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# NUMERICAL ANALYSIS OF A METHOD FOR SOLVING 2D LINEAR ISOTROPIC ELASTODYNAMICS WITH FREE BOUNDARY CONDITION USING POTENTIALS AND FINITE ELEMENTS

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ABSTRACT. When solving 2D linear elastodynamic equations in a homogeneous isotropic media, a Helmholtz decomposition of the displacement field decouples the equations into two scalar wave equations that only interact at the boundary. It is then natural to look for numerical schemes that independently solve the scalar equations and couple the solutions at the boundary. The case of rigid boundary condition was treated in [3, 2]. However in [4] the case of free surface boundary condition was proven to be unstable if a straightforward approach is used. Then an adequate functional framework as well as a time domain mixed formulation to circumvent these issues was presented. In this work we first review the formulation presented in [4] and propose a subsequent discretised formulation. We provide the complete stability analysis of the corresponding numerical scheme. Numerical results that illustrate the theory are also shown.

## 1. INTRODUCTION

The propagation of elastic waves in solids is a very classical problem and there exist already many numerical methods to simulate this phenomenon numerically. For instance, standard conforming finite elements in space (possibly of high order) can be used to solve the displacements formulations of elastodynamics equations in isotropic media (second order hyperbolic system). It is also possible to use mixed finite element methods to solve the equivalent velocity-stress formulation of the same system (first order differential system). On both cases, this space discretization is then coupled to a finite difference discretization in time. Then, if the discretization is explicit, the time step is subject to a CFL stability condition.

On the other hand, in many classical physics books, the well-known Helmholtz decomposition of vector fields is used to compute analytical solutions in homogeneous isotropic media: the vector field is written as the sum of a gradient and a curl of two scalar potentials. As mentioned in [4], such a decomposition relates elastodynamic equations to two wave equations and enlightens the decomposition

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of the wave field as the superposition of waves that propagate independently with different velocities: the pressure waves, or P-waves, that are gradients of a pressure potential  $\varphi_P$  and propagate with a velocity  $V_P$ ; the shear waves, or S-waves, that are curls of a shear potential  $\varphi_S$  and propagate with a velocity  $V_S < V_P$ .

In [15] a finite difference scheme is constructed with approximation properties independent of the ratio  $V_P/V_S$ . However, very few works have exploited the Helmholtz decomposition in the framework of finite elements even though it has been used in other domains of physics, in particular in fluid mechanics (current-vorticity formulations; see Chapter 2 and 3 in [14], [13] and [12]). However, as we have already shown in [3], [2] and [4] this idea can be successfully exploited for the numerical solution of elastodynamic equations in isotropic media (for simplicity, we have restricted ourselves to the 2D case where both pressure and shear potentials are scalar). In these works we have given an answer to the following question, of pure intellectual interest: could we write a stable formulation with potentials to solve isotropic elastodynamics with finite elements? However, there is a more relevant motivation that concerns the elastic media in which P-waves propagate much faster than S-waves. In such a case, it is known that explicit displacement-based methods, which do not distinguish both waves along the calculation process, are greatly penalized by large values of the ratio  $V_P/V_S$  due to the CFL condition (assuming that explicit time integrators are used). More precisely in the introduction of [4] we explained that the computational cost of standard explicit finite elements methods on a quasi-regular mesh satisfies the following rough estimates

$$(1.1) \quad \text{Cost} \propto \frac{L^d T}{h^d \Delta t} \propto \frac{L^d}{\lambda_S^d} \frac{T}{T_\star} \frac{V_P}{V_S},$$

where  $L$  is the size of the domain,  $\lambda_S$  the wavelength of the shear waves,  $[0, T]$  the time integration interval and  $T_\star$  the *minimal* time scale involved in the source term. The ratio  $L^d/\lambda_S^d$  represents the size of the problem in space while  $T/T_\star$  the size of the simulation in time. One clearly sees that  $V_P/V_S$  is a penalizing factor which is specific to the fact that elastodynamics is a coupled system of wave equations.

As pointed out in our previous works, potential formulations will authorize the use of different meshes for both the pressure and shear potentials in view of adapting the mesh size to each wavelength, which is smaller for  $S$  than for  $P$  waves. This would result in an important saving of the computation cost for large ratio  $V_P/V_S$ .

In a piecewise homogeneous media, the use of a Helmholtz decomposition is still valid locally and the different types of waves recouple at boundaries and interfaces. This is the main source of complexity because, contrary to the interior equations, boundary and transmission conditions are not easily expressed in terms of potentials. In [3, 2] we first addressed the case of a homogeneous medium with a clamped boundary, that is to say the Dirichlet boundary condition. However in [4], it was shown that a naive extension to the case of free surface boundary condition of what was done in [3, 2] led to an unstable formulation. To circumvent we proposed a stabilized formulation essentially based on a clever choice of functional framework.

In this work, we first recap in Section 2 how to reduce the resolution of 2D isotropic elastodynamic equations to two scalar wave equations and more importantly, explain how to treat the free surface boundary condition as it has been done in [4]. Then in Section 3 the space-time discretization is presented. More precisely, in Section 3.1 a detailed semi-discretization in space using a general Galerkin approach is given. Section 3.2 is devoted to its well-posedness and time stability.

In Section 3.3 we describe and analyze several approaches for the time discretization. In Section 4 we apply the results obtained in Section 3 to the case where  $\mathbb{P}_1$ -Lagrange finite elements are used for the space discretization. Special attention is paid to the well-posedness of the fully discrete scheme, CFL-stability conditions and mass lumping. In Section 5 we show some numerical results to illustrate the theory. In Section 6 we conclude and we mention some future research lines related to this topic. Finally, in Appendix 7 we provide some technical proofs.

## 2. MODEL PROBLEM AND POTENTIALS FORMULATION

This section has been added for pedagogical purpose, for making the paper self contained and for preparing Section 3.1. Section 2.1 recaps very standard material, Section 2.2 introduce the decomposition of the displacement field, while the sections 2.3 and 2.4 present some results obtained in [4] (where the reader can find the proofs that are not included here) and new material (in particular Theorems 2.9 and 2.10) useful for the analysis presented in Section 3.1.

### 2.1. The model problem. The displacements formulation.

*Preliminary notation.* Throughout the paper we will work in 2D and  $\mathbf{x} = (x_1, x_2)$  will denote the space variable. We shall use bold letters for representing vector fields such as  $\mathbf{u} = (u_1, u_2)$  for the displacement field in a elastic body or  $\mathbf{v} = (v_1, v_2)$  for the velocity field ( $\mathbf{v} = \partial_t \mathbf{u}$ ). Ordinary letters will be used for scalar fields such as the components of vector fields or scalar potentials. Finally, underlined bold letters will be used for  $2 \times 2$  tensor fields such as the deformation or strain tensor  $\underline{\boldsymbol{\varepsilon}}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))_{i,j}$  where  $1 \leq i, j \leq 2$  or the stress tensor  $\underline{\boldsymbol{\sigma}} = (\sigma_{ij})_{i,j}$ .

*Geometrical assumptions.* In the sequel we shall assume for simplicity that  $\Omega$  denotes a simply connected domain of  $\mathbb{R}^2$  that is locally on one side of its boundary  $\Gamma$ . Without any loss of generality, we shall assume that the origin  $\mathbf{0}$  is the center of gravity of  $\Omega$ . For the sequel we will introduce an origin  $\mathbf{o} \in \Gamma$  and a curvilinear abscissa  $s \in [0, L]$  along  $\Gamma$  (with length  $L$ ). We denote by  $\mathbf{n} = (n_1, n_2)$  the unit normal outward vector to  $\Gamma$ ,  $\boldsymbol{\tau} = (n_2, -n_1)$  the corresponding tangent vector, as well as, for any scalar field  $\varphi$ ,  $\partial_{\mathbf{n}}\varphi = \nabla\varphi \cdot \mathbf{n}$  the normal derivative and  $\partial_{\boldsymbol{\tau}}\varphi = \nabla\varphi \cdot \boldsymbol{\tau}$  the tangential derivative.

*Mathematical model.* The displacement field  $\mathbf{u}$  obeys the fundamental law of mechanics

$$(2.1) \quad \rho \partial_t^2 \mathbf{u} - \mathbf{div} \underline{\boldsymbol{\sigma}} = \mathbf{f}, \quad \text{in } \Omega,$$

where  $\mathbf{div} \underline{\boldsymbol{\sigma}}$  is defined by (using Einstein convention for repeated indices)

$$(\mathbf{div} \underline{\boldsymbol{\sigma}})_i = \partial_j \sigma_{ij}(\mathbf{u}),$$

where  $\rho = \rho(\mathbf{x})$  is the positive density and  $\mathbf{f}$  is the source term that, for simplicity, will be assumed to be compactly supported inside  $\Omega$ . In an *isotropic* medium, stresses and displacements are related by Hooke's law which involves the positive Lamé parameters  $\lambda(\mathbf{x})$  and  $\mu(\mathbf{x})$

$$(2.2) \quad \underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{\sigma}}(\mathbf{u}) := \lambda \operatorname{div} \mathbf{u} \mathbf{I} + 2 \mu \underline{\boldsymbol{\varepsilon}}(\mathbf{u}),$$

where  $\underline{\mathbf{I}}$  is the  $2 \times 2$  identity matrix and

$$\operatorname{div} \mathbf{u} = \partial_j u_j, \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (\partial_i u_j + \partial_j u_i), \quad 1 \leq i, j \leq 2.$$

Eliminating  $\underline{\boldsymbol{\sigma}}(\mathbf{u})$  using (2.2) and (2.1), we get a second order system in  $\mathbf{u}$ . In the homogeneous case, i.e. when  $\lambda$ ,  $\mu$  and  $\rho$  are constant, this system becomes

$$(2.3) \quad \partial_t^2 \mathbf{u} - V_P^2 \nabla \operatorname{div} \mathbf{u} + V_S^2 \mathbf{curl}(\operatorname{curl} \mathbf{u}) = \frac{\mathbf{f}}{\rho}, \quad \text{in } \Omega,$$

with  $V_P$  (resp.  $V_S$ ) the velocity of the P-waves (resp. S-waves) defined by

$$(2.4) \quad V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \text{P waves velocity}, \quad V_S = \sqrt{\frac{\mu}{\rho}}, \quad \text{S waves velocity},$$

and where we have used the two curl operators in 2D defined by

$$(2.5) \quad \begin{aligned} \operatorname{curl} \mathbf{u} &:= \partial_1 u_2 - \partial_2 u_1, & \text{for the scalar curl of a vector field } \mathbf{u}, \\ \mathbf{curl} \varphi &:= (\partial_2 \varphi, -\partial_1 \varphi), & \text{for the vector curl of a scalar field } \varphi. \end{aligned}$$

Equation (2.3) is completed with the free boundary condition

$$(2.6) \quad \underline{\boldsymbol{\sigma}}(\mathbf{u})\mathbf{n} = \mathbf{0}, \quad \text{on } \Gamma = \partial\Omega, \quad \text{where } \underline{\boldsymbol{\sigma}}(\mathbf{u})\mathbf{n} \equiv \lambda \operatorname{div} \mathbf{u} \mathbf{n} + 2\mu \underline{\boldsymbol{\varepsilon}}(\mathbf{u})\mathbf{n},$$

as well as null initial conditions

$$(2.7) \quad \mathbf{u}(\mathbf{x}, 0) = \partial_t \mathbf{u}(\mathbf{x}, 0) = 0, \quad \text{for all } \mathbf{x} \in \Omega.$$

For solving the problem (2.3, 2.6, 2.7) a particular role will be played by the space  $\mathbf{R}(\Omega)$  of rigid displacements satisfying  $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}$ , which results into

$$\mathbf{R}(\Omega) = \left\{ a(x_2, -x_1)^t + (b_1, b_2)^t, (a, (b_1, b_2)) \in \mathbb{R} \times \mathbb{R}^2 \right\}.$$

In the following, we shall assume that the source term  $\mathbf{f}$  satisfies

$$\mathbf{f}(t) \in \mathbf{L}_R^2(\Omega) = \left\{ \mathbf{w} \in \mathbf{L}^2(\Omega) / \int_{\Omega} \mathbf{w} \cdot \mathbf{w}_R \, d\mathbf{x} = 0, \forall \mathbf{w}_R \in \mathbf{R}(\Omega) \right\}.$$

Under this assumption, the solution  $\mathbf{u}$  remains at any time orthogonal to  $\mathbf{R}(\Omega)$ . Let us formulate this result as a theorem which requires to introduce the following spaces of displacement fields

$$(2.8) \quad \left\{ \begin{aligned} \mathbf{D} &:= \left\{ \mathbf{w} \in \mathbf{H}^1(\Omega) / -V_P^2 \nabla(\operatorname{div} \mathbf{w}) + V_S^2 \mathbf{curl}(\operatorname{curl} \mathbf{w}) \in \mathbf{L}^2(\Omega)^2 \right\}, \\ \mathbf{D}_R &:= \left\{ \mathbf{w} \in \mathbf{D} / \int_{\Omega} \mathbf{w} \cdot \mathbf{w}_R \, d\mathbf{x} = 0, \forall \mathbf{w}_R \in \mathbf{R}(\Omega) \right\}, \\ \mathbf{D}_N &:= \left\{ \mathbf{w} \in \mathbf{D}_R / \underline{\boldsymbol{\sigma}}(\mathbf{w})\mathbf{n} = \mathbf{0} \text{ on } \Gamma \right\}. \end{aligned} \right.$$

Of course,  $\mathbf{D}$  is a Hilbert space when equipped with the natural graph norm, and  $\mathbf{D}_R$ ,  $\mathbf{D}_N$  are closed subspaces of it. Then, we can state the following classical theorem that results, for instance, from the application of standard Hille-Yosida's theory for evolution problems.

**Theorem 2.1.** *Assume that  $\mathbf{f} \in C^1([0, T]; \mathbf{L}_R^2(\Omega))$ . Then, the problem (2.3, 2.6, 2.7) admits a unique strong solution:*

$$(2.9) \quad \mathbf{u} \in C^2([0, T]; \mathbf{L}^2(\Omega)) \cap C^1([0, T]; \mathbf{H}^1(\Omega)) \cap C^0([0, T]; \mathbf{D}_N).$$

**2.2. Introduction of the potentials.** From now, we assume that the source term  $\mathbf{f}(\cdot, t)$  is at any time supported in a fixed compact set of  $\Omega$ . We define two scalar potentials  $\varphi_P$  and  $\varphi_S$  by

$$(2.10) \quad \rho \partial_t \varphi_P = (\lambda + 2\mu) \operatorname{div} \mathbf{u}, \quad \rho \partial_t \varphi_S = -\mu \operatorname{curl} \mathbf{u}, \quad \varphi_P(\cdot, 0) = \varphi_S(\cdot, 0) = 0.$$

Then, one shows that (see also [4], Sect. 2.1),

$$(2.11) \quad \mathbf{v} = \nabla \varphi_P + \operatorname{curl} \varphi_S + \mathbf{g}, \quad \text{where} \quad \mathbf{g}(t) = \frac{1}{\rho} \int_0^t \mathbf{f}(\tau) d\tau,$$

where  $\varphi_P$  and  $\varphi_S$  satisfy the following two scalar wave equations

$$(2.12) \quad \frac{1}{V_P^2} \partial_t^2 \varphi_P - \Delta \varphi_P = \operatorname{div} \mathbf{g}, \quad \frac{1}{V_S^2} \partial_t^2 \varphi_S - \Delta \varphi_S = -\operatorname{curl} \mathbf{g}, \quad \text{in } \Omega.$$

We shall denote by  $\boldsymbol{\varphi} = (\varphi_P, \varphi_S)$  the 2D vector field with both potentials. The aim of the next two sections is to write a problem in  $\boldsymbol{\varphi}$  which is equivalent to (2.3, 2.6, 2.7). The first step consists in completing the equations (2.12) by boundary conditions for  $\boldsymbol{\varphi}$  equivalent to (2.6) (see Section 2.3) and deriving a variational formulation of the problem (see Section 2.4).

**2.3. The free boundary condition.** In terms of the velocity field, this condition writes,  $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \mathbf{0}$ . Using (2.11) and (2.6),  $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$  appears to be a linear combination of second order derivatives of the potentials. This is not adapted for a variational formulation of a second order problem as (2.12) and thus to a finite element approximation.

In order to overcome this difficulty, the trick proposed in [4], consists in:

- (1) Introducing (momentaneously) an auxiliary variable  $\mathbf{v}_\Gamma := \mathbf{v}|_\Gamma$ .
- (2) Interpret the step above as a non homogeneous Dirichlet condition,
- (3) Write an additional equation in  $(\mathbf{v}_\Gamma, \varphi_P, \varphi_S)$  equivalent to  $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \mathbf{0}$ .

The first two steps result into non homogeneous condition for  $\boldsymbol{\varphi} = (\varphi_P, \varphi_S)$ , more precisely, from (2.11), we deduce

$$(2.13) \quad \partial_{\mathbf{n}} \varphi_P = \partial_{\boldsymbol{\tau}} \varphi_S + \mathbf{v}_\Gamma \cdot \mathbf{n}, \quad \partial_{\mathbf{n}} \varphi_S = -\partial_{\boldsymbol{\tau}} \varphi_P + \mathbf{v}_\Gamma \cdot \boldsymbol{\tau}, \quad \text{on } \Gamma,$$

while, step 3 provides

$$(2.14) \quad \begin{cases} v_{\Gamma,2} + \frac{1}{2V_S^2} \mathcal{I}(\partial_t^2 \boldsymbol{\varphi} \cdot \mathbf{n}) \in \mathbb{P}_0(\Gamma), & \int_\Gamma \boldsymbol{\varphi} \cdot \mathbf{n} d\gamma = 0, \\ v_{\Gamma,1} - \frac{1}{2V_S^2} \mathcal{I}(\partial_t^2 \boldsymbol{\varphi} \cdot \boldsymbol{\tau}) \in \mathbb{P}_0(\Gamma), & \int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{\tau} d\gamma = 0, \end{cases}$$

where  $\mathcal{I}$  is a linear and continuous extension of the integral operator defined by

$$(2.15) \quad \begin{aligned} \mathcal{I}: L^2(\Gamma) &\longrightarrow L^2(\Gamma) \\ \eta &\longmapsto \mathcal{I}\eta(s) := \int_0^s \eta(\sigma) d\sigma. \end{aligned}$$

Notice that this operator maps  $\{\nu \in L^2(\Gamma) / \int_\Gamma \nu d\gamma = 0\}$  into  $H^1(\Gamma)$ , then if we introduce the following subspace of  $H^{-\frac{1}{2}}(\Gamma)$ ,

$$(2.16) \quad M := \left\{ \nu \in H^{-\frac{1}{2}}(\Gamma) / \int_\Gamma \nu d\gamma = 0 \right\},$$

it is clear that  $\mathcal{I}$  can be extended as a linear continuous operator  $\mathcal{I} \in \mathcal{L}(M, H^{1/2}(\Gamma))$ .

$$(2.17) \quad \text{In summary, } \underline{\boldsymbol{\sigma}}(\mathbf{v})\mathbf{n} = \mathbf{0} \iff (2.13) \text{ and } (2.14).$$

On the next section, the auxiliary variable  $\mathbf{v}_\Gamma$  will be eliminated when writing the variational formulation where the conditions in the second column in (2.14) will be treated as an essential condition in the variational space.

#### 2.4. The weak formulation.

2.4.1. *Preamble: a first naive formulation.* According to (2.11), it appears that, an appropriate functional space for the unknown  $\boldsymbol{\varphi}$  should be  $(\mathbf{v} \in \mathbf{L}^2(\Omega)$  since the kinetic energy must be finite)

$$(2.18) \quad \mathbf{V} := \{ \boldsymbol{\varphi} = (\varphi_P, \varphi_S) \in \mathbf{L}^2(\Omega) \text{ such that } \nabla \varphi_P + \mathbf{curl} \varphi_S \in \mathbf{L}^2(\Omega) \},$$

or equivalently, noticing that  $\nabla \varphi_P + \mathbf{curl} \varphi_S = (\text{div } \boldsymbol{\varphi}, -\text{curl } \boldsymbol{\varphi})^t$ ,

$$(2.19) \quad \mathbf{V} = H(\text{div}, \Omega) \cap H(\text{curl}, \Omega).$$

This space is a Hilbert space for the norm

$$(2.20) \quad \|\boldsymbol{\varphi}\|_{\mathbf{V}}^2 = \|\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{div } \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{curl } \boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}^2.$$

More precisely, to account for the linear constraints in the second column of (2.14), we shall work in the following closed subspace of  $\mathbf{V}$

$$(2.21) \quad \mathbf{V}_0 := \left\{ \boldsymbol{\varphi} \in \mathbf{V} \text{ s.t. } \int_{\Gamma} \boldsymbol{\varphi} \cdot \mathbf{n} \, d\gamma = \int_{\Gamma} \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \, d\gamma = 0 \right\}.$$

**Remark 2.2** (Gauge conditions and duality product). *The two equalities in the definition (2.21) of  $\mathbf{V}_0$  will be referred as the gauge conditions. Rigorously speaking, the two boundary integrals appearing there should be understood in the sense of duality brackets between  $H^{-\frac{1}{2}}(\Gamma)$  and  $H^{\frac{1}{2}}(\Gamma)$  and thus should be rewritten as*

$$(2.22) \quad \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma} = 0, \quad \langle \boldsymbol{\varphi} \cdot \boldsymbol{\tau}, 1 \rangle_{\Gamma} = 0.$$

*However, for simplicity and clarity, we shall continue to write them as boundary integrals in the sequel. Notice that, by Green's formula and our geometrical assumptions in  $\Omega$ , an alternative characterization of the space  $\mathbf{V}_0$  is*

$$(2.23) \quad \mathbf{V}_0 = \left\{ \boldsymbol{\varphi} \in \mathbf{V} \text{ s.t. } \int_{\Omega} \text{div } \boldsymbol{\varphi} \, d\mathbf{x} = \int_{\Omega} \text{curl } \boldsymbol{\varphi} \, d\mathbf{x} = 0 \right\}.$$

In the space  $\mathbf{V}_0$ , we can write a “natural” weak formulation (called naive in [4]) as follows. We first introduce the following continuous and symmetric bilinear forms (of mass type) in  $\mathbf{V}$

$$(2.24) \quad \begin{cases} m_{\Omega}(\boldsymbol{\varphi}, \boldsymbol{\psi}) & := \frac{1}{V_P^2} \int_{\Omega} \varphi_P \psi_P \, d\mathbf{x} + \frac{1}{V_S^2} \int_{\Omega} \varphi_S \psi_S \, d\mathbf{x}, \\ m_{\Gamma}(\boldsymbol{\varphi}, \boldsymbol{\psi}) & := \frac{1}{2V_S^2} \int_{\Gamma} (\mathcal{I}(\boldsymbol{\varphi} \cdot \mathbf{n}) \boldsymbol{\psi} \cdot \boldsymbol{\tau} + \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \mathcal{I}(\boldsymbol{\psi} \cdot \mathbf{n})) \, d\gamma, \\ m(\boldsymbol{\varphi}, \boldsymbol{\psi}) & := m_{\Omega}(\boldsymbol{\varphi}, \boldsymbol{\psi}) + m_{\Gamma}(\boldsymbol{\varphi}, \boldsymbol{\psi}), \end{cases}$$

as well as the stiffness bilinear form

$$(2.25) \quad \left\{ \begin{aligned} a(\boldsymbol{\varphi}, \boldsymbol{\psi}) &:= \int_{\Omega} (\nabla \varphi_P + \mathbf{curl} \varphi_S) \cdot (\nabla \psi_P + \mathbf{curl} \psi_S) \, d\mathbf{x} \\ &= \int_{\Omega} (\operatorname{div} \boldsymbol{\varphi} \operatorname{div} \boldsymbol{\psi} + \operatorname{curl} \boldsymbol{\varphi} \operatorname{curl} \boldsymbol{\psi}) \, d\mathbf{x}. \end{aligned} \right.$$

**Remark 2.3.** *Using an integration by parts along the boundary, one sees that an alternative expression for  $m_{\Gamma}(\cdot, \cdot)$  is*

$$(2.26) \quad m_{\Gamma}(\boldsymbol{\varphi}, \boldsymbol{\psi}) = -\frac{1}{2V_S^2} \int_{\Gamma} (\mathcal{I}(\boldsymbol{\varphi} \cdot \boldsymbol{\tau}) \boldsymbol{\psi} \cdot \mathbf{n} + \boldsymbol{\varphi} \cdot \mathbf{n} \mathcal{I}(\boldsymbol{\psi} \cdot \boldsymbol{\tau})) \, d\gamma.$$

Then, it can be shown (see [4]) that if  $\boldsymbol{\varphi}$  is a solution of (2.12, 2.13, 2.14), then it is solution of the variational problem (in which, by the way, the auxiliary unknown  $\mathbf{v}_{\Gamma}$  has disappeared)

$$(2.27) \quad \left\{ \begin{aligned} &\text{Find } \boldsymbol{\varphi} : [0, T] \rightarrow \mathbf{V}_0 \text{ such that} \\ &\frac{d^2}{dt^2} m(\boldsymbol{\varphi}(t), \boldsymbol{\psi}) + a(\boldsymbol{\varphi}(t), \boldsymbol{\psi}) = \ell(t, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \mathbf{V}_0, \end{aligned} \right.$$

where  $\ell(t, \cdot)$  is the continuous linear form in  $\mathbf{V}$  defined by

$$(2.28) \quad \ell(t, \boldsymbol{\psi}) := - \int_{\Omega} \mathbf{g}(\cdot, t) \cdot (\nabla \psi_P + \operatorname{curl} \psi_S) \, d\mathbf{x} \quad \forall \boldsymbol{\psi} \in \mathbf{V}.$$

Moreover, (2.27) is completed by initial conditions, coherent with (2.7), (2.10), (2.11) and  $\mathbf{g}(\cdot, 0) = 0$ , namely

$$(2.29) \quad \boldsymbol{\varphi}(\cdot, 0) = \partial_t \boldsymbol{\varphi}(\cdot, 0) = 0.$$

However, it is shown in [4] that (2.27) is not a good formulation to work with due to stability issues. This does not provide a well-posed problem because the quadratic form  $m(\cdot, \cdot)$  fails to be positive in  $\mathbf{V}_0$  due to  $m_{\Gamma}(\cdot, \cdot)$  (see Theorem 3.6 in [4] or Theorem 2.9 below): more precisely, the space  $\mathbf{V}_0$  is too big and must be replaced by a smaller space  $\mathbf{V}_N$  that we will describe below. The construction of  $\mathbf{V}_N$  is far from obvious and, in some sense, it is the main contribution in [4].

2.4.2. *The good functional setting.* First note that, since  $\mathbf{0}$  is the barycenter of  $\Omega$

$$\mathbf{R}_0(\Omega) := \left\{ \mathbf{w} \in \mathbf{R}(\Omega) / \int_{\Omega} \mathbf{w} \, d\mathbf{x} = 0 \right\} = \operatorname{span} [(x_2, -x_1)^t].$$

We next introduce the subspace  $\mathbf{K}$  of  $\mathbf{V}_0$  defined by

$$(2.30) \quad \mathbf{K} = \left\{ \boldsymbol{\psi} = (\psi_P, \psi_S) \in \mathbf{V}_0 / \nabla \psi_P + \mathbf{curl} \psi_S \in \mathbf{R}(\Omega) \right\},$$

or equivalently, thanks to (2.23),

$$(2.31) \quad \mathbf{K} = \left\{ \boldsymbol{\psi} \in \mathbf{V} / (\operatorname{div} \boldsymbol{\psi}, -\operatorname{curl} \boldsymbol{\psi})^t \in \mathbf{R}_0(\Omega) \right\}.$$

Even though  $m(\cdot, \cdot)$  is not a scalar product (indeed it is not positive), we can speak of  $m$ -orthogonality:

$$\boldsymbol{\varphi}, \boldsymbol{\psi} \text{ are } m\text{-orthogonal} \iff m(\boldsymbol{\varphi}, \boldsymbol{\psi}) = 0.$$



We can then define the Hilbert space  $\mathbf{V}_N$  as the  $m$ -orthogonal of  $\mathbf{K}$

$$(2.32) \quad \mathbf{V}_N := \mathbf{K}^{\perp, m} = \{\varphi \in \mathbf{V}_0 / m(\varphi, \psi) = 0, \forall \psi \in \mathbf{K}\}.$$

By continuity of  $m(\cdot, \cdot)$  in  $\mathbf{V}$ ,  $\mathbf{V}_N$  is a closed subspace of  $\mathbf{V}_0$  and, as a consequence, it is a Hilbert space for the norm (2.20). Initially, in [4], the space  $\mathbf{V}_N$  was not constructed as (2.32) but as

$$(2.33) \quad \mathbf{V}_N := \left\{ \left( V_P^2 \operatorname{div} \mathbf{w}, -V_S^2 \operatorname{curl} \mathbf{w} \right), \mathbf{w} \in \mathbf{D}_N \right\}.$$

We refer to Theorem 3.17 in [4] for the equivalence between (2.33) and (2.32). An important fact is that, since the solution  $\mathbf{u} \in C^0([0, T]; \mathbf{D}_N)$ , see (2.9), then the field  $\varphi = (\varphi_P, \varphi_S)$  defined by (2.10) satisfies

$$(2.34) \quad \varphi \in C^1([0, T]; \mathbf{V}_N).$$

For practical reasons, it is essential to have a more useful description of  $\mathbf{K}$  which, moreover, emphasizes the fact that  $\mathbf{K}$  is a *small* space compared to  $\mathbf{V}_0$ .

To do so, it is first useful to introduce the subspace  $\mathbf{K}_0$  of  $\mathbf{K}$  defined by

$$(2.35) \quad \mathbf{K}_0 = \left\{ \psi \in \mathbf{V}_0 / \operatorname{div} \psi = \operatorname{curl} \psi = 0 \right\}.$$

According to (2.23) (thanks to geometrical assumptions on  $\Omega$ ),  $\mathbf{V}_0$  can be replaced by  $\mathbf{V}$  in the above definition, i.e.,

$$(2.36) \quad \mathbf{K}_0 = \left\{ \psi \in \mathbf{V} / \operatorname{div} \psi = \operatorname{curl} \psi = 0 \right\},$$

which is known as the space of so-called *harmonic fields*. The space  $\mathbf{K}_0$  is closely related to the space of *harmonic functions*

$$(2.37) \quad \mathcal{H}(\Omega) := \left\{ q \in H^1(\Omega) / \Delta q = 0 \text{ in } \Omega \right\}.$$

It is well known that the space  $\mathcal{H}(\Omega)/\mathbb{R}$  is isomorphic to the space  $M$  (defined in (2.16)) of boundary functions. To emphasize this property, we introduce the so-called *harmonic lifting operator*

$$(2.38) \quad \nu \in M \quad \longrightarrow \quad p(\nu) \in \mathcal{H}(\Omega),$$

where  $p(\nu) \in H^1(\Omega)$  is the unique solution in  $\mathcal{H}(\Omega)$  of

$$(2.39) \quad -\Delta p(\nu) = 0, \quad \text{in } \Omega, \quad \partial_{\mathbf{n}} p(\nu) = \nu, \quad \text{in } \Gamma,$$

that satisfies (see Remark 2.4)

$$(2.40) \quad \mathcal{I}(\nabla p \cdot \boldsymbol{\tau}) = p.$$

**Remark 2.4.** *Let us draw the attention that, as it is well-known, (2.39) defines  $p$  up to an additive constant  $A$ . Since the function  $d := \mathcal{I}(\nabla p \cdot \boldsymbol{\tau}) - p$  is a constant for any  $p$ , ( $\partial_{\boldsymbol{\tau}} d = 0$ ) we can adjust the constant  $A$  in such a way that (2.40) holds.*

The following lemma is immediate.

**Lemma 2.5.** *The harmonic lifting operator  $\nu \rightarrow p(\nu)$  is an isomorphism from  $M$  in*

$$\{p \in \mathcal{H}(\Omega) / \mathcal{I}(\nabla p \cdot \boldsymbol{\tau}) = p\}.$$

In order to relate  $\mathbf{K}_0$  with  $\mathcal{H}(\Omega)$  (thus with  $M$  according to Lemma 2.5), we introduce the operator

$$(2.41) \quad \begin{cases} \nu \in M & \longrightarrow & \mathcal{E}(\nu) \in \mathbf{K}_0 \text{ such that} \\ \forall \nu \in M, & & \mathcal{E}(\nu) = \nabla p(\nu). \end{cases}$$

Note that the fact that  $\mathcal{E}(\nu)$  is curl free is a property of any gradient while the fact that it is divergence free follows from the first equation of (2.39). Thus  $\mathcal{E}(\nu) \in \mathbf{K}_0$ .

With this operator, we can define a projector on  $\mathbf{K}_0$  in the space  $\mathbf{V}_0$ , namely

$$(2.42) \quad \forall \boldsymbol{\varphi} \in \mathbf{V}_0, \quad \mathbf{P}_0 \boldsymbol{\varphi} := \mathcal{E}(\boldsymbol{\varphi} \cdot \mathbf{n}) = \nabla p(\boldsymbol{\varphi} \cdot \mathbf{n}).$$

**Lemma 2.6.** *The operator  $\mathbf{P}_0$  is a projector from  $\mathbf{V}_0$  onto  $\mathbf{K}_0$ .*

**Proof.** In order to check that  $\mathbf{P}_0$  is a projector on  $\mathbf{K}_0$ , it suffices to check that  $\mathbf{P}_0 \boldsymbol{\varphi} = \boldsymbol{\varphi}$  for any  $\boldsymbol{\varphi} \in \mathbf{K}_0$ . Since  $\text{curl } \boldsymbol{\varphi} = 0$  and  $\Omega$  is simply connected, by Theorem 2.9 in [14], we deduce that  $\boldsymbol{\varphi} = \nabla q$  with  $q \in H^1(\Omega)/\mathbb{R}$ . Since  $\text{div } \boldsymbol{\varphi} = 0$ , we have  $\Delta q = 0$  and moreover, by construction,  $\partial_{\mathbf{n}} q = \boldsymbol{\varphi} \cdot \mathbf{n}$ . Let  $p_{\boldsymbol{\varphi}} := p(\boldsymbol{\varphi} \cdot \mathbf{n})$ , we remark that  $\Delta(q - p_{\boldsymbol{\varphi}}) = 0$  and  $\partial_{\mathbf{n}}(q - p_{\boldsymbol{\varphi}}) = 0$ . Thus  $q - p_{\boldsymbol{\varphi}}$  is constant and  $\mathbf{P}_0 \boldsymbol{\varphi} := \nabla p_{\boldsymbol{\varphi}} = \nabla q = \boldsymbol{\varphi}$ . This concludes the proof.  $\blacksquare$

Then, a complete description of the space  $\mathbf{K}_0$  is provided by the following lemmas (which are the correct versions of Lemma 3.19 in [4], which contained a mistake).

**Lemma 2.7.** *The space  $\mathbf{K}_0$  is characterized as*

$$(2.43) \quad \mathbf{K}_0 = \nabla \mathcal{H}(\Omega).$$

*More precisely, the operator  $\mathcal{E}$  is an isomorphism from  $M$  in  $\mathbf{K}_0$ .*

**Proof.** The injectivity of  $\mathcal{E}$  follows from (2.39) ( $\nabla p(\nu) = 0 \implies \nu = 0$ ) and the surjectivity from the fact that, for any  $\boldsymbol{\varphi} \in \mathbf{K}_0$ ,  $\boldsymbol{\varphi} = \mathbf{P}_0 \boldsymbol{\varphi} \equiv \mathcal{E}(\boldsymbol{\varphi} \cdot \mathbf{n})$ .  $\blacksquare$

**Lemma 2.8.** *The space  $\mathbf{K}$  can be decomposed as the direct sum*

$$(2.44) \quad \mathbf{K} = \mathbf{K}_R \oplus \mathbf{K}_0,$$

*where  $\mathbf{K}_R$  is the 1 dimensional space  $\mathbf{K}_R = \text{span}\{\boldsymbol{\varphi}_R\}$ , with  $\boldsymbol{\varphi}_R = \frac{1}{2}(0, |\mathbf{x}|^2)^t$ .*

**Proof.** Due to the fact that the intersection of  $\mathbf{R}(\Omega)$  (of dimension 3) with  $\mathbf{V}_0$  (which includes two linear constraints), we expect that the co-dimension of  $\mathbf{K}_0$ , as a subspace of  $\mathbf{K}$ , is one.

First remark that since  $(\text{div } \boldsymbol{\varphi}_R, -\text{curl } \boldsymbol{\varphi}_R)^t = (x_2, -x_1)^t$ ,  $\boldsymbol{\varphi}_R$  belongs to  $\mathbf{K}$  (Definition (2.31)) but not to  $\mathbf{K}_0$  (Definition (2.35)). To conclude the proof, it suffices to show that for any  $\boldsymbol{\psi}$  in  $\mathbf{K}$  there exists  $a \in \mathbb{R}$  such that  $\boldsymbol{\psi} - a \boldsymbol{\varphi}_R \in \mathbf{K}_0$ . But, for any  $\boldsymbol{\psi}$  in  $\mathbf{K}$  (see (2.31)),  $(\text{div } \boldsymbol{\psi}, -\text{curl } \boldsymbol{\psi})^t \in \mathbf{R}_0(\Omega)$ , i.e.

$$(\text{div } \boldsymbol{\psi}, -\text{curl } \boldsymbol{\psi})^t = a (x_2, -x_1) \equiv a (\text{div } \boldsymbol{\varphi}_R, -\text{curl } \boldsymbol{\varphi}_R)^t.$$

It is then immediate to conclude.  $\blacksquare$

2.4.3. *Properties of the bilinear form  $m(\cdot, \cdot)$ .* In this section, we establish fundamental properties of the bilinear form  $m(\cdot, \cdot)$  that explain why the functional setting introduced in the previous section is the good one. These properties essentially concern the sign of the corresponding quadratic form when restricted to particular subspaces of  $\mathbf{V}_0$ .

The first property is a *negative* result that explains the instability of the naive formulation of Section 2.4.1. More precisely, the following result gives an alternative proof of the negativity result of Theorem 3.6 in [4]. It is also more precise in the sense that it emphasizes the “bad role” of  $\mathbf{K}_0$  and thus enlightens the need for removing this space in the construction of the “good” space  $\mathbf{V}_N$  (see (2.32)).

**Theorem 2.9.** *In the space  $\mathbf{K}_0$ , we have the formula*

$$(2.45) \quad \forall \varphi \in \mathbf{K}_0, \quad m_\Gamma(\varphi, \varphi) = -\frac{1}{V_S^2} \int_\Omega |\varphi|^2 \, d\mathbf{x}.$$

*As a consequence, the quadratic form  $m(\varphi, \varphi)$  is negative semi-definite in  $\mathbf{K}_0$ :*

$$(2.46) \quad \forall \varphi \in \mathbf{K}_0, \quad m(\varphi, \varphi) = -\frac{1}{V_*^2} \int_\Omega |\varphi_P|^2 \, d\mathbf{x}, \quad \frac{1}{V_*^2} := \frac{1}{V_S^2} - \frac{1}{V_P^2} > 0.$$

**Proof.** According to (2.42), for  $\varphi \in \mathbf{K}_0$ ,  $\varphi = \mathbf{P}_0 \varphi = \nabla p_\varphi$  with  $p_\varphi = p(\varphi \cdot \mathbf{n})$  (cf. (2.39, 2.40)). By definition of  $m_\Gamma(\cdot, \cdot)$

$$m_\Gamma(\varphi, \varphi) = \frac{1}{V_S^2} \int_\Gamma \mathcal{I}(\varphi \cdot \mathbf{n}) \varphi \cdot \boldsymbol{\tau} \, d\boldsymbol{\gamma} = \frac{1}{V_S^2} \int_\Gamma \mathcal{I}(\partial_n p_\varphi) \partial_\tau p_\varphi \, d\boldsymbol{\gamma}.$$

Next, using an integration by parts along the boundary and  $\mathcal{I}(\partial_\tau p_\varphi) = p_\varphi$  first, then Green’s formula and the fact that  $p_\varphi$  is harmonic, we get

$$m_\Gamma(\varphi, \varphi) \, d\boldsymbol{\gamma} = -\frac{1}{V_S^2} \int_\Gamma \partial_n p_\varphi p_\varphi \, d\boldsymbol{\gamma} = -\frac{1}{V_S^2} \int_\Omega |\nabla p_\varphi|^2 \, d\mathbf{x} = -\frac{1}{V_S^2} \int_\Omega |\varphi|^2 \, d\mathbf{x}.$$

since, for the last equality,  $\nabla p_\varphi = \varphi$ . It is then easy to conclude.  $\blacksquare$

Let us introduce the following functional space defined by

$$(2.47) \quad \mathbf{W}_N := (\mathbf{K}_0)^{\perp, m} = \{\varphi \in \mathbf{V}_0 / m(\varphi, \psi) = 0, \forall \psi \in \mathbf{K}_0\} \supset \mathbf{V}_N.$$

Our second and most important result is the positive counterpart of Theorem 2.9 and fully justifies the choice of  $\mathbf{V}_N$ . The fundamental property of the space  $\mathbf{V}_N$  is given by the following theorem.

**Theorem 2.10.** *In the space  $\mathbf{W}_N$ , the quadratic form  $m(\varphi, \varphi)$  takes the form, with  $p_\varphi = p(\varphi \cdot \mathbf{n})$  and  $1/V_*^2 := 1/V_S^2 - 1/V_P^2 > 0$ ,*

$$(2.48) \quad m(\varphi, \varphi) = \frac{1}{V_*^2} \int_\Omega |\partial_1 p_\varphi|^2 \, d\mathbf{x} + m_\Omega(\varphi - \nabla p_\varphi, \varphi - \nabla p_\varphi).$$

*In particular, it is positive definite in  $\mathbf{V}_N$*

$$(2.49) \quad \forall \varphi \in \mathbf{V}_N \setminus \{0\}, \quad m(\varphi, \varphi) > 0.$$

**Proof.** *Step 1: Proof of (2.48).*

For any  $\varphi \in \mathbf{W}_N$ , setting  $p_\varphi = p(\varphi \cdot \mathbf{n})$  (see (2.39)), we know that  $\nabla p_\varphi (\equiv \mathbf{P}_0 \varphi)$

belongs to  $\mathbf{K}_0$ , so that  $m(\boldsymbol{\varphi}, \nabla p_\varphi) = 0$  (see (2.47)), that is to say, by definition of  $m(\cdot, \cdot)$ ,

$$(2.50) \quad \begin{aligned} & \frac{1}{V_P^2} \int_{\Omega} \varphi_P \partial_1 p_\varphi \, d\mathbf{x} + \frac{1}{V_S^2} \int_{\Omega} \varphi_S \partial_2 p_\varphi \, d\mathbf{x} \\ & + \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\varphi} \cdot \mathbf{n}) \partial_\tau p_\varphi \, d\gamma + \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\partial_n p_\varphi) \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \, d\gamma = 0. \end{aligned}$$

As we are going to see, this equality provides a new expression for  $m_\Gamma(\cdot, \cdot)$  in  $\mathbf{W}_N$  that will lead to (2.48). For this, we transform the last two terms of (2.50) into volume integrals, using the fact that

$$(2.51) \quad -\Delta p_\varphi = 0, \quad \text{in } \Omega, \quad \partial_n p_\varphi = \boldsymbol{\varphi} \cdot \mathbf{n}, \quad \text{on } \Gamma.$$

Using an integration by parts along  $\Gamma$ ,  $\mathcal{I}(\partial_\tau p_\varphi) = p_\varphi$  and the fact that  $\int_{\Gamma} \boldsymbol{\varphi} \cdot \mathbf{n} \, d\gamma = 0$ , we obtain

$$\frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\varphi} \cdot \mathbf{n}) \partial_\tau p_\varphi \, d\gamma = -\frac{1}{2V_S^2} \int_{\Gamma} (\boldsymbol{\varphi} \cdot \mathbf{n}) p_\varphi \, d\gamma.$$

Therefore, using (2.51) and Green's formula,

$$(2.52) \quad \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\varphi} \cdot \mathbf{n}) \partial_\tau p_\varphi \, d\gamma = -\frac{1}{2V_S^2} \int_{\Omega} \partial_n p_\varphi p_\varphi \, d\mathbf{x} = -\frac{1}{2V_S^2} \int_{\Omega} |\nabla p_\varphi|^2 \, d\mathbf{x}.$$

On the other hand, since  $\partial_n p_\varphi = \boldsymbol{\varphi} \cdot \mathbf{n}$  on  $\Gamma$ , we simply have

$$(2.53) \quad \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\partial_n p_\varphi) \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \, d\gamma = \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\varphi} \cdot \mathbf{n}) \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \, d\gamma = \frac{1}{2} m_\Gamma(\boldsymbol{\varphi}, \boldsymbol{\varphi}).$$

Substituting (2.52) and (2.53) into (2.50), we obtain, after multiplication by 2,

$$m_\Gamma(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = \frac{1}{V_S^2} \int_{\Omega} |\nabla p_\varphi|^2 \, d\mathbf{x} - \frac{2}{V_P^2} \int_{\Omega} \varphi_P \partial_1 p_\varphi \, d\mathbf{x} - \frac{2}{V_S^2} \int_{\Omega} \varphi_S \partial_2 p_\varphi \, d\mathbf{x},$$

and therefore, adding  $m_\Omega(\boldsymbol{\varphi}, \boldsymbol{\varphi})$ ,

$$\begin{aligned} m(\boldsymbol{\varphi}, \boldsymbol{\varphi}) &= \frac{1}{V_S^2} \int_{\Omega} |\varphi_S|^2 \, d\mathbf{x} - \frac{2}{V_S^2} \int_{\Omega} \varphi_S \partial_2 p_\varphi \, d\mathbf{x} + \frac{1}{V_S^2} \int_{\Omega} |\partial_2 p_\varphi|^2 \, d\mathbf{x} \\ &+ \frac{1}{V_P^2} \int_{\Omega} |\varphi_P|^2 \, d\mathbf{x} - \frac{2}{V_P^2} \int_{\Omega} \varphi_P \partial_1 p_\varphi \, d\mathbf{x} + \frac{1}{V_S^2} \int_{\Omega} |\partial_1 p_\varphi|^2 \, d\mathbf{x}. \end{aligned}$$

Rearranging cleverly the above expression as a sum of squares then leads to (2.48).

*Step 2:  $m(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = 0$  and  $\boldsymbol{\varphi} \in \mathbf{V}_N$  implies  $\boldsymbol{\varphi} = 0$ .*

Indeed, using (2.48),  $m(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = 0$  implies  $\partial_1 p_\varphi = 0$  and  $\boldsymbol{\varphi} = \nabla p_\varphi$ , thus  $\varphi_P = 0$  and  $p_\varphi(x_1, x_2) = p(x_2)$ . Since  $p_\varphi$  is harmonic,  $p$  is simply an affine of function, thus  $\varphi_S \equiv \partial_2 p$  is a constant, denoted  $a$ . Written differently, we thus have

$$(2.54) \quad \boldsymbol{\varphi} = a \boldsymbol{\varphi}_*, \quad \boldsymbol{\varphi}_* := (0, 1)^t.$$

It remains to show that  $a = 0$ . This is where we are going to use the m-orthogonality to the space  $\mathbf{K}_R = \text{span}\{\boldsymbol{\varphi}_R\}$ .

Since  $m(\boldsymbol{\varphi}, \boldsymbol{\varphi}_R) = a m(\boldsymbol{\varphi}_*, \boldsymbol{\varphi}_R)$ , it suffices to check that  $m(\boldsymbol{\varphi}_*, \boldsymbol{\varphi}_R) \neq 0$ .

We firstly compute, with  $\varphi_R = \frac{1}{2}(0, |\mathbf{x}|^2)^t$  (cf. Lemma 2.8), that

$$(2.55) \quad m_\Omega(\varphi_*, \varphi_R) = \frac{\mathbf{J}}{2V_S^2}, \quad \mathbf{J} := \int_\Omega |\mathbf{x}|^2 d\mathbf{x} > 0 \quad (\text{inertia momentum of } \Omega).$$

Since  $\varphi_* = \nabla q_2 = -\mathbf{curl} q_1$  with  $q_2(\mathbf{x}) = x_2 + c_2$  and  $q_1(\mathbf{x}) = x_1 + c_1$ , we have

$$(2.56) \quad \varphi_* \cdot \mathbf{n} = \partial_\tau q_1, \quad \varphi_* \cdot \boldsymbol{\tau} = \partial_\tau q_2, \quad \text{on } \Gamma,$$

while

$$(2.57) \quad \varphi_R \cdot \mathbf{n} = \frac{1}{2} |\mathbf{x}|^2 n_2, \quad \varphi_R \cdot \boldsymbol{\tau} = -\frac{1}{2} |\mathbf{x}|^2 n_1, \quad \text{on } \Gamma.$$

Thus, since one can choose  $c_1$  such that  $\mathcal{I}(\partial_\tau q_1) = q_1$  (see Remark 2.4),

$$(2.58) \quad \begin{aligned} \int_\Gamma \mathcal{I}(\varphi_* \cdot \mathbf{n}) (\varphi_R \cdot \boldsymbol{\tau}) d\gamma &= \int_\Gamma q_1 (\varphi_R \cdot \boldsymbol{\tau}) d\gamma \\ &= -\frac{1}{2} \int_\Gamma (x_1 + c_1) |\mathbf{x}|^2 n_1 d\gamma = -\frac{1}{2} \int_\Gamma x_1 |\mathbf{x}|^2 n_1 d\gamma. \end{aligned}$$

On the other hand, choosing  $c_2$  such that  $\mathcal{I}(\partial_\tau q_2) = q_2$ , with an integration by parts along  $\Gamma$ , we get

$$(2.59) \quad \begin{aligned} \int_\Gamma (\varphi_* \cdot \boldsymbol{\tau}) \mathcal{I}(\varphi_R \cdot \mathbf{n}) d\gamma &= -\int_\Gamma \mathcal{I}(\varphi_* \cdot \boldsymbol{\tau}) (\varphi_R \cdot \mathbf{n}) d\gamma \\ &= -\int_\Gamma q_2 (\varphi_R \cdot \mathbf{n}) d\gamma = -\frac{1}{2} \int_\Gamma x_2 |\mathbf{x}|^2 n_2 d\gamma. \end{aligned}$$

In consequence, by adding the two previous equalities we obtain

$$m_\Gamma(\varphi_*, \varphi_R) = \frac{-1}{4V_S^2} \int_\Gamma \mathbf{x} \cdot \mathbf{n} |\mathbf{x}|^2 d\gamma = \frac{-1}{4V_S^2} \int_\Omega (\operatorname{div}(\mathbf{x}) |\mathbf{x}|^2 + \mathbf{x} \nabla |\mathbf{x}|^2) d\mathbf{x} = -\frac{\mathbf{J}}{V_S^2}.$$

Finally

$$(2.60) \quad m(\varphi_*, \varphi_R) = -\frac{\mathbf{J}}{2V_S^2} \neq 0,$$

which concludes the proof.  $\blacksquare$

**Remark 2.11.** In [4], Theorem 3.14, we even proved a stronger result, namely,

$$(2.61) \quad m(\varphi, \varphi) \geq C \int_\Omega |\varphi|^2 d\mathbf{x}, \quad \forall \varphi \in \mathbf{V}_N, \quad \text{for some } C > 0.$$

The proof used the characterization (2.33) of the space  $\mathbf{V}_N$  and Korn's inequality in  $\mathbf{D}_R$ . The interest of the proof of Theorem 2.10 is that it extends to the discrete setting as it will be shown later.

In the other hand, it is not clear how to deduce (2.61) from (2.48, 2.49). Note nevertheless that (2.48) yields the following "partial"  $L^2$  coercivity result, namely

$$(2.62) \quad m(\varphi, \varphi) \geq \alpha_* \int_\Omega |\varphi_P|^2 d\mathbf{x}, \quad \forall \varphi \in \mathbf{V}_N, \quad \text{for some } \alpha_* > 0.$$

2.4.4. *A stable weak formulation.* A first observation is that  $\varphi$  the solution of (2.12), (2.13), (2.14) and (2.29) not only satisfies (2.27) but also

$$(2.63) \quad \varphi(t) \in \mathbf{V}_N, \quad \forall t \in [0, T].$$

Indeed, since  $\varphi$  is related to the solution  $\mathbf{u}$  of (2.3), (2.6) and (2.7), by (see (2.10))

$$\partial_t \varphi_P = V_P^2 \operatorname{div} \mathbf{u}, \quad \partial_t \varphi_S = -V_S^2 \operatorname{curl} \mathbf{u},$$

where, at any time  $t$ ,  $\mathbf{u}(\cdot, t)$  belongs to  $\mathbf{D}_N$  (see Theorem 2.1). Then  $\partial_t \varphi(t) \in \mathbf{V}_N$  thanks to (2.33) (the same holds for  $\varphi$  due to the zero initial conditions).

As a consequence, since  $\mathbf{V}_N \subset \mathbf{V}_0$ , we deduce from (2.27) that  $\varphi$  is solution of

$$(2.64) \quad \begin{cases} \text{Find } \varphi : [0, T] \longrightarrow \mathbf{V}_N \text{ satisfying (2.29) and s.t.} \\ \frac{d^2}{dt^2} m(\varphi(t), \psi) + a(\varphi(t), \psi) = \ell(t, \psi), \quad \forall \psi \in \mathbf{V}_N, \end{cases}$$

which is now set in a nice variational framework due to (2.49) (see Theorem 2.10) and even more (2.61). We point out that the mapping  $\varphi \mapsto \sqrt{m(\varphi, \varphi)}$  is a norm in the space  $\mathbf{V}_N$  which satisfies

$$(2.65) \quad \mathcal{C}_1 \|\varphi\|_{\mathbf{L}^2(\Omega)}^2 \leq m(\varphi, \varphi) \leq \mathcal{C}_2 \|\varphi\|_{\mathbf{V}}^2.$$

However, this norm is not equivalent to the norm in  $\mathbf{V}_N$  that is  $\|\cdot\|_{\mathbf{V}}$  and is given by (2.20), nor to the  $\mathbf{L}^2(\Omega)$  norm. For this reason, problem (2.64) does not fit the standard theory for second order hyperbolic problems, see [8] for instance. However, the next theorem states that (2.64) has a unique solution. To state the result we recall that the source term in (2.64) is given by (2.28), and more precisely

$$(2.66) \quad \ell(t, \psi) = - \int_{\Omega} \mathbf{g} \cdot (\operatorname{div} \psi, -\operatorname{curl} \psi)^t \, d\mathbf{x} \text{ where } \mathbf{g}(t) := \frac{1}{\rho} \int_0^t \mathbf{f}(s) \, ds,$$

with  $\mathbf{f} \in L^1(0, T; \mathbf{L}_R^2(\Omega))$ . As the proof of this result is not central for the purpose of this article, we have chosen to delay it to the appendix.

**Theorem 2.12.** *Assume that  $\mathbf{f} \in L^1(0, T; \mathbf{L}_R^2(\Omega))$ . Then there exists a unique solution  $\varphi$  of (2.29, 2.64) such that*

$$(2.67) \quad \varphi \in L^2(0, T, \mathbf{V}_N), \quad \partial_t \varphi \in L^2(0, T, \mathbf{L}^2(\Omega)),$$

where the involved time derivatives are considered in the distributional sense.

2.4.5. *An equivalent mixed formulation.* We introduce additional unknowns that are Lagrange multipliers to take into account the property (2.63), that is to say, according to the definition (2.32) of  $\mathbf{V}_N$  as the  $m$ -orthogonal of  $\mathbf{K}$  and the characterization of  $\mathbf{K}$  via the lemmas 2.7 and 2.8. These additional unknowns are thus

- a boundary unknown  $\eta(t) \in M$  to treat the  $m$ -orthogonality to  $\mathbf{K}_0 \equiv \mathcal{E}(M)$ ,
- a scalar unknown  $\eta_r(t) \in \mathbb{R}$  to treat the  $m$ -orthogonality to  $\mathbf{K}_R$ .

This allows us to search the unknown  $\varphi$  directly in the space  $\mathbf{V}_0$ , which is much easier to discretize than the space  $\mathbf{V}_N$ . The mixed variational problem then reads

as follows

$$(2.68) \quad \begin{cases} \text{Find } (\varphi, \eta, \eta_r): [0, T] \longrightarrow \mathbf{V}_0 \times M \times \mathbb{R} \text{ satisfying (2.29) and s.t.} \\ \frac{d^2}{dt^2} m(\varphi, \psi) + a(\varphi, \psi) + b(\eta, \psi) + \eta_r m(\varphi_R, \psi) = \ell(t, \psi), \quad \forall \psi \in \mathbf{V}_0, \\ m(\varphi_R, \varphi) = 0, \quad \text{and} \quad b(\nu, \varphi) = 0, \quad \forall \nu \in M, \end{cases}$$

where the bilinear form  $b(\cdot, \cdot)$  is defined by

$$(2.69) \quad \forall (\nu, \psi) \in M \times \mathbf{V}_0, \quad b(\nu, \psi) := m(\mathcal{E}(\nu), \psi).$$

We have the following equivalence theorem between (2.64) and (2.68). We refer the reader to theorem 3.18 of [4] for a straightforward proof of this result

**Theorem 2.13.** *The problem (2.68) admits a unique solution  $(\varphi, \eta, \eta_r)$  where moreover,  $\varphi$  is the solution of (2.29, 2.64) with regularity (2.67), and  $\eta = \eta_r = 0$ .*

### 3. SPACE-TIME DISCRETIZATION

#### 3.1. Galerkin space semi-discretization.

*Discrete functional setting.* The consideration of adequate Galerkin approximation of problem (2.68) relies first on the introduction of finite dimensional approximations of the space  $\mathbf{V}_0$  denoted  $\mathbf{V}_{0,h}$  that will be naturally sought in the form  $\mathbf{V}_{0,h} = \mathbf{V}_0 \cap \mathbf{V}_h$  with

$$(3.1) \quad \mathbf{V}_h = V_{P,h} \times V_{S,h} \subset \mathbf{V},$$

where  $V_{P,h}$  and  $V_{S,h}$  are subspaces of  $H^1(\Omega)$  and  $h$  is an approximation parameter devoted to tend to 0. At this moment they are abstract spaces but in practice we aim at using typically standard Lagrange finite element approximations (see Section 4). They should satisfy the following approximation property, for all  $\psi \in \mathbf{H}^1(\Omega)$ ,

$$(3.2) \quad \lim_{h \rightarrow 0} \inf_{\psi_{P,h} \in V_{P,h}} \|\psi_{P,h} - \psi_P\|_{H^1(\Omega)} = 0, \quad \lim_{h \rightarrow 0} \inf_{\psi_{S,h} \in V_{S,h}} \|\psi_{S,h} - \psi_S\|_{H^1(\Omega)} = 0.$$

Thanks to the density of  $\mathbf{H}^1(\Omega)$  in  $\mathbf{V}$  (see [7]) and the continuous embedding of  $\mathbf{V}$  in  $\mathbf{H}^1(\Omega)$  this implies that,

$$(3.3) \quad \lim_{h \rightarrow 0} \inf_{\psi_h \in \mathbf{V}_h} \|\psi_h - \psi\|_{\mathbf{V}} = 0, \quad \forall \psi \in \mathbf{V},$$

which is a standard consistency condition for Galerkin approximation. We also introduce finite dimensional approximations of the space  $M$ ,

$$M_h \subset L^2(\Gamma) \cap M \subset M,$$

that satisfies the approximation property

$$(3.4) \quad \lim_{h \rightarrow 0} \inf_{\nu_h \in M_h} \|\nu_h - \nu\|_M = 0, \quad \forall \nu \in M.$$

A discrete bilinear form  $m_h(\cdot, \cdot)$ . In order to be able to mimic the proof of Theorem 2.10 which is closely related to stability properties that will be discussed later, it will be useful to make a non-conforming approximation of the bilinear form  $m(\cdot, \cdot)$ . As a matter of fact, the modification only concerns the bilinear form  $m_\Gamma(\cdot, \cdot)$ . We introduce for this the operator  $\Pi_h$ , the orthogonal projection from  $L^2(\Gamma)$  into  $M_h$ ,

$$(3.5) \quad (\Pi_h \nu) \in M_h / \int_\Gamma (\Pi_h \nu) \tilde{\nu}_h d\gamma = \int_\Gamma \nu \tilde{\nu}_h d\gamma, \quad \forall \tilde{\nu}_h \in M_h,$$

and we set, for all  $(\varphi_h, \psi_h) \in \mathbf{V}_h \times \mathbf{V}_h$

$$(3.6) \quad m_h(\varphi_h, \psi_h) := m_\Omega(\varphi_h, \psi_h) + m_{\Gamma,h}(\varphi_h, \psi_h),$$

where the symmetric bilinear form  $m_{\Gamma,h}(\cdot, \cdot)$  is defined consistently with the expression (2.26) for  $m_\Gamma(\cdot, \cdot)$ , with a clever use the projection operator  $\Pi_h$ , the interest of which will appear later (cf. Theorem 3.5),

$$(3.7) \quad m_{\Gamma,h}(\varphi_h, \psi_h) := -\frac{1}{2V_S^2} \int_\Gamma \mathcal{I}(\varphi_h \cdot \boldsymbol{\tau}) \Pi_h(\psi_h \cdot \mathbf{n}) d\gamma - \frac{1}{2V_S^2} \int_\Gamma \mathcal{I}(\psi_h \cdot \boldsymbol{\tau}) \Pi_h(\varphi_h \cdot \mathbf{n}) d\gamma.$$

**Remark 3.1.** Note that if the space of normal traces of functions in  $\mathbf{V}_h$ , namely

$$\mathbf{V}_{n,h}(\Gamma) = \{\boldsymbol{\psi}_h|_\Gamma \cdot \mathbf{n}, \boldsymbol{\psi}_h \in \mathbf{V}_h\} \subset L^2(\Gamma),$$

is included in  $M_h$  then  $m_{\Gamma,h}(\cdot, \cdot) = m_\Gamma(\cdot, \cdot)$  since then,  $\Pi_h \boldsymbol{\nu}_h = \boldsymbol{\nu}_h$ ,  $\forall \boldsymbol{\nu}_h \in \mathbf{V}_{n,h}(\Gamma)$ .

A discrete bilinear form  $b_h(\cdot, \cdot)$ . Due to the operator  $\mathcal{E}(\cdot)$ , it is not possible to deal with the bilinear form  $b(\nu, \boldsymbol{\psi})$  directly at the discrete level. For this reason we build a discrete version. A first step consists in providing an alternative expression of  $b(\cdot, \cdot)$  (see (2.69)). To get this expression we remark that if  $p \equiv p(\nu)$ , see (2.39) and (2.40), thus  $\nabla p \cdot \mathbf{n} = \nu$  and since  $b(\nu, \boldsymbol{\psi}) = m(\mathcal{E}(\nu), \boldsymbol{\psi})$  with  $\mathcal{E}(\nu) = \nabla p$ , we have

$$(3.8) \quad \left| \begin{aligned} b(\nu, \boldsymbol{\psi}) &= \frac{1}{V_P^2} \int_\Omega \partial_1 p \psi_P d\mathbf{x} + \frac{1}{V_S^2} \int_\Omega \partial_2 p \psi_S d\mathbf{x} \\ &\quad - \frac{1}{2V_S^2} \int_\Gamma \mathcal{I}(\boldsymbol{\psi} \cdot \boldsymbol{\tau}) \nu d\gamma - \frac{1}{2V_S^2} \int_\Gamma \mathcal{I}(\nabla p \cdot \boldsymbol{\tau}) \boldsymbol{\psi} \cdot \mathbf{n} d\gamma. \end{aligned} \right.$$

Since  $\mathcal{I}(\nabla p \cdot \boldsymbol{\tau}) = p$ , we have

$$(3.9) \quad \left| \begin{aligned} b(\nu, \boldsymbol{\psi}) &= \frac{1}{V_P^2} \int_\Omega \partial_1 p \psi_P d\mathbf{x} + \frac{1}{V_S^2} \int_\Omega \partial_2 p \psi_S d\mathbf{x} \\ &\quad - \frac{1}{2V_S^2} \int_\Gamma \mathcal{I}(\boldsymbol{\psi} \cdot \boldsymbol{\tau}) \nu d\gamma - \frac{1}{2V_S^2} \int_\Gamma p \boldsymbol{\psi} \cdot \mathbf{n} d\gamma. \end{aligned} \right.$$

As said above we need to make a non-conforming approximation of the mixed-problem (2.68) because in practice we can not manage with the operator  $\mathcal{E}(\cdot)$  itself and we need to replace it by a discrete version  $\mathcal{E}_h(\cdot)$  which is defined by the Galerkin approximation of problem (2.39, 2.40). For this we introduce another finite dimensional subspace of  $H^1(\Omega)$ , denoted  $P_h$ , satisfying the same approximation property than (3.2) and that must contain the constant functions. More precisely we define  $\mathcal{E}_h \in \mathcal{L}(M_h, \mathbf{L}^2(\Omega))$  such that  $\mathcal{E}_h \nu_h = \nabla p_h$  where  $p_h \equiv p_h(\nu_h)$  is defined as the



unique solution of

$$(3.10) \quad \left\{ \begin{array}{l} \text{Find } p_h \in P_h \text{ such that} \\ \int_{\Omega} \nabla p_h \cdot \nabla q_h \, d\mathbf{x} = \int_{\Gamma} \nu_h q_h \, d\gamma, \quad \forall q_h \in P_h, \\ \mathcal{I}(\nabla p_h \cdot \boldsymbol{\tau}) = p_h. \end{array} \right.$$

Note that  $\nabla p_h$  does not belong to  $\mathbf{V}$  in general but belongs to  $\mathbf{L}^2(\Omega)$ . We define  $b_h(\cdot, \cdot)$  for all  $\nu_h \in M_h$  and  $\boldsymbol{\psi}_h \in \mathbf{V}_h$  by, replacing  $p(\nu)$  with  $p_h(\nu_h)$  in the expression (3.9) of  $b(\cdot, \cdot)$ , making use again of the operator  $\Pi_h$  in order to counter-balance its presence in the definition of  $m_{\Gamma,h}(\cdot, \cdot)$  (cf. (3.7)),

$$(3.11) \quad \left\{ \begin{array}{l} b_h(\nu_h, \boldsymbol{\psi}_h) := \frac{1}{V_P^2} \int_{\Omega} \partial_1 p_h \psi_{P,h} \, d\mathbf{x} + \frac{1}{V_S^2} \int_{\Omega} \partial_2 p_h \psi_{S,h} \, d\mathbf{x} \\ - \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\psi}_h \cdot \boldsymbol{\tau}) \nu_h \, d\gamma - \frac{1}{2V_S^2} \int_{\Gamma} p_h \Pi_h(\boldsymbol{\psi}_h \cdot \mathbf{n}) \, d\gamma. \end{array} \right.$$

Note  $b_h(\cdot, \cdot)$  is well defined in  $M_h \times \mathbf{V}_h$  because the normal and tangential traces of  $\boldsymbol{\psi}_h \in \mathbf{V}_h$  are in  $L^2(\Gamma)$  which is the domain of definition of the integral operator  $\mathcal{I}(\cdot)$  (see (2.15)).

**Remark 3.2.** *Note that in particular  $p_h$ , defined by (3.10), is a discrete harmonic function in the sense that it satisfies*

$$\int_{\Omega} \nabla p_h \cdot \nabla q_h \, d\mathbf{x} = 0, \quad \forall q_h \in P_h \cap H_0^1(\Omega),$$

which is discrete equivalent of the property

$$\int_{\Omega} \nabla p \cdot \nabla q \, d\mathbf{x} = 0, \quad \forall q \in H_0^1(\Omega),$$

that characterizes harmonic functions that belongs to  $H^1(\Omega)$ .

*The semi-discrete approximation of the mixed problem.* We consider the semi-discrete equivalent of the initial condition (2.29),

$$(3.12) \quad \boldsymbol{\varphi}_h(\cdot, 0) = \partial_t \boldsymbol{\varphi}_h(\cdot, 0) = 0,$$

and, in accordance to the previous paragraphs we consider the following approximation of (2.68)

$$(3.13) \quad \left\{ \begin{array}{l} \text{Find } (\boldsymbol{\varphi}_h, \eta_h, \eta_r) : [0, T] \longrightarrow \mathbf{V}_{0,h} \times M_h \times \mathbb{R} \text{ satisfying (3.12) and} \\ \frac{d^2}{dt^2} m_h(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + a(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) \\ + b_h(\eta_h, \boldsymbol{\psi}_h) + \eta_r m(\boldsymbol{\varphi}_R, \boldsymbol{\psi}_h) = \ell(t, \boldsymbol{\psi}_h), \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_{0,h}, \\ m(\boldsymbol{\varphi}_R, \boldsymbol{\varphi}_h) = 0, \quad \text{and} \quad b_h(\nu_h, \boldsymbol{\varphi}_h) = 0, \quad \forall \nu_h \in M_h. \end{array} \right.$$

In order to work in the space  $\mathbf{V}_h$  instead of  $\mathbf{V}_{0,h}$ , we choose to treat the gauge conditions in the definition of  $\mathbf{V}_0$  (see (2.21)) in a weak (but exact) way by introducing two new scalar unknowns  $(\eta_{\mathbf{n}}, \eta_{\boldsymbol{\tau}}) \in \mathbb{R}^2$  that are Lagrange multipliers associated to the gauge conditions in the definition of  $\mathbf{V}_0$ . This provides the following mixed

formulation (the equivalence with (2.68) will be straightforward after existence and uniqueness which will be discussed in section 3.2)

$$(3.14) \quad \left\{ \begin{array}{l} \text{Find } (\boldsymbol{\varphi}_h, \eta_h, \eta_r, \eta_{\mathbf{n}}, \eta_{\boldsymbol{\tau}}) : [0, T] \rightarrow \mathbf{V}_h \times M_h \times \mathbb{R}^3 \text{ satisfying (3.12) and} \\ \frac{d^2}{dt^2} m_h(\boldsymbol{\varphi}_h(t), \boldsymbol{\psi}_h) + a(\boldsymbol{\varphi}_h(t), \boldsymbol{\psi}_h) + b_h(\eta_h(t), \boldsymbol{\psi}_h) + \eta_r(t) m(\boldsymbol{\varphi}_R, \boldsymbol{\psi}_h) \\ \quad + \eta_{\mathbf{n}}(t) \int_{\Gamma} \boldsymbol{\psi}_h \cdot \mathbf{n} \, d\boldsymbol{\gamma} + \eta_{\boldsymbol{\tau}}(t) \int_{\Gamma} \boldsymbol{\psi}_h \cdot \boldsymbol{\tau} \, d\boldsymbol{\gamma} = \ell(t, \boldsymbol{\psi}_h), \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_h, \\ b_h(\nu_h, \boldsymbol{\varphi}_h(t)) = 0, \quad \forall \nu_h \in M_h, \\ m(\boldsymbol{\varphi}_R, \boldsymbol{\varphi}_h(t)) = 0, \quad \int_{\Gamma} \boldsymbol{\varphi}_h \cdot \mathbf{n} \, d\boldsymbol{\gamma} = 0, \quad \int_{\Gamma} \boldsymbol{\varphi}_h \cdot \boldsymbol{\tau} \, d\boldsymbol{\gamma} = 0. \end{array} \right.$$

Moreover, using appropriate basis for the spaces  $\mathbf{V}_h$  and  $M_h$  of respective dimension  $N$  and  $M$  and denoting by  $\boldsymbol{\Phi}_h \in \mathbb{R}^N$ , and  $\mathbf{E}_h \in \mathbb{R}^M$  the vectors of degrees of freedom of  $\boldsymbol{\varphi}_h$  and  $\eta_h$  in this basis, problem (3.14) results into the following algebraic system of ODEs

$$(3.15) \quad \left\{ \begin{array}{l} \mathbb{M}_h \frac{d^2 \boldsymbol{\Phi}_h}{dt^2} + \mathbb{A}_h \boldsymbol{\Phi}_h + \mathbb{B}_h \mathbf{E}_h + \mathbb{G}_r \eta_r + \mathbb{G}_{\mathbf{n}} \eta_{\mathbf{n}} + \mathbb{G}_{\boldsymbol{\tau}} \eta_{\boldsymbol{\tau}} = \mathbf{F}_h, \\ \mathbb{B}_h^T \boldsymbol{\Phi}_h = \mathbf{0}, \\ \mathbb{G}_r^T \boldsymbol{\Phi}_h = 0, \quad \mathbb{G}_{\mathbf{n}}^T \boldsymbol{\Phi}_h = 0, \quad \mathbb{G}_{\boldsymbol{\tau}}^T \boldsymbol{\Phi}_h = 0. \end{array} \right.$$

Note that  $\mathbb{M}_h$  and  $\mathbb{A}_h$  are  $N \times N$  symmetric matrices (moreover  $\mathbb{A}_h$  is positive semi-definite),  $\mathbb{B}_h$  is a  $N \times M$  matrix, while  $(\mathbb{G}_r, \mathbb{G}_{\mathbf{n}}, \mathbb{G}_{\boldsymbol{\tau}})$  are  $N \times 1$  matrices.

The formulation (3.14) has the advantage of well separating the role of the Lagrange multiplier  $\eta$  from  $(\eta_r, \eta_{\mathbf{n}}, \eta_{\boldsymbol{\tau}})$ . However, for the sequel, it will be useful to have a more compact writing of (3.14) by introducing a generalized Lagrange multiplier

$$(3.16) \quad \widehat{\eta}_h := (\eta_h, \eta_r, \eta_{\boldsymbol{\tau}}, \eta_{\mathbf{n}}) \in \widehat{M}_h = M_h \times \mathbb{R}^3.$$

Then (3.14) rewrites

$$(3.17) \quad \left\{ \begin{array}{l} \text{Find } (\boldsymbol{\varphi}_h, \widehat{\eta}_h) : [0, T] \rightarrow \mathbf{V}_h \times \widehat{M}_h \text{ satisfying (3.12) and} \\ \frac{d^2}{dt^2} m_h(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + a(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + \widehat{b}_h(\widehat{\eta}_h, \boldsymbol{\psi}_h) = \ell(t, \boldsymbol{\psi}_h), \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_h, \\ \widehat{b}_h(\widehat{\nu}_h, \boldsymbol{\varphi}_h) = 0, \quad \forall \widehat{\nu}_h \in \widehat{M}_h. \end{array} \right.$$

where the bilinear form  $\widehat{b}(\cdot, \cdot) : \widehat{M}_h \times \mathbf{V}_h \rightarrow \mathbb{R}$  is defined for  $(\widehat{\nu}_h, \boldsymbol{\psi}_h) \in \widehat{M}_h \times \mathbf{V}_h$  by

$$(3.18) \quad \widehat{b}_h(\widehat{\nu}_h, \boldsymbol{\psi}_h) = b_h(\nu_h, \boldsymbol{\psi}_h) + \nu_r m(\boldsymbol{\varphi}_R, \boldsymbol{\psi}_h) + \nu_{\mathbf{n}} \int_{\Gamma} \boldsymbol{\psi}_h \cdot \mathbf{n} \, d\boldsymbol{\gamma} + \nu_{\boldsymbol{\tau}} \int_{\Gamma} \boldsymbol{\psi}_h \cdot \boldsymbol{\tau} \, d\boldsymbol{\gamma}.$$

In the same way, introducing

$$\widehat{\mathbf{E}}_h = (\mathbf{E}_h, \eta_r, \eta_{\mathbf{n}}, \eta_{\boldsymbol{\tau}})^t \in \mathbb{R}^{M+3},$$

the algebraic system (3.15) can be rewritten

$$(3.19) \quad \begin{cases} \mathbb{M}_h \frac{d^2 \Phi_h}{dt^2} + \mathbb{A}_h \Phi_h + \widehat{\mathbb{B}}_h \widehat{\mathbf{E}}_h = \mathbf{F}_h, \\ \widehat{\mathbb{B}}_h^T \Phi_h = \mathbf{0}, \end{cases}$$

where  $\widehat{\mathbb{B}}_h$  is the  $N \times (M+3)$  matrix given by

$$\widehat{\mathbb{B}}_h = (\mathbb{B}_h \ \mathbb{G}_R \ \mathbb{G}_n \ \mathbb{G}_\tau).$$

Of course, according to (3.12), (3.19) is completed with the initial conditions

$$(3.20) \quad \Phi_h(0) = \frac{d\Phi_h}{dt}(0) = \mathbf{0}.$$

With the assumption we have on the source term  $\mathbf{g}$ , we will look for solutions of (3.19) with the regularity

$$\Phi_h \in C^2([0, T]; \mathbb{R}^N), \quad \widehat{\mathbf{E}}_h \in C^0([0, T]; \mathbb{R}^{M+3}),$$

which going back to (3.14), corresponds to

$$\varphi_h \in C^2([0, T]; \mathbf{V}_h), \quad \eta_h \in C^0([0, T]; M_h), \quad (\eta_r, \eta_n, \eta_\tau) \in C^0([0, T]; \mathbb{R}^3).$$

### 3.2. Well-posedness and stability analysis.

3.2.1. *Preliminary considerations.* The problem (3.17) or equivalently (3.19) raises several theoretical questions. Indeed the first one, to which we shall restrict ourselves in this paper, is the well-posedness and the time stability of this problem. In standard cases, this analysis relies on two properties:

i) The discrete coercivity of the bilinear form  $m_h(\cdot, \cdot)$  in  $\mathbf{V}_{N,h}$ , namely,

$$(3.21) \quad m_h(\psi_h, \psi_h) > 0, \quad \forall \psi_h \in \mathbf{V}_{N,h} \setminus \{0\},$$

where  $\mathbf{V}_{N,h}$  is a discrete equivalent of the space  $\mathbf{V}_N$  defined as

$$(3.22) \quad \mathbf{V}_{N,h} = \{\psi_h \in \mathbf{V}_h, \widehat{b}_h(\widehat{v}_h, \psi_h) = 0, \forall \widehat{v}_h \in \widehat{M}_h\}.$$

This corresponds, at the algebraic level, to

$$(3.23) \quad \mathbb{M}_h \Psi_h \cdot \Psi_h > 0, \quad \forall \Psi_h \neq \mathbf{0} \text{ s.t. } \widehat{\mathbb{B}}_h^T \Psi_h = \mathbf{0}.$$

ii) A discrete inf-sup condition namely  $\forall \widehat{v}_h \in \widehat{M}_h$

$$(3.24) \quad \left[ \widehat{b}_h(\widehat{v}_h, \psi_h) = 0, \forall \psi_h \in \mathbf{V}_h \right] \Rightarrow \widehat{v}_h = \mathbf{0}.$$

This corresponds to  $\text{Ker } \widehat{\mathbb{B}}_h = \{\mathbf{0}\}$ .

**Remark 3.3.** *Since the spaces  $\mathbf{V}_{N,h}$  and  $\widehat{M}_h$  are finite dimensional spaces, the conditions (3.21) and (3.24) are equivalent to the existence of constants  $\kappa_h > 0$  and  $\delta_h > 0$  such that*

$$(3.25) \quad m_h(\psi_h, \psi_h) \geq \kappa_h m_\Omega(\psi_h, \psi_h), \quad \forall \psi_h \in \mathbf{V}_{N,h},$$

$$(3.26) \quad \forall \widehat{v}_h \in \widehat{M}_h, \exists \psi_h \in \mathbf{V}_h, \widehat{b}_h(\widehat{v}_h, \psi_h) \geq \delta_h \|\widehat{v}_h\|_{\widehat{M}} \|\psi_h\|_{\mathbf{V}}.$$

*In particular (3.25) can be seen as a discrete equivalent of (2.61). Note that we do not know if the positive constant  $\kappa_h$  can be made independent of  $h$ . Of course, we conjecture that this is the case for any “reasonable” choice of the approximation*

spaces but the proof of such result, which is fundamental for the convergence theory, appears to be quite challenging.

The same remark applies to inequality (3.26), for which we even do not know (at this moment) if there exists an equivalent at the continuous level.

As we shall see later condition i) is a sufficient condition for ensuring the existence of solution of problem (3.14). It is also a necessary condition for time stability. Notice that the Lagrange multiplier  $\widehat{\eta}_h(t)$  is a-priori only defined up to any function of time with values in (see Theorem 3.9)

$$\widehat{M}_{N,h} = \{\nu_h \in \widehat{M}_h, \widehat{b}_h(\nu_h, \boldsymbol{\psi}_h) = 0, \forall \boldsymbol{\psi}_h \in \mathbf{V}_h\}.$$

The condition ii) ensures that this space is reduced to  $\{\mathbf{0}\}$ , or equivalently  $\text{Ker } \widehat{\mathbb{B}}_h = \{\mathbf{0}\}$ , thus that  $\widehat{\eta}_h(t)$  is uniquely defined.

While it is clear that condition i) is of fundamental importance, ii) is much less crucial since we are essentially interested in  $\boldsymbol{\varphi}_h$ . Moreover in practice, the failure of ii) can be circumvented by selecting the solution  $\widehat{\eta}_h$  of minimal norm (e.g. in  $L^2(\Gamma) \times \mathbb{R}^3$ ) via pseudo-inverse procedure that we shall describe in Section 3.2.3. The rest of this section is devoted to:

- a) Provide an adequate compatibility conditions between  $P_h$  and  $V_{S,h}$  in order to ensure i). These are the conditions that must be verified when studding the case of finite element approximation in Section 4.
- b) Show that condition i) implies the well-posedness of (3.17) in the sense explained above.

3.2.2. *On the positivity condition (3.21).* Let us introduce the space (see (2.47) for its analogue at the continuous level)

$$(3.27) \quad \mathbf{W}_{N,h} = \left\{ \boldsymbol{\psi}_h \in \mathbf{V}_{0,h}, b_h(\nu_h, \boldsymbol{\psi}_h) = 0, \forall \nu_h \in M_h \right\} \supset \mathbf{V}_{N,h}.$$

Our analysis will strongly rely on the following ad hoc compatibility condition between the spaces  $P_h$  and  $V_{S,h}$ .

**Assumption 3.4** (Weak compatibility assumption). *For any  $p_h \in P_h$  such that  $\partial_1 p_h = 0$  and  $\partial_2 p_h \in V_{S,h}$  then*

$$\int_{\Omega} \partial_2 p_h \partial_2 q_h \, d\mathbf{x} = 0, \forall q_h \in P_h \cap H_0^1(\Omega) \quad \Rightarrow \quad \exists (a, b) \in \mathbb{R}^2 \text{ s.t. } p_h = a x_2 + b.$$

**Theorem 3.5.** *In the space  $\mathbf{W}_{N,h}$ , The quadratic form  $m_h(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h)$  takes the form*

$$(3.28) \quad m_h(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) = \frac{1}{V_*^2} \int_{\Omega} |\partial_1 p_h(\nu_h)|^2 \, d\mathbf{x} + m_{\Omega}(\boldsymbol{\psi}_h - \nabla p_h(\nu_h), \boldsymbol{\psi}_h - \nabla p_h(\nu_h)),$$

where  $\nu_h = \Pi_h(\boldsymbol{\psi}_h \cdot \mathbf{n}) \in M_h$  and  $1/V_*^2 := 1/V_S^2 - 1/V_P^2 > 0$ . Furthermore, if Assumption 3.4 is satisfied then, (3.21) holds true.

**Proof.** The proof is done by mimicking the proof of Theorem 2.10 at the semi-discrete level.

*Step 1: Proof of (3.28)*

By definition (3.27) and (3.11) of  $\mathbf{W}_{N,h}$  and  $b_h(\cdot, \cdot)$ ,  $\boldsymbol{\psi} \in \mathbf{W}_{N,h}$  means that, for any function  $\nu_h$  in  $M_h$ ,

$$(3.29) \quad \begin{aligned} & \frac{1}{V_P^2} \int_{\Omega} \partial_1 p_h(\nu_h) \psi_{P,h} \, d\mathbf{x} + \frac{1}{V_S^2} \int_{\Omega} \partial_2 p_h(\nu_h) \psi_{S,h} \, d\mathbf{x} \\ & - \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\psi}_h \cdot \boldsymbol{\tau}) \nu_h \, d\boldsymbol{\gamma} - \frac{1}{2V_S^2} \int_{\Gamma} p_h(\nu_h) \Pi_h(\boldsymbol{\psi}_h \cdot \mathbf{n}) \, d\boldsymbol{\gamma} = 0, \end{aligned}$$

where we recall that  $p_h(\nu_h)$  is the discrete harmonic function defined as the solution of (3.10). Let us choose  $\nu_h = \Pi_h(\boldsymbol{\psi}_h \cdot \mathbf{n}) \in M_h$ . This is where the operator  $\Pi_h$  is needed when the assumption of Remark 3.1 does not hold. We first observe, from the definition of  $m_{\Gamma,h}(\cdot, \cdot)$  given by (3.7), that

$$-\frac{1}{V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\psi}_h \cdot \boldsymbol{\tau}) \nu_h \, d\boldsymbol{\gamma} = -\frac{1}{V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\psi}_h \cdot \boldsymbol{\tau}) \Pi_h(\boldsymbol{\psi}_h \cdot \mathbf{n}) \, d\boldsymbol{\gamma} = m_{\Gamma,h}(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h).$$

Since  $p_h(\nu_h)$  is the solution of (3.10), we have (take  $q_h = p_h(\nu_h)$ ) in (3.10)

$$\frac{1}{2V_S^2} \int_{\Gamma} p_h(\nu_h) \Pi_h(\boldsymbol{\psi}_h \cdot \mathbf{n}) \, d\boldsymbol{\gamma} = \frac{1}{2V_S^2} \int_{\Gamma} p_h(\nu_h) \nu_h \, d\boldsymbol{\gamma} = \frac{1}{2V_S^2} \int_{\Omega} |\nabla p_h(\nu_h)|^2 \, d\mathbf{x}.$$

Substituting the last two equalities into (3.29), we obtain

$$(3.30) \quad \begin{aligned} m_{\Gamma,h}(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) &= \frac{1}{V_S^2} \int_{\Omega} |\nabla p_h(\nu_h)|^2 \, d\mathbf{x} \\ &\quad - \frac{2}{V_P^2} \int_{\Omega} \psi_{P,h} \partial_1 p_h(\nu_h) \, d\mathbf{x} - \frac{2}{V_S^2} \int_{\Omega} \psi_{S,h} \partial_2 p_h(\nu_h) \, d\mathbf{x}. \end{aligned}$$

Thus, proceeding as in the proof of Theorem 2.10, one sees that  $m_h(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h)$  defined by (3.6) can be rewritten as a sum of squares, which leads to (3.28). Details are omitted.

*Step 2:*  $m_h(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) = 0$  and  $\boldsymbol{\psi}_h \in \mathbf{V}_{N,h}$  imply  $\boldsymbol{\psi}_h = 0$ .

This is where Assumption 3.4 is needed. Indeed, using (3.28) (notice that we have the inclusion  $\mathbf{V}_{N,h} \subset \mathbf{W}_{N,h}$ ),  $m_h(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) = 0$  implies  $\partial_1 p_h = 0$  and  $\boldsymbol{\psi}_h = \nabla p_h$ . We immediately deduce that,  $\psi_{P,h} = 0$  and  $\partial_2 p_h \in V_{S,h}$ . Moreover since,  $p_h$  is a discrete harmonic function, i.e.,

$$(3.31) \quad \int_{\Omega} \partial_2 p_h \partial_2 q_h = 0, \quad \forall q_h \in P_h \cap H_0^1(\Omega),$$

we can therefore use Assumption 3.4 to state that  $p_h$  is an affine function of  $x_2$  and that  $\psi_{S,h}$  is constant. We now use the orthogonality of  $\boldsymbol{\psi}_h$  to  $\mathbf{K}_R$  to conclude that  $\boldsymbol{\psi}_h = 0$  as in the proof of Theorem 2.10.  $\blacksquare$

Assumption 3.4 appears as a compatibility condition between space  $P_h$  with respect to the space  $V_{S,h}$  but, since,  $V_{S,h} \subset H^1(\Omega)$ , this assumption is implied by the following assumption that only involves the space  $P_h$ .

**Assumption 3.6** (Strong compatibility assumption). *For any  $p_h \in P_h$  such that  $\partial_1 p_h = 0$  and  $\partial_2 p_h \in H^1(\Omega)$  then*

$$\int_{\Omega} \partial_2 p_h \partial_2 q_h = 0, \quad \forall q_h \in P_h \cap H_0^1(\Omega) \quad \Rightarrow \quad \exists (a, b) \in \mathbb{R}^2 \text{ s.t. } p_h = a x_2 + b$$

This assumption can be interpreted as follows: *every discrete harmonic functions of  $P_h$ , as defined in Remark 3.2, whose derivative  $\partial_2 p_h$  has  $H^1(\Omega)$ -regularity and that is independent of  $x_1$  is an affine function of  $x_2$ .* The reader will observe that this condition is the discrete equivalent of the trivial condition

$$\Delta p = 0, \quad \partial_1 p = 0 \quad \implies \quad \exists (a, b) \in \mathbb{R}^2 \text{ s.t. } p = a x_2 + b,$$

that was used in the proof of Theorem 2.10.

**3.2.3. Well-posedness of the semi-discrete problem.** We choose to analyse the problem (3.17) in its algebraic form, that is, (3.19). As it was said before, the major difficulty is that the matrix  $\mathbb{M}_h$  is not positive definite. To circumvent this problem we shall propose an exact penalized version of (3.19). The basic ingredient is the following result in linear algebra.

**Lemma 3.7.** *Let  $\mathbb{M} \in \mathbb{R}^{N,N}$  be a symmetric matrix and  $\mathbb{B} \in \mathbb{R}^{M,N}$  a rectangular matrix such that*

$$\mathbb{M} \Psi \cdot \Psi > 0, \quad \forall \Psi \text{ s.t. } \mathbb{B}^T \Psi = \mathbf{0}.$$

*Then, there exists a sufficiently small  $\varepsilon > 0$  such that*

$$(3.32) \quad \left( \mathbb{M} + \frac{1}{\varepsilon} \mathbb{B} \mathbb{B}^T \right) \Psi \cdot \Psi > 0, \quad \forall \Psi \in \mathbb{R}^N.$$

**Proof.** In this proof we denote  $\Phi \cdot \Psi$  the usual inner product in  $\mathbb{R}^N$  and  $\|\cdot\|_2$  the corresponding euclidean norm. We proceed by contradiction. Assume that (3.32) is not true. Then, there exists a sequence  $\Psi_k$  such that  $\|\Psi_k\|_2 = 1$  and

$$(3.33) \quad \left( \mathbb{M} + k \mathbb{B} \mathbb{B}^T \right) \Psi_k \cdot \Psi_k \leq 0.$$

Since  $\Psi_k$  is a bounded sequence in a finite dimensional space, up to the extraction of a subsequence, we can assume that it converges towards  $\Psi$ . Moreover, dividing by  $k$ , equation (3.33), we obtain, at the limit  $k$  goes to infinity, that  $\mathbb{B}^T \Psi = \mathbf{0}$  with  $\|\Psi\|_2 = 1$  and we have, since  $\mathbb{B} \mathbb{B}^T$  is positive,

$$\left( \mathbb{M} + k \mathbb{B} \mathbb{B}^T \right) \Psi_k \cdot \Psi_k \leq 0 \quad \implies \quad \mathbb{M} \Psi_k \cdot \Psi_k \leq 0 \quad \implies \quad \mathbb{M} \Psi \cdot \Psi \leq 0,$$

which is a contradiction. ■

We can apply Lemma 3.7 to  $\mathbb{M} = \mathbb{M}_h$  and  $\mathbb{B} = \widehat{\mathbb{B}}_h$  thanks to (3.23). Thus, there exists  $\varepsilon$  sufficiently small such that

$$(3.34) \quad \mathbb{M}_h^\varepsilon := \mathbb{M}_h + \frac{1}{\varepsilon} \widehat{\mathbb{B}}_h \widehat{\mathbb{B}}_h^T \text{ is positive definite.}$$

Thus, we introduce the penalised system

$$(3.35) \quad \begin{cases} \mathbb{M}_h^\varepsilon \frac{d^2 \Phi_h^\varepsilon}{dt^2} + \mathbb{A}_h \Phi_h^\varepsilon + \widehat{\mathbb{B}}_h \widehat{\mathbf{E}}_h^\varepsilon = \mathbf{F}_h, \\ \widehat{\mathbb{B}}_h^T \Phi_h^\varepsilon = \mathbf{0}. \end{cases}$$

A priori the solution depends on  $\varepsilon$  but in fact we did not change anything (this is an “exact penalisation”) as shown in the following obvious lemma.

**Lemma 3.8.** *Any solution of (3.19, 3.20) is a solution of (3.35, 3.20) and reciprocally.*

Thanks to the penalisation we enter the more standard framework already encountered in [6] for the case of static problems and as a consequence we can state the theorem below. For the sake of completeness, and to make the article self content, we provide a proof.

**Theorem 3.9.** *Assume that  $\mathbf{F}_h \in C^0([0, T]; \mathbb{R}^N)$ . Then, the problem (3.19, 3.20) admits a unique solution*

$$(\Phi_h, \bar{\mathbf{E}}_h) \in C^2([0, T]; \mathbb{R}^N) \times C^0([0, T]; (\text{Ker } \widehat{\mathbb{B}}_h)^\perp).$$

*Any other solution is of the form  $(\Phi_h, \bar{\mathbf{E}}_h + \mathbf{E}_h^0)$  where  $\mathbf{E}_h^0 \in C^0([0, T]; \text{Ker } \widehat{\mathbb{B}}_h)$  is arbitrary.*

**Proof.** By Lemma 3.8, it is equivalent to consider the penalised problem (3.35) and we drop the superscript  $\varepsilon$  for the solutions from now on. The way we proceed to prove existence (and uniqueness) of the solution  $(\Phi_h, \bar{\mathbf{E}}_h)$  is as follows.

In a first step we assume that a solution  $(\Phi_h, \widehat{\mathbf{E}}_h)$  exists and we construct a well-posed problem in  $\Phi_h$  alone (see (3.40)), by eliminating  $\widehat{\mathbf{E}}_h$ . In a second step, with the solution  $\Phi_h$  of this problem, we construct a particular  $\bar{\mathbf{E}}_h \in C^0(0, T; (\text{Ker } \widehat{\mathbb{B}}_h)^\perp)$  (see (3.38)) and prove that  $(\Phi_h, \bar{\mathbf{E}}_h)$  solves (3.35).

Note that this process proves in particular the uniqueness of  $\Phi_h$ .

*Step 1.* We first obtain an equation for  $\widehat{\mathbf{E}}_h$  only. For this, we multiply the first equation of (3.35) by  $\widehat{\mathbb{B}}_h^T (\mathbb{M}_h^\varepsilon)^{-1}$ , which exists thanks to (3.34), and obtain

$$(3.36) \quad \mathbb{C}_h^\varepsilon \widehat{\mathbf{E}}_h = \widehat{\mathbb{B}}_h^T \widehat{\mathbf{F}}_h$$

where we have set

$$(3.37) \quad \mathbb{C}_h^\varepsilon := \widehat{\mathbb{B}}_h^T (\mathbb{M}_h^\varepsilon)^{-1} \widehat{\mathbb{B}}_h \quad \text{and} \quad \widehat{\mathbf{F}}_h := (\mathbb{M}_h^\varepsilon)^{-1} (\mathbf{F}_h - \mathbb{A}_h \Phi_h) \in C^0([0, T]; \mathbb{R}^N).$$

From a classical result in linear algebra (see [9], Section 8.1 for instance) we have

$$\text{Im}(\widehat{\mathbb{B}}_h^T) = \text{Im}(\mathbb{C}_h^\varepsilon),$$

so that, given  $\widehat{\mathbf{F}}_h$ , there exists a solution  $\widehat{\mathbf{E}}_h$  of (3.36), which is unique up to an element of  $\text{Ker } \widehat{\mathbb{B}}_h$ . We can restore uniqueness by the so-called pseudo-inverse procedure that selects the (unique) particular solution  $\bar{\mathbf{E}}_h$  of minimum euclidian norm. This is equivalent to imposing that  $\bar{\mathbf{E}}_h$  is orthogonal to  $\text{Ker } \widehat{\mathbb{B}}_h$  or equivalently, imposing  $\mathbb{P}_h \bar{\mathbf{E}}_h = \mathbf{0}$  where  $\mathbb{P}_h$  the orthogonal projection operator in  $\mathbb{R}^{M+3}$  on  $\text{Ker } \widehat{\mathbb{B}}_h$ . It is well known (see [1] for instance) that this solution is given by

$$(3.38) \quad \bar{\mathbf{E}}_h = (\widehat{\mathbb{C}}_h^\varepsilon)^{-1} \widehat{\mathbb{B}}_h^T \widehat{\mathbf{F}}_h, \quad \widehat{\mathbb{C}}_h^\varepsilon = (\mathbb{C}_h^\varepsilon + (\mathbb{P}_h)^T \mathbb{P}_h).$$

Since  $\widehat{\mathbb{B}}_h \widehat{\mathbf{E}}_h = \widehat{\mathbb{B}}_h \bar{\mathbf{E}}_h$ , from the first equation of (3.35), we see that  $\Phi_h$  must satisfy

$$(3.39) \quad \mathbb{M}_h^\varepsilon \frac{d^2 \Phi_h}{dt^2} + \mathbb{A}_h \Phi_h + \widehat{\mathbb{B}}_h (\widehat{\mathbb{C}}_h^\varepsilon)^{-1} \widehat{\mathbb{B}}_h^T \widehat{\mathbf{F}}_h = \mathbf{F}_h.$$

Using (3.37), the equation above can be rewritten as

$$(3.40) \quad \left| \begin{aligned} \mathbb{M}_h^\varepsilon \frac{d^2 \Phi_h}{dt^2} + \mathbb{A}_h \Phi_h - \widehat{\mathbb{B}}_h (\widehat{\mathbb{C}}_h^\varepsilon)^{-1} \widehat{\mathbb{B}}_h^T (\mathbb{M}_h^\varepsilon)^{-1} \mathbb{A}_h \Phi_h \\ = \mathbf{F}_h - \widehat{\mathbb{B}}_h (\widehat{\mathbb{C}}_h^\varepsilon)^{-1} \widehat{\mathbb{B}}_h^T (\mathbb{M}_h^\varepsilon)^{-1} \mathbf{F}_h. \end{aligned} \right.$$

Since  $\mathbb{M}_h^\varepsilon$  is positive the system above has a unique solution  $\Phi_h \in C^2([0, T]; \mathbb{R}^N)$ .

*Step 2.* We now prove that the couple  $(\Phi_h, \bar{\mathbf{E}}_h)$  is solution of (3.35), where  $\Phi_h$  is solution of (3.40) and  $\bar{\mathbf{E}}_h$  is given by (3.38). First, by definition of  $\bar{\mathbf{E}}_h$  we have  $\bar{\mathbf{E}}_h \in C^0([0, T]; \mathbb{R}^{M+3})$  and, from (3.39),

$$\mathbb{M}_h^\varepsilon \frac{d^2 \Phi_h}{dt^2} + \mathbb{A}_h \Phi_h + \widehat{\mathbb{B}}_h \bar{\mathbf{E}}_h = \mathbf{F}_h.$$

We need to check that  $\widehat{\mathbb{B}}_h^T \Phi_h = \mathbf{0}$ . To do so, we multiply the equation above by  $\widehat{\mathbb{B}}_h^T (\mathbb{M}_h^\varepsilon)^{-1}$  and get

$$\widehat{\mathbb{B}}_h^T \frac{d^2 \Phi_h}{dt^2} = \widehat{\mathbb{B}}_h^T (\mathbb{M}_h^\varepsilon)^{-1} (\mathbf{F}_h - \mathbb{A}_h \Phi_h) - \widehat{\mathbb{B}}_h^T (\mathbb{M}_h^\varepsilon)^{-1} \widehat{\mathbb{B}}_h \bar{\mathbf{E}}_h = \widehat{\mathbb{B}}_h^T \widehat{\mathbf{F}}_h - \mathbb{C}_h^\varepsilon \bar{\mathbf{E}}_h = \mathbf{0}.$$

One concludes using the initial conditions (3.20).  $\blacksquare$

**Remark 3.10.** Notice that if  $\mathbf{f} \in L^1(0, T; \mathbf{V}_{N,h})$  then  $\mathbf{F}_h \in C^0([0, T]; \mathbb{R}^N)$  thanks to (2.66).

**3.2.4. Time stability analysis.** In this section, we suppose that (3.21) is satisfied, for instance through Assumption 3.4.

We introduce the energy  $\mathcal{E}_h(t)$  of the semi-discrete solution as

$$(3.41) \quad \mathcal{E}_h(t) = \frac{1}{2} m_h \left( \frac{\partial \varphi_h}{\partial t}, \frac{\partial \varphi_h}{\partial t} \right) + \frac{1}{2} a(\varphi_h, \varphi_h),$$

which we know to be a positive energy thanks to the positivity property (3.21) (see Theorem 3.5) since  $\varphi_h(t)$  belongs to  $\mathbf{V}_{N,h}$  (cf. the third line of (3.17) and the definition (3.22)).

**Theorem 3.11.** *There exists a constant  $\mathcal{C}$ , independent of  $T$  and  $h$ , such that the unique solution  $\varphi_h \in C^2([0, T]; \mathbf{V}_{N,h})$  of (3.14, 3.12) satisfies*

$$(3.42) \quad \sup_{t \in [0, T]} \mathcal{E}_h^{1/2}(t) \leq \mathcal{C} \|\mathbf{f}\|_{L^1(0, T; L^2(\Omega))}.$$

**Proof.** Choosing  $\psi_h = \partial_t \varphi_h(t) \in \mathbf{V}_{N,h}$  as a test function in the weak formulation (3.14), we easily obtain (details are left to the reader)

$$(3.43) \quad m_h \left( \frac{\partial^2 \varphi_h}{\partial t^2}, \frac{\partial \varphi_h}{\partial t} \right) + a \left( \varphi_h, \frac{\partial \varphi_h}{\partial t} \right) = \ell \left( t, \frac{\partial \varphi_h}{\partial t} \right).$$

After integrating (3.43) in time and using the initial conditions (3.12), we get, for all  $t \in [0, T]$ ,

$$(3.44) \quad \mathcal{E}_h(t) = \int_0^t \ell \left( s, \frac{\partial \varphi_h}{\partial t}(s) \right) ds.$$

Following the proof of Theorem 2.12 in Appendix, the right hand side of equation (3.44) reads

$$(3.45) \quad \left| \begin{aligned} \int_0^t \ell \left( s, \frac{\partial \varphi_h}{\partial t}(s) \right) ds &= \frac{1}{\rho} \int_0^t \int_{\Omega} \mathbf{f} \cdot (\operatorname{div}(\varphi_h(s)), -\operatorname{curl}(\varphi_h(s)))^t d\mathbf{x} ds \\ &\quad - \int_{\Omega} \mathbf{g} \cdot (\operatorname{div}(\varphi_h(t)), -\operatorname{curl}(\varphi_h(t)))^t d\mathbf{x}. \end{aligned} \right.$$



Hence we obtain, using Cauchy-Schwartz inequality, as well as the definition of  $a(\cdot, \cdot)$  given by (2.25), and of  $\mathbf{g}$  given by (2.11),

$$\left| \int_0^t \ell\left(s, \frac{\partial \varphi_h}{\partial t}(s)\right) ds \right| \leq \frac{2}{\rho} \sup_{s \in [0, t]} \sqrt{a(\varphi_h(s), \varphi_h(s))} \|\mathbf{f}\|_{L^1(0, t; L^2(\Omega))}.$$

Again, as in the proof of Theorem 2.12 we deduce

$$\sup_{t \in [0, T]} \mathcal{E}_h(t) \leq \frac{2\sqrt{2}}{\rho} \sup_{t \in [0, T]} \mathcal{E}_h^{1/2}(t) \|\mathbf{f}\|_{L^1(0, T; L^2(\Omega))},$$

which directly gives the result of the theorem.  $\blacksquare$

**Corollary 3.12.** *There exists a constant  $\mathcal{C}$ , independent of  $T$  and  $h$ , such that the unique solution  $\varphi_h \in C^2([0, T]; \mathbf{V}_{N, h})$  of (3.14) satisfies, for all  $t \in [0, T]$ ,*

$$(3.46) \quad \begin{aligned} (a) \quad & \sqrt{a(\varphi_h(t), \varphi_h(t))} \leq \mathcal{C} \|\mathbf{f}\|_{L^1(0, T; L^2(\Omega))} \\ (b) \quad & \sqrt{m_h(\varphi_h(t), \varphi_h(t))} \leq \mathcal{C} T \|\mathbf{f}\|_{L^1(0, T; L^2(\Omega))}. \end{aligned}$$

**Proof.** Estimation (3.46)(a) is a direct consequence of (3.41, 3.21, 3.42). Thanks to the vanishing initial condition (3.20) we can write

$$\varphi_h(t) = \int_0^t \frac{\partial \varphi_h}{\partial t}(s) ds$$

Since  $\sqrt{m_h(\cdot, \cdot)}$  is a norm on  $\mathbf{V}_{N, h}$ , Jensen's inequality (a continuous version of the triangular inequality) implies

$$\sqrt{m_h(\varphi_h(t), \varphi_h(t))} \leq \int_0^t \sqrt{m_h\left(\frac{\partial \varphi_h}{\partial t}(s), \frac{\partial \varphi_h}{\partial t}(s)\right)} ds \leq \sqrt{2} t \sup_{t \in [0, T]} \mathcal{E}_h^{1/2}(t).$$

Then we use Theorem 3.11 to obtain (3.46)(b).  $\blacksquare$

The relative weakness of the above result is that it is not clear whether estimation (3.46) guarantees an uniform (in  $h$ ) estimate of  $\varphi_h(t)$  in  $H(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega)$ : it provides only a control of the  $L^2$  norm of the divergence and the curl of  $\psi_h$  (by definition of  $a(\cdot, \cdot)$ ) and also, thanks to the expression (3.28) of  $m_h(\cdot, \cdot)$  in  $\mathbf{V}_{N, h}$ , a control of the  $L^2$ -norm of  $\varphi_{P, h}(t)$ . However, it does not give any control of  $L^2$ -norm of  $\varphi_{S, h}(t)$ .

**3.3. Time discretization.** In this section, we investigate the well-known  $\theta$ -schemes (Section 3.3.1) and a mixed explicit-implicit scheme (Section 3.3.2). The  $\theta$ -scheme will be considered as the reference scheme and the mixed implicit-explicit scheme as an improvement and an extension of the scheme that was already proposed for the case of the Dirichlet condition [3]. Its advantages will be outlined in terms of the CFL condition.

Throughout this section,  $\Delta t$  will denote a time step such that  $N\Delta t = T$ ,  $(\Phi_h^n, \widehat{\mathbf{E}}_h^n)$  denotes the approximation of  $(\Phi_h(n\Delta t), \widehat{\mathbf{E}}_h(n\Delta t))$ , where  $(\Phi_h, \widehat{\mathbf{E}}_h)$  is the solution of (3.19) and  $\mathbf{F}_h^n = \mathbf{F}_h(n\Delta t)$ .

3.3.1. *Theta-schemes.* The schemes we propose in this section are a family of conservative centered integrators, parameterized by  $\theta \in \mathbb{R}$  and that belong to the class of Newmark schemes. Applied to (3.19), these schemes lead to

$$(3.47) \quad \begin{cases} \mathbb{M}_h \frac{\Phi_h^{n+1} - 2\Phi_h^n + \Phi_h^{n-1}}{\Delta t^2} + \mathbb{A}_h \{\Phi_h^n\}_\theta + \widehat{\mathbb{B}}_h \widehat{\mathbf{E}}_h^n = \mathbf{F}_h^n, \\ \widehat{\mathbb{B}}_h^T \Phi_h^n = \mathbf{0}, \end{cases}$$

completed with  $\Phi_h^0 = \Phi_h^1 = \mathbf{0}$ . In (3.47), we have used the notation

$$(3.48) \quad \{\Phi_h^n\}_\theta := \theta \Phi_h^{n+1} + (1 - 2\theta) \Phi_h^n + \theta \Phi_h^{n-1}.$$

At each iteration,  $(\Phi_h^{n-1}, \Phi_h^n)$  are known and we compute  $(\Phi_h^{n+1}, \widehat{\mathbf{E}}_h^n)$  by solving the linear problem

$$(3.49) \quad \mathbb{L}_{\theta,h}(\Phi_h^{n+1}, \widehat{\mathbf{E}}_h^n)^t = (\mathbf{F}_h^n + \mathbb{M}_h \frac{2\Phi_h^n - \Phi_h^{n-1}}{\Delta t^2} - (1 - 2\theta)\mathbb{A}_h \Phi_h^n - \theta \mathbb{A}_h \Phi_h^{n-1}, \mathbf{0})^t$$

with

$$(3.50) \quad \mathbb{L}_{\theta,h} = \begin{pmatrix} \frac{\mathbb{M}_h}{\Delta t^2} + \theta \mathbb{A}_h & \widehat{\mathbb{B}}_h \\ \widehat{\mathbb{B}}_h^T & \mathbb{O} \end{pmatrix}.$$

**Remark 3.13.** When  $\theta = 0$ , we get the "explicit scheme", frequently called the "Leap Frog" scheme. Assuming  $\widehat{\mathbf{E}}_h^n$  known, the computation of  $\Phi_h^{n+1}$  is reduced to the inversion of the mass matrix  $\mathbb{M}_h$  which might be easy (when  $\theta \neq 0$  the matrix to be inverted involves  $\mathbb{A}_h$ ), reason why it is called explicit. The quasi-explicit nature of the scheme will become more clear when Lagrange finite element will be used (see Section 4.3): in such a case, the use of mass lumping techniques will produce a mass matrix which is "almost diagonal" and thus easy to invert.

We describe below the theoretical properties of this scheme.

*Well-posedness analysis.* Not surprisingly, if the properties (3.21) (discrete coercivity) and (3.24) (discrete inf-sup) are satisfied (see Section 3.2.1), it is easily seen that the matrix  $\mathbb{L}_{\theta,h}$  is invertible. This can be done along the same lines of the analysis of the semi-discrete scheme (see Section 3.2.3 and Theorem 3.9). Let us sketch the proof. Note that the invertibility of the matrix  $\mathbb{L}_{\theta,h}$  is equivalent (this is exact penalization) to the one of the matrix

$$(3.51) \quad \mathbb{L}_{\theta,h}^\varepsilon = \begin{pmatrix} \frac{\mathbb{M}_h}{\Delta t^2} + \frac{\widehat{\mathbb{B}}_h \widehat{\mathbb{B}}_h^T}{\varepsilon} + \theta \mathbb{A}_h & \widehat{\mathbb{B}}_h \\ \widehat{\mathbb{B}}_h^T & \mathbb{O} \end{pmatrix},$$

where  $\varepsilon$  is chosen small enough so that (this is possible thanks to Lemma 3.7)

$$\frac{\mathbb{M}_h}{\Delta t^2} + \frac{\widehat{\mathbb{B}}_h \widehat{\mathbb{B}}_h^T}{\varepsilon} \quad \text{is positive definite.}$$

Since  $\mathbb{A}_h$  is non negative, the first block of  $\mathbb{L}_{\theta,h}^\varepsilon$  in (3.51) is also positive definite. Then, a standard Schur complement procedure yields the invertibility of  $\mathbb{L}_{\theta,h}^\varepsilon$ .

**Remark 3.14.** If only (3.21) (discrete coercivity) holds, but not (3.24), we have

$$\text{Ker } \mathbb{L}_{\theta,h} = \{(\mathbf{0}, \widehat{\mathbf{E}}_h)^t, \widehat{\mathbf{E}}_h \in \text{Ker } \widehat{\mathbb{B}}_h\}.$$

*Stability analysis.* Let us use energy techniques. Let  $\Phi_h^{n+\frac{1}{2}} := \frac{1}{2}(\Phi_h^{n+1} + \Phi_h^n)$

$$(3.52) \quad \mathcal{E}_{h,\theta}^{n+\frac{1}{2}} = \frac{1}{2} \mathbb{M}_{h,\Delta t} \frac{\Phi_h^{n+1} - \Phi_h^n}{\Delta t} \cdot \frac{\Phi_h^{n+1} - \Phi_h^n}{\Delta t} + \frac{1}{2} \mathbb{A}_h \Phi_h^{n+\frac{1}{2}} \cdot \Phi_h^{n+\frac{1}{2}},$$

be the discrete energy, with  $\mathbb{M}_{h,\Delta t} := \mathbb{M}_h - \Delta t^2 \left( \frac{1-4\theta}{4} \right) \mathbb{A}_h$  the modified mass matrix.

We show below that the stability of the scheme can be proven if this energy is a positive quadratic functional on the kernel of  $\mathbb{B}_h^T$ . Since  $\mathbb{A}_h$  is a positive semi-definite matrix, it is sufficient to have that

$$(3.53) \quad \mathbb{B}_h^T \Psi_h = 0 \implies \mathbb{M}_{h,\Delta t} \Psi_h \cdot \Psi_h \geq 0.$$

If  $\theta \geq 1/4$  then (3.53) holds thanks to (3.23) (the algebraic version of (3.21)). On the other hand, when  $0 \leq \theta < 1/4$ , (3.53) requires a condition on  $\Delta t$ :

**Assumption 3.15.** [CFL condition - Theta-scheme] If  $0 \leq \theta < 1/4$ ,  $\Delta t$  satisfies

$$(3.54) \quad \alpha < 1 \quad \text{with} \quad \alpha = \Delta t^2 \left( \frac{1-4\theta}{4} \right) \max_{\psi_h \in \mathbf{V}_{N,h} \setminus \{0\}} \frac{a(\psi_h, \psi_h)}{m_h(\psi_h, \psi_h)}.$$

We have the following stability result.

**Theorem 3.16.** Assume (3.21), (3.24) as well as the stability condition (3.15). Then, there exists a constant  $\mathcal{C}$ , independent of  $T$  and  $h$ , such that the unique solution  $\Phi_h^n$  of (3.47) satisfies

$$(3.55) \quad \max_{n \in \{0, \dots, N\}} (\mathcal{E}_{h,\theta}^{n+\frac{1}{2}})^{\frac{1}{2}} \leq \mathcal{C} \|\mathbf{f}\|_{L^1(0,T;L^2(\Omega))}.$$

**Proof.** The first part of the proof is standard and we just recall the main steps for the sake of completeness. Taking the euclidian scalar product of (3.47) with  $(\Phi_h^{n+1} - \Phi_h^{n-1})/2\Delta t$  we obtain

$$(3.56) \quad \frac{\mathcal{E}_{h,\theta}^{n+\frac{1}{2}} - \mathcal{E}_{h,\theta}^{n-\frac{1}{2}}}{\Delta t} = \mathbf{F}_h^n \cdot \frac{\Phi_h^{n+1} - \Phi_h^{n-1}}{2\Delta t},$$

where we have used the symmetry of  $\mathbb{A}_h$  and  $\mathbb{M}_h$ . In particular, after summation over discrete times, we deduce that

$$\begin{aligned} \mathcal{E}_{h,\theta}^{n+\frac{1}{2}} &= \Delta t \sum_{m=1}^n \mathbf{F}_h^m \cdot \frac{\Phi_h^{m+1} - \Phi_h^{m-1}}{2\Delta t} \\ &= -\Delta t \sum_{m=1}^{n-1} \frac{\mathbf{F}_h^{m+1} - \mathbf{F}_h^m}{\Delta t} \cdot \frac{\Phi_h^{m+1} + \Phi_h^m}{2} + \mathbf{F}_h^n \cdot \frac{\Phi_h^{n+1} + \Phi_h^n}{2}, \end{aligned}$$

which is nothing but a discrete equivalent of the identity (3.44) for the semi-discrete case. To conclude, it suffices to mimick the end of the proof of Theorem 3.11. We omit the details.  $\blacksquare$

**Corollary 3.17.** Assume (3.21), (3.24). Then there exists  $\mathcal{C} > 0$ , independent of  $T$  and  $h$ , such that the solution of (3.47) satisfies, for all  $n \in \{0, \dots, N\}$ ,

$$(3.57) \quad \left( \mathbb{M}_h \Phi_h^{n+1} \cdot \Phi_h^{n+1} \right)^{\frac{1}{2}} \leq \frac{\mathcal{C}}{\sqrt{1-\alpha}} T \|\mathbf{f}\|_{L^1(0,T;L^2(\Omega))},$$

and

$$(3.58) \quad \left( \mathbb{A}_h \Phi_h^{n+\frac{1}{2}} \cdot \Phi_h^{n+\frac{1}{2}} \right)^{\frac{1}{2}} \leq C \|f\|_{L^1(0,T;L^2(\Omega))},$$

**Proof.** First the estimate (3.58) directly follows from (3.52, 3.53, 3.55). To obtain (3.57) we write

$$\Phi_h^{n+1} = \Delta t \sum_{m=0}^n \frac{\Phi_h^{m+1} - \Phi_h^m}{\Delta t},$$

and using classical manipulation (omitted here) we obtain (3.55). Simply note that the presence of  $\sqrt{1-\alpha}$  where  $\alpha$  is defined by (3.54) results from a lower bound of the modified mass matrix  $\mathbb{M}_{h,\Delta t}$ .  $\blacksquare$

3.3.2. *A mixed explicit-implicit scheme.* Following [3] we introduce an explicit-implicit time discretisation where we only treat implicitly a part of the bilinear form  $a(\cdot, \cdot)$ . More precisely, starting from (2.25) and since  $V_{P,h}$  and  $V_{S,h}$  are subspaces of  $H^1(\Omega)$  it is possible to write, for all  $\varphi_h$  and  $\psi_h \in \mathbf{V}_h$ , (see [4] or [3])

$$(3.59) \quad a(\varphi_h, \psi_h) = a_\Omega(\varphi_h, \psi_h) + a_\Gamma(\varphi_h, \psi_h),$$

where

$$(3.60) \quad a_\Omega(\varphi_h, \psi_h) = \int_\Omega \nabla \varphi_{P,h} \cdot \nabla \psi_{P,h} \, d\mathbf{x} + \int_\Omega \nabla \varphi_{S,h} \cdot \nabla \psi_{S,h} \, d\mathbf{x},$$

and the "boundary part"

$$(3.61) \quad a_\Gamma(\varphi_h, \psi_h) = \int_\Gamma (\partial_\tau \varphi_{P,h} \psi_{S,h} - \partial_\tau \varphi_{S,h} \psi_{P,h}) \, d\gamma.$$

Let  $\mathbb{A}_h^\Omega$  and  $\mathbb{A}_h^\Gamma$  be the matrices associated to the bilinear forms  $a_\Omega$  and  $a_\Gamma$ , i.e.

$$\mathbb{A}_h = \mathbb{A}_h^\Omega + \mathbb{A}_h^\Gamma, \quad \mathbb{A}_h^\Omega \Psi_h \cdot \Phi_h = a_\Omega(\varphi_h, \psi_h), \quad \mathbb{A}_h^\Gamma \Psi_h \cdot \Phi_h = a_\Gamma(\varphi_h, \psi_h).$$

The explicit-implicit scheme for (3.19) is obtained by treating implicitly with  $\theta = \frac{1}{4}$  the "boundary part"  $\mathbb{A}_h^\Gamma$  of  $\mathbb{A}_h$ , and explicitly the remaining part:

$$(3.62) \quad \begin{cases} \mathbb{M}_h \frac{\Phi_h^{n+1} - 2\Phi_h^n + \Phi_h^{n-1}}{\Delta t^2} + \mathbb{A}_h^\Omega \Phi_h^n + \mathbb{A}_h^\Gamma \{\Phi_h^n\}_{\frac{1}{4}} + \widehat{\mathbb{B}}_h \widehat{\mathbf{E}}_h^n = \mathbf{F}_h^n, \\ \widehat{\mathbb{B}}_h^T \Phi_h^n = 0, \end{cases}$$

completed with  $\Phi_h^0 = \Phi_h^1 = \mathbf{0}$ . At each iteration,  $(\Phi_h^{n-1}, \Phi_h^n)$  are known and we compute  $(\Phi_h^{n+1}, \widehat{\mathbf{E}}_h^n)$  by solving

(3.63)

$$\mathbb{L}_{\Gamma,h} \left( \Phi_h^{n+1}, \widehat{\mathbf{E}}_h^n \right)^t = \left( \mathbf{F}_h^n + \mathbb{M}_h \frac{2\Phi_h^n - \Phi_h^{n-1}}{\Delta t^2} - \mathbb{A}_h^\Omega \Phi_h^n - \frac{1}{2} \mathbb{A}_h^\Gamma \Phi_h^n - \frac{1}{4} \mathbb{A}_h \Phi_h^{n-1}, 0 \right)^t$$

with

$$\mathbb{L}_{\Gamma,h} = \begin{pmatrix} \mathbb{M}_h & \mathbb{A}_h^\Gamma \\ \Delta t^2 + \frac{\mathbb{A}_h^\Gamma}{4} & \widehat{\mathbb{B}}_h \\ \widehat{\mathbb{B}}_h^T & \mathbb{O} \end{pmatrix}.$$

The reason why we call this scheme explicit-implicit will become more clear when applied with Lagrange finite elements (see Section 4). Note that, since  $\mathbb{A}_h^\Gamma$  has no sign, it is not clear that (3.63) can be solved. We show below that a unique  $\Phi_h^{n+1}$  solution of (3.63) exists provided an appropriate stability condition (in other words solvability and stability are proven together).

*Analysis.* Again, we proceed by energy techniques. The we introduce the discrete energy

$$(3.64) \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \left( \mathbb{M}_h - \frac{\Delta t^2}{4} \mathbb{A}_h^\Omega \right) \frac{\Phi_h^{n+1} - \Phi_h^n}{\Delta t} \cdot \frac{\Phi_h^{n+1} - \Phi_h^n}{\Delta t} + \frac{1}{2} \mathbb{A}_h \Phi_h^{n+\frac{1}{2}} \cdot \Phi_h^{n+\frac{1}{2}}.$$

Note that an important difference between (3.52) and (3.64), is that, in the first term of the sum,  $\mathbb{A}_h$  has been replaced by  $\mathbb{A}_h^\Omega$  (and, less important,  $1 - 4\theta$  by 1).

As in [3], with the same approach as for proving Theorem 3.16, we prove an equivalent for (3.62) of the identity (3.56) for the  $\theta$ -scheme.

**Lemma 3.18.** *Any solution  $(\Phi_h^n, E_h^n)$  of (3.62) satisfies the identity*

$$(3.65) \quad \frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = F_h^n \cdot \frac{\Phi_h^{n+1} - \Phi_h^{n-1}}{2\Delta t}.$$

**Assumption 3.19.** *[CFL condition - Implicit-Explicit scheme] The time step  $\Delta t$  satisfies*

$$(3.66) \quad \alpha < 1 \quad \text{with} \quad \alpha = \frac{\Delta t^2}{4} \max_{\psi_h \in \mathbf{V}_{N,h} \setminus \{0\}} \frac{a_\Omega(\psi_h, \psi_h)}{m_h(\psi_h, \psi_h)}.$$

**Remark 3.20.** *Let us explain here why we consider that the stability condition (3.66) is better than the condition for the Leap Frog, i.e. (3.54) for  $\theta = 0$ , at least for the Dirichlet problem treated in [3]. The reason is that the bilinear form  $a_\Omega(\cdot, \cdot)$  is, contrary to  $a(\cdot, \cdot)$ , a "decoupled" bilinear form (see (3.60)) in  $(\varphi_P, \varphi_S)$  or, equivalently, that  $\mathbb{A}_h^\Omega$  is  $2 \times 2$  block diagonal. For the Dirichlet problem, we consider  $m_\Gamma \equiv 0$ , that is to say  $m_h(\cdot, \cdot) \equiv m_\Omega(\cdot, \cdot)$ , which is also decoupled, and the stability condition is*

$$\alpha_D < 1, \quad \alpha_D = \frac{\Delta t^2}{4} \max(A_P, A_S), \quad A_Q = V_Q^2 \max_{\psi_h \in \mathbf{V}_{Q,h}} \frac{\int_\Omega |\nabla \psi_h|^2}{\int_\Omega |\psi_h|^2}, \quad Q \in \{P, S\}.$$

Note that  $A_P$  only depends on the discretization space  $\mathbf{V}_{P,h}$  and  $A_S$  only depends on the discretization space  $\mathbf{V}_{S,h}$ . As a consequence, one can fully benefit of choosing different spaces  $\mathbf{V}_{P,h}$  and  $\mathbf{V}_{S,h}$  is such a way that  $A_P \simeq A_S$ , so that the CFL condition is no longer penalized by a large ratio  $V_P/V_S$ . We shall see that, despite of the presence of  $m_\Gamma$  for the Neumann problem, a similar nice property can be recovered with Lagrange finite elements (see Section 4.2).

We have the following main result.

**Theorem 3.21.** *Let Assumption 3.19 hold, then there exists a unique  $\Phi_h^n$  solution of (3.62) and there exists a constant  $\mathcal{C}$ , independent of  $T$  and  $h$ , such that it satisfies*

$$(3.67) \quad \max_{n \in \{0, \dots, N\}} \left( \mathcal{E}_h^{n+\frac{1}{2}} \right)^{\frac{1}{2}} \leq \mathcal{C} \|f\|_{L^1(0,T;L^2(\Omega))}.$$

As a consequence, the a priori estimates (3.57) and (3.58) hold true.

**Proof.** The a priori estimates are established as in Theorem 3.16 and Corollary 3.17. Since the problem is linear and one works in finite dimensional subspaces, these yields the solvability of the scheme. Details are omitted.  $\blacksquare$

4. APPLICATION TO  $\mathbb{P}_1$ -FINITE ELEMENT APPROXIMATIONS

In this section we assume that  $\Omega$  is a polygonal domain with (open) edges denoted  $\gamma_m, 1 \leq m \leq M$ :

$$\Gamma = \bigcup_{m=1}^M \bar{\gamma}_m, \quad \gamma_k \cap \gamma_\ell = \emptyset \text{ if } k \neq \ell.$$

We consider three Lagrange quasi-uniform (see Remark 4.1) families of conforming (non-degenerate, see Remark 4.2) triangulations of  $\Omega$  (see [10] or [5])  $\mathcal{T}_{h_S}^S, \mathcal{T}_{h_P}^P$  and  $\mathcal{T}_h$  (with respective sizes  $h_S, h_P$  and  $h$ ) that will serve respectively for the discretization of the potentials  $\varphi_S, \varphi_P$  and the auxiliary variable  $p = p(\nu)$  (see (2.39, 2.40)) involved in the definition of the operator  $\mathcal{E}$  and the bilinear form  $b(\cdot, \cdot)$  (see (2.41) and (2.69)). We define the following spaces

$$(4.1) \quad \begin{aligned} V_{P,h} &:= \{ \varphi_{P,h} \in H^1(\Omega) / \forall K \in \mathcal{T}_{h_P}^P, \varphi_{P,h}|_K \in \mathbb{P}_1(K) \}, \\ V_{S,h} &:= \{ \varphi_{S,h} \in H^1(\Omega) / \forall K \in \mathcal{T}_{h_S}^S, \varphi_{S,h}|_K \in \mathbb{P}_1(K) \}, \\ P_h &:= \{ p_h \in H^1(\Omega) / \forall K \in \mathcal{T}_h, p_h|_K \in \mathbb{P}_1(K) \}, \end{aligned}$$

and we recall that  $\mathbf{V}_h = V_{P,h} \times V_{S,h}$ . In section (4.1), we investigate the positivity property (3.21) and the discrete inf-sup condition (3.24), provided an adequate choice of the space  $M_h$ . Then, in Section 4.2, we study in more details the stability conditions (3.54) and (3.66).

**Remark 4.1.** *A family of triangulations  $\mathcal{T}_h$  of step size  $h := \sup\{\text{diam } K, K \in \mathcal{T}_h\}$  is said to be quasi-uniform (see [10] or [5]) if there exists a number  $\rho > 0$  (by definition the quasi-uniformity factor of the family), independent of  $h$ , such that*

$$(4.2) \quad \forall K \in \mathcal{T}_h, \rho_K \geq \rho h, \text{ where } \rho_K \text{ is the radius of the largest disk inside } K.$$

**Remark 4.2.** *By non degenerate triangulations, we mean triangulations that have at least one interior node in  $\Omega$ . The reader will notice that, in practice, this is almost not a constraint.*

## 4.1. Positivity and inf-sup condition.

*Positivity property.* Our first (and easy) result expresses the fact that using  $\mathbb{P}_1$  finite elements for the auxiliary space  $P_h$  is sufficient for guaranteeing the positivity property (3.21), whatever is the choice of the spaces  $V_{P,h}$  and  $V_{S,h}$ .

**Theorem 4.3.** *With the choice of the space  $P_h$  given by (4.1) the strong compatibility assumption 3.6 holds hence the positivity property (3.21) holds.*

**Proof.** For any  $p_h \in P_h$  since  $p_h|_K \in \mathbb{P}_1(K)$ , in particular  $\partial_2 p_h$  is constant in  $K$ . The condition  $\partial_2 p_h \in H^1(\Omega)$  is then sufficient, since  $\Omega$  is connected, to conclude that this constant is the same in all triangles, i.e.  $\partial_2 p_h = a \in \Omega$  for some  $a \in \mathbb{R}$ . ■

*Construction of  $M_h$  and inf-sup condition.* A less important but nevertheless desirable property is the discrete inf-sup property (3.24). It appears that proving this property is slightly more difficult than proving the positivity property (3.21). This is where the construction of  $M_h$  is involved. Our proof of (3.24) will use a weaker inf-sup condition, that appears for instance in the analysis of mortar element methods (see [16] for instance)

$$(4.3) \quad \nu_h \in M_h, \quad \int_{\Gamma} \nu_h q_h \, d\gamma = 0 \quad \forall q_h \in P_h \quad \Rightarrow \quad \nu_h = 0.$$

Several choices of space  $M_h$  would make property (4.3) true. We have chosen to construct  $M_h$  as the trace space of functions in  $P_h$ , except at the vertices of  $\Omega$ , where we introduce a discontinuity while, at the same time, reducing by 1 the polynomial order of functions in  $M_h$  along the two adjacent edges of the mesh that share this vertex. This leads to introduce a mesh  $\mathcal{T}_h(\Gamma)$  as the trace of  $\mathcal{T}_h$

$$\mathcal{T}_h(\Gamma) = \{\partial K \cap \partial\Omega / K \in \mathcal{T}_h\},$$

and to separate the edges of  $\mathcal{S}_h$  into two groups,  $\mathcal{T}_h(\Gamma) = \mathcal{T}_h^i(\Gamma) \cup \mathcal{T}_h^b(\Gamma)$  with  $\mathcal{T}_h^i(\Gamma) \cap \mathcal{T}_h^b(\Gamma) = \emptyset$ . The edges in  $\mathcal{T}_h^i(\Gamma)$  do not share any extremity with the edges  $\gamma_m$ . Saying otherwise, no end point of any edge  $e$  of  $\mathcal{T}_h^i(\Gamma)$  coincide with a vertex of  $\Omega$ . We next define the finite element space  $M_h$  as

$$(4.4) \quad M_h := \left\{ \nu_h \in \prod_{m=1}^M H^1(\gamma_m) \quad / \quad \forall e \in \mathcal{T}_h^i(\Gamma) \quad \nu_h|_e \in \mathbb{P}_1(e), \right. \\ \left. \forall e \in \mathcal{T}_h^b(\Gamma) \quad \nu_h|_e \in \mathbb{P}_0(e) \right\}.$$

With this choice it can be proven that (4.3) holds (see again [16]). Moreover it will be useful later to use the property that, since  $\Omega$  is a polygonal domain, components of the outward normal  $\mathbf{n} = (n_1, n_2)$  belong to  $M_h$ , and therefore, if  $\Pi_h$  is the projection from  $L^2(\Gamma)$  on  $M_h$ , (c.f. (3.5)),

$$(4.5) \quad \Pi_h(n_1) = n_1, \quad \Pi_h(n_2) = n_2.$$

For proving the inf-sup condition, we shall use two technical conditions. The first concerns the triangulation  $\mathcal{T}_h$ .

**Assumption 4.4** (Admissible triangulation). *The triangulation  $\mathcal{T}_h$  of  $\Omega$  is admissible if all the edges parallel to the  $x_2$  axis not belonging to the boundary have at least one extremity inside  $\Omega$ .*

Note that Assumption 4.4 is not very demanding since, even when it fails (see for instance Figure 1, left picture), a small modification of  $\mathcal{T}_h$  permits to recover it (Figure 1, right picture)).

The second condition is a compatibility condition between the space  $V_{S,h}$  and the triangulation  $\mathcal{T}_h$ .

**Assumption 4.5** (Compatibility). *For each element  $K \in \mathcal{T}_h$  there exists a function  $\psi_{S,K} \in V_{S,h}$  with support included in  $K$  and such that  $\int_K \psi_{S,K} \, d\mathbf{x} \neq 0$ .*

The reader will easily verify that this assumption holds if  $\mathcal{T}_{h_S}^S$  is a submesh of  $\mathcal{T}_h$  so that any open triangle  $K$  of  $\mathcal{T}_h$  contains a node of  $\mathcal{T}_{h_S}^S$ , as illustrated by fig. (1).

In the rest of this section,

- (i) We first prove an intermediate inf-sup condition for the bilinear form  $b_h(\cdot, \cdot)$ , using the "mortar" inf-sup condition (4.3). This is Lemma 4.6.
- (ii) We show that the proof of inf-sup condition for the augmented bilinear form  $\widehat{b}_h(\cdot, \cdot)$  can be more or less reduced to the use of Lemma 4.6. This is Theorem 4.7.

**Lemma 4.6.** *If the assumptions 4.4 and 4.5 hold, then*

$$(4.6) \quad b_h(\nu_h, \psi_h) = 0, \quad \forall \psi_h \in \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega) \quad \Rightarrow \quad \nu_h = 0.$$

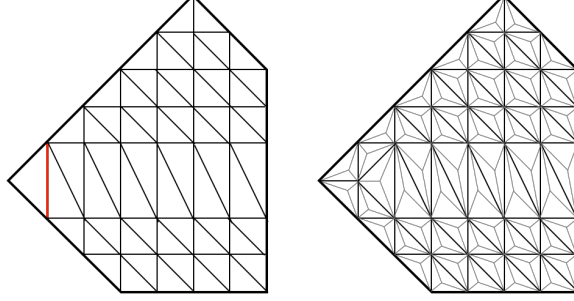


FIGURE 1. Left: sketch of a triangulation  $\mathcal{T}_h$  that does not satisfies Assumption 4.4 because of the leftmost interior edge (in red). Right: sketch of a triangulation  $\mathcal{T}_h = \mathcal{T}_{h_P}^P$  (in black) and  $\mathcal{T}_{h_S}^S$  (in grey) satisfying both assumptions 4.4 and 4.5.

**Proof.** Assume that  $b_h(\nu_h, \psi_h) = 0$  for all  $\psi_h \in \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$ . Using the definition (3.11) of  $b_h(\cdot, \cdot)$  as long as  $\psi_h \in \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$ , we have, since boundary terms vanish,

$$(4.7) \quad b_h(\nu_h, \psi_h) = \frac{1}{V_P^2} \int_{\Omega} \partial_1 p_h \psi_{P,h} \, d\mathbf{x} + \frac{1}{V_S^2} \int_{\Omega} \partial_2 p_h \psi_{S,h} \, d\mathbf{x},$$

where  $p_h \equiv p_h(\nu_h)$  is defined by (3.10). We split the rest of the proof in two steps.

*Step 1 :* we prove that  $\nabla p_h(\nu_h) = 0$ . Let  $K$  be any triangle of  $\mathcal{T}_h$ . Using Assumption 4.5, we can take  $\psi_h = (0, \psi_{S,K})$  in (4.6) and get

$$\int_K \partial_2 p_h \psi_{S,K} \, d\mathbf{x} = \partial_2 p_h|_K \int_K \psi_{S,K} \, d\mathbf{x} = 0,$$

since  $\partial_2 p_h$  is constant on each element  $K$ . Thus  $\partial_2 p_h|_K = 0$  and therefore

$$p_h(\mathbf{x}) = a_K x_1 + b_K, \quad \forall x \in K.$$

Next, let  $K$  and  $K'$  in  $\mathcal{T}_h$  share an edge  $e$ . We also have

$$p_h(\mathbf{x}) = a_{K'} x_1 + b_{K'}, \quad \forall x \in K'.$$

We now prove that  $a_K = a_{K'}$  and  $b_K = b_{K'}$ . To do so we distinguish two mutually exclusive cases.

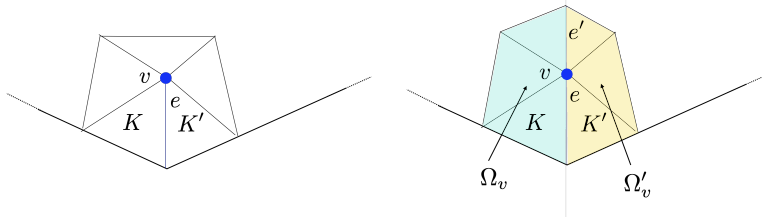


FIGURE 2. Left: sketch of the configuration B(i). Right: sketch of the configuration B(ii). Each time the element  $K$  is chosen adjacent to the boundary  $\Gamma$ .



- A. If the edge  $e$  is not parallel to the  $x_2$  axis, then the continuity of  $p_h$  across  $e$  yields  $a_K = a_{K'}$  and  $b_K = b_{K'}$ : argue on the equality of two affine functions of the real variable along a non empty interval of  $\mathbb{R}$ , namely the projection of  $e$  on the real axis (this is where the fact that  $e$  is not parallel to the  $x_2$  axis comes into play).
- B. If the edge  $e$  is parallel to the  $x_2$  axis, then thanks to the admissibility condition of Assumption 4.4, there exists an extremity  $v$  of  $e$  that is interior to  $\Omega$ . Therefore, the corresponding Lagrange basis function  $q_v$  belongs to  $P_h \cap H_0^1(\Omega)$ . Then, as illustrated in the Figure 2 below, two cases arise.
- (i) There does not exist any other edge having  $v$  as a vertex that is parallel to the  $x_2$  axis. In that case, arguing several times as in point A (turning around  $v$  without crossing  $e$ , see Figure 2), we conclude again that  $a_K = a_{K'}$  and  $b_K = b_{K'}$ .
  - (ii) There exists another edge  $e'$  that has  $v$  as a vertex and is parallel to the  $x_2$  axis. Then, the vertical segment  $\overline{e \cup e'}$  splits the support of  $q_v$  into two parts:

$$\text{supp } q_v = \overline{\Omega_v} \cup \overline{\Omega'_v}, \quad K \subset \Omega_v, \quad K' \subset \Omega'_v.$$

Arguing as in point A,  $\partial_1 p_h = a_K$  in  $\Omega_v$  and  $\partial_1 p_h = a_{K'}$  in  $\Omega'_v$ . We now use the property that  $p_h$  is a discrete harmonic function. Choosing  $q_h = q_v$  in (3.10) we obtain

$$\int_{\Omega} \partial_1 p_h \partial_1 q_v \, d\mathbf{x} = 0 \quad \Rightarrow \quad a_K \int_{\Omega_v} \partial_1 q_v \, d\mathbf{x} + a_{K'} \int_{\Omega'_v} \partial_1 q_v \, d\mathbf{x} = 0.$$

Using Green's formula we obtain  $(a_K - a_{K'}) \int_{e \cup e'} q_v \, d\gamma = 0$  which implies  $a_K = a_{K'}$ .

Finally, the continuity of  $p_h$  across  $e$  also implies  $b_K = b_{K'}$ .

Repeating the above argument, by connexity of  $\Omega$ , we deduce that  $p_h$  is an affine function in all  $\Omega$ ,

$$p_h(x) = a x_1 + b, \quad \forall x \in \Omega.$$

Finally, the triangulation  $\mathcal{T}_h$  being non degenerate, there exists a function  $\psi_{P,\Omega} \in V_{P,h}$  whose integral over  $\Omega$  is positive. Choosing now  $\boldsymbol{\psi}_h = (\psi_{P,\Omega}, 0) \in \mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$  as a test function, we obtain

$$b_h(\nu_h, \boldsymbol{\psi}_h) \equiv \frac{1}{V_P^2} \int_{\Omega} \partial_1 p_h \psi_{P,\Omega} \, d\mathbf{x} = a \int_{\Omega} \psi_{P,\Omega} \, d\mathbf{x} \quad \text{which implies} \quad a = 0.$$

This completes the proof of Step 1.

*Step 2:* we prove that  $\nu_h = 0$ . Going back to the definition (3.10) of  $p_h(\nu_h)$ , we see that  $\nabla p_h(\nu_h) = 0$  implies

$$\int_{\Gamma} \nu_h q_h \, d\gamma = 0, \quad \forall q_h \in P_h,$$

hence  $\nu_h = 0$ , thanks to the inf-sup condition (4.3). ■

We are now in position to prove the following theorem.

**Theorem 4.7.** *If the assumptions 4.4 and 4.5 hold then (3.24) holds, i.e.*

$$(4.8) \quad \widehat{b}_h(\widehat{\nu}_h, \boldsymbol{\psi}_h) = 0, \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_h \quad \implies \quad \widehat{\nu}_h = \mathbf{0}.$$

**Proof.** By definition (3.18) of  $\widehat{b}_h(\cdot, \cdot)$ , the assumption at the l. h. s. of (4.8) is

$$(4.9) \quad b_h(\nu_h, \boldsymbol{\psi}_h) + \nu_r m(\boldsymbol{\varphi}_R, \boldsymbol{\psi}_h) + \nu_n \int_{\Gamma} \boldsymbol{\psi}_h \cdot \mathbf{n} \, d\gamma + \nu_{\boldsymbol{\tau}} \int_{\Gamma} \boldsymbol{\psi}_h \cdot \boldsymbol{\tau} \, d\gamma = 0,$$

for any  $\boldsymbol{\psi}_h \in \mathbf{V}_h$ . In the following proof, we show first that  $\nu_r = 0$ , then that  $\nu_h = 0$ , finally that  $\nu_n = \nu_{\boldsymbol{\tau}} = 0$ .

In order to isolate  $\nu_r$ , we first get rid of  $\nu_h$  by building a function  $\boldsymbol{\psi}_h$  in the kernel of the bilinear form  $b_h(\cdot, \cdot)$ . The reader will easily verify that, at the continuous level,  $\boldsymbol{\varphi} := (0, 1)$  is in the kernel of  $b(\cdot, \cdot)$ . The first step is to check that this property is preserved at the discrete level.

*Step 1:* the field  $\boldsymbol{\varphi} := (0, 1) \in \mathbf{V}_h$  satisfies  $b_h(\nu_h, \boldsymbol{\varphi}) = 0, \quad \forall \nu_h \in M_h$ .

Indeed by definition (3.11) of  $b_h(\cdot, \cdot)$ , we have, with  $p_h = p_h(\nu_h)$ ,

$$\begin{aligned} b_h(\nu_h, \boldsymbol{\varphi}) &= \frac{1}{V_S^2} \int_{\Omega} \partial_2 p_h \, d\mathbf{x} - \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\varphi} \cdot \boldsymbol{\tau}) \nu_h \, d\gamma - \frac{1}{2V_S^2} \int_{\Gamma} p_h \Pi_h(\boldsymbol{\varphi} \cdot \mathbf{n}) \, d\gamma, \\ &= \frac{1}{V_S^2} \int_{\Gamma} p_h n_2 \, d\gamma - \frac{1}{2V_S^2} \int_{\Gamma} \mathcal{I}(\boldsymbol{\varphi} \cdot \boldsymbol{\tau}) \nu_h \, d\gamma - \frac{1}{2V_S^2} \int_{\Gamma} p_h \Pi_h(\boldsymbol{\varphi} \cdot \mathbf{n}) \, d\gamma, \end{aligned}$$

where we have used Green's formula for the second line.

Since  $\boldsymbol{\varphi} = \nabla x_2$ ,  $\mathcal{I}(\boldsymbol{\varphi} \cdot \boldsymbol{\tau}) = x_2$ . Moreover, by (4.5),  $\Pi_h(\boldsymbol{\varphi} \cdot \mathbf{n}) = n_2$ . Therefore

$$(4.10) \quad b_h(\nu_h, \boldsymbol{\varphi}) = \frac{1}{2V_S^2} \int_{\Gamma} p_h n_2 \, d\gamma - \frac{1}{2V_S^2} \int_{\Gamma} x_2 \nu_h \, d\gamma.$$

On the other hand, taking  $q_h = x_2$  in (3.10) (for  $p_h(\nu_h)$ ) yields

$$\frac{1}{V_S^2} \int_{\Gamma} p_h n_2 \, d\gamma = \frac{1}{V_S^2} \int_{\Gamma} x_2 \nu_h \, d\gamma.$$

This equality, coupled with (4.10), completes the proof of Step 1.

*Step 2:*  $\nu_r = 0$ . We choose  $\boldsymbol{\psi}_h = \boldsymbol{\varphi}$  in (4.9). Noting that, since  $\Gamma = \partial\Omega$  is a closed loop and  $\boldsymbol{\varphi} = \nabla x_2$ ,

$$\int_{\Gamma} \boldsymbol{\varphi} \cdot \mathbf{n} \, d\gamma = \int_{\Gamma} \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \, d\gamma = 0,$$

we simply get  $\nu_r m(\boldsymbol{\varphi}_R, \boldsymbol{\varphi}) = 0$  which yields  $\nu_r = 0$  since we proved (see (2.55)) that  $m(\boldsymbol{\varphi}_R, \boldsymbol{\varphi}) \neq 0$ .

*Step 3:*  $\nu_h = 0$ . We simply take in (4.9) any element  $\boldsymbol{\psi}_h$  of  $\mathbf{V}_h \cap \mathbf{H}_0^1(\Omega)$ . Of course, boundary terms vanish and we simply get  $b_h(\nu_h, \boldsymbol{\psi}_h) = 0$  for all these  $\boldsymbol{\psi}_h$ . Then, Lemma 4.6 allows us to conclude.

*Step 4:*  $\nu_n = \nu_{\boldsymbol{\tau}} = 0$ . It suffices to take in (4.9)  $\boldsymbol{\psi}_h = \boldsymbol{\psi}_n := (x_1, x_2) \equiv \frac{1}{2} \nabla |\mathbf{x}|^2 \in$

$\mathbf{V}_h$  which satisfies

$$\int_{\Gamma} \psi_{\mathbf{n}} \cdot \mathbf{n} \, d\gamma = \frac{1}{2} \int_{\Omega} 2 \, dx > 0 \quad \text{and} \quad \int_{\Gamma} \psi_{\mathbf{n}} \cdot \boldsymbol{\tau} \, d\gamma = 0.$$

to conclude that  $\nu_{\mathbf{n}} = 0$ . In the same way, taking  $\psi_h = \psi_{\boldsymbol{\tau}} := (x_2, -x_1) = \frac{1}{2} \mathbf{curl} |\mathbf{x}|^2$  gives  $\nu_{\boldsymbol{\tau}} = 0$ .  $\blacksquare$

#### 4.2. About the CFL condition for leap-frog and implicit/explicit schemes.

4.2.1. *Preamble.* For the acoustic wave equation with velocity  $c$ , it is well known that finite element approximations (on a quasi-uniform family of meshes of step-size  $h$ ) in space coupled with explicit time discretization (Leap Frog scheme) lead to a CFL stability condition of the form

$$(4.11) \quad \frac{c\Delta t}{h} \leq C(\rho),$$

where the constant  $C(\rho)$  does not depend on the velocity  $c$  but only on the coefficient  $\rho$  appearing in the characterization of the meshes (see Remark 4.1). More precisely, it is possible to show that there exists a *sufficient stability condition* of the form (4.11) and that, in the particular case of regular meshes, such a condition is *asymptotically necessary* (when  $h \rightarrow 0$ ).

Note that this means that  $\Delta t$  is constrained only by the ratio  $h/c$ , that is to say by the *number of mesh points per wavelength*: take for instance the case of a time harmonic source with period  $T$ , that generates a wavelength  $\Lambda = cT$ , then  $h/c = T(h/\Lambda)$ , where the ratio  $h/\Lambda$  is (for a regular mesh for instance) nothing but the inverse of number of mesh points per wavelength and determinates in practice the accuracy of the approximation.

If we extrapolate the above observation to systems involving several velocities between  $c_{min}$  and  $c_{max} > c_{min}$ , like the elastodynamics system, the accuracy leads to constrain  $h$  by  $c_{min}$  (associated to the smallest wavelength). In the other hand, the stability of the scheme constrains the time step  $\Delta t$  by the inverse of  $c_{max}$ . In consequence, the computational time is proportional to  $c_{max}/c_{min}$ .

For elastodynamics, with our method, we want to take advantage of having two meshes with step-sizes  $h_P$  and  $h_S$  respectively, that we can adjust in such a way that  $h_P/V_P \simeq h_S/V_S$ , so that the number of mesh points per wavelength is about the same for the two types of waves, which are then approximated with a similar accuracy.

To fully benefit from this, for a given "explicit" time discretization, we would like to obtain a (sufficient) stability condition of the form

$$(4.12) \quad \Delta t \leq C(\rho) \min \left( \frac{h_P}{V_P}, \frac{h_S}{V_S} \right).$$

If this holds, the corresponding computational time would not depend on  $V_P/V_S$ .

4.2.2. *Assumptions and main results.* Before going into more details, let us first explain why it is not so easy to obtain a sufficient stability condition of the form (4.12), for example for the implicit-explicit scheme (3.62). A natural approach, that we recall for completeness, would consist in starting from the following classical local inequality to get corresponding global inequalities.

**Lemma 4.8.** *For all  $v \in \mathbb{P}_1(K)$*

$$\int_K |\nabla v|^2 \, d\mathbf{x} \leq \frac{72}{\rho_K^2} \int_K |v|^2 \, d\mathbf{x},$$

where  $\rho_K$  is the radius of the largest disk included in  $K$ .

**Proof.** We give a short proof for the sake of completeness. Following the notation of [10], let  $F_K(\hat{\mathbf{x}}) = B_K \hat{\mathbf{x}} + b_K$  denote the affine mapping from the reference element  $\hat{K}$  (the triangle with vertices  $(0,0)^t$ ,  $(0,1)^t$  and  $(1,0)^t$ ) to the element  $K$ . Then (as in Theorem 3.12 in [10]) one can easily deduce

$$(4.13) \quad \begin{aligned} \int_K |\nabla v|^2 \, d\mathbf{x} &\leq \|B_K^{-1}\|_2^2 |\det B_K| \int_{\hat{K}} |\nabla_{\hat{\mathbf{x}}}(v \circ F_K)|^2 \, d\hat{\mathbf{x}}, \\ \int_K |v|^2 \, d\mathbf{x} &= |\det B_K| \int_{\hat{K}} |v \circ F_K|^2 \, d\hat{\mathbf{x}}, \end{aligned}$$

where, as usual,  $\nabla_{\hat{\mathbf{x}}}$  is the gradient operator in the local coordinates  $\hat{\mathbf{x}}$  and  $\|\cdot\|_2$  also denotes the matrix norm induced by the euclidian norm. Moreover one can show that

$$(4.14) \quad \int_{\hat{K}} |\nabla_{\hat{\mathbf{x}}}(v \circ F_K)|^2 \, d\hat{\mathbf{x}} \leq 36 \int_{\hat{K}} |v \circ F_K|^2 \, d\hat{\mathbf{x}},$$

since one can verify that the inequality above is an equality for the eigenfunction  $\hat{w} = -2 + 3\hat{x}_1 + 3\hat{x}_2$  in  $\mathbb{P}_1(\hat{K})$  that is associated to the largest eigenvalue  $\lambda = 36$  of the eigenvalue problem,

$$\int_{\hat{K}} \nabla \hat{w} \cdot \nabla \hat{v} \, d\hat{\mathbf{x}} = \lambda \int_{\hat{K}} \hat{w} \hat{v} \, d\hat{\mathbf{x}}, \quad \forall \hat{v} \in \mathbb{P}_1(\hat{K}).$$

Note that the other eigenpairs are  $(\lambda = 0, \hat{w} = 1)$  and  $(\lambda = 12, \hat{w} = \hat{x}_1 - \hat{x}_2)$ . Equation (4.13) and (4.14) imply that

$$\int_K |\nabla v|^2 \, d\mathbf{x} \leq 36 \|B_K^{-1}\|_2^2 \int_K |v|^2 \, d\mathbf{x},$$

and, finally, from Theorem 3.1.3 in [10] we have  $\|B_K^{-1}\|_2 \leq \sqrt{2}/\rho_K$  hence the result.  $\blacksquare$

Since the triangulations  $\mathcal{T}_{h_P}^P$  and  $\mathcal{T}_{h_S}^S$  are quasi-uniform (cf. Remark 4.1), one easily show the following result.

**Theorem 4.9.** *Let  $A(\rho) = 72/\rho^2$ , then for any  $(\psi_{P,h}, \psi_{S,h}) \in V_{P,h} \times V_{S,h}$ ,*

$$\int_{\Omega} |\nabla \psi_{P,h}|^2 \, d\mathbf{x} \leq \frac{A(\rho)}{h_P^2} \int_{\Omega} |\psi_{P,h}|^2 \, d\mathbf{x}, \quad \int_{\Omega} |\nabla \psi_{S,h}|^2 \, d\mathbf{x} \leq \frac{A(\rho)}{h_S^2} \int_{\Omega} |\psi_{S,h}|^2 \, d\mathbf{x}.$$

As a consequence, we deduce immediately

$$\forall \boldsymbol{\psi}_h \in \mathbf{V}_h, \quad a_{\Omega}(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) \leq A(\rho) \max\left(\frac{V_P^2}{h_P^2}, \frac{V_S^2}{h_S^2}\right) m_{\Omega}(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h)$$

However, the quantity involved in the abstract CFL condition (3.66) is

$$(4.15) \quad \begin{aligned} \max_{\psi_h \in \mathbf{V}_{N,h} \setminus \{0\}} \frac{a_\Omega(\psi_h, \psi_h)}{m_h(\psi_h, \psi_h)} &\leq \kappa_h^{-1} \max_{\psi_h \in \mathbf{V}_{N,h} \setminus \{0\}} \frac{a_\Omega(\psi_h, \psi_h)}{m_\Omega(\psi_h, \psi_h)} \\ &\leq \kappa_h^{-1} A(\rho) \max\left(\frac{V_P^2}{h_P^2}, \frac{V_S^2}{h_S^2}\right), \end{aligned}$$

where, according to Remark 3.3, we have define,

$$(4.16) \quad \kappa_h := \max_{\psi_h \in \mathbf{V}_h \setminus \{0\}} \frac{m_h(\psi_h, \psi_h)}{m_\Omega(\psi_h, \psi_h)}.$$

Using (4.15) in (3.66), we see that a sufficient stability condition for the scheme is

$$(4.17) \quad \Delta t^2 < 4 \kappa_h A(\rho)^{-1} \min\left(\frac{h_P^2}{V_P^2}, \frac{h_S^2}{V_S^2}\right).$$

We would have reached our goal if we would be able to prove that  $\kappa_h$  is bounded from below by a quantity independent of  $(h_P, h_S)$ , as well as  $(V_P, V_S)$ . Even though we believe this is true, this is precisely what we did not succeed to prove (cf. Remark 3.3).

This is why we need to proceed differently which leads us, for technical reasons, to assume the following condition of the meshes triangulations  $\mathcal{T}_{h_P}^P$ ,  $\mathcal{T}_{h_S}^S$  and  $\mathcal{T}_h$

**Assumption 4.10.** *The triangulations  $\mathcal{T}_{h_P}^P$  and  $\mathcal{T}_{h_S}^S$  are subtriangulations of  $\mathcal{T}_h$  hence the approximation spaces satisfy  $P_h \subset V_{h,P}$  and  $P_h \subset V_{h,S}$  (by saying  $\mathcal{T}'_h$  is a subtriangulation of  $\mathcal{T}_h$ , we mean that any triangle of  $\mathcal{T}_h$  is a finite union of triangles of  $\mathcal{T}'_h$ ).*

**Theorem 4.11.** *Suppose that the families of triangulations  $\mathcal{T}_h$ ,  $\mathcal{T}_{h_P}^P$  and  $\mathcal{T}_{h_S}^S$  are quasi-uniform with quasi-uniformity factor  $\rho$  and that Assumption 4.10 holds then:*

- (i) *A sufficient stability condition for the implicit-explicit scheme is*

$$(4.18) \quad \Delta t < \Delta t_{EI} \text{ with } \Delta t_{EI} = \frac{\rho}{3\sqrt{2}} \min\left(\frac{h_P}{V_P}, \frac{h_S}{V_S}\right).$$

- (ii) *A sufficient stability condition for the  $\theta$ -scheme is*

$$(4.19) \quad \Delta t < \frac{\Delta t_{LF}}{(1-4\theta)^{\frac{1}{2}}} \text{ with } \Delta t_{LF} = \Delta t_{EI}/\sqrt{2}.$$

**Remark 4.12.** *We suspect that the above assumption on the triangulations, that we use of course in our proof, is not really needed for guaranteeing sufficient stability conditions similar to those of Theorem 4.11. Note however, that it is not so constraining in practice.*

**Remark 4.13.** *If we compare the sufficient CFL condition for the implicit-explicit scheme with the one of the explicit Leap Frog Scheme, we see - this is not surprising - that the obtained CFL for the Leap-Frog scheme ( $\theta = 0$ ) is  $\sqrt{2}$  more restrictive.*

4.2.3. *Proof of Theorem 4.11.* We shall restrict ourselves to prove the point (i) of Theorem 4.11, i.e. to establish the sufficient CFL condition for the mixed explicit-implicit scheme (3.62). The proof of point (ii) is quite similar and left to the reader (see however Remark 4.15 for useful comments).

The idea is to obtain a global bound in the whole domain  $\Omega$  from local bounds in each triangle  $K$ . This leads us to introduce the “broken” discrete spaces

$$(4.20) \quad \begin{aligned} V_{P,h}^b &:= \{ \varphi_{P,h} \in L^2(\Omega) / \forall K \in \mathcal{T}_{h_p}^P, \varphi_{P,h}|_K \in \mathbb{P}_1(K) \}, \\ V_{S,h}^b &:= \{ \varphi_{S,h} \in L^2(\Omega) / \forall K \in \mathcal{T}_{h_s}^S, \varphi_{S,h}|_K \in \mathbb{P}_1(K) \}, \\ \mathbf{V}_h^b &:= V_{P,h}^b \times V_{S,h}^b, \end{aligned}$$

as well as the “broken” bilinear form  $a_\Omega^b$  defined by

$$a_\Omega^b(\varphi_h, \psi_h) := \sum_{K \in \mathcal{T}_h} a_{\Omega,K}(\varphi_h, \psi_h), \quad \forall (\varphi_h, \psi_h) \in \mathbf{V}_h^b \times \mathbf{V}_h^b.$$

where we have defined

$$a_{\Omega,K}(\varphi_h, \psi_h) := \int_K (\nabla \varphi_{P,h} \cdot \nabla \psi_{P,h} + \nabla \varphi_{S,h} \cdot \nabla \psi_{S,h}) \, d\mathbf{x}.$$

Remark that  $\mathbf{V}_h \subset \mathbf{V}_h^b$  and that  $a_\Omega(\cdot, \cdot)$  and  $a_\Omega^b(\cdot, \cdot)$  coincide in  $\mathbf{V}_h$ . Moreover if Assumption 4.10 holds then  $\nabla P_h \subset \mathbf{V}_h^b$ . We have then the following result:

**Lemma 4.14.** *Let Assumption 4.10 hold, then the condition (3.66) holds if*

$$(4.21) \quad m_\Omega(\psi_h, \psi_h) - \frac{\Delta t^2}{4} a_\Omega^b(\psi_h, \psi_h) \geq 0, \quad \forall \psi_h \in \mathbf{V}_h^b.$$

**Proof.** First observe that, since second order derivatives of functions in  $P_h$  vanish in each  $K$ , for any  $q_h \in P_h$  and  $\varphi_h \in \mathbf{V}_h^b$  we have,

$$a_\Omega^b(\varphi_h, \varphi_h) = a_\Omega^b(\varphi_h - \nabla q_h, \varphi_h - \nabla q_h).$$

Moreover thanks to our assumptions on the approximation spaces, for all  $\varphi_h \in \mathbf{V}_h^b$  and all  $q_h \in P_h$  we can choose  $\psi_h = \varphi_h - \nabla q_h$  in (4.21) and obtain

$$(4.22) \quad m_\Omega(\varphi_h - \nabla q_h, \varphi_h - \nabla q_h) - \frac{\Delta t^2}{4} a_\Omega^b(\varphi_h, \varphi_h) \geq 0, \quad \forall (\varphi_h, q_h) \in \mathbf{V}_h^b \times P_h.$$

Thanks to (3.28) (see Theorem 3.5) we know that, for any  $\varphi_h \in \mathbf{V}_{N,h}$ ,

$$(4.23) \quad m_h(\varphi_h, \varphi_h) \geq m_\Omega(\varphi_h - \nabla p_h(\nu_h), \varphi_h - \nabla p_h(\nu_h)), \quad \nu_h = \Pi_h(\varphi_h \cdot \mathbf{n}).$$

Then applying (4.22) with  $q_h = p_h(\nu_h)$ , we obtain

$$m_h(\varphi_h, \varphi_h) - \frac{\Delta t^2}{4} a_\Omega^b(\varphi_h, \varphi_h) \geq 0, \quad \forall \varphi_h \in \mathbf{V}_{N,h},$$

which is nothing but (3.66) since the bilinear forms  $a_\Omega^b$  and  $a_\Omega$  coincide in  $\mathbf{V}_{N,h}$ . ■

To conclude the proof of Theorem 4.11, point (i), it suffices to proceed in the same way that we explained in Section 4.2.1. The details are left to the reader.

**Remark 4.15.** *The proof of (ii) for the  $\theta$ -scheme is done along the same lines, introducing the broken version  $a^b(\cdot, \cdot)$  of the bilinear form  $a(\cdot, \cdot)$  and establishing the equivalent of Lemma. Let us simply mention that the  $\sqrt{2}$  factor appearing in the right hand side of is a consequence of the obvious (but sharp) inequality between the*

quadratic forms associated to  $a(\cdot, \cdot)$  and  $a_\Omega(\cdot, \cdot)$  (and the same holds for the broken versions)

$$\forall \varphi \in \mathbf{V}, \quad a(\varphi, \varphi) \leq 2 a_\Omega(\varphi, \varphi).$$

**4.3. About mass lumping.** Let us recall that the mass lumping procedure for finite element methods consists in computing approximately the mass matrix via appropriate quadrature formulas on triangles in such a way that the approximate mass matrix becomes diagonal and the order of accuracy of the approximation is preserved. This is essential for making explicit schemes really explicit computationally.

In our case, the approximation concerns only the volumic part of the mass matrix, that is to say the bilinear form  $m_\Omega(\cdot, \cdot)$  that is computed using the triangular trapezoidal rule, that is exact for polynomials of degree 1. Therefore we make the approximation

$$m_h(\varphi_h, \psi_h) \sim m_{\Omega,h}(\varphi_h, \psi_h) + m_{\Gamma,h}(\varphi_h, \psi_h),$$

where  $m_{\Omega,h}(\cdot, \cdot)$  holds for the approximation of  $m_\Omega(\cdot, \cdot)$  via quadrature. This leads to replace, in the algebraic semi-discrete scheme, the mass matrix  $\mathbb{M}_h$  by the approximate one

$$\tilde{\mathbb{M}}_h = \tilde{\mathbb{M}}_h^\Omega + \mathbb{M}_h^\Gamma$$

where the "boundary part" of the original mass matrix is kept exact. The fact that the matrix  $\tilde{\mathbb{M}}_h^\Omega$  is diagonal results from the position of the quadrature points that coincide with the Lagrange degrees of freedom. For  $\mathbb{P}_1$  finite elements, it is known that the accuracy of the approximation is preserved [11] and that the diagonal elements  $\tilde{\mathbb{M}}_h^\Omega$  are nothing but the sum of the elements of the same line in the original matrix  $\mathbb{M}_h^\Omega$  (which are, by the way all strictly positive since  $\mathbb{P}_1$  shape functions are positive)

$$(4.24) \quad (\tilde{\mathbb{M}}_h^\Omega)_{ii} = \sum_j (\mathbb{M}_h^\Omega)_{ij}, \quad (\tilde{\mathbb{M}}_h^\Omega)_{ij} = 0 \text{ for } j \neq i.$$

thus, the resolution of the corresponding mixed explicit / implicit scheme

$$(4.25) \quad \begin{cases} (\mathbb{M}_h^\Gamma + \tilde{\mathbb{M}}_h^\Omega) \frac{\Phi_h^{n+1} - 2\Phi_h^n + \Phi_h^{n-1}}{\Delta t^2} + \mathbb{A}_h^\Omega \Phi_h^n + \mathbb{A}_h^\Gamma \{\Phi_h^n\}_{\frac{1}{4}} + \widehat{\mathbb{B}}_h \widehat{\mathbf{E}}_h^n = \mathbf{F}_h^n, \\ \widehat{\mathbb{B}}_h^T \Phi_h^n = \mathbf{0}, \end{cases}$$

is implicit for the boundary degrees of freedom and explicit for the interior degrees of freedom. Let us clarify this by setting

$$\Phi_h^n = (\Phi_{\mathbf{i},h}^n, \Phi_{\mathbf{b},h}^n)^t,$$

where  $\Phi_{\mathbf{i},h}^n$  is the vector of Lagrange degrees of freedom associated to the interior nodes of the meshes and  $\Phi_{\mathbf{b},h}^n$  vector of Lagrange degrees of freedom associated to the corresponding boundary nodes. If we introduce the corresponding block decompositions of  $\tilde{\mathbb{M}}_h^\Omega$ ,  $\mathbb{M}_h$  and  $\mathbb{A}_h^\Gamma$

$$\tilde{\mathbb{M}}_h^\Omega = \begin{pmatrix} \tilde{\mathbb{M}}_{\mathbf{ii},h}^\Omega & 0 \\ 0 & \tilde{\mathbb{M}}_{\mathbf{bb},h}^\Omega \end{pmatrix}, \quad \tilde{\mathbb{A}}_h^\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{A}_{\mathbf{bb},h}^\Gamma \end{pmatrix}, \quad \mathbb{M}_h^\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\mathbb{M}}_{\mathbf{bb},h}^\Gamma \end{pmatrix},$$

the inspection of the scheme shows that the updating of  $(\Phi_h^n, \mathbf{E}_h^n)$  from the previous time steps (hidden below in the terms *r.h.s.*) is obtained by inverting the following system (we consider the explicit-implicit scheme introduced in Section 3.3.2)

$$(4.26) \quad \begin{cases} \tilde{\mathbb{M}}_{\mathbf{ii},h}^\Omega \Phi_{\mathbf{i},h}^{n+1} = r.h.s., & (i) \\ (\tilde{\mathbb{M}}_{\mathbf{bb},h}^\Omega + \tilde{\mathbb{M}}_{\mathbf{bb},h}^\Gamma + \frac{\Delta t^2}{4} \mathbb{A}_{\mathbf{bb},h}^\Gamma) \Phi_{\mathbf{b},h}^{n+1} = r.h.s., & (ii) \\ \hat{\mathbb{B}}_h (\tilde{\mathbb{M}}_{\mathbf{bb},h}^\Omega + \tilde{\mathbb{M}}_{\mathbf{bb},h}^\Gamma + \frac{\Delta t^2}{4} \mathbb{A}_{\mathbf{bb},h}^\Gamma)^{-1} \hat{\mathbb{B}}_h^T \hat{\mathbf{E}}_h^{n+1} = r.h.s., & (iii) \end{cases}$$

Thus (4.26)(i) shows that  $\Phi_{\mathbf{i},h}^{n+1}$  is computed explicitly while  $\hat{\mathbf{E}}_h^{n+1}$  (resp.  $\Phi_{\mathbf{b},h}^{n+1}$ ) is computed via the resolution of a small - since reduced to boundary degrees of freedom - and dense linear systems (4.26)(ii). This explains the denomination explicit / implicit.

Finally, it is worthwhile mentioning that mass lumping not only simplifies the numerical computations but also improves the CFL condition. This seems to be rather specific to  $\mathbb{P}_1$  finite elements and relies on the following lemma (that we suspect to be known but did not find in the literature):

**Lemma 4.16.** *In the sense of symmetric matrices (i.e. of the corresponding quadratic forms), one has the inequality*

$$(4.27) \quad \tilde{\mathbb{M}}_h^\Omega \geq \mathbb{M}_h^\Omega.$$

**Proof.** Let us introduce  $\mathbb{D}_h = \tilde{\mathbb{M}}_h^\Omega - \mathbb{M}_h^\Omega$ . Proving (4.27) amounts to proving that the eigenvalues of  $\mathbb{D}_h$  are positive. For this, we can use the Gershgorin circle theorem, that, for real symmetric matrices, says that the spectrum  $\sigma(\mathbb{D}_h)$  is included in a finite union of intervals:

$$\sigma(\mathbb{D}_h) \subset \bigcup_i \left[ (\mathbb{D}_h)_{ii} - \sum_{j \neq i} |(\mathbb{D}_h)_{ij}|, (\mathbb{D}_h)_{ii} + \sum_{j \neq i} |(\mathbb{D}_h)_{ij}| \right]$$

Thus, to conclude, it suffices to show that the lower extrema of the above intervals is positive. In fact, each of them is 0 since, thanks to (4.24),

$$(\mathbb{D}_h^\Omega)_{ii} = \sum_{j \neq i} (\mathbb{M}_h^\Omega)_{ij},$$

while, since  $\tilde{\mathbb{M}}_h^\Omega$  is diagonal and all the entries of  $\mathbb{M}_h^\Omega$  are positive,

$$\sum_{j \neq i} |(\mathbb{D}_h)_{ij}| = \sum_{j \neq i} |(\mathbb{M}_h^\Omega)_{ij}| = \sum_{j \neq i} (\mathbb{M}_h^\Omega)_{ij}.$$

■

**Corollary 4.17.** *The stability condition for the mass lumped explicit/implicit scheme (4.25) is less restrictive than the stability condition for the explicit/implicit scheme (3.62) without mass lumping. In other words, if the scheme (3.62) is stable, the scheme (4.25) is stable too.*

**Proof.** The stability condition (3.66) for the scheme (3.62) means that

$$\mathbb{M}_h - \frac{\Delta t^2}{4} \mathbb{A}_h^\Omega \text{ is a positive matrix.}$$



Thus, by Lemma 4.16,  $\tilde{M}_h - \frac{\Delta t^2}{4} \mathbb{A}_h^\Omega$  is positive too, thus the stability of (4.25). ■

## 5. NUMERICAL RESULTS

In this section we present illustrative numerical results to exhibit the good behaviour of the method. The examples we provide correspond to the propagation of transient waves in an isotropic homogeneous domain which is bounded by a free boundary.

### 5.1. Preparatory definitions.

*The model problem.* We are interested in finding  $\mathbf{u}(\mathbf{x}, t)$  solution of the following elastodynamic problem (which is completed with vanishing initial data)

$$(5.1) \quad \begin{cases} \rho \partial_t^2 \mathbf{u} - (\lambda + 2\mu) \nabla \operatorname{div} \mathbf{u} + \mu \operatorname{curl} \operatorname{curl} \mathbf{u} = \mathbf{f} & \text{in } \Omega \times [0, T], \\ \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n} = \mathbf{0} & \text{on } \Gamma \times [0, T], \end{cases}$$

where we choose the final time to be  $T = 5$  and the computational domain  $\Omega = [-5, 5] \times [-5, 5]$  which centre of gravity is at the origin. We also consider the physical properties given by

$$\rho = 1, \quad \lambda = 56 \quad \text{and} \quad \mu = 4$$

and the medium to be excited by the following source (see Figure 3)

$$(5.2) \quad \mathbf{f}(\mathbf{x}, t) = g(t) \mathbf{h}(\mathbf{x}) \quad \text{where} \quad g(t) = \partial_t \exp(-50(t-1)^2) \quad \text{and} \quad \mathbf{h} \in \mathbf{L}_R^2(\Omega).$$

We present three experiments depending on the value of  $\mathbf{h}$ .

*First Experiment.* We consider a source that generates only pressures waves up to the interaction with the boundary. In this case,  $\mathbf{h}$  it is chosen as the gradient of a smooth compactly supported function centered at  $\mathbf{x}_0 = \mathbf{0}$ .

$$\mathbf{h}(\mathbf{x}) = \nabla s(\mathbf{x}) \quad \text{where} \quad s(\mathbf{x}) = \exp\left(1 + \frac{1}{|2\mathbf{x}|^2 - 1}\right), \quad \text{if } |\mathbf{x}| < 0.5,$$

and zero elsewhere. We shall remark that  $\mathbf{h} \in \mathbf{L}_R^2(\Omega)$  since  $s$  is compactly supported and

$$\int_{\Omega} \nabla s \cdot \mathbf{w}_R \, d\mathbf{x} = - \int_{\Omega} s \operatorname{div}(\mathbf{w}_R) \, d\mathbf{x} = 0, \quad \forall \mathbf{w}_R \in \mathbf{R}(\Omega).$$

*Second Experiment.* In a second case we consider a source that only generates shear waves up to the interaction with the boundary.

$$\mathbf{h}(\mathbf{x}) = \operatorname{curl}(x_2 s(\mathbf{x})).$$

We remark that  $\mathbf{h} \in \mathbf{L}_R^2(\Omega)$  since,

$$\int_{\Omega} \operatorname{curl}(x_2 s) \cdot \mathbf{w}_R \, d\mathbf{x} = \int_{\Omega} x_2 s \operatorname{curl}(\mathbf{w}_R) \, d\mathbf{x} = 0 \quad \forall \mathbf{w}_R \in \mathbf{R}(\Omega),$$

because  $\operatorname{curl}(\mathbf{w}_R)$  is constant and  $x_2 s(\mathbf{x})$  has zero average in  $\Omega$ .

*Third Experiment.* The source we consider is localized near the boundary and generates both shear and pressure waves.

$$\mathbf{h}(\mathbf{x}) = (\partial_2 s(\mathbf{x} - \mathbf{x}_0), \partial_1 s(\mathbf{x} - \mathbf{x}_0))^t \quad \text{where} \quad \mathbf{x}_0 = (3.5, 0)^t.$$

Notice that in consequence,  $\mathbf{h} \in \mathbf{L}_R^2(\Omega)$  since

$$\begin{aligned} \int_{\Omega} \begin{pmatrix} \partial_2 s \\ \partial_1 s \end{pmatrix} \cdot \mathbf{w}_R \, d\mathbf{x} &= \int_{\Omega} \mathbf{curl} \, s \cdot \begin{pmatrix} w_{R,1} \\ -w_{R,2} \end{pmatrix} \, d\mathbf{x} \\ &= \int_{\Omega} s \, \mathbf{curl} \begin{pmatrix} w_{R,1} \\ -w_{R,2} \end{pmatrix} \, d\mathbf{x} = 0, \quad \forall \mathbf{w}_R \in \mathbf{R}(\Omega). \end{aligned}$$

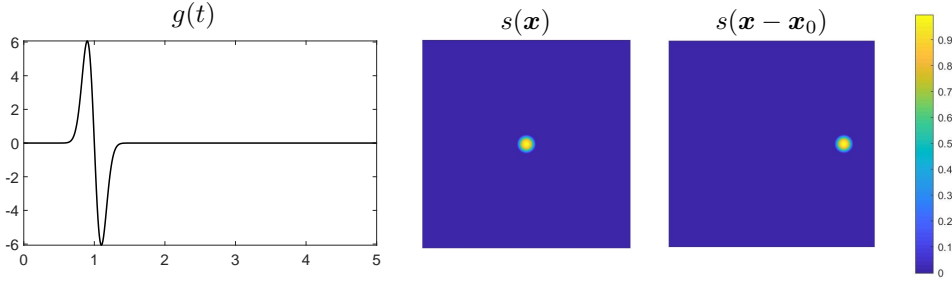


FIGURE 3. Left: time dependence of the source. Right: space dependence of the source.

*Space discretisation of the potential formulation.* According to the technique presented in this article, the finite elements spaces  $V_{P,h}$ ,  $V_{S,h}$ ,  $P_h$  and  $M_h$  must be adequately chosen. In Section 4 we have detailed a particular choice in order to automatically guarantee the good properties of the method. Therefore, we consider the first order Lagrange finite elements spaces defined in (4.1) and (4.4) where the triangulations involved ( $\mathcal{T}_{h_P}^P$ ,  $\mathcal{T}_{h_S}^S$  and  $\mathcal{T}_h$ ) will satisfy assumptions 4.4 and 4.5. Moreover, we shall also notice that  $V_P/V_S = 4$ , thus for computational reasons we would like to consider  $h_P \simeq 4h_S$ . In consequence, we first introduce the triangulation  $\mathcal{T}_h$  given in Figure 4 (Left) which satisfies Assumption 4.4. Then, on the one hand we choose  $\mathcal{T}_{h_P}^P = \mathcal{T}_h$ , while on the other hand we choose  $\mathcal{T}_{h_S}^S$  given by the second refinement of  $\mathcal{T}_h$  (see Figure 4 Right) in order to verify Assumption 4.5 and the property  $h_P \simeq 4h_S$ .

*Time discretisation.* We approximate the solution of problem (5.1) by considering the mixed explicit-implicit scheme (3.62). The maximum time step allowed  $\Delta t^*$  is computed as in Remark 3.20 and then the time step  $\Delta t$  for the simulations is chosen as the maximum time step such that  $\Delta t < \Delta t^*$  and  $0.01/\Delta t$  is an integer.

*Computation of a reference displacement solution.* To obtain a reference solution we compute the velocity (i.e.  $\mathbf{v} = \partial_t \mathbf{u}$ ) using a classical method. More precisely, we compute  $\mathbf{v}_h$ , a numerically approximation of  $\mathbf{v} = \partial_t \mathbf{u}$ , by solving a discretized version of the problem

$$(5.3) \quad \begin{cases} \rho \partial_t^2 \mathbf{v} - (\lambda + 2\mu) \nabla \operatorname{div} \mathbf{v} + \mu \mathbf{curl} \operatorname{curl} \mathbf{v} = \partial_t \mathbf{f} & \text{in } \Omega \times [0, T], \\ \underline{\boldsymbol{\sigma}}(\mathbf{v}) \mathbf{n} = \mathbf{0} & \text{on } \Gamma \times [0, T]. \end{cases}$$

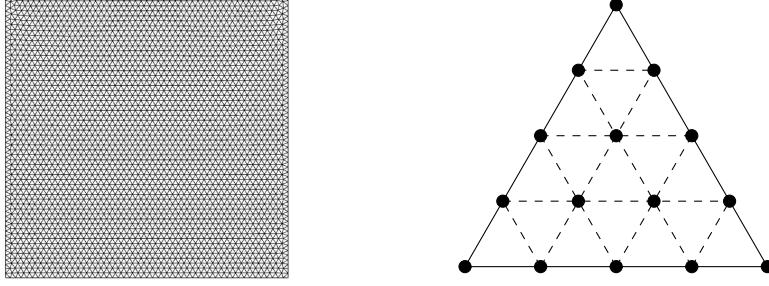


FIGURE 4. On the left we observe the triangulation  $\mathcal{T}_h$  (which is equal to  $\mathcal{T}_{h_P}^P$ ). On the right we plot how each element is divided in order to compute  $\mathcal{T}_{h_S}^S$  as the second refinement of  $\mathcal{T}_h$ .

In particular we consider first order Lagrange finite elements in space (considering the mesh  $\mathcal{T}_{h_S}^S$ ) and finite differences in time, more precisely, a leap-frog scheme with the largest possible times step  $\Delta\tau$  such that the scheme is stable and  $0.01/\Delta\tau$  is an integer.

*Procedure of comparison: Potential / Displacement.* In order to validate the results we take into account the relation

$$\partial_t \mathbf{u} = \nabla \varphi_P + \mathbf{curl} \varphi_S + \mathbf{g} \quad \text{with} \quad \mathbf{g}(\mathbf{x}, t) = \frac{\mathbf{h}(\mathbf{x})}{\rho} \int_0^t g(s) ds,$$

and we compare with respect to a solution obtained by a classical method in displacement. The method presented provides a numerical approximation of the potentials  $\varphi_P$  and  $\varphi_S$ , thus in order to compare the obtained solution with the solution generated with the displacement formulation (5.3) we shall also introduce the orthogonal projection  $\Pi_{S,h}$  from  $\mathbf{L}^2(\Omega)$  into  $V_{S,h} \times V_{S,h}$  that for every  $\phi \in \mathbf{L}^2(\Omega)$  assigns  $\Pi_{S,h}(\phi)$  satisfying

$$\int_{\Omega} (\Pi_{S,h}(\phi) - \phi) \cdot \Psi_h = 0 \quad \forall \Psi_h \in V_{S,h} \times V_{S,h}.$$

In consequence, in the following, in order to validate the numerical results obtained with the potentials approach, we shall represent

$$\mathbf{v}_h \text{ compared with respect to } \tilde{\mathbf{v}}_h := \Pi_{S,h}(\nabla \varphi_{P,h}) + \Pi_{S,h}(\mathbf{curl} \varphi_{S,h}) + \Pi_{S,h}(\mathbf{g}).$$

**5.2. First experiment.** Since the source is given by a gradient, we observe in Figures 5 and 6 that the source only generates P-waves that propagate in the interior of the domain. Then, when the P-waves reach the boundary, S-waves are generated and propagate in the interior of the domain. It is interesting to observe how both type of waves are continuously interacting along the boundary. Finally, in Figures 7 and 8 we validate qualitatively the results obtained when using potentials formulation by comparing with respect to the solution obtained with a classical formulation.

**5.3. Second experiment.** Contrary to the previous example, now the source is given by a rotational and therefore, as we observe in Figures 9 and 10, only S-waves are generated by the source. Similarly to the previous case, we now observe

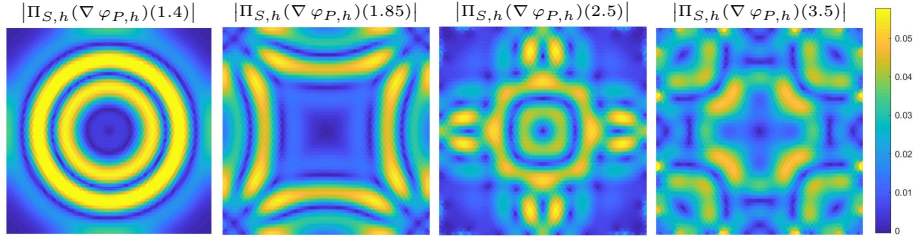


FIGURE 5. Snapshots of the reconstructed P-waves (first experiment).

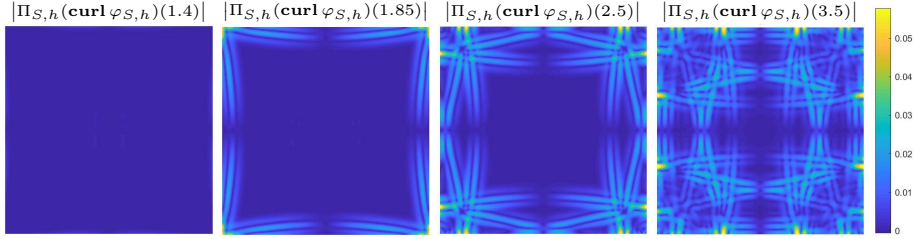


FIGURE 6. Snapshots of the reconstructed S-waves (first experiment).

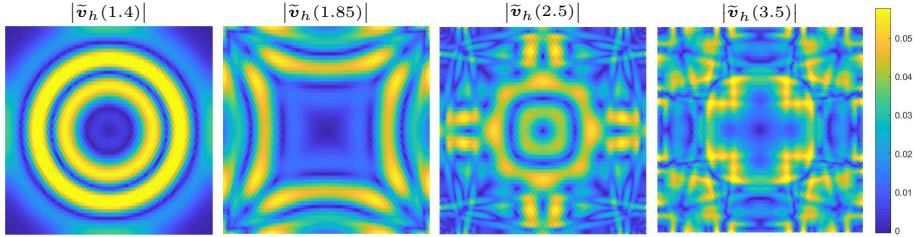


FIGURE 7. Snapshots of the reconstructed velocity field (first experiment).

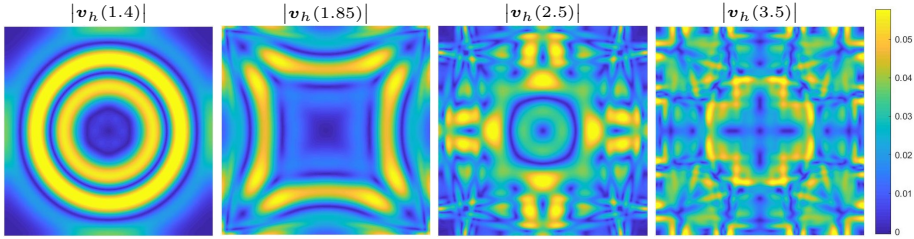


FIGURE 8. Snapshots of the velocity field (first experiment).

that S-waves propagate in the interior of the domain and only when they reach the boundary we observe that P-waves are generated. In order to validate qualitatively the results, we compare in Figures 11 and 12 the results obtained when using potentials with respect to the solution obtained with a classical method.

**5.4. Third experiment.** In this case, both types of waves are generated by the source as we can see in Figures 13 and 14. Then we observe again that both types of waves propagate independently in the interior of the domain while on the boundary

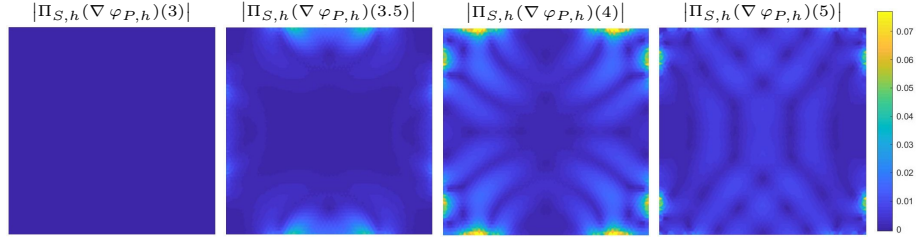


FIGURE 9. Snapshots of the reconstructed P-waves (second experiment).

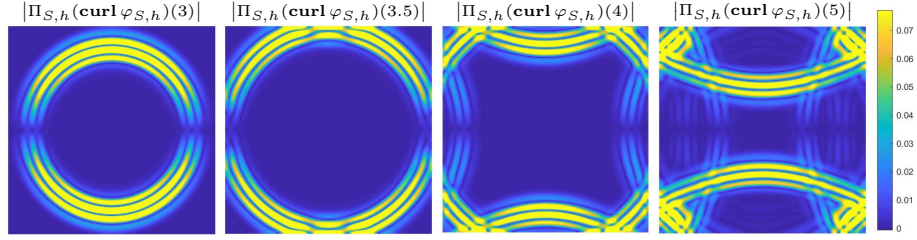


FIGURE 10. Snapshots of the reconstructed S-waves (second experiment).

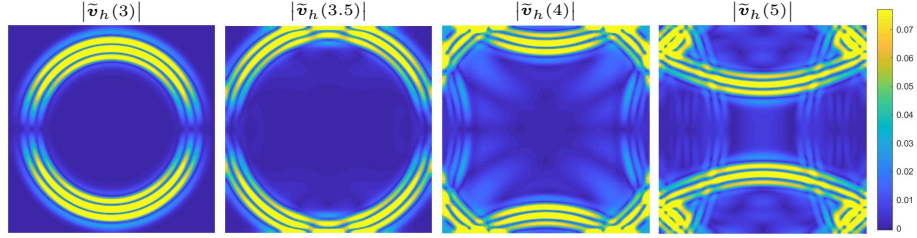


FIGURE 11. Snapshots of the reconstructed velocity field (second experiment).

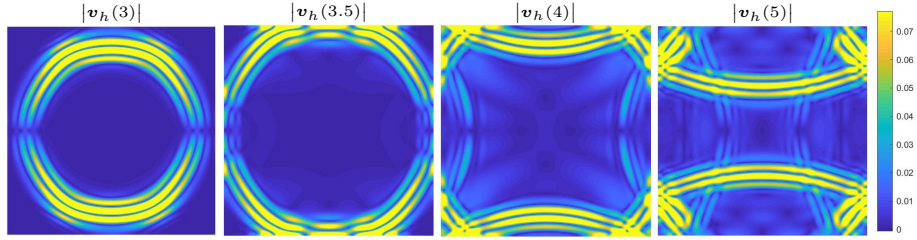


FIGURE 12. Snapshots of the velocity field (second experiment).

the interaction of both waves is stronger than in the previous cases. In consequence, the well known Rayleigh waves are observed (specially) in Figure 13. Finally, as in the previous case, we compare in Figures 15 and 16 the reconstructed velocity field obtained using potentials formulation with respect to the solution obtained with a classical formulation.

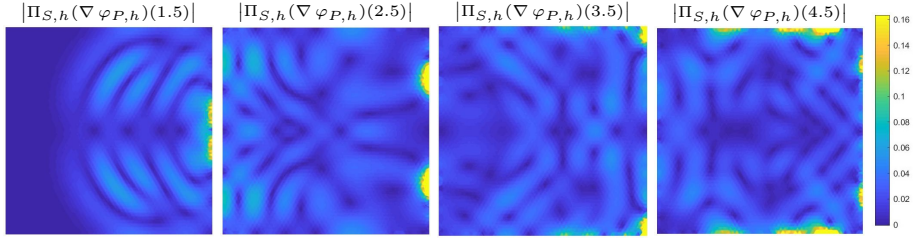


FIGURE 13. Snapshots of the reconstructed P-waves (third experiment).

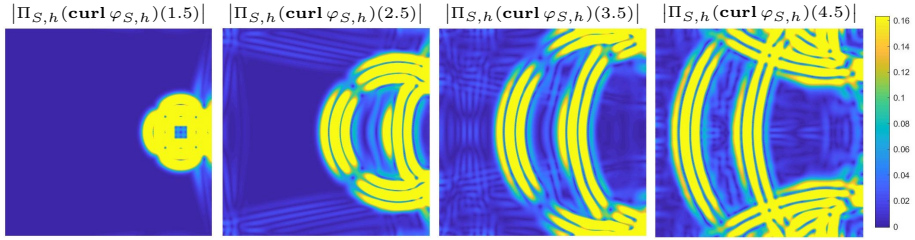


FIGURE 14. Snapshots of the reconstructed S-waves (third experiment).

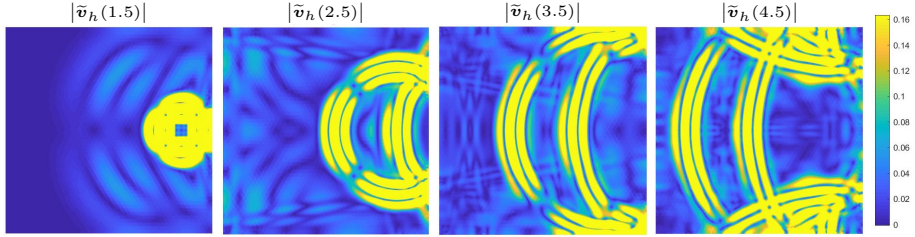


FIGURE 15. Snapshots of the reconstructed velocity field (third experiment).

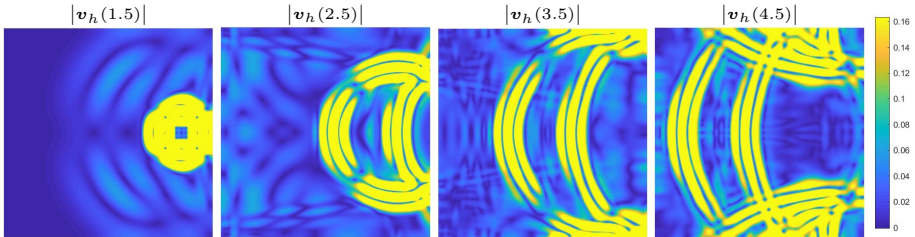


FIGURE 16. Snapshots of the velocity field (third experiment).

## 6. CONCLUSIONS AND PERSPECTIVES

In this work we recapped and completed the method presented in [4] for solving linear isotropic homogeneous elastodynamics with free boundary conditions by means of scalar potentials resulting from a Helmholtz decomposition of the displacement field. We presented in detail a space-time discretization of the formulation which allows to use different meshes for the discretization of P and S waves. We

proved that, under some reasonable compatibility assumptions on the discretization spaces, the fully discrete scheme is well-posed and stable under the usual CFL-stability conditions for the scalar wave equations (the time step not being penalized by the coupling at the boundary). Numerical experiments confirm the good properties of the method.

As future work, first, we would like to address the full convergence analysis of the method (that is still an open question) and to address the question of efficiency of the overall method. Secondly, we would like to extend the method for the transmission problem between two homogeneous isotropic media. Finally, notice that the extension to the 3D case is not trivial since the potential corresponding to the shear waves is vector-valued and a gauge condition should be taken into account.

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## 7. APPENDIX: PROOF OF THE WELL-POSEDNESS OF THEOREM 2.12

**Existence.** We start by introducing the regularized  $\epsilon$ -dependent problem

$$(7.1) \quad \begin{cases} \text{Find } \varphi_\epsilon : [0, T] \longrightarrow \mathbf{V}_N \subset \mathbf{L}^2(\Omega) \text{ s.t.} \\ \frac{d^2}{dt^2} m_\epsilon(\varphi_\epsilon(t), \boldsymbol{\psi}) + a(\varphi_\epsilon(t), \boldsymbol{\psi}) = \ell(t, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \mathbf{V}_N, \\ \varphi_\epsilon(t=0) = 0, \quad \partial_t \varphi_\epsilon(t=0) = 0, \end{cases}$$

where

$$(7.2) \quad m_\epsilon(\boldsymbol{\varphi}, \boldsymbol{\psi}) := m(\boldsymbol{\varphi}, \boldsymbol{\psi}) + \epsilon a(\boldsymbol{\varphi}, \boldsymbol{\psi}),$$

and

$$\ell(t, \boldsymbol{\psi}) = - \int_{\Omega} \mathbf{g} \cdot (\operatorname{div} \boldsymbol{\psi}, -\operatorname{curl} \boldsymbol{\psi})^t \, d\mathbf{x} \text{ where } \mathbf{g}(t) := \frac{1}{\rho} \int_0^t \mathbf{f}(s) \, ds$$

with  $\mathbf{f} \in L^1(0, T; \mathbf{L}_R^2(\Omega))$ . Since the mapping  $\boldsymbol{\varphi} \mapsto \sqrt{m_\epsilon(\boldsymbol{\varphi}, \boldsymbol{\varphi})}$  is a norm in  $\mathbf{V}_N$  equivalent to the norm in  $\mathbf{V}$  and  $\mathbf{f}$  belongs to  $L^1(0, T; \mathbf{L}^2(\Omega))$ , we deduce that there exists a unique solution of the regularized problem

$$\varphi_\epsilon \in C^2([0, T]; \mathbf{V}_N).$$

Next we obtain estimates for  $\varphi_\epsilon$ . Plugging  $\boldsymbol{\psi} = \partial_t \varphi_\epsilon \in \mathbf{V}_N$  in (7.1) leads, after time integration, to

$$(7.3) \quad \left| \begin{aligned} m(\partial_t \varphi_\epsilon(t), \partial_t \varphi_\epsilon(t)) + \epsilon a(\partial_t \varphi_\epsilon(t), \partial_t \varphi_\epsilon(t)) + a(\varphi_\epsilon(t), \varphi_\epsilon(t)) \\ = 2 \int_0^t \ell(s, \partial_t \varphi_\epsilon(s)) \, ds, \end{aligned} \right.$$

where we have used the vanishing initial conditions. Next we rewrite the contribution of the right hand side as follows,

$$\left| \begin{aligned} \int_0^t \ell(s, \partial_t \varphi_\epsilon(s)) \, ds &= - \int_0^t \int_{\Omega} \mathbf{g}(s) \cdot (\operatorname{div} (\partial_t \varphi_\epsilon(s)), -\operatorname{curl} (\partial_t \varphi_\epsilon(s)))^t \, d\mathbf{x} \, ds \\ &= \int_0^t \int_{\Omega} \frac{1}{\rho} \mathbf{f}(s) \cdot (\operatorname{div} (\varphi_\epsilon(s)), -\operatorname{curl} (\varphi_\epsilon(s)))^t \, d\mathbf{x} \, ds \\ &\quad - \int_{\Omega} \mathbf{g}(t) \cdot (\operatorname{div} (\varphi_\epsilon(t)), -\operatorname{curl} (\varphi_\epsilon(t)))^t \, d\mathbf{x}, \end{aligned} \right.$$

to obtain the following bounds

$$\left| \begin{aligned} \left| \int_0^t \ell(s, \partial_t \varphi_\epsilon(s)) \, ds \right| &\leq \frac{1}{\rho} \int_0^t \int_{\Omega} \left| \mathbf{f}(s) \cdot (\operatorname{div} (\varphi_\epsilon(s)), -\operatorname{curl} (\varphi_\epsilon(s)))^t \right| \, d\mathbf{x} \, ds \\ &\quad + \frac{1}{\rho} \int_0^t \int_{\Omega} \left| \mathbf{f}(s) \cdot (\operatorname{div} (\varphi_\epsilon(t)), -\operatorname{curl} (\varphi_\epsilon(t)))^t \right| \, d\mathbf{x} \, ds \\ &\leq \frac{2}{\rho} \sup_{s \in [0, T]} \sqrt{a(\varphi_\epsilon(s), \varphi_\epsilon(s))} \|\mathbf{f}\|_{L^1(0, t; \mathbf{L}^2(\Omega))}. \end{aligned} \right.$$

Applying Young's inequality we obtain, using the norm inequalities (2.65), that

$$(7.4) \quad \left| \begin{aligned} \sup_{0 \leq t \leq T} (\|\partial_t \varphi_\epsilon(t)\|_{\mathbf{L}^2(\Omega)}^2 + \epsilon a(\partial_t \varphi_\epsilon(t), \partial_t \varphi_\epsilon(t)) + a(\varphi_\epsilon(t), \varphi_\epsilon(t))) \\ \leq \mathcal{C} \|\mathbf{f}\|_{L^1(0, T; \mathbf{L}^2(\Omega))}^2, \end{aligned} \right.$$



where  $\mathcal{C}$  is a positive scalar independent of  $\epsilon$ . We conclude that the sequence  $\partial_t \varphi_\epsilon$  is bounded in  $L^2(0, T; \mathbf{L}^2(\Omega))$ . Therefore the same holds true for  $\varphi_\epsilon$ , and as a consequence  $\varphi_\epsilon$  is also bounded in  $L^2(0, T; \mathbf{V}_N)$ . Since  $\mathbf{V}_N$  is a closed subspace of  $\mathbf{V}$ , we can extract a subsequence (that will still be denoted by  $\varphi_\epsilon$ ) such that

$$(7.5) \quad \begin{aligned} \varphi_\epsilon &\rightharpoonup \varphi \in L^2(0, T; \mathbf{V}_N), & \varphi_\epsilon &\rightharpoonup \varphi \in L^2(0, T; \mathbf{L}^2(\Omega)), \\ \partial_t \varphi_\epsilon &\rightharpoonup \mathbf{d}\varphi \in L^2(0, T; \mathbf{L}^2(\Omega)), \end{aligned}$$

where the convergence holds in the weak topology. Furthermore, it can be easily proven that  $\partial_t \varphi = \mathbf{d}\varphi$  where the time derivative is considered in the distributional sense. Finally we will prove that

$$\frac{d^2}{dt^2} m(\varphi(t), \psi) + a(\varphi(t), \psi) = \ell(t, \psi), \quad \forall \psi \in \mathbf{V}_N,$$

where the time derivative are, once again, considered in the distributional sense. Let  $\Theta \in \mathcal{D}(-\infty, T)$ , we have from (7.1)

$$\int_0^T (m(\varphi_\epsilon(t), \psi) + \epsilon a(\varphi_\epsilon(t), \psi)) \Theta''(t) dt + \int_0^T a(\varphi_\epsilon(t), \psi) \Theta(t) dt = \int_0^T \ell(t, \psi) \Theta(t) dt.$$

The above-mentioned weak convergence, together with (7.4), allows to easily pass to the limit in each term and prove that

$$\int_0^T m(\varphi(t), \psi) \Theta''(t) dt + \int_0^T a(\varphi(t), \psi) \Theta(t) dt = \int_0^T (\mathbf{f}(t), \psi)_{\mathbf{L}^2(\Omega)} \Theta(t) dt.$$

This concludes the existence proof.

**Uniqueness.** Let us first prove uniqueness in

$$(7.6) \quad \mathcal{C}^2([0, T]; \mathbf{L}^2(\Omega)) \cap \mathcal{C}^1([0, T]; \mathbf{V}_N).$$

Assume that  $\varphi_1$  and  $\varphi_2$  are solution of (2.64) both belonging to this space. Then,  $\varphi := \varphi_1 - \varphi_2$  satisfies (2.64) with vanishing data. Plugging  $\psi = \partial_t \varphi$  into the variational formulation

$$0 = m(\partial_{tt} \varphi, \partial_t \varphi) + a(\varphi(t), \partial_t \varphi) = \frac{d}{dt} \left\{ \frac{1}{2} m(\partial_t \varphi, \partial_t \varphi) + \frac{1}{2} a(\varphi, \varphi) \right\}.$$

Integrating in time we get

$$m(\partial_t \varphi(t), \partial_t \varphi(t)) + a(\varphi(t), \varphi(t)) = 0.$$

We have thus  $\partial_t \varphi(t) = 0$  hence  $\varphi(t) = \varphi(0) = 0$ , leading to the uniqueness. Let us finally assume that we have two solutions  $\varphi_i$ ,  $i \in \{1, 2\}$  such that

$$\varphi_i \in L^2(0, T; \mathbf{V}_N), \quad \text{and} \quad \partial_t \varphi_i \in L^2(0, T; \mathbf{L}^2(\Omega)).$$

Then  $\Phi$  defined as the second anti-derivative of  $\varphi = \varphi_1 - \varphi_2$  belongs to the space in (7.6) and, by linearity, satisfies (2.64) with vanishing data. In consequence,  $\Phi = \mathbf{0}$  which leads to  $\varphi_1 = \varphi_2$ .

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