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# Dynamic equivalence and flatness of control systems: some results and open questions

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Everything is  $C^\omega$ .

## Control systems

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, 0 < m \leq n, \quad (*)$$

studied locally around some  $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$ ,  
or around some germ/jet of solution  $t \mapsto (\bar{x}(t), \bar{u}(t))$ .

**“Geometry”**:  $f$  fiber preserving map  $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{T}\mathbb{R}^n$ .

$f(\mathbb{T}\mathbb{R}^n)$  is a

sub-bundle of  $\mathbb{T}\mathbb{R}^n$  if  $\text{Rank } \frac{\partial f}{\partial u}$  constant,

affine sub-bundle if  $f(x, u) = X_0(x) + \sum_1^m u_k X_k(0)$ ,

distribution if  $X_0 = 0$ ...

**Underdetermined system of ODEs**: the general solution  
 $t \mapsto (x(t), u(t))$  depends on  $m$  arbitrary functions of time  $u(\cdot)$  and  
 $n$  arbitrary constants  $x(0)$ . Important object:

“Set of solutions” for under-determined ODEs

$\mathcal{B}$  = set of all (germs of)  $t \mapsto (x(t), u(t))$  solution of (\*).

(behavior?) (tribute to J. Willems)

## “Static” equivalence

$$\begin{aligned}(\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\(\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, v \in \mathbb{R}^{m'}\end{aligned}$$

### Definition (local static equivalence)

$(\Sigma)$  and  $(\Sigma')$  are locally static equivalent at  $(\bar{x}, \bar{u})/(\bar{z}, \bar{v})$  iff there is a diffeo  $(\varphi, \psi) : \mathcal{O}_{(\bar{x}, \bar{u})} \rightarrow \mathcal{O}_{(\bar{z}, \bar{v})}$ , of the form  $(x, u) \mapsto (\varphi(x), \psi(x, u))$ , (bundle isomorphism) that conjugates  $f$  to  $g$ .

If control affine & constant control rank, then

- local in state only,
- $\varphi : \mathcal{O}_{\bar{x}} \rightarrow \mathcal{O}_{\bar{z}}$  such that  $\varphi_*$  conjugates affine sub-bundles.

### Alternative definition:

$(\varphi, \psi) : \mathcal{O}_{(\bar{x}, \bar{u})} \rightarrow \mathcal{O}_{(\bar{z}, \bar{v})}$  maps (by composition), germs of solutions of  $\Sigma$  to germs of solutions of  $\Sigma'$ , and vice versa.

## Deciding static equivalence

**It is feasible.** Given  $\Sigma$  and  $\Sigma'$ , there is in principle a finite algorithm to write PDEs for  $\varphi, \psi$  and decide (differential elimination) whether there is an obstruction of system formally integrable, Cauchy-Kovalevska, analytic solution.

**Geometric study.** Invariants of distributions, affine sub-bundles. In non-affine case, affine geometry for submanifolds of  $\mathbb{T}\mathbb{R}^n$ .

A lot of litterature.

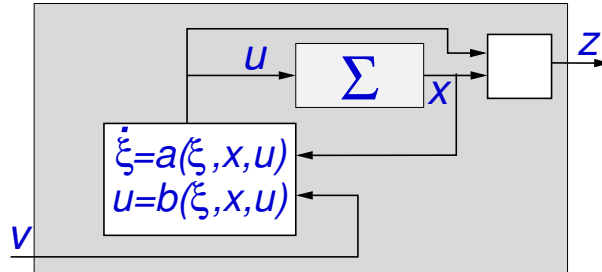
Functionnal moduli, hence equivalence is a “rare” property, or classes are very thin.

**Exact linearization:** characterization of static equivalence to a linear controllable system [Jakubczyk-Respondek, 80].

## Dynamic feedback linearization

Since Static linearization is very restrictive, try dynamic ! (1980's)

Pre-compensator such that  $v \rightarrow z$  linear controllable:



[Isidori-Moog-de Luca 86]: performing dynamic decoupling, full linearization may occur.

Decide pre-compensator, check for static feedback linearizability

[Charlet-Lévine-Marino, 91]

► Sufficient conditions.

## Linear controllable systems, linearization

**Linear controllable system** ( $\Sigma$ ) :  $\dot{x} = Ax + Bu$ .

$$\text{Rank}\{B, AB, \dots\} = n.$$

Transformation  $z = Px$ ,  $v = Kx + Qu$

with  $z = (z_{1,1}, \dots, z_{1,r_1}, \dots, z_{m,1}, \dots, z_{m,r_m})$   $\sum r_k = n$

yields  $z_{k,1}^{(r_k)} = v_k$ ,  $1 \leq k \leq m$ . Brunovsky canonical form.

The general solution is uniquely defined by  $m$  arbitrary functions of time  $z_{1,1}, z_{2,1}, \dots, z_{m,1}$  (and no initial conditions).

[Fliess-Lévine-Martin-Rouchon 91]: system  $\dot{x} = f(x, u)$  is “**flat**”

(at  $\bar{x}, \bar{u}, \dots, \bar{u}^{(j)}$ ) iff there is a formula giving the general solution a function of arbitrary  $y_1, \dots, y_m$  and time-derivatives **and**  $y_1, \dots, y_m$  may also be recovered from  $x, u$  and derivatives.

$(y_1, \dots, y_m)$  is called a **flat output**.

This yields the dynamic precompensator.

# Equivalence

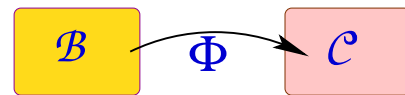
[É. Cartan, “sur l'équivalence absolue...”, 1914]:

La première idée qui vient à l'esprit, et qu'il s'agira de préciser, est la suivante : *deux systèmes seront dits « absolument équivalents » lorsqu'on pourra établir une correspondance univoque (au moins dans un champ fonctionnel suffisamment petit) entre les solutions de ces deux systèmes.*

$$\begin{aligned}
 (\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad \mathcal{B} = \{\text{solutions}\} \\
 (\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, v \in \mathbb{R}^{m'}, \quad \mathcal{C} = \{\text{solutions}\}
 \end{aligned}$$

## Definition

Two systems are “**equivalent**” iff their (germs of) solutions are in **one-to-one correspondence**.



The nature of  $\Phi$  matters a lot !!

## Dynamic Feedback Transformations

$$\begin{array}{ccc}
 \Sigma & & \Sigma' \\
 \dot{x} = f(x, u) & \begin{array}{c} \left( \begin{array}{c} x \\ u \\ \dot{u} \\ \ddot{u} \\ \vdots \end{array} \right) & \begin{array}{c} (z, v) = \phi(x, u, \dots, u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x, u) = \psi(z, v, \dots, v^{(K')}) \end{array} & \left( \begin{array}{c} z \\ v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{array} \right) & \dot{z} = g(z, v)
 \end{array}
 \end{array}$$

**(Local) dynamic equivalence:**  $\exists \varphi : \mathcal{O}_{(\bar{x}, \bar{u}, \bar{u}^{(K)})} \rightarrow \mathbb{R}^{n'} \times \mathbb{R}^{m'}$ ,  
 $\psi : \mathcal{O}_{(\bar{z}, \bar{v}, \bar{v}^{(K')})} \rightarrow \mathbb{R}^n \times \mathbb{R}^m$  such that the above induces a univocal transformation on solutions.

## Flatness

**Flatness:**  $\Sigma$  is flat if this holds with  $\Sigma'$  trivial:

$$\Sigma \quad \begin{pmatrix} x \\ u \\ \dot{u} \\ \vdots \end{pmatrix} \quad \begin{array}{c} v = \phi(x, u, \dots, u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x, u) = \psi(v, \dots, v^{(K')}) \end{array} \quad \begin{pmatrix} v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{pmatrix} \quad \Sigma' \quad \text{no relation}$$

Initially, [Fliess & al] stated this in terms of differential fields extensions. Obviously more restrictive ( $\phi, \psi$  should be algebraic).

Equivalence can however be translated into isomorphism of differential algebras or conjugation of a vector field on a (infinite) jet manifold.

## Further characterizations

Differential algebra:  $\mathcal{A}_\Sigma = \{(\text{germs of}) C^\omega \text{ functions of a finite number of variables among } x, u, \dot{u}, \dots\}$ ,

dérivation:  $F = f(x, u) \frac{\partial}{\partial x} + \dot{u} \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}} + \dots$

[Jakubczyk 1993]:  $\Sigma$  is equivalent to  $\Sigma'$  if and only  $\mathcal{A}_\Sigma$  and  $\mathcal{A}_{\Sigma'}$  are isomorphic diff. algebras.

[Fliess & al.], [JBP] 1992: it also translates into a diffeomorphism between “manifolds” where coordinates are  $x, u, \dot{u}, \dots$  and  $z, v, \dot{v}, \dots$ , that conjugates  $F$  to  $G$ .

Similar to Lie-Bäcklund transformations.

## Checkable conditions ?

How to decide flatness of a system  $\Sigma$ ?

or equivalence of  $\Sigma$  to  $\Sigma'$  ?

Invariants ?

- ① In principle, if  $K$  and  $K'$  are fixed, one may decide upon existence of  $\phi$  and  $\psi$ , that depend on a fixed number of variables. (see static feedback)
- ② ... **but no known a priori bound on  $K, K'$  !!**
- ③ However, many physical systems are flat and this is useful.
- ④ A lot of current work on all possible choices of flat outputs (they are far from unique), respecting symmetries, etc... see [Respondek, Nicolau et al], [Murray et al.], [Rouchon, Martin et al.] ...

## Checkable conditions ?

Difficult point is: necessary conditions.

If no  $\phi, \psi$  for some  $K$ , why not for  $K + 1$  ?

Invariants:

- ①  $m$  (number of inputs) is an invariant of dynamic equivalence.
- ② If  $m = 1$ , dynamic equivalence is static equivalence. [Charlet et al., 1991], [JBP 1993].
- ③ Singular curves are an obstruction to flatness: if  $t \mapsto (x(t), u(t))$  is singular, flatness fails at any truncated jet  $(x(0), u(0), \dot{u}(0), \dots, u^{(j)}(0))$  [common knowledge ?]
- ④ Ruled manifold criterium

# Ruled systems

## Regularity assumption

$\Sigma$  and  $\Sigma'$  define smooth ( $C^\omega$ ) sub-bundles of  $T\mathbb{R}^n$  and  $T\mathbb{R}^{n'}$ .

E.g.  $\text{Rank} \frac{\partial f}{\partial u} = m$ ,  $\text{Rank} \frac{\partial g}{\partial v} = m'$

- $\Sigma \rightarrow \mathbb{R}^n$  sub-bundle of  $T\mathbb{R}^n \rightarrow \mathbb{R}^n$ .  
 $\Sigma_x = \{f(x, u), u \in \mathbb{R}^m\}$  sub-manifold of the linear space  $T_x\mathbb{R}^n$ .
- A submanifold of an affine space is called (locally) **ruled** if it is a union of straight lines (locally of open segments).

## Definition (ruled system)

$\Sigma$  is ruled iff each  $\Sigma_x$  is a ruled submanifold of  $T_x\mathbb{R}^n (\approx \mathbb{R}^n)$ .

Note: This property is preserved by static equivalence.

## Necessary condition for flatness

$$(\Sigma) \quad \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (\Sigma') \text{ "trivial" .}$$

## Theorem ([Rouchon], [Sluis], 1992)

*A flat system must be ruled*

What if

$$\begin{array}{ll} (\Sigma) & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ (\Sigma') & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, \quad v \in \mathbb{R}^{m'} \end{array} \quad \text{dynamic equivalent?}$$



## Necessary condition for equivalence

$$\begin{aligned}(\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\(\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, v \in \mathbb{R}^{m'}\end{aligned}$$

### Theorem (JBP, 2009)

If  $\Sigma$  and  $\Sigma'$  are dynamic equivalent and  $n = n'$ , then, locally,

- ① either they are **static equivalent**,
- ② or they are **both ruled**.

If  $n < n'$ ,  $\Sigma'$  must be ruled.

In the  $C^\infty$  case, ① and ② may both occur for the same  $\Sigma, \Sigma'$ .

Note: the condition for flatness is a consequence (trivial system is ruled).

## Flatness with 2 controls and 3 states

$$\dot{z} = f(x, y, z, \dot{x}, \dot{y})$$

[Avanessof, JBP 2007].

We look for a parameterization  $x, y, z$  function of  $(u, \dot{u}, \dots, u^{(k)}, v, \dots, v^{(\ell)})$ ,  $k \leq \ell$ , such that all solutions are covered. one has

- Either no parameterization because of ruled criterion of non-controllability,
- or parameterization with  $(k, \ell) = (1, 2)$ ,
- or there could be a parameterization, but  $k$  and  $\ell$  have to be no smaller than 3 and the largest one no smaller than 4.

## Examples

**Example 1:**  $\dot{z} = y + (\dot{y} - z\dot{x})\dot{x}$ . Parameterization of order (1, 2) around jets such that  $\ddot{x} + \dot{x}^3 \neq 1$ , given by :

$$x = v, \quad y = \frac{\dot{v}^2 u + \dot{v}}{\ddot{v} + \dot{v}^3 - 1}, \quad z = \frac{(1 - \ddot{v})u + \dot{v}\dot{v}}{\ddot{v} + \dot{v}^3 - 1}. \quad (*)$$

Formulas can be “inverted” by  $u = -z + y\dot{x}$ ,  $v = x$ , hence flatness. Note on singularity:  $\ddot{x} + \dot{x}^3 = 1$  is the equation of the singular curves. There is a parameterization of higher order at jets of order 3 such that  $x^{(3)} + 3\dot{x}^2\ddot{x} \neq 0$ .

**Example 2:**  $\dot{z} = y + (\dot{y} - z\dot{x})^2\dot{x}$ : if there is a parameterization, it has order at least (3, 4). Conjecture: no parameterization.

## Open questions

- ① Given  $\Sigma$ , give an a priori bound on the number  $K$  of derivatives.
- ② Prove that at least one system for which above mentioned necessary conditions do not work is not flat, or does not admit a parameterization.
- ③ Does existence of a parameter imply flatness ? (exogenous implies exogenous, according to [Fliess et al.])

Thank you all for attention, and  
thank you for your action in the community, Bronek !