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# High-order DG-SE approximations of elastodynamic wave problems with stable Perfectly Matched Layers<sup>†</sup>

Hélène Barucq<sup>1</sup>, Henri Calandra<sup>2</sup>, Aurélien Citrain<sup>3,1</sup>, Julien Diaz<sup>1</sup> and Christian Gout<sup>3</sup>

<sup>1</sup> Team project Magique.3D, INRIA, E2S UPPA, CNRS, Pau, France.

<sup>2</sup> TOTAL SA, CSTJF, Pau, France.

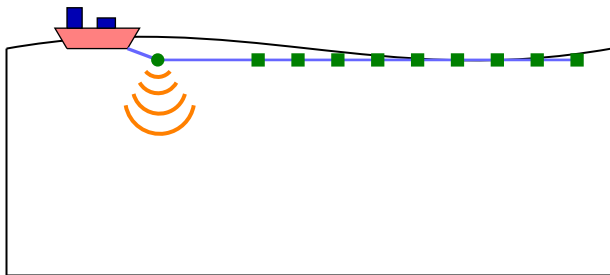
<sup>3</sup> LMI, Normandie Université, INSA ROUEN, 76000 Rouen, France.

Mathias 2019

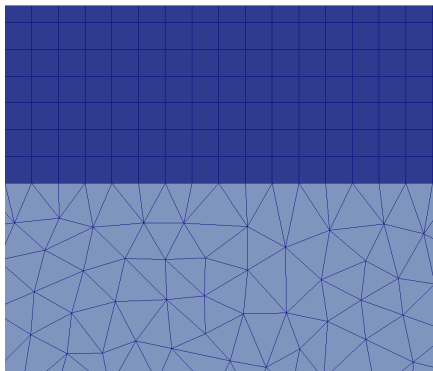
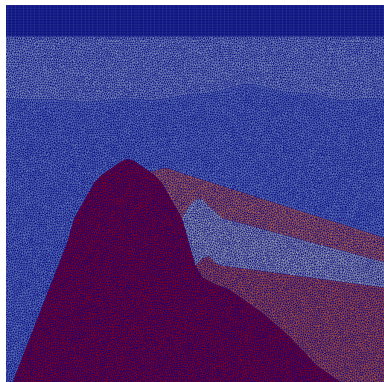


*The authors thank the M2NUM project which is co-financed by the European Union with the European regional development fund (ERDF, HN0002137) and by the Normandie Regional Council.*

<sup>†</sup> *This work is dedicated to the memory of Dimitri Komatitsch.*

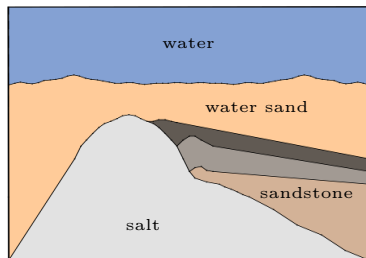
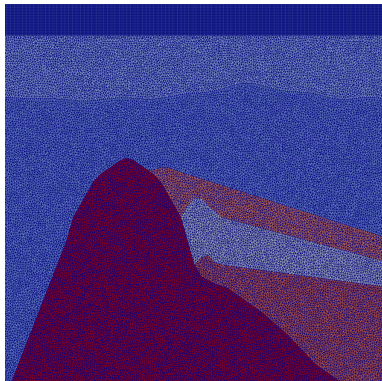


## Why using hybrid meshes?



- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water).
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.

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$x \in \Omega \subset \mathbb{R}^d$ ,  $t \in [0, T]$ ,  $T > 0$  :

$$\begin{cases} \rho(x) \frac{\partial v}{\partial t}(x, t) &= \nabla \cdot \underline{\underline{\sigma}}(x, t), \\ \frac{\partial \underline{\underline{\sigma}}}{\partial t}(x, t) &= \underline{\underline{C}}(x) \underline{\underline{\epsilon}}(v(x, t)). \end{cases}$$

With:

- $\rho(x)$  the density,
- $\underline{\underline{C}}(x)$  the elasticity tensor,
- $\underline{\underline{\epsilon}}(x, t)$  the deformation tensor,
- $v(x, t)$ , the wavespeed,
- $\underline{\underline{\sigma}}(x, t)$  the strain tensor.

Software written in **Fortran** for wave propagation simulation in the **time domain**

## Features

Simulation:

- on various types of meshes (**unstructured triangles and tetrahedra**),
  - on **heterogeneous media (acoustic, elastic and elasto-acoustic)**.
- 
- **Discontinuous Galerkin (DG)** based on **unstructured triangles and unstructured tetrahedra**,
  - with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**,
  - with **multi-order** computation(**p-adaptivity**)...

- 1 DGm and SEM
- 2 DG/SEM coupling
- 3 Perfectly Matched Layer(PML)



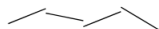
Use discontinuous functions :



mesh

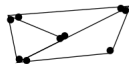


continuous

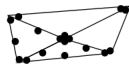


discontinuous

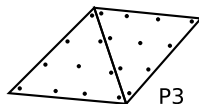
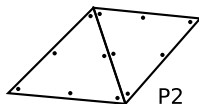
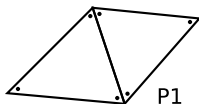
$h$  adaptivity :



$p$  adaptivity :

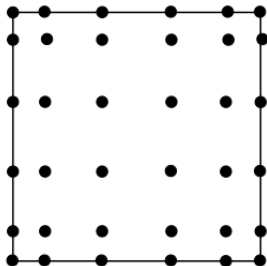


Degrees of freedom necessary on each cell :



## General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature,
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on  $L^2$ -norm).



- $\int f(x) dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j),$
- $\varphi_i(\xi_j) = \delta_{ij}.$

## DG

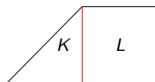
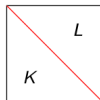
- Element per element computation ( *hp*-adaptivity).
- Time discretization quasi explicit (block diagonal mass matrix).
- Simple to parallelize.
- Robust to brutal changes of physics and geometry

## SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).

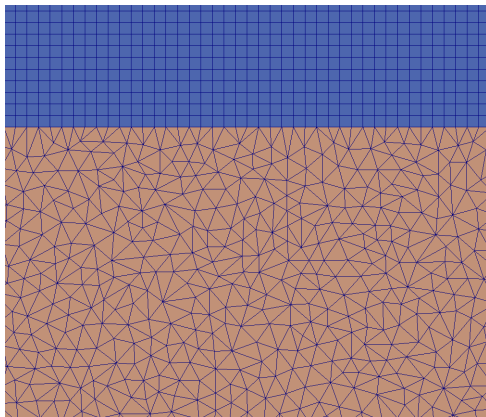
- Aim at coupling  $P_k$  and  $Q_k$  structures.
- Need to extend or split some structures (e.g. neighbour indices).
- Define new face matrices:

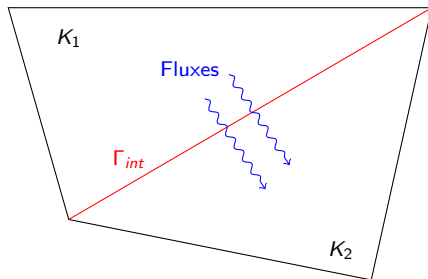
$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L.$$



## Global context

- Domain in two parts :  $\Omega_{h,1}$  (**structured quadrangles + SEM**),  $\Omega_{h,2}$  (**unstructured triangles + DG**).





## Definitions

- Jump and average.

$$[[u]] = (u_{K_1} \mathbf{n}_{K_1} + u_{K_2} \mathbf{n}_{K_2})$$

$$\{\{u\}\} = \frac{1}{2}(u_{K_2} + u_{K_1})$$

SEM variational formulation :

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 : \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1, \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = \int_{\Omega_{h,1}} (C \xi_1) : \nabla v_1. \end{cases}$$

DG variational formulation :

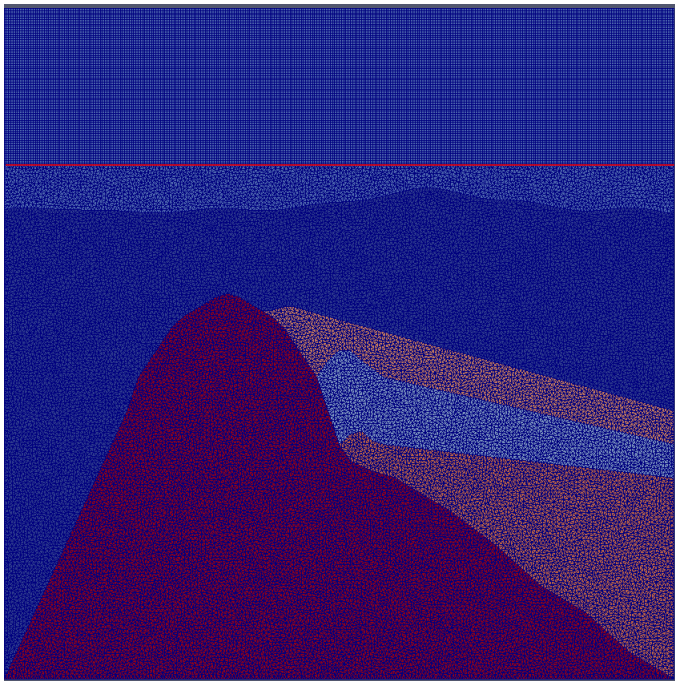
$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 : \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\} : [[w_2]], \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla \cdot (C \xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C \xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\} \cdot [[C \xi_2]]. \end{cases}$$

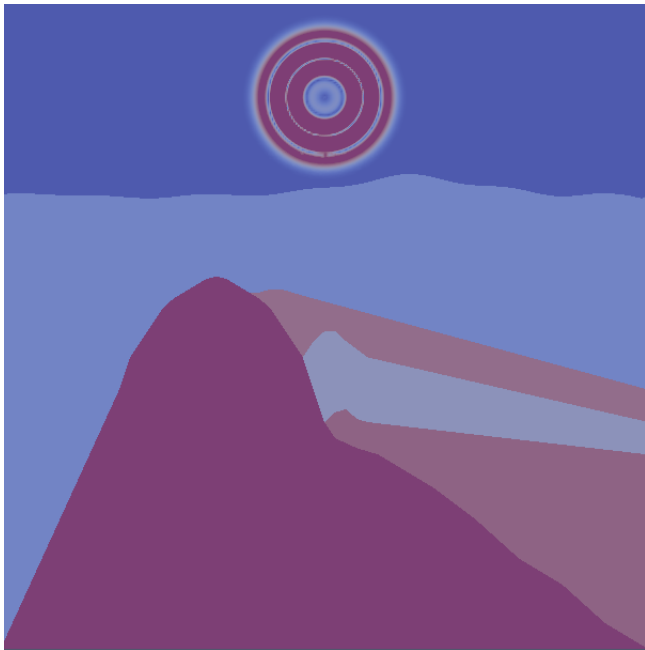
$$\left\{ \begin{aligned}
 & \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,1}} \sigma_1 : \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 : \nabla w_2 \\
 & + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} : [ [ w_2 ] ] \\
 & + \int_{\Gamma_{1/2}} [ [ \sigma w ] ], \\
 & \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = \int_{\Omega_{h,1}} (C \xi_1) : \nabla v_1 - \int_{\Omega_{h,2}} (\nabla \cdot (C \xi_2)) \cdot v_2 \\
 & + \int_{\Gamma_{out,2}} (C \xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} \cdot [ [ C \xi_2 ] ] \\
 & + \int_{\Gamma_{1/2}} [ [ (C \xi) v ] ].
 \end{aligned} \right.$$

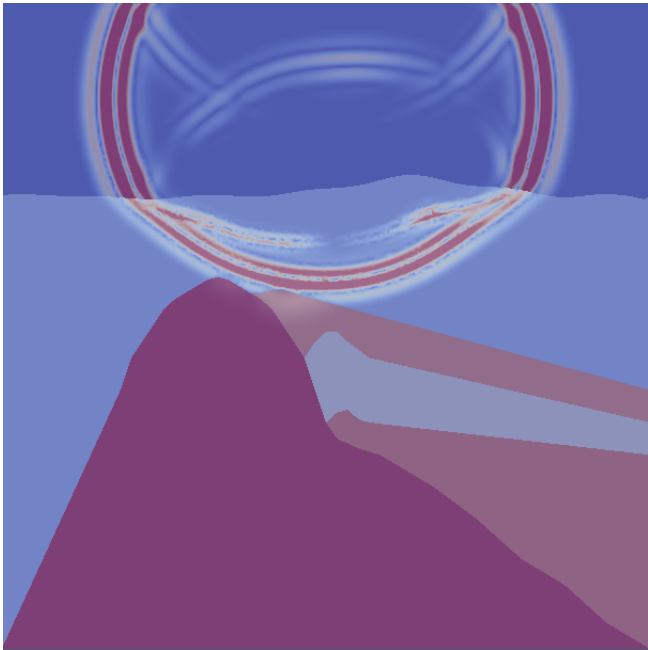


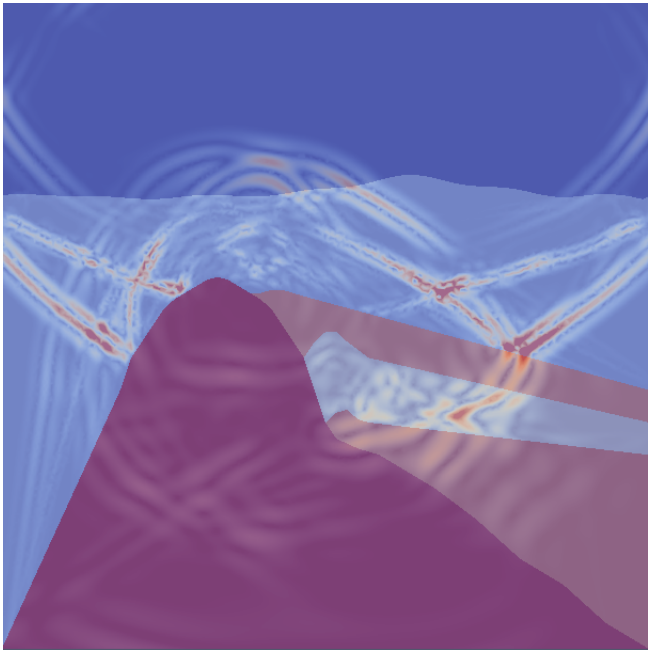
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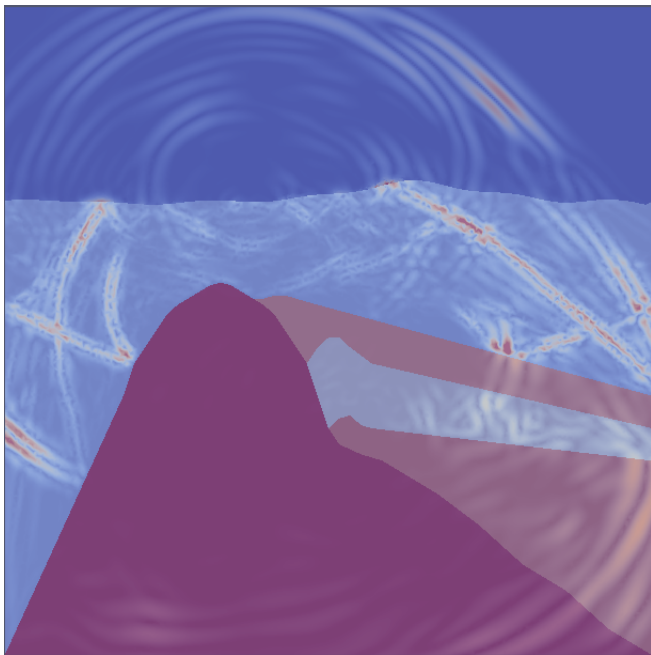
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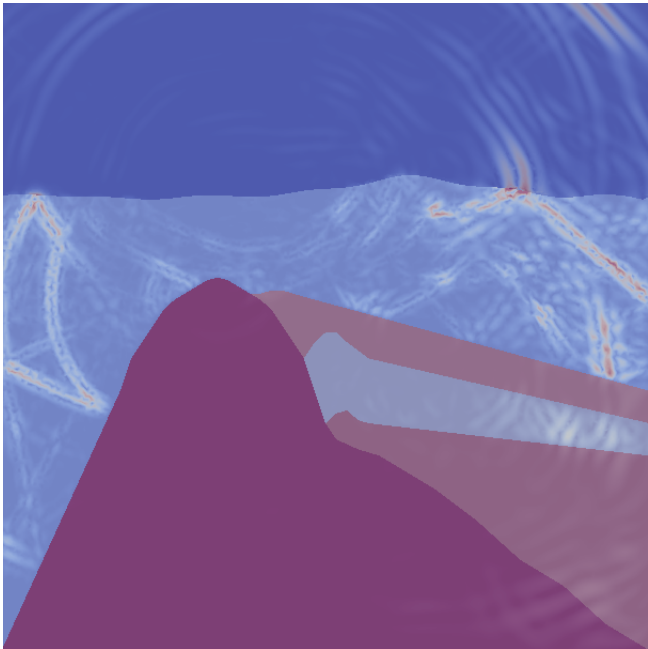






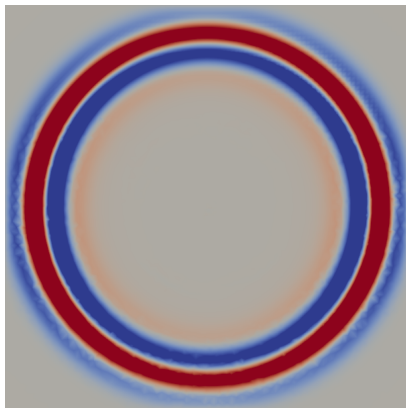




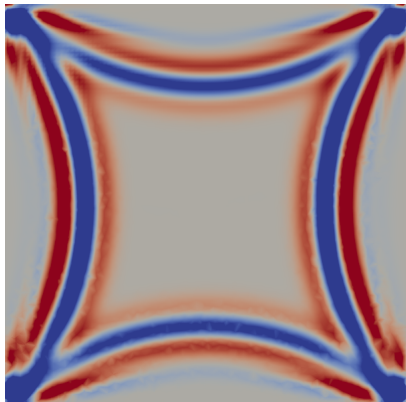




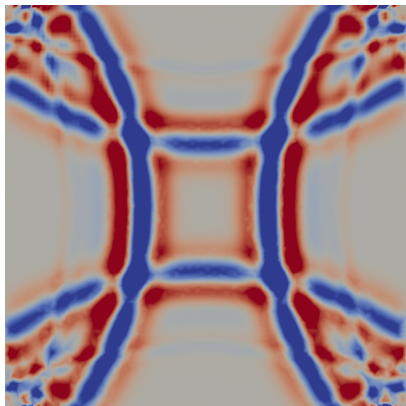
## What is a PML and why using it?



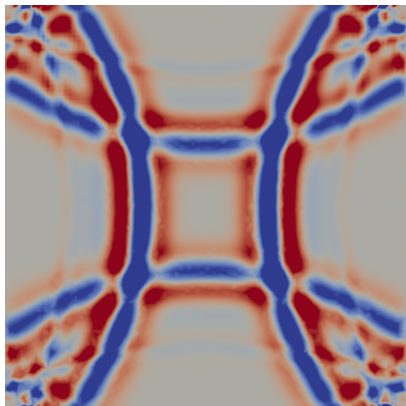
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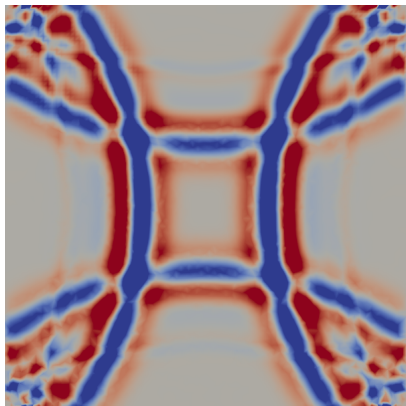


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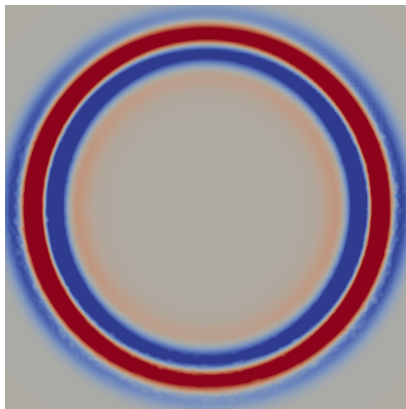
- Easy to implement

## What is a PML and why using it?



- Easy to implement
- Not really adapted to the simulation of "infinite" domain

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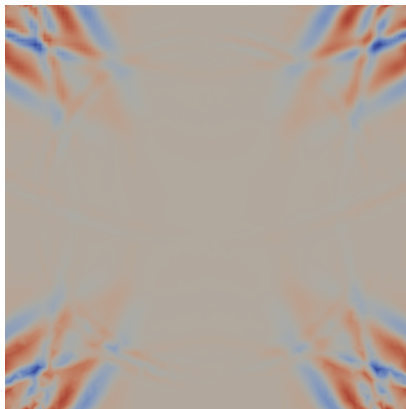
- Possibility to simulate an infinite domain

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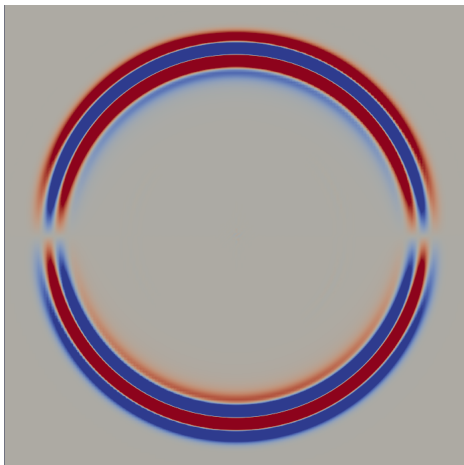
- Possibility to simulate an infinite domain
- **Complicated to implement when the order of the condition increase**

## What is a PML and why using it?

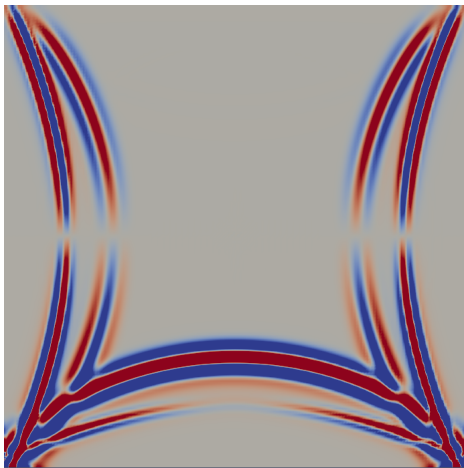


- Possibility to simulate an infinite domain
- Complicated to implement when the order of the condition increase
- Apparition of reflection when it comes to waves with grazing incidence

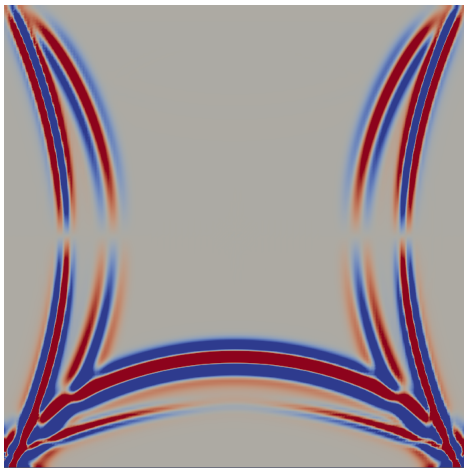
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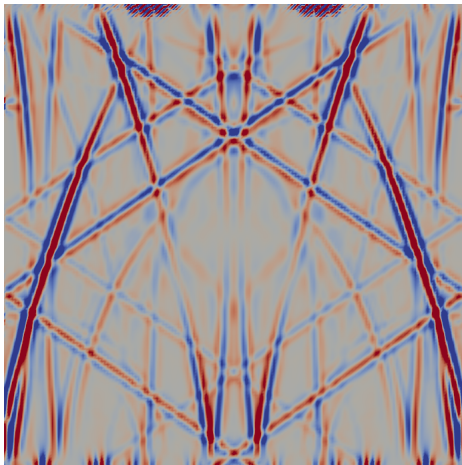


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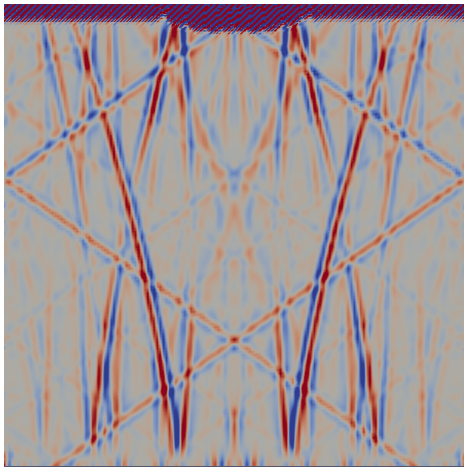
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## What is a PML and why using it?



- Possibility to simulate an infinite domain

## What is a PML and why using it?



- Possibility to simulate an infinite domain
- Some stability problems on elastic case using DGM



Step 1: Rewrite the system in the frequency domain

$$\begin{cases} i\omega\rho\mathbf{v} = \nabla \cdot \underline{\underline{\sigma}} \\ i\omega\underline{\underline{\sigma}} = \underline{\underline{C}} : \nabla\mathbf{v} \end{cases}$$

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Step 2: Introduce a new system of complex coordinates

Define  $d_z$  the damping :

$$\begin{cases} d_z > 0 & \text{if } z \in \Omega_{PML} \\ d_z = 0 & \text{if } z \notin \Omega_{PML} \end{cases}$$

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Define the associate differential operator :

$$\partial_{\tilde{z}} = \frac{i\omega}{i\omega + d_z} \partial_z = 1 - \frac{d_z}{i\omega + d_z} \partial_z$$

## Step 3: Rewrite the system

$$\begin{aligned}i\omega\rho v_x &= \partial_x\sigma_{xx} + \partial_{\bar{z}}\sigma_{xz}, \\i\omega\rho v_z &= \partial_x\sigma_{xz} + \partial_{\bar{z}}\sigma_{zz}, \\i\omega\sigma_{xx} &= (\lambda + 2\mu)\partial_x v_x + \lambda\partial_{\bar{z}} v_z, \\i\omega\sigma_{zz} &= \lambda\partial_x v_x + (\lambda + 2\mu)\partial_{\bar{z}} v_z, \\i\omega\sigma_{xz} &= \mu(\partial_x v_z + \partial_{\bar{z}} v_x).\end{aligned}$$



Martin, Roland and Komatitsch, Dimitri and Gedney, Stephen D and Bruthiaux, Emilien and others

A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)  
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of  $2^d + 1$  new variables  $\psi_\star \in H^1 \rightarrow$  Simplify the implementation BUT add new equation to the linear system



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Back to original coordinates



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↓

Back to original coordinates

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$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \frac{d_z}{d_z + i\omega} \partial_z \sigma_{zz}$$

↓

Back to time domain



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A high-order time and space formulation of the unsplit perfectly matched layer for the seismic wave equation using Auxiliary Differential Equations (ADE-PML)  
Computer Modeling in Engineering and Sciences (CMES),2010

Introduction of  $2^d + 1$  new variables  $\psi_* \in H^1 \rightarrow$  Simplify the implementation BUT add new equation to the linear system

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz}$$

↓ Back to original coordinates

$$i\omega\rho v_z = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \frac{d_z}{d_z + i\omega} \partial_z \sigma_{zz}$$

↓ Back to time domain

$$\rho \partial_t v_x = \partial_x \sigma_{xz} + \partial_z \sigma_{zz} - \frac{d_z}{d_z + \partial_t} \partial_z \sigma_{zz}$$

Definition of the memory variable  $\psi_{\sigma_{zz}}$ :

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ADE with C-PML (Convolutional PML)

$$\partial_{\bar{z}} = \left( 1 - \frac{d_z}{\alpha_z + i\omega + d_z} \right) \partial_z$$



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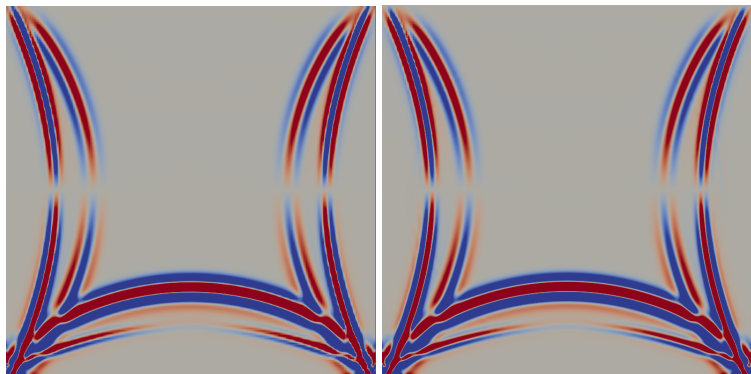
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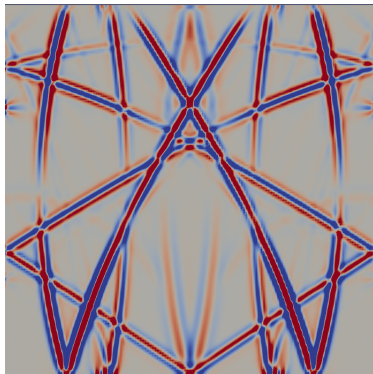
$$\partial_{\bar{z}} = \left( 1 - \frac{d_z}{\alpha_z + i\omega + d_z} \right) \partial_z$$

$$\partial_t \psi_{\sigma_{zz}} = -d_z \partial_z \sigma_{zz} - (d_z + \alpha_z) \psi_{\sigma_{zz}}$$

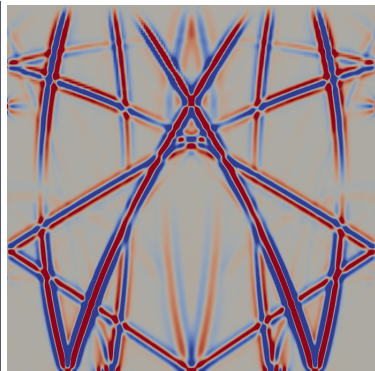


(a)  $t=1.5s$

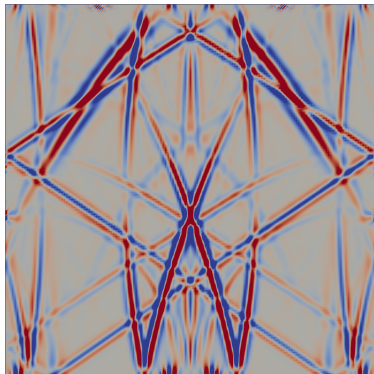
(b)  $t=1.5s$



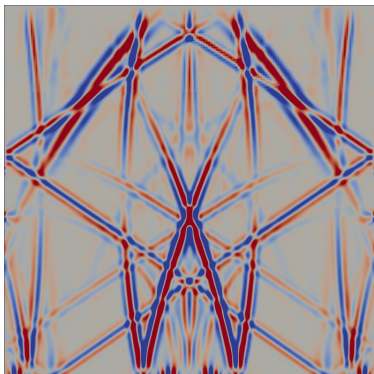
(c)  $t=3s$



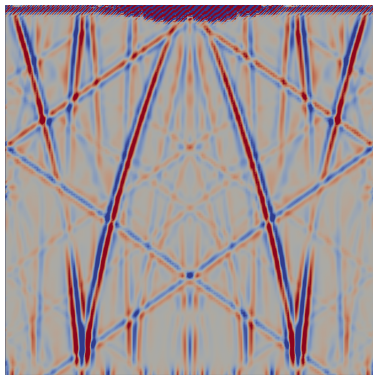
(d)  $t=3s$



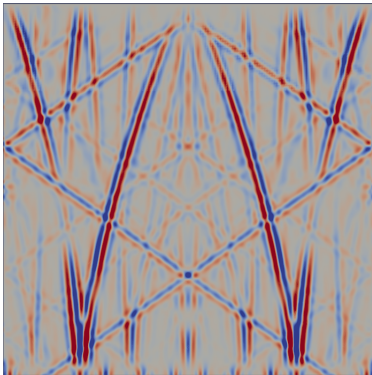
(e)  $t=5s$



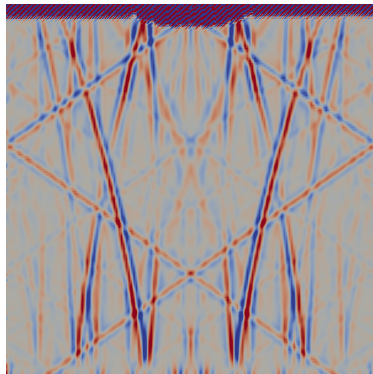
(f)  $t=5s$



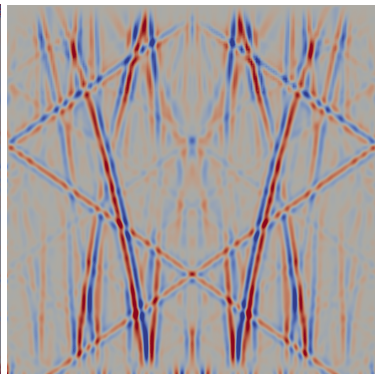
(g)  $t=8s$



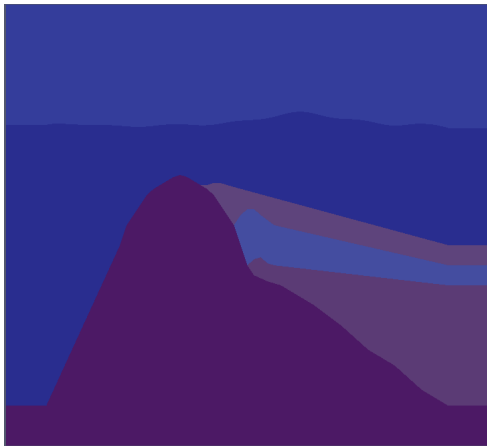
(h)  $t=8s$

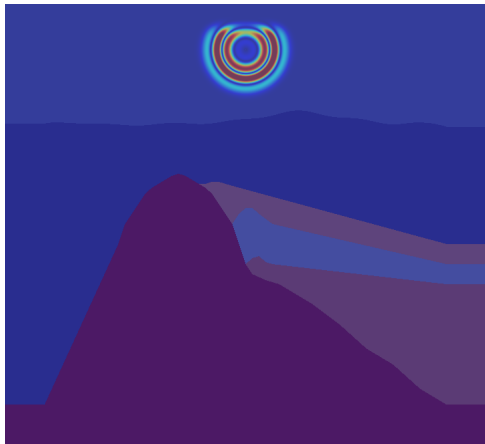


(i)  $t=10s$



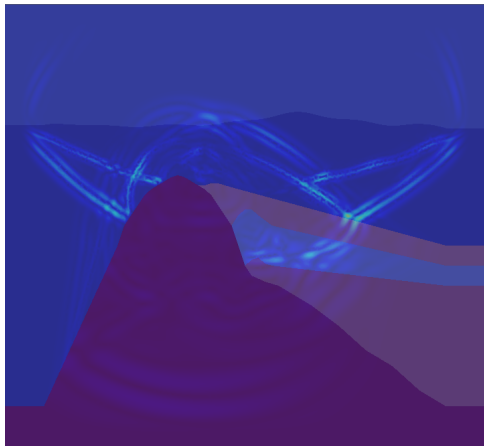
(j)  $t=10s$

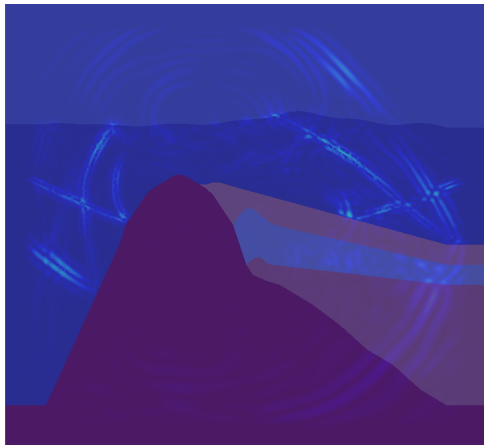












## Conclusion

- 1 SEM is more efficient on structured quadrangle mesh than DG
- 2 Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
- 3 Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

## Achievements

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Develop PML in the hexahedral part ✓

*Thank you for your attention !*

Questions?