



HAL
open science

High-order DG-SE approximations of elastodynamic wave problems with stable Perfectly Matched Layers

Hélène Barucq, Henri Calandra, Aurélien Citrain, Julien Diaz, Christian Gout

► To cite this version:

Hélène Barucq, Henri Calandra, Aurélien Citrain, Julien Diaz, Christian Gout. High-order DG-SE approximations of elastodynamic wave problems with stable Perfectly Matched Layers. MATHIAS 2019, Oct 2019, Serris, France. hal-02425960

HAL Id: hal-02425960

<https://inria.hal.science/hal-02425960v1>

Submitted on 7 Jan 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

High-order DG-SE approximations of elastodynamic wave problems with stable Perfectly Matched Layers[†]

Hélène Barucq¹, Henri Calandra², Aurélien Citrain^{3,1}, Julien Diaz¹ and Christian Gout³

¹ Team project Magique.3D, INRIA, E2S UPPA, CNRS, Pau, France.

² TOTAL SA, CSTJF, Pau, France.

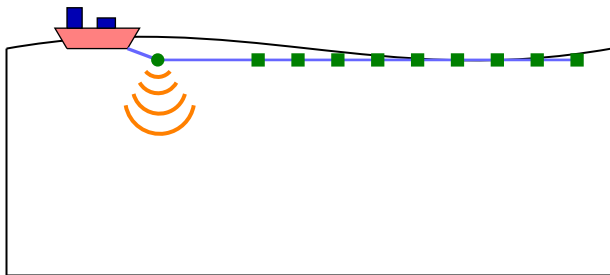
³ LMI, Normandie Université, INSA ROUEN, 76000 Rouen, France.

Mathias 2019

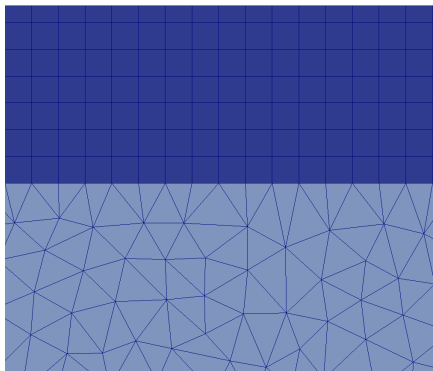
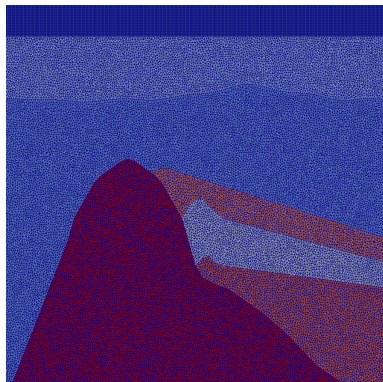


The authors thank the M2NUM project which is co-financed by the European Union with the European regional development fund (ERDF, HN0002137) and by the Normandie Regional Council.

[†] *This work is dedicated to the memory of Dimitri Komatitsch.*

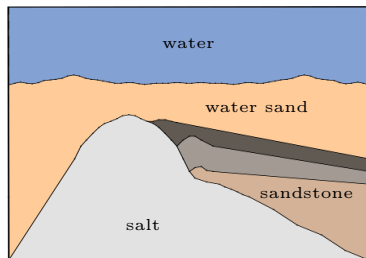
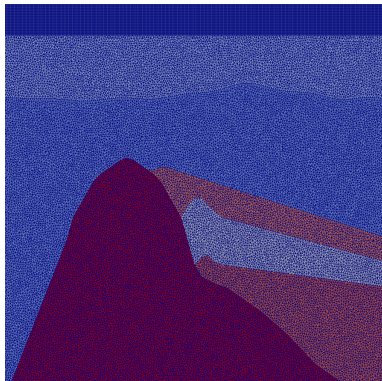


Why using hybrid meshes?



- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water).
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.

Why using hybrid meshes?



- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water).
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.

$x \in \Omega \subset \mathbb{R}^d$, $t \in [0, T]$, $T > 0$:

$$\begin{cases} \rho(x) \frac{\partial v}{\partial t}(x, t) &= \nabla \cdot \underline{\underline{\sigma}}(x, t), \\ \frac{\partial \underline{\underline{\sigma}}}{\partial t}(x, t) &= \underline{\underline{C}}(x) \underline{\underline{\epsilon}}(v(x, t)). \end{cases}$$

With:

- $\rho(x)$ the density,
- $\underline{\underline{C}}(x)$ the elasticity tensor,
- $\underline{\underline{\epsilon}}(x, t)$ the deformation tensor,
- $v(x, t)$, the wavespeed,
- $\underline{\underline{\sigma}}(x, t)$ the strain tensor.

Software written in **Fortran** for wave propagation simulation in the **time domain**

Features

Simulation:

- on various types of meshes (**unstructured triangles and tetrahedra**),
 - on **heterogeneous media (acoustic, elastic and elasto-acoustic)**.
-
- **Discontinuous Galerkin (DG)** based on **unstructured triangles and unstructured tetrahedra**,
 - with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**,
 - with **multi-order** computation(**p-adaptivity**)...

- 1 DGm and SEM
- 2 DG/SEM coupling
- 3 Perfectly Matched Layer(PML)

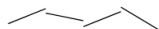
Use discontinuous functions :



mesh

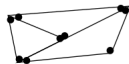


continuous

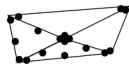


discontinuous

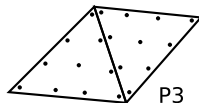
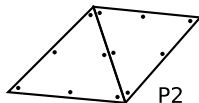
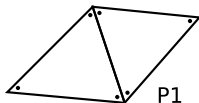
h adaptivity :



p adaptivity :

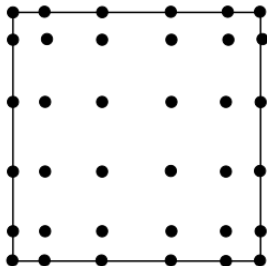


Degrees of freedom necessary on each cell :



General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature,
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on L^2 -norm).



- $\int f(x) dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j),$
- $\varphi_i(\xi_j) = \delta_{ij}.$

DG

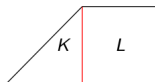
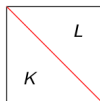
- Element per element computation (*hp*-adaptivity).
- Time discretization quasi explicit (block diagonal mass matrix).
- Simple to parallelize.
- Robust to brutal changes of physics and geometry

SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).

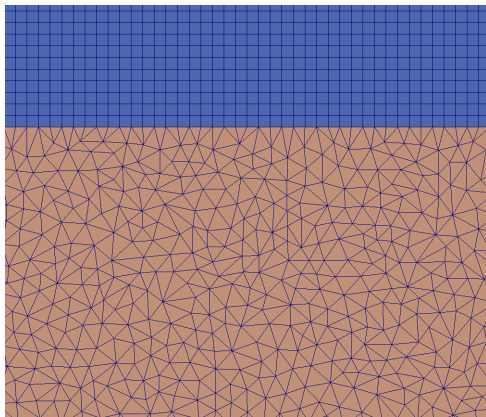
- Aim at coupling P_k and Q_k structures.
- Need to extend or split some structures (e.g. neighbour indices).
- Define new face matrices:

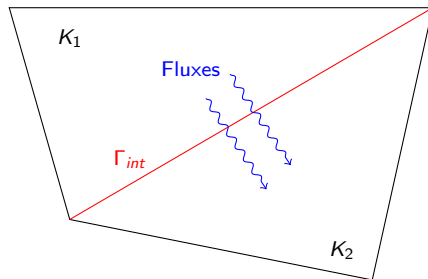
$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L.$$



Global context

- Domain in two parts : $\Omega_{h,1}$ (**structured quadrangles + SEM**), $\Omega_{h,2}$ (**unstructured triangles + DG**).





Definitions

- Jump and average.

$$[[u]] = (u_{K_1} \mathbf{n}_{K_1} + u_{K_2} \mathbf{n}_{K_2})$$

$$\{\{u\}\} = \frac{1}{2}(u_{K_2} + u_{K_1})$$

SEM variational formulation :

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 : \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1, \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = \int_{\Omega_{h,1}} (C \xi_1) : \nabla v_1. \end{cases}$$

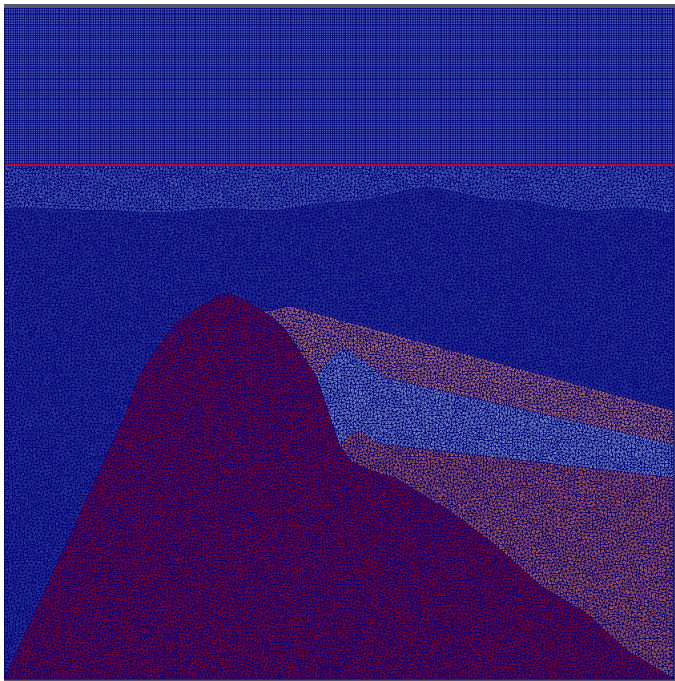
DG variational formulation :

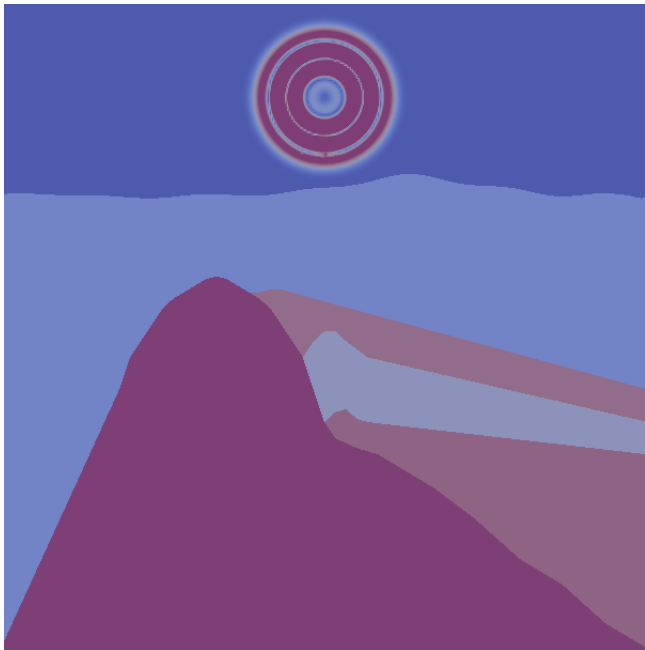
$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 : \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\} : [[w_2]], \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla \cdot (C \xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C \xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\} \cdot [[C \xi_2]]. \end{cases}$$

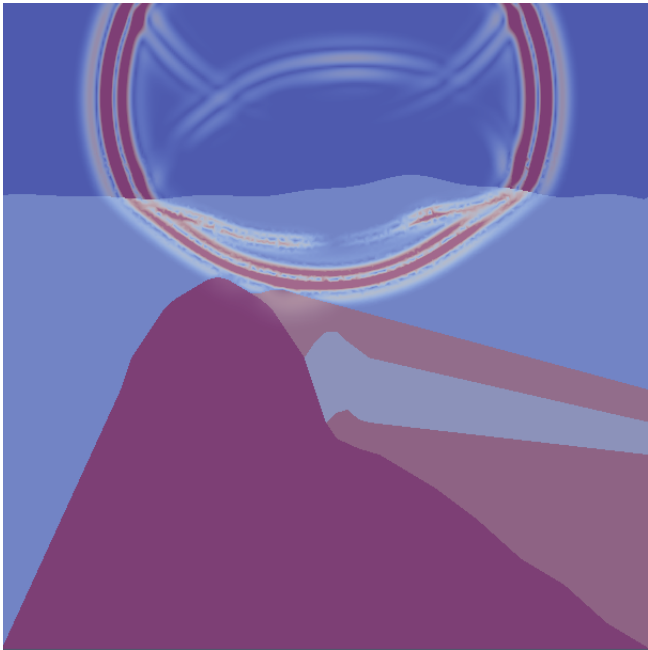
$$\left\{ \begin{aligned}
 & \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,1}} \sigma_1 : \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 : \nabla w_2 \\
 & + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} : [[w_2]] \\
 & + \int_{\Gamma_{1/2}} [[\sigma w]], \\
 & \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = \int_{\Omega_{h,1}} (C \xi_1) : \nabla v_1 - \int_{\Omega_{h,2}} (\nabla \cdot (C \xi_2)) \cdot v_2 \\
 & + \int_{\Gamma_{out,2}} (C \xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} \cdot [[C \xi_2]] \\
 & + \int_{\Gamma_{1/2}} [[(C \xi) v]].
 \end{aligned} \right.$$

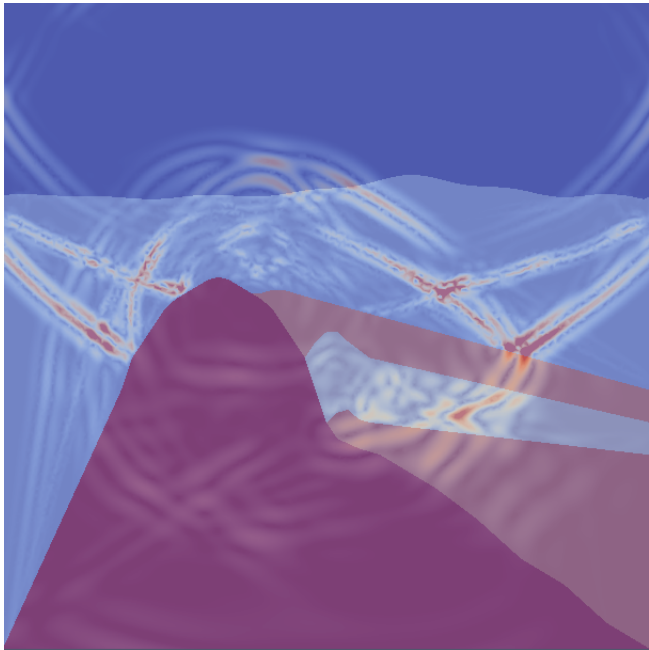
$$\left\{ \begin{aligned}
 & \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,1}} \sigma_1 : \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 : \nabla w_2 \\
 & + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\} : [[w_2]] \\
 & + \int_{\Gamma_{1/2}} \{\{\sigma\}\} : [[w]] + [[\sigma]] \cdot \{\{w\}\}, \\
 & \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = \int_{\Omega_{h,1}} (C \xi_1) : \nabla v_1 - \int_{\Omega_{h,2}} (\nabla \cdot (C \xi_2)) \cdot v_2 \\
 & + \int_{\Gamma_{out,2}} (C \xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\} \cdot [[C \xi_2]] \\
 & + \int_{\Gamma_{1/2}} \{\{v\}\} \cdot [[C \xi]] + \{\{C \xi\}\} : [[v]].
 \end{aligned} \right.$$

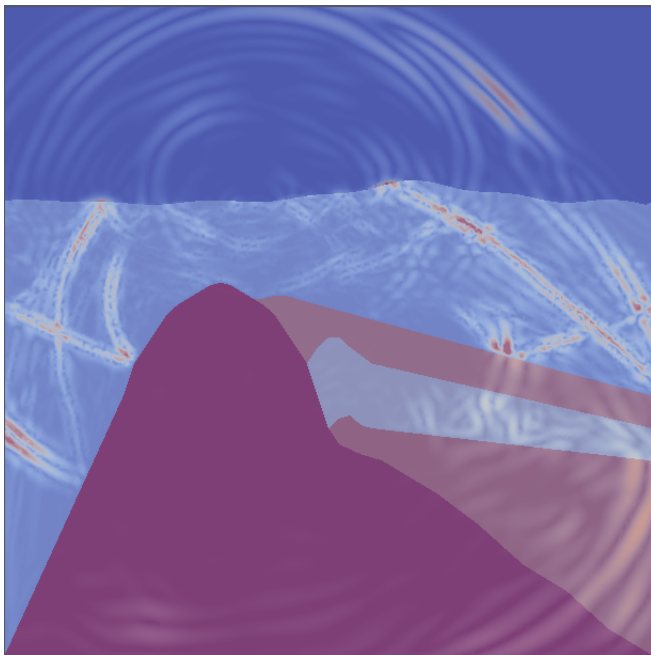
$$\left\{ \begin{aligned}
 & \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,1}} \sigma_1 : \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 : \nabla w_2 \\
 & + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} : [[w_2]] \\
 & + \int_{\Gamma_{1/2}} \{ \{ \sigma \} \} : [[w]] \quad \color{red} \cancel{+ [[\sigma]] \cdot \{ \{ w \} \}}, \\
 & \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = \int_{\Omega_{h,1}} (C \xi_1) : \nabla v_1 - \int_{\Omega_{h,2}} (\nabla \cdot (C \xi_2)) \cdot v_2 \\
 & + \int_{\Gamma_{out,2}} (C \xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} \cdot [[C \xi_2]] \\
 & + \int_{\Gamma_{1/2}} \{ \{ v \} \} \cdot [[C \xi]] \quad \color{red} \cancel{+ \{ \{ C \xi \} \} : [[v]]}.
 \end{aligned} \right.$$

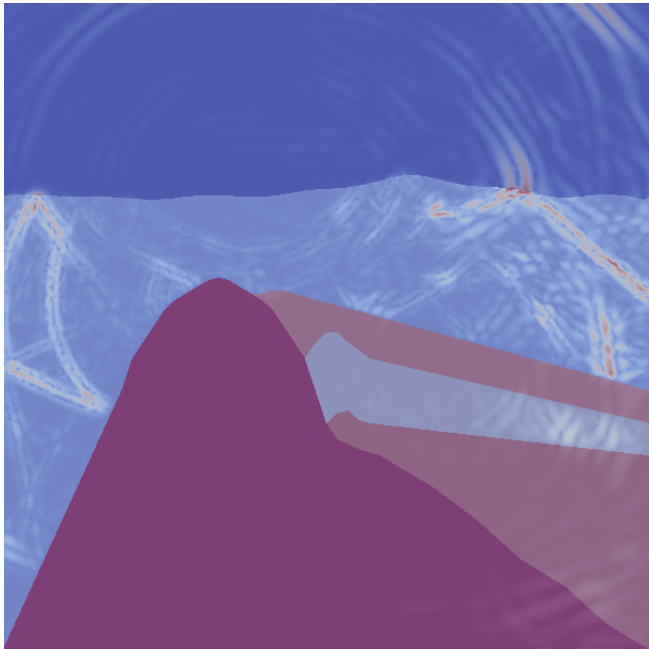




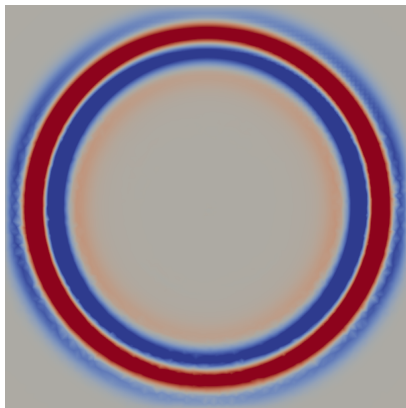




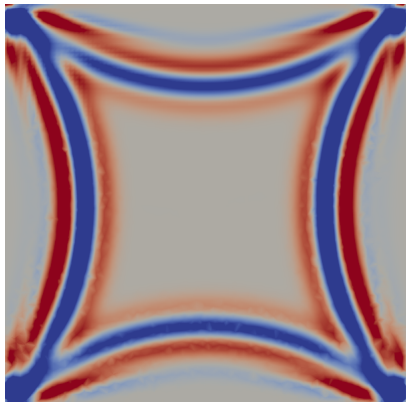




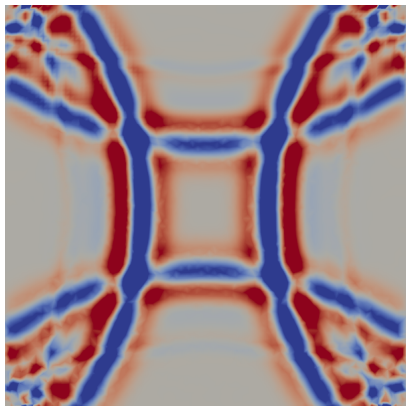
What is a PML and why using it?



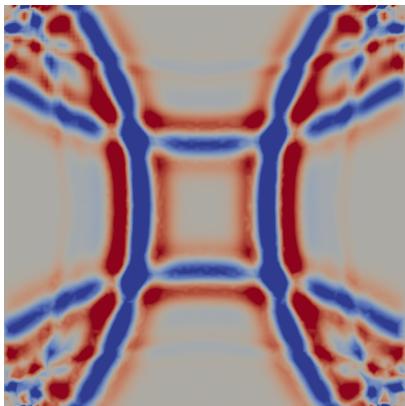
What is a PML and why using it?



What is a PML and why using it?

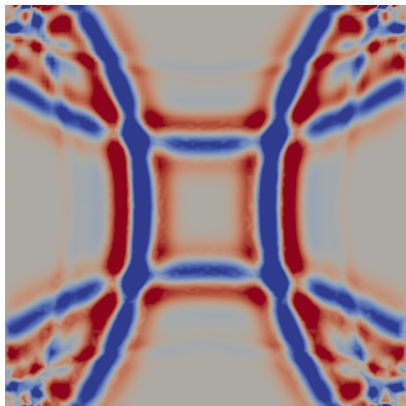


What is a PML and why using it?



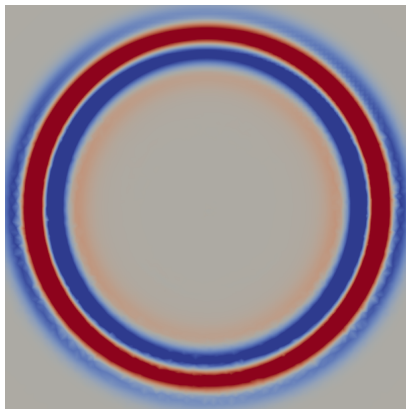
- Easy to implement

What is a PML and why using it?



- Easy to implement
- Not really adapted to the simulation of "infinite" domain

What is a PML and why using it?



What is a PML and why using it?



What is a PML and why using it?



What is a PML and why using it?



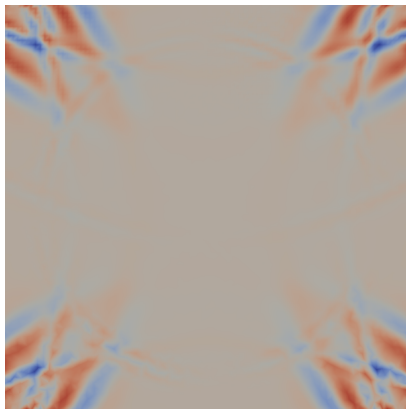
- Possibility to simulate an infinite domain

What is a PML and why using it?



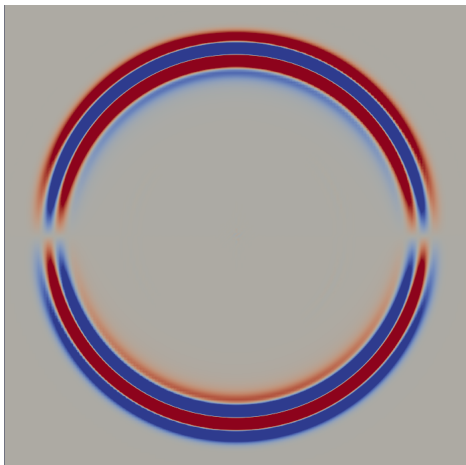
- Possibility to simulate an infinite domain
- Complicated to implement when the order of the condition increase

What is a PML and why using it?

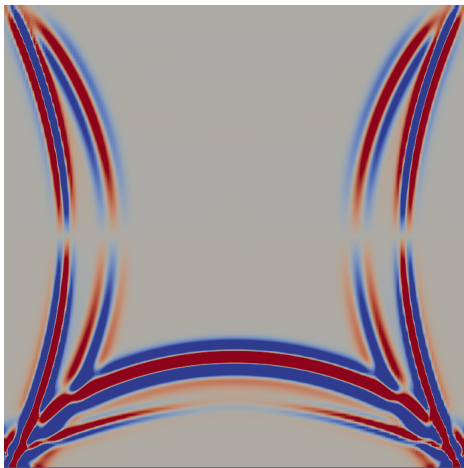


- Possibility to simulate an infinite domain
- Complicated to implement when the order of the condition increase
- Apparition of reflection when it comes to waves with grazing incidence

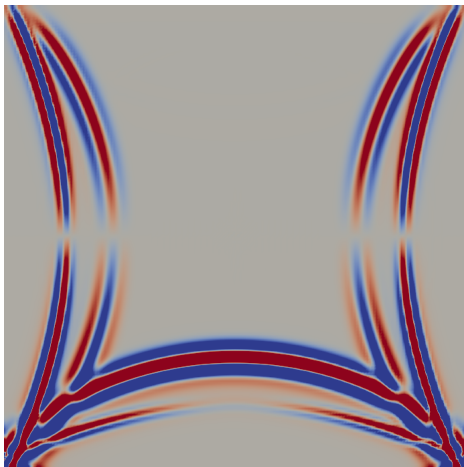
What is a PML and why using it?



What is a PML and why using it?

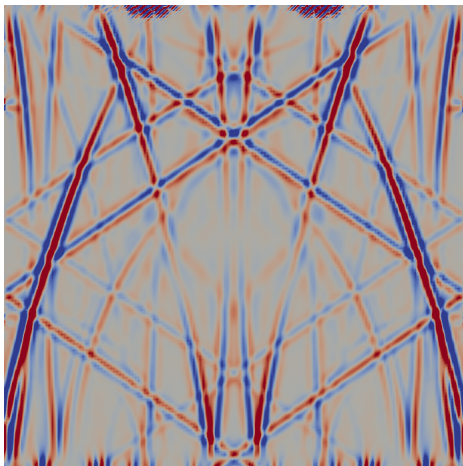


What is a PML and why using it?



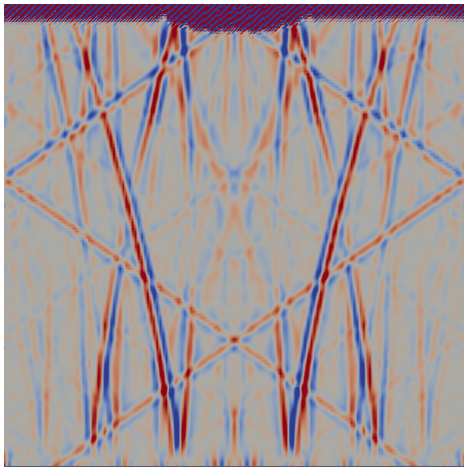
- Possibility to simulate an infinite domain

What is a PML and why using it?



- Possibility to simulate an infinite domain

What is a PML and why using it?



- Possibility to simulate an infinite domain
- Some stability problems on elastic case using DGM

Step 1: Rewrite the system in the frequency domain

$$\begin{cases} i\omega\rho\mathbf{v} = \nabla \cdot \underline{\underline{\sigma}} \\ i\omega\underline{\underline{\sigma}} = \underline{\underline{C}} : \nabla\mathbf{v} \end{cases}$$

Step 1: Rewrite the system in the frequency domain

$$\begin{cases} i\omega \rho \mathbf{v} = \nabla \cdot \underline{\underline{\sigma}} \\ i\omega \underline{\underline{\sigma}} = \underline{\underline{C}} : \nabla \mathbf{v} \end{cases}$$

Step 2: Introduce a new system of complex coordinates

Define d_z the damping :

$$\begin{cases} d_z > 0 & \text{if } z \in \Omega_{PML} \\ d_z = 0 & \text{if } z \notin \Omega_{PML} \end{cases}$$

Step 2: Introduce a new system of complex coordinates

Define d_z the damping :

$$\begin{cases} d_z > 0 & \text{if } z \in \Omega_{PML} \\ d_z = 0 & \text{if } z \notin \Omega_{PML} \end{cases}$$

Step 2: Introduce a new system of complex coordinates

Define d_z the damping :

$$\begin{cases} d_z > 0 & \text{if } z \in \Omega_{PML} \\ d_z = 0 & \text{if } z \notin \Omega_{PML} \end{cases}$$

$$\tilde{z}(z) = z - \frac{i}{\omega} \int_0^z d_z(s) ds$$

Step 2: Introduce a new system of complex coordinates

Define d_z the damping :

$$\begin{cases} d_z > 0 & \text{if } z \in \Omega_{PML} \\ d_z = 0 & \text{if } z \notin \Omega_{PML} \end{cases}$$

$$\tilde{z}(z) = z - \frac{i}{\omega} \int_0^z d_z(s) ds$$

Define the associate differential operator :

$$\partial_{\tilde{z}} = \frac{i\omega}{i\omega + d_z} \partial_z = 1 - \frac{d_z}{i\omega + d_z} \partial_z$$

Step 3: Rewrite the system

$$\begin{aligned}i\omega\rho v_x &= \partial_x\sigma_{xx} + \partial_{\bar{z}}\sigma_{xz}, \\i\omega\rho v_z &= \partial_x\sigma_{xz} + \partial_{\bar{z}}\sigma_{zz}, \\i\omega\sigma_{xx} &= (\lambda + 2\mu)\partial_x v_x + \lambda\partial_{\bar{z}} v_z, \\i\omega\sigma_{zz} &= \lambda\partial_x v_x + (\lambda + 2\mu)\partial_{\bar{z}} v_z, \\i\omega\sigma_{xz} &= \mu(\partial_x v_z + \partial_{\bar{z}} v_x).\end{aligned}$$



Martin, Roland and Komatitsch, Dimitri and Gedney, Stephen D and Bruthiaux, Emilien and others

A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of $2^d + 1$ new variables $\psi_\star \in H^1 \rightarrow$ Simplify the implementation BUT add new equation to the linear system



Martin, Roland and Komatitsch, Dimitri and Gedney, Stephen D and Bruthiaux, Emilien and others

A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of $2^d + 1$ new variables $\psi_\star \in H^1 \rightarrow$ Simplify the implementation BUT add new equation to the linear system

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz}$$



Martin, Roland and Komatitsch, Dimitri and Gedney, Stephen D and Bruthiaux, Emilien and others

A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of $2^d + 1$ new variables $\psi_\star \in H^1 \rightarrow$ Simplify the implementation BUT add new equation to the linear system

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz}$$



Back to original coordinates



Martin, Roland and Komatitsch, Dimitri and Gedney, Stephen D and Bruthiaux, Emilien and others

A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of $2^d + 1$ new variables $\psi_* \in H^1 \rightarrow$ Simplify the implementation BUT add new equation to the linear system

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz}$$

↓

Back to original coordinates

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \frac{d_z}{d_z + i\omega} \partial_z \sigma_{zz}$$



Martin, Roland and Komatitsch, Dimitri and Gedney, Stephen D and Bruthiaux, Emilien and others

A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of $2^d + 1$ new variables $\psi_* \in H^1 \rightarrow$ Simplify the implementation BUT add new equation to the linear system

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz}$$

↓

Back to original coordinates

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \frac{d_z}{d_z + i\omega} \partial_z \sigma_{zz}$$

↓

Back to time domain



Martin, Roland and Komatitsch, Dimitri and Gedney, Stephen D and Bruthiaux, Emilien and others

A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of $2^d + 1$ new variables $\psi_* \in H^1 \rightarrow$ Simplify the implementation BUT add new equation to the linear system

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz}$$

↓ Back to original coordinates

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \frac{d_z}{d_z + i\omega} \partial_z \sigma_{zz}$$

↓ Back to time domain

$$\rho \partial_t v_x = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \frac{d_z}{d_z + \partial_t} \partial_z \sigma_{zz}$$

Definition of the memory variable $\psi_{\sigma_{zz}}$:

$$\psi_{\sigma_{zz}} = -\frac{d_z}{d_z + \partial_t} \partial_z \sigma_{zz}$$

Definition of the memory variable $\psi_{\sigma_{zz}}$:

$$\psi_{\sigma_{zz}} = -\frac{d_z}{d_z + \partial_t} \partial_z \sigma_{zz}$$

$$\rho \partial_t v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} + \psi_{\sigma_{zz}}$$

Definition of the memory variable $\psi_{\sigma_{zz}}$:

$$\psi_{\sigma_{zz}} = -\frac{d_z}{d_z + \partial_t} \partial_z \sigma_{zz}$$

$$\rho \partial_t v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} + \psi_{\sigma_{zz}}$$

$$\partial_t \psi_{\sigma_{zz}} = -d_z \partial_z \sigma_{zz} - d_z \psi_{\sigma_{zz}}$$

Definition of the memory variable $\psi_{\sigma_{zz}}$:

$$\psi_{\sigma_{zz}} = -\frac{d_z}{d_z + \partial_t} \partial_z \sigma_{zz}$$

$$\rho \partial_t v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} + \psi_{\sigma_{zz}}$$

$$\partial_t \psi_{\sigma_{zz}} = -d_z \partial_z \sigma_{zz} - d_z \psi_{\sigma_{zz}}$$

ADE with C-PML (Convolutional PML)

$$\partial_{\bar{z}} = \left(1 - \frac{d_z}{\alpha_z + i\omega + d_z} \right) \partial_z$$

Definition of the memory variable $\psi_{\sigma_{zz}}$:

$$\psi_{\sigma_{zz}} = -\frac{d_z}{d_z + \partial_t} \partial_z \sigma_{zz}$$

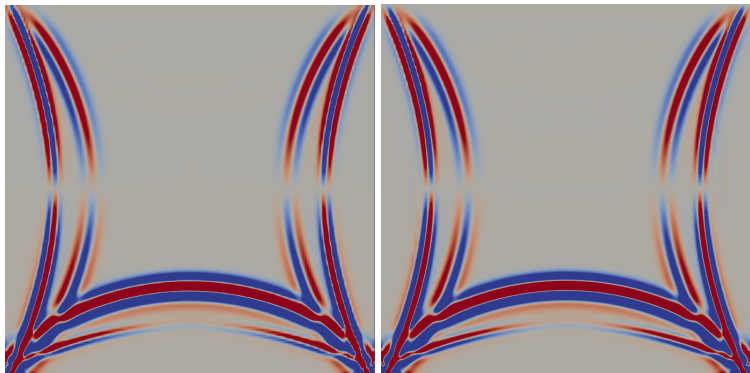
$$\rho \partial_t v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} + \psi_{\sigma_{zz}}$$

$$\partial_t \psi_{\sigma_{zz}} = -d_z \partial_z \sigma_{zz} - d_z \psi_{\sigma_{zz}}$$

ADE with C-PML (Convolutional PML)

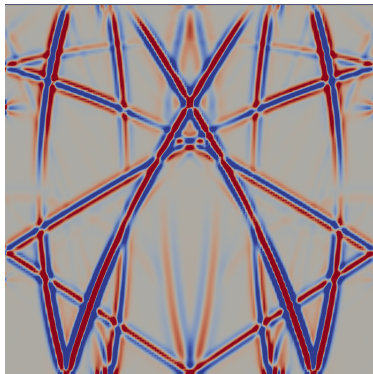
$$\partial_{\bar{z}} = \left(1 - \frac{d_z}{\alpha_z + i\omega + d_z} \right) \partial_z$$

$$\partial_t \psi_{\sigma_{zz}} = -d_z \partial_z \sigma_{zz} - (d_z + \alpha_z) \psi_{\sigma_{zz}}$$

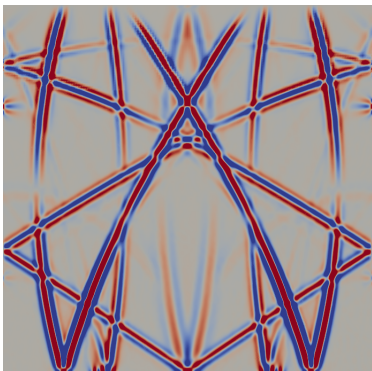


(a) $t=1.5s$

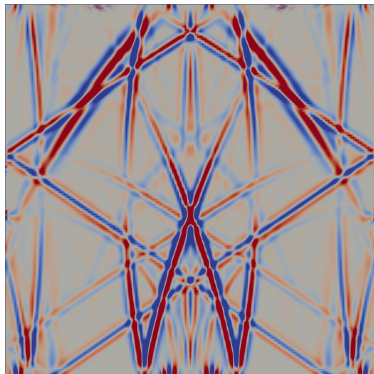
(b) $t=1.5s$



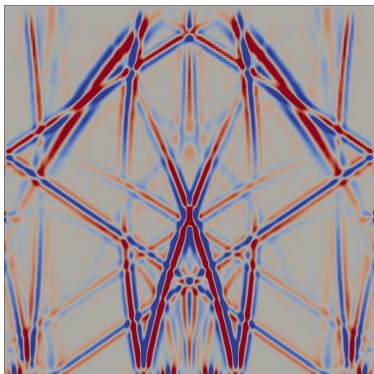
(c) $t=3s$



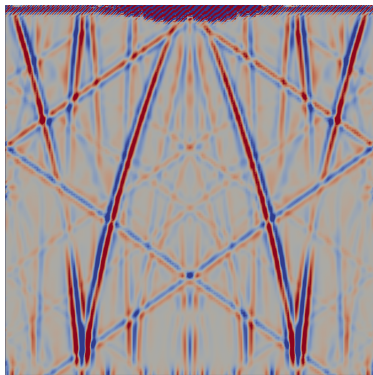
(d) $t=3s$



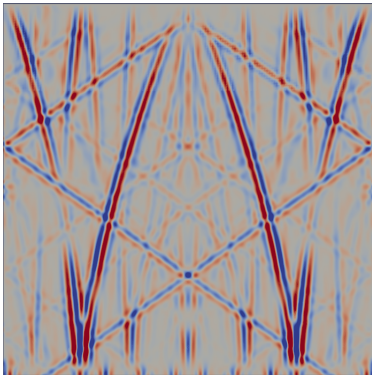
(e) $t=5s$



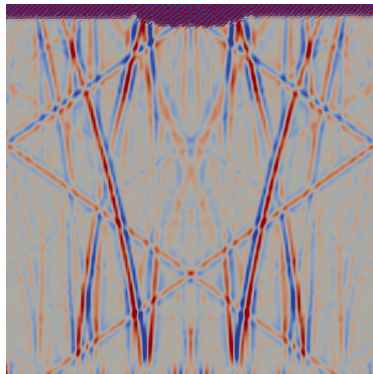
(f) $t=5s$



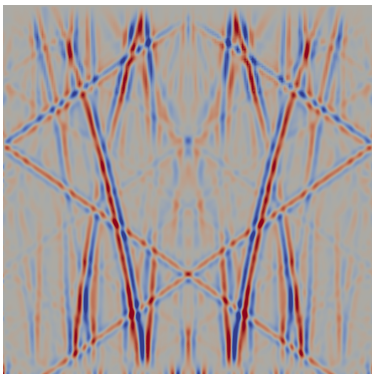
(g) $t=8s$



(h) $t=8s$

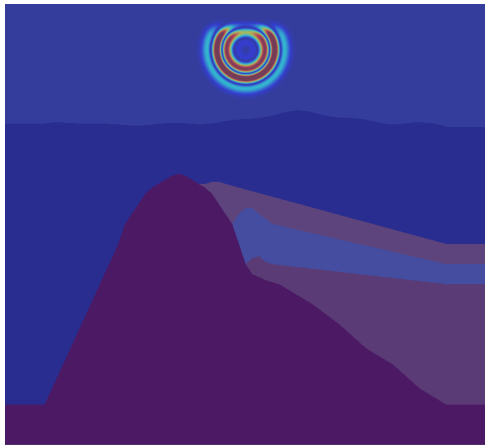


(i) $t=10s$



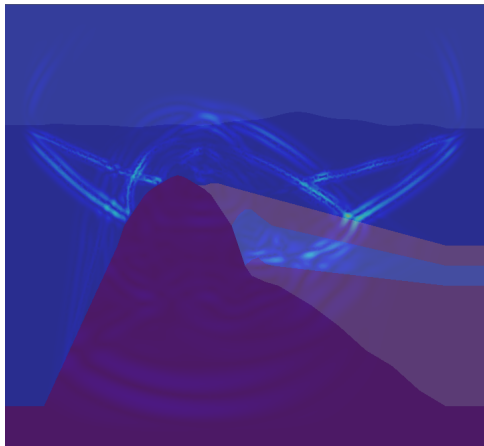
(j) $t=10s$





Salt dome surround by SEM-PML







Conclusion

- 1 SEM is more efficient on structured quadrangle mesh than DG
- 2 Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
- 3 Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

Achievements

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Develop PML in the hexahedral part ✓

Thank you for your attention !

Questions?