous of a degree* $\nu\in\mathbb{R}$ *if the domain* $\mathcal{D}(f)$ *(resp.* $\mathcal{D}(h)$*) is a* $\mathfrak{d}$*-homogeneous cone and*

$$\begin{array}{c} e^{\nu s}\mathfrak{d}(s)f(u)=f(\mathfrak{d}(s)u),\quad \forall s\in\mathbb{R},\quad \forall u\in\mathcal{D}(f),\\ (resp.\ h(\mathfrak{d}(s)u)=e^{\nu s}h(u),\quad \forall s\in\mathbb{R},\quad \forall u\in\mathcal{D}(h)) \end{array} \tag{3.5}$$

where $\mathfrak{d}$ *is a group of linear invertible operators in* $\mathbb{B}$.

The canonical homogeneous is the simplest example of the $\mathfrak{d}$-homogeneous functional $\mathbb{B}\to\mathbb{R}$ of the degree 1.

Example 8 ([25]). *Simple computations show the Laplace operator* $\Delta: H^2(\mathbb{R}^n,\mathbb{R})\subset L^2(\mathbb{R}^n,\mathbb{R})\to L^2(\mathbb{R}^n,\mathbb{R})$ *is* $\mathfrak{d}$*-homogeneous of the degree* 2β *with respect to the dilation* $\mathfrak{d}$ *introduced in the Example* 1, *where* $\Delta x=\sum_{i=1}^n\frac{\partial^2 x}{\partial z_i^2}, z_i\in\mathbb{R}, x\in H^2(\mathbb{R}^n,\mathbb{R})$ *and the derivatives are understood in the weak sense.*

Homogeneity allows local properties of nonlinear operators (such as regularity) to be extend globally [25, Chapter 7]. We say that an evolution equation is $\mathfrak{d}$-homogeneous of a degree $\mu\in\mathbb{R}$ if its right-hand side is a $\mathfrak{d}$-homogeneous operator of the degree μ.

3.3. Symmetry of homogeneous evolution equations. Let us consider the nonlinear system

$$\dot{x}=Ax+f(x),\quad t>0,\quad x(0)=x_0 \tag{3.6}$$

where $x(t)\in\mathbb{B}$ is a system state at the time instant t, $x_0\in\mathbb{B}$ is an initial state, $A:\mathcal{D}(A)\subset\mathbb{B}\to\mathbb{B}$ is a linear (possibly unbounded) closed densely defined operator which generates a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{B}$, $f:\mathcal{D}(f)\subset\mathbb{B}\to\mathbb{B}$ is a non-linear (possibly unbounded) closed densely defined operator such that $f(\mathbf{0})=\mathbf{0}$ and $\mathcal{D}(f)\subset\mathbb{B}$ is a linear subspace dense in $\mathbb{B}$, $\mathcal{D}(A)\subset\mathcal{D}(f)$.

The non-linear evolution equations are well-studied in the literature (see, for example, [23, 9]), where the notion of solution is introduced using the theory of evolution semigroups.

Definition 3.8. *A continuous function $x : [0,T) \to \mathbb{B}$ is said to be a mild solution of* (3.6) *if $f(x(\cdot)) \in L^1((0,T),\mathbb{B})$ and*

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)f(x(s))ds, \quad \forall t \in [0,T).$$

If x satisfies (3.6) *for (almost) all $t \in (0,T)$ then x is called classical (strong) solution of* (3.6).

The above integral is understood in the sense of Bochner (see e.g. [8], page 187). The assumption $f(\mathbf{0}) = \mathbf{0}$ implies the existence of the zero solution of (3.6) with $x_0 = \mathbf{0}$.

Definition 3.9. *A closed densely defined non-linear operator $f : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B}$ is said to be M-regular if there exists a linear closed operator $M : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B}$ with a bounded inverse $M^{-1} : \mathbb{B} \to \mathcal{D}(f)$ such that the nonlinear mapping $x \to f(M^{-1}x)$ is locally Lipschitz continuous in $x \in \mathbb{B}\backslash\{\mathbf{0}\}$. More precisely, for any $r > 0$ there exists $L_r > 0$ such that*

$$\|f(M^{-1}x_1) - f(M^{-1}x_2)\|_{\mathbb{B}} \leq L_r\|x_1 - x_2\|_{\mathbb{B}} \tag{3.7}$$

for all $x_i \in K_{\mathbb{B}}(r)$, where $\gamma \in (0,1]$ and $i = 1,2$.

For example, if $f(x) = g(Bx, x)$, where $B : \mathcal{D}(B) \subset \mathbb{B} \to \mathbb{B}$ is a closed densely defined linear unbounded operator and $g : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is locally Lipschitz continuous, then f is M-regular with $M = B - \lambda I$ provided that $\lambda \in \mathbb{R}$ belongs to the resolvent set of B. For a well-posedness of (3.6) with $x_0 \neq \mathbf{0}$, we assume that f admits some "M-regularization" consistent with A.

Assumption 1. *Let $f : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B}$ be M-regular, M commutes with Φ:*

$$\Phi(t)Mx_0 = M\Phi(t)x_0 \quad \textit{for all } t \geq 0 \textit{ and all } x_0 \in \mathcal{D}(f),$$

the linear operator $M\Phi(t) : \mathbb{B} \to \mathbb{B}$ is bounded for any $t > 0$ and there exists a continuous function $\omega : (0,+\infty) \to \mathbb{R}_+$ such that

$$\|M\Phi(t)\|_{\mathbb{B}} \leq \omega(t) \quad \textit{and} \quad \int_0^t \omega(\sigma)d\sigma < +\infty, \forall t \in (0,+\infty).$$

If $\mathcal{D}(f) = \mathbb{B}$ then $M = I$ is the identity operator and Assumption 1 simply asks the regularity of f on $\mathbb{B}\backslash\{\mathbf{0}\}$. If A is a generator of an analytic semigroup Φ then Assumption 1 can be fulfilled for an operator M being a fractional power of the operator A (see e.g. [23], page 195). The function ω has the form $\omega(t) = t^{-\alpha}C, \alpha \in (0,1), C \geq 1$ in this case.

Assumption 1 guarantees that for any $x_0 \in \mathcal{D}(f)\backslash\{\mathbf{0}\}$ the evolution equation (3.6) has a unique mild solution $x_{x_0} : [0,T) \to D(f)$ (see e.g. [23, page 195] or [26, Lemma 2] for more details). If the Banach space $\mathbb{B}$ is reflexive (e.g. if $\mathbb{B} = \mathbb{H}$) and $x_0 \in M^{-1}\mathcal{D}(A)$ then the corresponding mild solution is strong (see e.g. [23, page 109] or [26, Corollary 3] for more details).

A semigroup generated by a closed densely defined linear homogeneous operator in $\mathbb{B}$ is homogeneous as well [25, Lemma 8.1]. The same can be proven for solutions of nonlinear homogeneous evolution equations.

Theorem 3.10 ([25], Theorem 8.1). *Let $\mathfrak{d}$ be a group of linear bounded invertible operators on $\mathbb{B}$. Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ generate a strongly continuous semigroup Φ of linear bounded operators on $\mathbb{B}$ and $f : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B}$. Let A and f be $\mathfrak{d}$-homogeneous operators of a degree $\nu \in \mathbb{R}$. If $x : [0, T) \to \mathcal{D}(f)$ is a mild solution of (3.6) then for any $s \in \mathbb{R}$ the function $x^s : [0, e^{-\nu s}T) \to \mathcal{D}(f)$ given by $x^s(t) := \mathfrak{d}(s)x(e^{\nu s}t)$, $t \in [0, e^{-\nu s}T)$ is a mild solution of the evolution equation (3.6) as well.*

The latter theorem proves the symmetry of solutions of the evolution equation (3.6) with $\mathfrak{d}$-homogeneous operators A and f:

$$x_{\mathfrak{d}(s)x_0}(t) = \mathfrak{d}(s)x_{x_0}(e^{\mu s}t), \quad s \in \mathbb{R}, \quad t \geq 0, \tag{3.8}$$

where x_z denotes a solution of (3.6) with the initial data $x(0) = z$.

Example 9. *The system $\dot{x} = \Delta x - (x \cdot \nabla)x$, $t \geq 0$, $x \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ is $\mathfrak{d}$-homogeneous of the degree 2 with respect to the dilation $(\mathfrak{d}(s)x)(z) = e^s x(e^s z), z \in \mathbb{R}^n, s \in \mathbb{R}$, where $\Delta : H^2(\mathbb{R}^n, \mathbb{R}^n) \subset L^2(\mathbb{R}^n, \mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathbb{R}^n)$ is the Laplace operator and $f : H^1(\mathbb{R}^n, \mathbb{R}^n) \subset L^2(\mathbb{R}^n, \mathbb{R}^n) \to L^2(\mathbb{R}^n, \mathbb{R}^n)$ given by $f(x) = (x \cdot \nabla)x$ is a non-linear M-regular operator with $M = (-\Delta)^{1/2}$ being the fractional Laplacian. According to Theorem 3.10, mild solutions of the considered system are symmetric and satisfy (3.8).*

The dilation symmetry expands globally any local property of solutions. For example, *if the origin of (3.6) is locally stable then, from (3.8) and the limit property of $\mathfrak{d}$, we immediately derive global stability.* Similarly, the existence of solutions for small initial data implies the existence of solutions for large initial data and so on.

4. Homogeneous Galerkin Projection.

As a motivating example let us consider the Burgers equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z}, \quad t > 0, \quad x(0, z) = x_0(z), \quad z \in \mathbb{R}, \quad x_0 \in L^2 \tag{4.1}$$

for which the classical Galerkin projection is given by a reduced order model of the form (see e.g. [6] for more details)

$$\frac{d}{dt}\tilde{x}(t) = A_n\tilde{x}(t) - B(\tilde{x}(t))\tilde{x}(t), \quad t > 0, \tag{4.2}$$

where $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)^\top \in \mathbb{R}^n$ approximates the solution of the original system $x(t) \in L^2$ as follows

$$x(t) \approx \sum_{i=1}^{n} \tilde{x}_i(t)h_i, \quad t \geq 0$$

with $h_i \in L^2, i = 1, 2, ..., n$ being a (sufficiently smooth) orthonormal family in L^2, $A_n \in \mathbb{R}^{n\times n}$ and $\tilde{x} \to B(\tilde{x}) \in \mathbb{R}^{n\times n}$ is a linear mapping.

Notice that the Burgers equation in $L^2(\mathbb{R}, \mathbb{R})$ is $\mathfrak{d}$-homogeneous of the degree 2 with respect to the dilation (2.1) (see Example 9). The latter implies the symmetry of solutions of the Burgers equation (see Theorem 3.10). This symmetry is destroyed by the conventional

Galerkin projection. Indeed, the first term in the right-hand side of (4.2) is $\mathfrak{d}$-homogeneous (with $\mathfrak{d}(s) = e^s I_n$) of the degree 0:

$$A_n \mathfrak{d}(s)\tilde{x} = e^{0s}\mathfrak{d}(s)A_n\tilde{x}, \quad \forall s \in \mathbb{R}, \forall \tilde{x} \in \mathbb{R}^n,$$

and the second one is $\mathfrak{d}$-homogeneous of the degree 1:

$$B(\mathfrak{d}(s)\tilde{x})\mathfrak{d}(s)\tilde{x} = e^{1s}\mathfrak{d}(s)B(\tilde{x})\tilde{x}, \quad \forall s \in \mathbb{R}, \forall \tilde{x} \in \mathbb{R}^n,$$

while the right-hand side (4.2) is not $\mathfrak{d}$-homogeneous, so solutions of (4.2) are not symmetric with respect to homogeneous scaling of the initial condition. Therefore, the classical Galerkin method does not preserve the dilation symmetry of the Burgers equation. The same conclusions can be made for the Galerkin projection of Navier–Stokes equations.

There are two possible ways to preserve the homogeneity in the finite-dimensional approximated model of the Burgers equation. The first one is to use the Lie-symmetry-preserving numerical integration schemes based on finite-difference approximations [5] which would immediately provide some nonlinear discrete-time approximate model of the system. The second way is to develop a Galerkin-like projection method which would preserve a dilation symmetry of the Burgers equation in its finite-dimensional reduced order model (ODE). Since most of control design and analysis methods are developed for continuous-time homogeneous ODE models, the second option could be more useful for the modern control systems theory.

It seems that the problem of a homogeneous Galerkin projection was studied in [25, Chapter 10] for the first time, where the following two step procedure was suggested: initially, an approximation of a linear dilation $\mathfrak{d}$ is obtained, next, an approximation of a non-linear $\mathfrak{d}$-homogeneous operator is designed. This paper refines the suggested procedure.

4.1. Approximation of the dilation group. Notice that the generator of a dilation in $\mathbb{H}$ is a linear closed densely defined operator due to Hille-Yosida theorem [23]. This allows the conventional Galerkin projection method to be utilized for the approximation of the dilation.

Since any linear dilation $\mathfrak{d}(s)$ in a real Hilbert space $\mathbb{H}$ is defined by it generator then for any $x_0 \in \mathcal{D}(G_\mathfrak{d})$ the function $s \to x(s) := \mathfrak{d}(s)x_0$ satisfies

$$\dot{x}(s) = G_\mathfrak{d} x(s), \quad s \in \mathbb{R}, \quad x(0) = x_0.$$

In this section we deal only with the Hilber space $\mathbb{H}$ and, for shortness, *we omit the index* $\mathbb{H}$ *in notations of norms and the inner products* if the context is clear.

Recall that the classical Galerkin approximation deals the following weak formulation of the latter linear equation:

$$\begin{array}{c} \textbf{find } x_v \in C(\mathbb{R}, V) \textbf{ such that} \\ \langle \dot{x}_v(s) - G_\mathfrak{d} x_v(s), v\rangle = 0, \quad \forall v \in V, \quad \forall s \in \mathbb{R}, \\ \langle x(0), v\rangle_\mathbb{H} = \langle x_0, v\rangle, \quad \forall v \in V, \end{array} \tag{4.3}$$

where $V \subset \mathbb{H}$ is a linear subspace of $\mathbb{H}$.

Let $h_i \in \mathcal{D}(G_{\mathfrak{d}}), i = 1, 2, ..., n$ be an orthonormal family of vectors from a real Hilbert space $\mathbb{H}$. Let the operator $\Pi_n : \mathbb{H} \to \mathbb{R}^n$ be defined as follows

$$\Pi_n x = \begin{pmatrix} \langle x, h_1 \rangle \\ \langle x, h_2 \rangle \\ ... \\ \langle x, h_n \rangle \end{pmatrix}. \tag{4.4}$$

Hence, repeating the standard arguments for $V = \text{span}\{h_1, h_2, ..., h_n\}$ we derive

$$x_v(s) = \sum_{i=1}^{n} \tilde{x}_i(s) h_i$$

is the unique solution of (4.3) provided that

$$\frac{d}{ds}\tilde{x}(s) = G_{\mathfrak{d}_n}\tilde{x}(s), \quad \tilde{x}(0) = \Pi_n x_0, \quad \tilde{x}(s) = (\tilde{x}_1(s), ..., \tilde{x}_n(s))^\top \in \mathbb{R}^n, \quad s \in \mathbb{R}$$

where

$$G_{\mathfrak{d}_n} = \begin{pmatrix} \langle G_{\mathfrak{d}} h_1, h_1 \rangle & \langle G_{\mathfrak{d}} h_2, h_1 \rangle & ... & \langle G_{\mathfrak{d}} h_n, h_1 \rangle \\ \langle G_{\mathfrak{d}} h_1, h_2 \rangle & \langle G_{\mathfrak{d}} h_2, h_2 \rangle & ... & \langle G_{\mathfrak{d}} h_n, h_2 \rangle \\ ... & ... & ... & ... \\ \langle G_{\mathfrak{d}} h_1, h_n \rangle & \langle G_{\mathfrak{d}} h_2, h_n \rangle & ... & \langle G_{\mathfrak{d}} h_n, h_n \rangle \end{pmatrix} \in \mathbb{R}^{n \times n} \tag{4.5}$$

is the matrix of a linear operator $V \to V$ being the conventional Galerkin projection of the generator $G_{\mathfrak{d}}$ to a finite-dimensional linear subspace V. Notice that $\tilde{x}(s) = e^{sG_{\mathfrak{d}_n}}\Pi_n x_0$, so the one-parameter group

$$\mathfrak{d}_n(s) = e^{sG_{\mathfrak{d}_n}} \in \mathbb{R}^{n \times n}, \quad s \in \mathbb{R} \tag{4.6}$$

can be interpreted as *a finite-dimensional projection of the dilation* $\mathfrak{d}$. For any strictly monotone dilation $\mathfrak{d}$ satisfying $\langle G_{\mathfrak{d}} v, v \rangle \geq \gamma \langle v, v \rangle, \forall v \in V$ we obviously have

$$G_{\mathfrak{d}_n} + G_{\mathfrak{d}_n}^\top \succeq 2\gamma I_n. \tag{4.7}$$

Moreover, in this case, the Galerkin projection $G_{\mathfrak{d}_n} \in \mathbb{R}^{n \times n}$ of the generator $G_{\mathfrak{d}}$ always admits the representation

$$G_{\mathfrak{d}_n} = \Lambda + \Xi, \tag{4.8}$$

where $\Lambda \in \mathbb{R}^{n \times n}$ *is a diagonal positive definite matrix and* $\Xi \in \mathbb{R}^{n \times n}$ *is a skew-symmetric matrix, provided that* $h_i \in D(G_{\mathfrak{d}})$ *are properly selected.* Indeed, since

$$G_{\mathfrak{d}} = \frac{G_{\mathfrak{d}} + G_{\mathfrak{d}}^*}{2} + \frac{G_{\mathfrak{d}} - G_{\mathfrak{d}}^*}{2}$$

where $G_{\mathfrak{d}}^*$ is the adjoint operator to $G_{\mathfrak{d}}$, then $G_{\mathfrak{d}}$ always admits the representation (4.8) with Λ being symmetric Galerkin projection of $\frac{G_{\mathfrak{d}}+G_{\mathfrak{d}}^*}{2}$ and Ξ being skew-symmetric Galerkin projection $\frac{G_{\mathfrak{d}}-G_{\mathfrak{d}}^*}{2}$, respectively. Since

$$\left\langle \frac{G_{\mathfrak{d}} + G_{\mathfrak{d}}^*}{2} x, x \right\rangle = \langle G_{\mathfrak{d}} x, x \rangle \geq \gamma \langle x, x \rangle, \quad \gamma > 0,$$

then $\frac{G_{\mathfrak{d}}+G_{\mathfrak{d}}^*}{2}$ is a positive symmetric operator. Such an operator always have an orthonormal family of eigenfunction $h_i \in D(G_{\mathfrak{d}}), i = 1, 2, 3, ...$ being a basis in the Hilbert space $\mathbb{H}$, so the corresponding Galerkin projection gives a diagonal positive definite matrix $\Lambda \succeq \gamma I_n$.

Example 10. *Let us consider the dilation $\mathfrak{d}$ in $L^2(\mathbb{R}, \mathbb{R})$ given by (2.1) recalled here as*

$$(\mathfrak{d}(s)x)(z) = e^{\alpha s} x(e^{\beta s} z), \quad z, s \in \mathbb{R}, \quad x \in L^2(\mathbb{R}, \mathbb{R}),$$

where $\alpha > \beta/2$. The generator of the dilation $\mathfrak{d}$ is given by (see Example 3)

$$G_{\mathfrak{d}} x = \alpha x + \beta z \frac{\partial x}{\partial z}, \quad x \in \mathcal{D}(G_{\mathfrak{d}}) \subset L^2(\mathbb{R}, \mathbb{R}), z \in \mathbb{R},$$

where $\mathcal{D}(G_{\mathfrak{d}})$ is a closure of $C_c^{\infty}(\mathbb{R}, \mathbb{R})$ with respect to the norm $\|x\| = \|x\|_{L^2} + \|G_{\mathfrak{d}} x\|_{L^2}$.

Let us consider the Hermite functions

$$h_i(z) = \frac{(-1)^{i-1}}{\sqrt{2^{i-1}(i-1)!\sqrt{\pi}}} e^{\frac{z^2}{2}} \frac{d^{i-1}}{dy^{i-1}} e^{-z^2}, \quad z \in \mathbb{R}, i = 1, 2, ... \tag{4.9}$$

which are known to be a smooth orthonormal basis in $L^2(\mathbb{R}, \mathbb{R})$: $\langle h_i, h_j \rangle = \delta_{ij}$ where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. It is easy to see that $h_i \in \mathcal{D}(G_{\mathfrak{d}})$.

The finite-dimensional projection of the dilation $\mathfrak{d}$ can be given by (4.5), where $G_{\mathfrak{d}_n} \in \mathbb{R}^{n\times n}$ is the generator of the dilation $\mathfrak{d}_n$ in $\mathbb{R}^n$ with the following elements

$$\langle G_{\mathfrak{d}} h_i, h_j \rangle_{L^2} = \int_{\mathbb{R}} (G_{\mathfrak{d}} h_i)(z) h_j(z) dz = \int_{\mathbb{R}} \left(\alpha h_i(z) + \beta z \frac{\partial h_i(z)}{\partial z} \right) h_j(z) dz =$$

$$\alpha \delta_{ij} + \int_{\mathbb{R}} \beta z \frac{\partial h_i(z)}{\partial z} h_j(z) dz.$$

Taking into account

$$\frac{\partial h_i(z)}{\partial z} = \sqrt{\frac{i-1}{2}} h_{i-1}(z) - \sqrt{\frac{i}{2}} h_{i+1}(z) \tag{4.10}$$

we obtain

$$\langle G_{\mathfrak{d}} h_i, h_j \rangle_{L^2} = \alpha \delta_{ij} + \beta \sqrt{\frac{i-1}{2}} \langle h_{i-1}, y h_j \rangle - \beta \sqrt{\frac{i}{2}} \langle h_{i+1}, y h_j \rangle.$$

Finally, using the identity

$$y h_j(y) = \sqrt{\frac{j-1}{2}} h_{j-1}(y) + \sqrt{\frac{j}{2}} h_{j+1}(y)$$

we derive

$$\langle G_{\mathfrak{d}} h_i, h_j \rangle_{L^2} = \tfrac{2\alpha-\beta}{2} \delta_{ij} + \tfrac{\beta\sqrt{(i-1)j}}{2} \delta_{i-1\, j+1} - \tfrac{\beta\sqrt{i(j-1)}}{2} \delta_{i+1\, j-1}.$$

Therefore, the matrix $G_{\mathfrak{d}_n}$ has the form

$$G_{\mathfrak{d}_n} = \begin{pmatrix} \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{\frac{1}{2}} & 0 & 0 & 0 & \dots \\ 0 & \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{\frac{3}{2}} & 0 & 0 & \dots \\ -\beta\sqrt{\frac{1}{2}} & 0 & \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{3} & 0 & \dots \\ 0 & -\beta\sqrt{\frac{3}{2}} & 0 & \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{5} & \dots \\ 0 & 0 & -\beta\sqrt{3} & 0 & \frac{2\alpha-\beta}{2} & 0 & \dots \\ 0 & 0 & 0 & -\beta\sqrt{5} & 0 & \frac{2\alpha-\beta}{2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{4.11}$$

and the finite-dimensional projection of the dilation $\mathfrak{d}$ is given by

$$\mathfrak{d}_n(s) = e^{sG_{\mathfrak{d}_n}} = e^{(\alpha-0.5\beta)s}e^{s\Xi}, \quad s \in \mathbb{R}, \quad \Xi = -\Xi^\top = G_{\mathfrak{d}_n} - (\alpha - 0.5\beta)I_n.$$

Notice that $e^{s\Xi}$ is the orthogonal matrix: $e^{s\Xi}e^{s\Xi^\top} = e^{s\Xi^\top}e^{s\Xi} = I_n$. The latter means that $\|\tilde{x}\|_{\mathfrak{d}_n} = \|\tilde{x}\|_{\mathbb{R}^n}^{\frac{1}{\alpha-0.5\beta}}$ for any $\tilde{x} \in \mathbb{R}^n$.

4.2. Homogeneous Galerkin Projection. Assume that the dilation $\mathfrak{d}$ is strongly continuous and strictly monotone and the operators A, f be $\mathfrak{d}$-homogeneous of a degree $\nu \in \mathbb{R}$. A behavior of any $\mathfrak{d}$-homogeneous evolution system is completely defined by its operator on a unit sphere [25] and the homogeneity degree of the system. Indeed, using the $\mathfrak{d}$-homogeneity it can be easily checked that for any strong solution of (3.6) $x : [0, T) \to \mathbb{H}$ the following quantities $\phi(t) = \mathfrak{d}(-\ln \|x(t)\|_{\mathfrak{d}})x(t)$, $r(t) = \|x(t)\|_{\mathfrak{d}}$ satisfy the equation

$$r^{-\nu}(t)\mathfrak{d}(-\ln r(t))\tfrac{d(\mathfrak{d}(\ln r(t))\phi(t))}{dt} \overset{a.e.}{=} A\phi(t)+f(\phi(t)), \quad t\in(0,T), \quad \phi(0)=\mathfrak{d}(-\ln \|x_0\|_{\mathfrak{d}})x_0, \tag{4.12}$$

Using (3.3) and $\mathfrak{d}$-homogeneity of A, f we derive

$$\dot{r}(t) \overset{a.e.}{=} r^{\nu+1}(t)\tfrac{\langle A\phi(t)+f(\phi(t)),\phi(t)\rangle}{\langle G_{\mathfrak{d}}\phi(t),\phi(t)\rangle}, \quad t \in (0, T), \quad r(0) = \|x_0\|_{\mathfrak{d}}. \tag{4.13}$$

Notice that the operator $x \to \mathfrak{d}(-\ln \|x\|_{\mathfrak{d}})x$ is a $\mathfrak{d}$-homogeneous projector of a vector x on the unit sphere in $\mathbb{H}$, i.e. $\|\mathfrak{d}(-\ln \|x\|_{\mathfrak{d}})x\| = 1$ for $\forall x \in \mathbb{H}$. Therefore, the system (4.12), (4.13) is an equivalent representation of the evolution equation (3.6) in *polar homogeneous coordinates* [30], [25]. The weak formulation of (4.12)-(4.13) is

$$\begin{array}{c} \textbf{find } \phi_v \in C([0,T), V) \textbf{ and } \tilde{r} \in C([0,T), \mathbb{R}_+) \textbf{ such that} \\ \left\langle \tilde{r}^{-\nu}(t)\mathfrak{d}(-\ln \tilde{r}(t))\tfrac{d}{dt}(\mathfrak{d}(\ln \tilde{r}(t))\phi_v(t)) - A\phi_v(t) - f(\phi_v(t)), v\right\rangle \overset{a.e.}{=} 0, \quad \forall v\in V, \forall t\in(0,T), \\ \tfrac{d}{dt}\tilde{r}(t) \overset{a.e.}{=} \tilde{r}^{\nu+1}(t)\tfrac{\langle A\phi_v(t)+f(\phi_v(t)),\phi_v(t)\rangle}{\langle G_{\mathfrak{d}}\phi_v(t),\phi_v(t)\rangle}, \quad \forall t \in (0, T) \\ \langle \phi_v(0), v\rangle = \langle \mathfrak{d}(-\ln \|x_0\|_{\mathfrak{d}})x_0, v\rangle, \forall v \in V, \quad \text{and} \quad \tilde{r}(0) = \|x_0\|_{\mathfrak{d}}, \end{array} \tag{4.14}$$

where $V \subset \mathbb{H}$ is a linear subspace of $\mathbb{H}$ and $S_V = \{z \in \mathbb{V} : \|z\|_{\mathbb{H}} = 1\}$ is the unit sphere in V.

The pair $\phi_v(t), \tilde{r}(t)$ is a Galerkin-like projection of the pair $(\phi(t), r(t)) \in S_{\mathbb{H}} \times \mathbb{R}_+$ on $S_V \times \mathbb{R}_+$, and

$$x_v(t) = \mathfrak{d}(\ln \tilde{r}(t))\phi_v(t), \quad t \in [0, T) \tag{4.15}$$

is a Galerkin-like projection of the strong solution $x : [0,T) \to \mathbb{H}$ of the system (3.6) on the $\mathfrak{d}$-homogeneous cone

$$\mathcal{D}_V := \bigcup_{s\in\mathbb{R}} \mathfrak{d}(s) S_V. \tag{4.16}$$

Notice that $\mathcal{D}_v$ is not a linear subspace of $\mathbb{H}$ in the general case. If S_V is dense in $S_{\mathbb{H}}$ then $\mathcal{D}_V$ is dense in due to $\mathbb{H}$ [25, Lemma 6]. Obviously, $\|x_v(t)\|_{\mathfrak{d}} = \tilde{r}(t), t \geq 0$ and $x_v \in C([0,T), \mathcal{D}_V)$.

Theorem 4.1 (On existence of a homogeneous Galerkin projection). *Let $\mathfrak{d}$ be a strongly continuous strictly monotone dilation in $\mathbb{H}$, the operators A, f be $\mathfrak{d}$-homogeneous of the degree $\nu \in \mathbb{R}$, $\mathcal{D}(A) \subset \mathcal{D}(f)$, the operator f be M-regular and $h_i \in \mathcal{D}(A) \cap \mathcal{D}(G_{\mathfrak{d}}), i = 1, 2, ..., n$ be an orthonormal basis in $V = \mathrm{span}\{h_1, ..., h_n\}$. Then for any $x_0 \in \bigcup_{s\in\mathbb{R}} \mathfrak{d}(s) S_V$ there exists a pair $\phi_v, \tilde{r}$ satisfying (4.14) such that*

$$\phi_v(t) = \sum_{i=1}^{n} \tilde{\phi}_i(t) h_i, \quad t \in [0,T),$$

where the pair $\tilde{\phi}(t) = (\tilde{\phi}_1(t), ..., \tilde{\phi}_n(t))^\top \in \mathbb{R}^n$, $\tilde{r}(t) \in \mathbb{R}_+$ is the unique classical solution of the following ODE

$$\begin{cases} \frac{d}{dt}\tilde{\phi}(t) = \tilde{r}^\nu(t)\left(A_n\tilde{\phi}(t) + \tilde{f}(\tilde{\phi}(t))\right) - \tilde{r}^\nu(t)\frac{\tilde{\phi}^\top(t)A_n\tilde{\phi}(t) + \tilde{\phi}^\top(t)\tilde{f}(\tilde{\phi}(t))}{\tilde{\phi}^\top(t)G_{\mathfrak{d}_n}\tilde{\phi}(t)} G_{\mathfrak{d}_n}\tilde{\phi}(t), \\ \frac{d}{dt}\tilde{r}(t) = \tilde{r}^{\nu+1}(t)\frac{\tilde{\phi}^\top(t)A_n\tilde{\phi}(t) + \tilde{\phi}^\top\tilde{f}(\tilde{\phi}(t))}{\tilde{\phi}^\top(t)G_{\mathfrak{d}_n}\tilde{\phi}(t)}, & t \in (0,T), \\ \tilde{\phi}(0) = \Pi_n \mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0, \quad \tilde{r}(0) = \|x_0\|_{\mathfrak{d}}, \end{cases} \tag{4.17}$$

where $G_{\mathfrak{d}_n}$ is given by (4.5),

$$\tilde{A}_n = \begin{pmatrix} \langle Ah_1,h_1\rangle & \langle Ah_2,h_1\rangle & ... & \langle Ah_n,h_1\rangle \\ \langle Ah_1,h_2\rangle & \langle Ah_2,h_2\rangle & ... & \langle Ah_n,h_2\rangle \\ ... & ... & ... & ... \\ \langle Ah_1,h_n\rangle & \langle Ah_2,h_n\rangle & ... & \langle Ah_n,h_n\rangle \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{4.18}$$

$$\tilde{f}(\tilde{\phi}) = \begin{pmatrix} \left\langle f\left(\sum_{k=1}^{n}\tilde{\phi}_k h_k\right), h_1\right\rangle \\ \left\langle f\left(\sum_{k=1}^{n}\tilde{\phi}_k h_k\right), h_2\right\rangle \\ ... \\ \left\langle f\left(\sum_{k=1}^{n}\tilde{\phi}_k h_k\right), h_n\right\rangle \end{pmatrix}, \quad \tilde{\phi} = (\tilde{\phi}_1, ..., \tilde{\phi}_n)^\top \in \mathbb{R}^n, \tag{4.19}$$

and $T \leq +\infty$ such that
- $T = +\infty$ *if* $\nu \geq 0, \tilde{\phi}^\top(A_n\tilde{\phi} + \tilde{f}(\tilde{\phi})) \leq 0, \forall\tilde{\phi} \in S_{\mathbb{R}^n}$ *or* $\nu \leq 0, \tilde{\phi}^\top(A_n\tilde{\phi} + \tilde{f}(\tilde{\phi})) \geq 0, \forall\tilde{\phi} \in S_{\mathbb{R}^n}$;
- $T < +\infty : \lim_{t\to T}\tilde{r}(t) = 0$ *if* $\nu < 0, \tilde{\phi}^\top(A_n\tilde{\phi} + \tilde{f}(\tilde{\phi})) < 0, \forall\tilde{\phi} \in S_{\mathbb{R}^n}$;
- $T < +\infty : \lim_{t\to T}\tilde{r}(t) = +\infty$ *if* $\nu > 0, \tilde{\phi}^\top(A_n\tilde{\phi} + \tilde{f}(\tilde{\phi})) > 0, \forall\tilde{\phi} \in S_{\mathbb{R}^n}$.

Proof. Since h_i is an orthonormal basis in V then any $v \in V$ can be written as $v = \sum_{i=1}^{n} \tilde{v}_i h_i$ with some $\tilde{v} = (\tilde{v}_1, ..., \tilde{v}_n)^\top \in \mathbb{R}^n$ and any $\phi_v : [0,T) \to S_V$ can be rewritten as $\phi_v(t) = \sum_{i=1}^{n} \tilde{\phi}_i(t) h_i$ with some $\tilde{\phi} : [0,T) \to S_{\mathbb{R}^n}$. Taking into account

$$\left\langle \tilde{r}^{-\nu}(t)\mathfrak{d}(-\ln \tilde{r}(t))\frac{d}{dt}\left(\mathfrak{d}(\ln \tilde{r}(t))\phi_v(t)\right) - A\phi_v(t) - f(\phi_v(t)), v\right\rangle =$$

$$\left\langle \tilde{r}^{-\nu}(t)\mathfrak{d}(-\ln \tilde{r}(t))\frac{d}{dt}\left(\mathfrak{d}(\ln \tilde{r}(t))\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k\right) - A\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k - f\left(\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k\right), v\right\rangle =$$

$$\sum_{i=1}^{n}\tilde{v}_i\left\langle \tilde{r}^{-\nu}(t)\mathfrak{d}(-\ln \tilde{r}(t))\frac{d}{dt}\left(\mathfrak{d}(\ln \tilde{r}(t))\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k\right) - A\sum_{i=1}^{n}\tilde{\phi}_k(t)h_k - f\left(\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k\right), h_i\right\rangle$$

we conclude that the problem (4.14) can be rewritten as follows

$$\begin{cases} \textbf{find } \tilde{\phi} : [0,T) \to S_{\mathbb{R}^n} \subset \mathbb{R}^n \textbf{ and } \tilde{r} : [0,T) \to \mathbb{R} \textbf{ such that} \\ \sum\limits_{i=1}^{n}\tilde{v}_i\left\langle \frac{\mathfrak{d}(-\ln \tilde{r}(t))}{\tilde{r}^{\nu}(t)}\frac{d\left(\mathfrak{d}(\ln \tilde{r}(t))\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right)}{dt} - A\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k - f\left(\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right), h_i\right\rangle = 0, \forall \tilde{v} \in \mathbb{R}^n, \\ \frac{d}{dt}\tilde{r}(t) \overset{a.e.}{=} \tilde{r}^{\nu+1}(t)\frac{\left\langle A\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k + f(\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k), \sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right\rangle}{\left\langle G_{\mathfrak{d}}\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k, \sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right\rangle}, \quad \forall t \in (0,T), \\ \sum\limits_{i=1}^{n}\tilde{v}_i\left\langle \sum\limits_{k=1}^{n}\tilde{\phi}_k(0)h_k - \mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0, h_i\right\rangle = 0, \forall \tilde{v} \in \mathbb{R}^n, \quad \text{and} \quad \tilde{r}(0) = \|x_0\|_{\mathfrak{d}}, \end{cases}$$

or, equivalently,

$$\begin{cases} \textbf{find } \tilde{\phi} : [0,T) \to S_{\mathbb{R}^n} \subset \mathbb{R}^n \textbf{ and } \tilde{r} : [0,T) \to \mathbb{R} \textbf{ such that} \\ \left\langle \frac{\mathfrak{d}(-\ln \tilde{r}(t))}{\tilde{r}^{\nu}(t)}\frac{d\left(\mathfrak{d}(\ln \tilde{r}(t))\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right)}{dt} - A\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k - f\left(\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right), h_i\right\rangle = 0, i = 1, 2, ..., n, \\ \frac{d}{dt}\tilde{r}(t) \overset{a.e.}{=} \tilde{r}^{\nu+1}(t)\frac{\left\langle A\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k + f(\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k), \sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right\rangle}{\left\langle G_{\mathfrak{d}}\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k, \sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right\rangle}, \quad \forall t \in (0,T), \\ \left\langle \sum\limits_{k=1}^{n}\tilde{\phi}_k(0)h_k - \mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0, h_i\right\rangle = 0, i = 1, 2, ..., n \quad \text{and} \quad \tilde{r}(0) = \|x_0\|_{\mathfrak{d}}, \end{cases}$$

Since $h_i \in \mathcal{D}(G_{\mathfrak{d}})$ implies $\frac{d}{ds}\mathfrak{d}(s)h_i = \mathfrak{d}(s)G_{\mathfrak{d}}h_i$ then

$$\left\langle \frac{\mathfrak{d}(-\ln \tilde{r}(t))}{\tilde{r}^{\nu}(t)}\frac{d\left(\mathfrak{d}(\ln \tilde{r}(t))\sum\limits_{k=1}^{n}\tilde{\phi}_k(t)h_k\right)}{dt} - A\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k - f\left(\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k\right), h_i\right\rangle =$$

$$\left\langle \sum_{k=1}^{n}\frac{\mathfrak{d}(-\ln \tilde{r}(t))}{\tilde{r}^{\nu}(t)}\left(\mathfrak{d}(\ln \tilde{r}(t))h_k\frac{d\tilde{\phi}_k(t)}{dt} + \tilde{\phi}_k(t)\frac{d\mathfrak{d}(\ln \tilde{r}(t))h_k}{dt}\right) - \tilde{\phi}_k(t)Ah_k - f\left(\sum_{k=1}^{n}\tilde{\phi}_k(t)h_k\right), h_i\right\rangle =$$

$$\left\langle \sum_{k=1}^{n} \frac{1}{\tilde r^{\nu}(t)}\frac{d\tilde\phi_k(t)}{dt}h_k + \frac{\frac{d\tilde r(t)}{dt}}{\tilde r^{\nu+1}(t)}\tilde\phi_k(t)G_{\mathfrak{d}}h_k - \tilde\phi_k(t)Ah_k - f\left(\sum_{k=1}^{n}\tilde\phi_k(t)h_k\right), h_i\right\rangle.$$

Hence, we derive that the problem (4.14) can be rewritten as follows

$$\textbf{find } \tilde\phi : [0,T)\to S_{\mathbb{R}^n}\subset\mathbb{R}^n \textbf{ and } \tilde r : [0,T)\to\mathbb{R} \text{ satisfying (4.17).}$$

Since f is M-regular and $h_i\in\mathcal{D}(A)\subset\mathcal{D}(f)$ then $\tilde f$ is locally Lipschitz continuous on $\mathbb{R}^n\backslash\{\mathbf{0}\}$. Since due to Proposition 3.3 we have $\langle G_{\mathfrak{d}}\phi_v(t),\phi_v(t)\rangle = \tilde\phi^\top(t)G_{\mathfrak{d}_n}\tilde\phi(t)\geq\gamma\langle\phi_v(t),\phi_v(t)\rangle = \gamma\tilde\phi^\top(t)\tilde\phi(t)$ then the right-hand side of (4.17) locally Lipschitz continuous on $(\mathbb{R}^n\backslash\{\mathbf{0}\})\times(\mathbb{R}_+\backslash\{0\})$. The latter means that for any $x_0\in\mathbb{H}:\Pi_n\mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0\neq\mathbf{0}$ the system has a unique classical solution defined as long as $\tilde r(t)>0$ and $\tilde z(t)\neq\mathbf{0}$. Moreover, if $x_0\in\bigcup_{s\in\mathbb{R}}\mathfrak{d}(s)S_V$ then $\|\tilde\phi(0)\|_{\mathbb{R}^n}=1$ and $\|\tilde\phi(t)\|_{\mathbb{R}_n}=1$ for all $t\in(0,T)$ due to

$$\frac{d}{dt}\left(\tilde\phi^\top(t)\tilde\phi(t)\right) = 2\left(\tilde\phi^\top(t)\frac{d}{dt}\tilde\phi(t)\right) = 0.$$

Since $h_i\in\mathcal{D}(A)\subset D(f)$ and $\|\tilde\phi^\top(t)\|=1$ then $|\tilde\phi^\top(t)(\tilde A_n\tilde\phi(t)+\tilde f(\tilde\phi(t)))|/|\tilde\phi^\top(t)G\mathfrak{d}_n\tilde\phi(t)|\leq C_n/\gamma$ for some $C_n>0$ and $r(t)$ converges to zero in a finite time $T<+\infty$ if $\nu<0, \tilde\phi^\top(t)(A_n\tilde\phi(t)+\tilde f(\tilde\phi(t)))<0$ or $r(t)$ blows up in a finite time $T<+\infty$ if $\nu>0, \tilde\phi^\top(t)(A_n\tilde\phi(t)+\tilde f(\tilde\phi(t)))>0$. For $\nu\geq 0, \tilde\phi^\top(t)(A_n\tilde\phi(t)+\tilde f(\tilde\phi(t)))\leq 0$ or $\nu\leq 0, \tilde\phi^\top(t)(A_n\tilde\phi(t)+\tilde f(\tilde\phi(t)))\geq 0$ we have $T=+\infty$. ■

Introducing the notation

$$\tilde x(t) = \mathfrak{d}_n(\ln\tilde r(t))\tilde\phi(t)$$

we derive that the system (4.17) is homeomorphic on $\mathbb{R}^n$ and diffeomorphic on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ to

$$\text{(4.20)}\qquad \begin{cases} \frac{d\tilde x(t)}{dt} = \|\tilde x(t)\|_{\mathfrak{d}_n}^{\nu}\mathfrak{d}_n(\ln\|\tilde x(t)\|_{\mathfrak{d}})\left(\tilde A_n\mathfrak{d}_n(-\ln\|\tilde x\|_{\mathfrak{d}_n})\tilde x(t) + \tilde f\left(\mathfrak{d}_n(-\ln\|\tilde x(t)\|_{\mathfrak{d}_n})\tilde x(t)\right)\right), \\ \tilde x(0) = \mathfrak{d}_n(\ln\|x_0\|_{\mathfrak{d}})\Pi_n\mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0, \quad t\in(0,T). \end{cases}$$

Taking into account $\mathfrak{d}$-homogeneity of the operator $A+f$ we conclude that the system (4.20) can be treated as a finite-dimensional projection of the $\mathfrak{d}$-homogeneous evolution equation (3.6), since

$$Ax(t)+f(x(t)) = \|x(t)\|_{\mathfrak{d}}^{\nu}\mathfrak{d}(\ln\|x(t)\|_{\mathfrak{d}})\left(A\mathfrak{d}(-\ln\|x(t)\|_{\mathfrak{d}})x(t) + f(\mathfrak{d}(-\ln\|x(t)\|_{\mathfrak{d}})x(t))\right).$$

It can obtained by means of sequential approximations of the dilation group $\mathfrak{d}$, the canonical homogeneous norm $\|x\|_{\mathfrak{d}}$ and the operators A, f. The formula (4.15) can be rewritten as

$$x_v(t) = \mathfrak{d}(\ln\|\tilde x(t)\|_{\mathfrak{d}_n})\sum_{i=1}^{n}[\mathfrak{d}_n(-\ln\|\tilde x(t)\|_{\mathfrak{d}_n})\tilde x(t)]_i h_i, \quad t\geq 0.$$

Since $\|\mathfrak{d}_n(-\ln\|\tilde x(t)\|_{\mathfrak{d}_n})\tilde x(t)\|_{\mathbb{R}_n}=1$ then $\|\sum_{i=1}^{n}[\mathfrak{d}_n(-\ln\|\tilde x(t)\|_{\mathfrak{d}_n})\tilde x(t)]_i h_i\|_{\mathbb{H}}=1$ and

$$\|x_v(t)\|_{\mathfrak{d}} = \|\tilde x(t)\|_{\mathfrak{d}_n}, \quad t\geq 0.$$

Notice that, for the standard dilation $\mathfrak{d}(s)=e^sI$ and the homogeneity degree $\nu=0$, (4.20) becomes the conventional Galerkin projection of (3.6): $\frac{d\tilde{x}(t)}{dt}=\tilde{A}_n\tilde{x}(t)+\tilde{f}(\tilde{x}(t)), \tilde{x}(0)=\Pi_n x_0$.

The system (4.20) is, obviously, $\mathfrak{d}_n$-homogeneous of the degree ν. Therefore, *the proposed homogeneous Galerkin projection preserves the homogeneity of the original system as well as its homogeneity degree.*

Corollary 4.2. *Let all conditions of Theorem 4.1 be fulfilled and $x_v\in C([0,T),\mathcal{D}_V)$ be a solution of (4.14) for $x_0\in\mathcal{D}_V$. Then $\forall s\in\mathbb{R}^n$ the function $x_v^s\in C([0,e^{-\nu s}T),\mathbb{V})$ given by*

$$x_v^s(t)=\mathfrak{d}(s)x_v(e^{\nu s})$$

is the solution of (4.14) with the scaled initial condition $x(0)=\mathfrak{d}(s)x_0$.

Proof. Let $\tilde{x}$ be the solution of (4.20) for some $x_0\in\mathcal{D}_V$. Denote $x_0^s=\mathfrak{d}(s)x_0$. Since $\mathcal{D}_V$ is a $\mathfrak{d}$-homogeneous cone then $x_0^s\in\mathcal{D}_V$. Since $\mathfrak{d}(-\ln\|x_0^s\|_{\mathfrak{d}})\mathfrak{d}(s)x_0^s=\mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0$ and $\|x_0^s\|_{\mathfrak{d}}=e^s\|x_0\|_{\mathfrak{d}}$ then

$$\mathfrak{d}_n(\ln\|x_0^s\|_{\mathfrak{d}})\Pi_n\mathfrak{d}(-\ln\|x_0^s\|_{\mathfrak{d}})x_0^s=\mathfrak{d}_n(s)\mathfrak{d}_n(\ln\|x_0\|_{\mathfrak{d}})\Pi_n\mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0=\mathfrak{d}(s)\tilde{x}(0).$$

Due to Theorem 3.10, the solution of (4.20) for the case when x_0 is replaced with x_0^s is given by $\tilde{x}^s(t)=\mathfrak{d}(s)\tilde{x}(e^{\nu s}t)$. Hence, the solution of (4.14) with the scaled initial condition $x(0)=\mathfrak{d}(s)x_0$ has the form

$$x_v^s(t)=\mathfrak{d}(\ln\|\tilde{x}^s(t)\|_{\mathfrak{d}_n})\sum_{i=1}^n[\mathfrak{d}_n(-\ln\|\tilde{x}^s(t)\|_{\mathfrak{d}_n})\tilde{x}^s(t)]_ih_i=.$$

$$\mathfrak{d}(s)\mathfrak{d}(\ln\|\tilde{x}(e^{\nu s}t)\|_{\mathfrak{d}_n})\sum_{i=1}^n[\mathfrak{d}_n(-\ln\|\tilde{x}(e^{\nu s}t)\|_{\mathfrak{d}_n})\tilde{x}(e^{\nu s}t)]_ih_i=\mathfrak{d}(s)x_v(e^{\nu s}t).$$

■

The obtained finite-dimensional projection of the nonlinear evolution equation (3.6) preserve some stability properties of the original system. Indeed, if

$$\exists\rho\geq 0:\quad\langle A\phi+f(\phi),\phi\rangle\leq-\rho\langle G_{\mathfrak{d}}\phi,\phi\rangle,\quad\forall\phi\in\mathcal{D}(A)\cap\mathcal{D}(G_{\mathfrak{d}}):\|\phi\|=1$$

then we have

$$\langle A\phi_v+f(\phi_v),\phi_v\rangle=\tilde{\phi}^\top A_n\tilde{\phi}+\tilde{\phi}^\top f(\tilde{\phi})\leq-\rho\langle G_{\mathfrak{d}}\phi_v,\phi_v\rangle=-\rho\tilde{\phi}^\top G_{d_n}\tilde{\phi},\quad\forall\tilde{\phi}\in S_{\mathbb{R}^n},$$

where $\phi_v=\sum_{i=1}^n\tilde{\phi}_ih_i\in S_V$. The latter means

$$\frac{d}{dt}\|\tilde{x}(t)\|_{\mathfrak{d}_n}\leq-\rho\|\tilde{x}(t)\|_{\mathfrak{d}_n}^{\nu+1},$$

i.e. the canonical homogeneous norm is the Lyapunov function for the system (4.20). Due to the formula (3.3) the inequality $\langle A\phi+f(\phi),\phi\rangle\leq-\rho,\forall\phi\in\mathcal{D}(A)\cap\mathcal{D}(G_{\mathfrak{d}}):\|\phi\|=1$ implies

$$\frac{d}{dt}\|x(t)\|_{\mathfrak{d}}\overset{a.e.}{\leq}-\rho\|x(t)\|_{\mathfrak{d}}^{\nu+1}$$

for any strong solution x of (3.6), i.e. the canonical homogeneous norm $\|\cdot\|_{\mathfrak{d}}$ is the Lypaunov function of the original system (3.6). Moreover, the convergence rates of the original and the projected systems are identical, namely, both systems are finite-time stable if $\nu < 0$; both systems are exponentially stable if $\nu = 0$; both systems are nearly fixed-time stable if $\nu > 0$ (see e.g. [25, Chapter 8] for more details about convergence rates of the homogeneous systems).

5. Example: Homogeneous Galerkin Projection of Burgers Equation. For comparison reasons, let us consider the Burgers equation (4.1). This equation can be the represented in the form (3.6) with $A = \frac{\partial^2}{\partial z^2}$, $x(t), x_0 \in \mathbb{H} = L^2(\mathbb{R},\mathbb{R})$, $\mathcal{D}(A) = H^2(\mathbb{R},\mathbb{R})$, $f(x) = -x\frac{\partial x}{\partial z}$ and $\mathcal{D}(f) = \mathbb{Y} = H^1(\mathbb{R},\mathbb{R})$. Notice that f is M-regular with $M = \frac{\partial}{\partial z} + I$ and $(M^{-1}y)(z) = \int_{-\infty}^{z} e^{-(z-s)}y(s)ds$, $y \in \mathbb{B}, z \in \mathbb{R}$, respectively. The considered system satisfies Assumption 1 with $(\Phi(t)y)(z) = \frac{1}{\sqrt{4\pi t}}\int_{\infty}^{+\infty} e^{-\frac{(z-s)^2}{4t}} y(s)ds, y \in \mathbb{B}$. Moreover, it admits the explicit solution

$$[x(t)](z) = -2\frac{\partial}{\partial z}\ln\left\{\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{(z-\sigma)^2}{4t}-\frac{1}{2}\int_0^\sigma x_0(s)ds}d\sigma\right\} \tag{5.1}$$

due to the Hopf-Cole transformation. This exact solution is utilized below in order to compare the approximation precision of the classical and homogeneous Galerkin method.

The system (4.1) is $\mathfrak{d}$-homogeneous of the degree $\nu = 2$ with respect to the dilation $(\mathfrak{d}(s)x)(z) = e^s x(e^s z)$, $z \in \mathbb{R}, x \in \mathbb{B}$. The dilation $\mathfrak{d}$ is strictly monotone and strongly continuous in $\mathbb{B}$ and $\mathbb{Y}$ (see Example 1).

Taking the Hermite functions (4.9) as an orthonormal basis $h_1, h_2, ..., h_n$ we derive the following homogeneous Galerkin projection of the Burgers equation

$$\begin{cases} \frac{d\tilde{x}(t)}{dt} = \|\tilde{x}(t)\|_{\mathfrak{d}_n}^{\nu}\mathfrak{d}_n(\ln\|\tilde{x}(t)\|_{\mathfrak{d}})\left(\tilde{A}_n + B\left(\mathfrak{d}_n(-\ln\|\tilde{x}(t)\|_{\mathfrak{d}_n})\tilde{x}(t)\right)\right)\mathfrak{d}_n(-\ln\|\tilde{x}\|_{\mathfrak{d}_n})\tilde{x}(t), \\ \tilde{x}(0) = \mathfrak{d}_n(\ln\|x_0\|_{\mathfrak{d}})\Pi_n\mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0, \quad t > 0, \end{cases} \tag{5.2}$$

where $G_n$and $\mathfrak{d}_n$ are defined in Example 4.5, $A_n \in \mathbb{R}^n$ with the elements

$$(A_n)_{ij} = \langle Ah_i, h_j\rangle = \left\langle \frac{d^2}{dy^2}h_i, h_j\right\rangle,$$

and the linear matrix-valued function $\tilde{x} \to B(\tilde{x}) \in \mathbb{R}^{n\times n}$ is defined as follows

$$[B(\tilde{x})]_{i,j} = \frac{1}{2}\sum_{k=1}^{n}\tilde{x}_k\left\langle h_k h_j, \frac{\partial h_i}{\partial y}\right\rangle, \quad \tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_k)^\top \in \mathbb{R}^n, \quad i,j = 1,2,...,n.$$

The matrix valued function $\tilde{x} \to B(\tilde{x}), \tilde{x} \in \mathbb{R}^n$ is obtained from (4.19) taking into account the identity

$$-\left\langle\left(\sum_{k=1}^n \tilde{x}_k h_k\right)\frac{\partial}{\partial z}\left(\sum_{k=1}^n \tilde{x}_k h_k\right), h_i\right\rangle = \frac{1}{2}\left\langle\left(\sum_{k=1}^n \tilde{x}_k h_k\right)^2, \frac{\partial}{\partial z}h_i\right\rangle,$$

which comes from the integration by parts. Using the identity

$$\frac{d^2}{dz^2}h_i(z) = -(2i-1)h_i(z) + y^2 h_i(z), \quad z \in \mathbb{R},$$

we derive

$$(A_n)_{ij} = -(2i-1)\delta_{ij} + \langle yh_i, yh_j \rangle ,$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. From the identity

$$zh_j(z) = \sqrt{\frac{j-1}{2}} h_{j-1}(z) + \sqrt{\frac{j}{2}} h_{j+1}(z), \quad z \in \mathbb{R}$$

we conclude

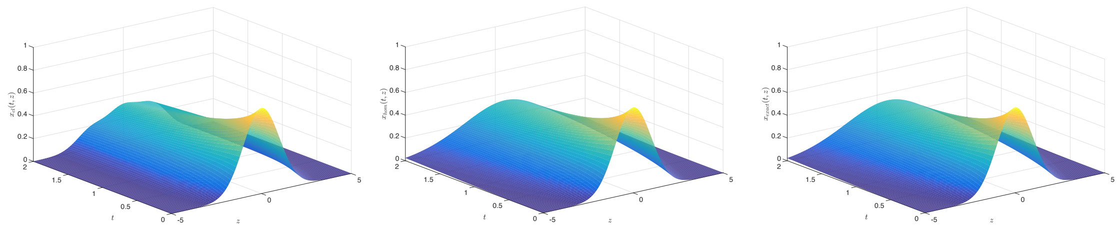
$$(A_n)_{ij} = -(i-1/2)\delta_{ij} + \sqrt{\frac{i-1}{2}}\sqrt{\frac{j}{2}}\delta_{i-1\,j+1} + \sqrt{\frac{i}{2}}\sqrt{\frac{j-1}{2}}\delta_{i+1\,j-1}.$$

Notice that the classical Galerking projection gives the following ODE

$$\begin{cases} \frac{d\tilde{x}^{cl}(t)}{dt} = \left(\tilde{A}_n + B\left(\tilde{x}^{cl}(t)\right)\right)\tilde{x}^{cl}(t), \quad t > 0, \\ \tilde{x}^{cl}(0) = \Pi_n x_0, \end{cases} \tag{5.3}$$

which approximates the solution for the Burgers equation as follows

$$x_{cl}(t) = \sum_{i=1}^{n} \tilde{x}^{cl}_i(t) h_i, t \geq 0.$$



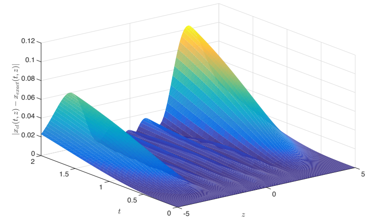
(a) Classical Galerkin (b) Homogeneous Galerkin (c) Exact solution

Figure 5.1: Approximate and exact solutions for $x_0 = h_1$

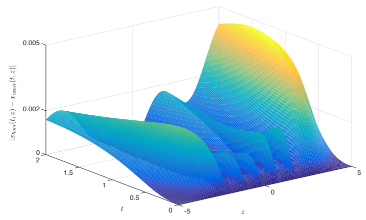
Figures 5.1 and 5.2 presents the comparison results of the exact solution x_{exact} of the Burgers equation (given by (5.1)) with the approximate solutions x_{cl} and

$$x_{hom}(t) = \mathfrak{d}(\ln \|\tilde{x}(t)\|_{\mathfrak{d}_n}) \sum_{i=1}^{n} [\mathfrak{d}_n(-\ln \|\tilde{x}(t)\|_{\mathfrak{d}_n})\tilde{x}(t)]_i h_i,$$

obtained by means of the classical and the homogeneous Galerkin methods, respectively. The initial condition was selected as follows $x_0 = h_1 \in V$. The number of basis functions was identical for both classical and homogeneous Galerkin projections: $n = 5$.



(a) The classical Galerkin method



(b) The homogeneous Galerkin method

Figure 5.2: Approximation errors for $x_0 = h_1$

In both cases, the numerical simulations were made using semi-implicit Euler scheme with the time step $\tau = 0.01$:

$$\frac{\tilde{x}_{cl}^{[p+1]} - \tilde{x}_{cl}^{[p]}}{\tau} = \left(\tilde{A}_n + B\left(\tilde{x}_{cl}^{[p]}\right)\right)\tilde{x}_{cl}^{[p+1]}, \quad \tilde{x}_{cl}^{[0]} = \Pi_n x_0, \quad p = 0, 1, \ldots$$

In the view of the results [28], the semi-implicit discretization is more appropriate for homogeneous systems with positive degrees. The equivalent representation (4.17) of the system (5.2) is utilized for numerical simulations, since it is computationally less consuming:

$$\begin{cases} \frac{\tilde{\phi}^{[p+1]}-\tilde{\phi}^{[p]}}{\tau} = \left(\tilde{r}^{[p]}\right)^{\nu}\left(\tilde{A}_n + B\left(\tilde{\phi}^{[p]}\right) - \frac{(\tilde{\phi}^{[p]})^{\top}\left(\tilde{A}_n + B\left(\tilde{\phi}^{[p]}\right)\right)\tilde{\phi}^{[p]}}{(\tilde{\phi}^{[p]})^{\top} G_{\mathfrak{d}_n}\tilde{\phi}^{[p]}} G_{\mathfrak{d}_n}\right)\tilde{\phi}^{[p+1]}, \\ \frac{\tilde{r}^{[p+1]}-\tilde{r}^{[p]}}{\tau} = \left(\tilde{r}^{[p]}\right)^{\nu} \frac{(\tilde{\phi}^{[p]})^{\top}\left(\tilde{A}_n + B\left(\tilde{\phi}^{[p]}\right)\right)\tilde{\phi}^{[p]}}{(\tilde{\phi}^{[p]})^{\top} G_{\mathfrak{d}_n}\tilde{\phi}^{[p]}} \tilde{r}^{[p+1]}, & p = 0, 1, \ldots \\ \tilde{\phi}^{[0]} = \Pi_n \mathfrak{d}(-\ln\|x_0\|_{\mathfrak{d}})x_0, \quad \tilde{r}^{[0]} = \|x_0\|_{\mathfrak{d}}. \end{cases}$$

Since $\tilde{\phi}(t) \in S_{\mathbb{R}^n}, \forall t \geq 0$, then $\phi^{[p+1]}$ is projected on the unit sphere $S_{\mathbb{R}^n}$ on each discretization step. Therefore, $\tilde{x}^{[p]} = \mathfrak{d}(\ln \tilde{r}^{[p]})\tilde{\phi}^{[p]}$, where $\tilde{x}^{[p]}$ is the discrete-time approximation of $\tilde{x}(t)$.

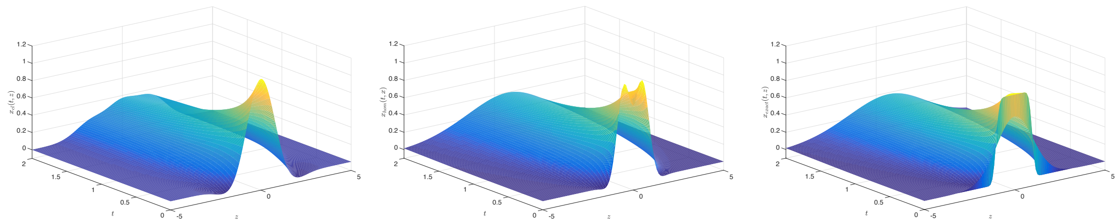
The simulations for $x_0 = h_1$ show that the homogeneous Galerkin method has 23 times better precision in L^{∞} norm and 12 times better precision in L^2-norm than the classical one. Notice that such a good approximate solution was obtained by the homogeneous Galerkin method with just 5 basis functions! To obtain the similar precision on $(t, z) \in [0, 2] \times [-5, 5]$, the classical Galerkin method needs at least 20 basis functions.

Figures 5.3 and 5.4 present comparison results for the initial condition $x_0 = \xi \notin \mathbb{V}$:

$$\xi(z) = \begin{cases} 1 & \text{if} \quad |z| \leq 1, \\ 0 & \text{if} \quad |z| > 1, \end{cases} \qquad z \in \mathbb{R}.$$

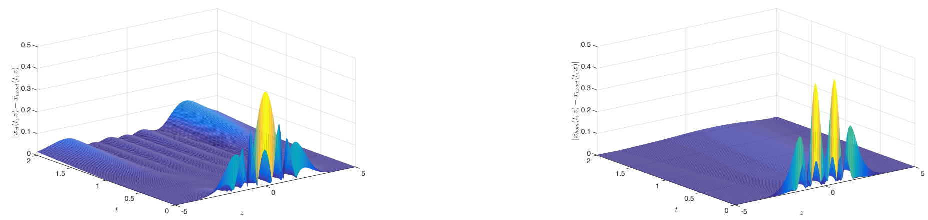
The homogeneous Galerkin projection method again shows much better approximation precision than the classical one.

6. Conclusions. In the paper, a new Galerkin-type projection method for a class of homogeneous evolution equations with possibly unbounded nonlinear operators in the right-hand sides is introduced. The proposed method preserves the dilation symmetry of a system in its finite-dimensional projection. The latter allows the stability and convergence rates properties



(a) Classical Galerkin (b) Homogeneous Galerkin (c) Exact solution

Figure 5.3: Approximate and exact solutions for $x_0 = \xi$



(a) The classical Galerkin method (b) The homogeneous Galerkin method

Figure 5.4: Approximation errors for $x_0 = \xi$

of the infinite dimensional system to be preserved as well. The numerical simulations for the Burgers equation have shown significant improvement of the approximation precision of the homogeneous Galerkin projection in comparison with the classical Galerkin projection. For control systems analysis and design it is also important to know if some other useful properties of homogeneous systems (such as input-to-state stability [1], [2]) can be preserved by means of homogeneous Galerkin projection. This is an interesting problem for future research.

7. Acknowledgments. The work is partially supported by ANR DIGITSLID (ANR-18-CE40-0008).

References.

[1] V. Andrieu, L. Praly, and A. Astolfi. Homogeneous Approximation, Recursive Observer Design, and Output Feedback. *SIAM Journal of Control and Optimization*, 47(4):1814–1850, 2008.

[2] E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti. Verification of ISS, iISS and IOSS properties applying weighted homogeneity. *System & Control Letters*, 62(12):1159–1167, 2013.

[3] S. P. Bhat and D. S. Bernstein. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17:101–127, 2005.

[4] A. Bihlo, J. Jackaman, and F. Valiquette. On the development of symmetry-preserving finite element schemes for ordinary differential equations. *Journal of Computational Dynamics*, 7(2):339–368, 2020.

[5] A. Bihlo and Francis V. Symmetry-preserving numerical schemes. In *Symmetries and Integrability of Difference Equations*, pages 261–324. Springer, 2017.
[6] A.M. Davies. Application of the galerkin method to the solution of burgers' equation. *Computer Methods in Applied Mechanics and Engineering*, 14(3):305–321, 1978.
[7] V. A. Dorodnitsyn. Transformation groups in net spaces,. *Journal of Soviet Mathematics*, 55(1):1490–1517, translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya 43 (1989), 149–191, 1991.
[8] B.K. Driver. *Analysis Tools with Applications*. Springer, 2003.
[9] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer Verlag: Berlin, Heidelberg, New York, 2000.
[10] V. Fischer and M. Ruzhansky. *Quantization on Nilpotent Lie Groups*. Springer, 2016.
[11] G. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Arkiv for Matematik*, 13(1-2):161–207, 1975.
[12] L. Grune. Homogeneous state feedback stabilization of homogeneous systems. *SIAM Journal of Control and Optimization*, 38(4):1288–1308, 2000.
[13] Wanner G. Hairer, E. and C. Lubich. Geometric numerical integration: Structure-preserving algorithms for ordinary differential equations. In *Springer Series in Computational Mathematics*, volume 31. Springer-Verlag Berlin Heidelberg, 2006.
[14] H. Hermes. Nilpotent approximations of control systems and distributions. *SIAM Journal of Control and Optimization*, 24(4):731, 1986.
[15] L.S. Husch. Topological Characterization of The Dilation and The Translation in Frechet Spaces. *Mathematical Annals*, 190:1–5, 1970.
[16] M. Kawski. Homogeneous stabilizing feedback laws. *Control Theory and Advanced Technology*, 6(4):497–516, 1990.
[17] M. Kawski. Families of dilations and asymptotic stability. *Analysis of Controlled Dynamical Systems*, pages 285–294, 1991.
[18] V. V. Khomenuk. On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica (in Russian).*, 3(22):157–164, 1961.
[19] A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41(5):823–830, 2005.
[20] F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time observer design: Implicit Lyapunov function approach. *Automatica*, 87(1):52–60, 2018.
[21] H. Nakamura, Y. Yamashita, and H. Nishitani. Smooth Lyapunov functions for homogeneous differential inclusions. In *Proceedings of the 41st SICE Annual Conference*, pages 1974–1979, 2002.
[22] Y. Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal of Control and Optimization*, 43(4):1253–1271, 2005.
[23] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.
[24] A. Polyakov. Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, 29(3):682–701, 2018.
[25] A. Polyakov. *Generalized Homogeneity in Systems and Control*. Springer, 2020.

[26] A. Polyakov. Homogeneous Lyapunov Functions for Homogeneous Infinite Dimensional Systems with Unbounded Nonlinear Operators. *Systems & Control Letters*, (submitted)(https://hal.inria.fr/hal-02921426), 2020.

[27] A. Polyakov, J.-M. Coron, and L. Rosier. On homogeneous finite-time control for linear evolution equation in hilbert space. *IEEE Transactions on Automatic Control*, 63(9):3143–3150, 2018.

[28] A. Polyakov, D. Efimov, and B. Brogliato. Consistent discretization of finite-time and fixed-time stable systems. *SIAM Journal of Control and Optimization*, 57(1):78–103, 2019.

[29] A. Polyakov, D. Efimov, E. Fridman, and W. Perruquetti. On homogeneous distributed parameters equations. *IEEE Transactions on Automatic Control*, 61(11):3657–3662, 2016.

[30] L. Praly. Generalized weighted homogeneity and state dependent time scale for linear controllable systems. In *Proc. IEEE Conference on Decision and Control*, pages 4342–4347, San Diego, USA, 1997.

[31] G.R.W. Quispel and D.I. McLaren. A new class of energy-preserving numerical integration methods,. *Journal of Physics A: Mathematical and Theoretical*, 41:045206, 2008.

[32] L. Rosier. Etude de quelques problèmes de stabilization. *PhD Thesis, Ecole Normale Superieure de Cachan (France)*, 1993.

[33] V.I. Zubov. On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica (in Russian).*, 1:80–88, 1958.