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# Proofs for ”Discretization of Homogeneous Systems Using Euler Method with a State-Dependent Step”

Denis Efimov <sup>a,b</sup>, Andrey Polyakov <sup>a,b</sup>, Alexander Aleksandrov <sup>c</sup>

<sup>a</sup>*Inria, Univ. Lille, CNRS, UMR 9189 - CRISTAL, F-59000 Lille, France*

<sup>b</sup>*ITMO University, 49 av. Kronverkskiy, 197101 Saint Petersburg, Russia*

<sup>c</sup>*Saint Petersburg State University, 7-9 Universitetskaya nab., 199034 Saint Petersburg, Russia*

## Abstract

This note contains some proofs for the paper ”Discretization of Homogeneous Systems Using Euler Method with a State-Dependent Step” of the same authors.

*Key words:* Homogeneous systems; Discretization; Euler method.

## 1 Proof of Theorem 9

Let us take a discretization step  $h > 0$  and consider the behavior of a  $\mathbf{r}$ -homogeneous Lyapunov function  $V$  (satisfying (7)) on the sequence generated by (2). For this purpose define  $x_i = \Lambda_r(\|x_i\|_r)y_i$  for some  $y_i \in S_r(1)$ :

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= V(x_i + \|x_i\|_r^{-\nu} h f(x_i)) - V(x_i) \\ &= \|x_i\|_r^\mu [V(y_i + h f(y_i)) - V(y_i)]. \end{aligned}$$

Since  $V$  is twice continuously differentiable, then by the Taylor expansion theorem with Lagrange remainder [1] there is  $\theta \in (0, 1)$  such that

$$\begin{aligned} V(y_i + h f(y_i)) &= V(y_i) + \frac{\partial V(\xi)}{\partial \xi} \Big|_{\xi=y_i} h f(y_i) \\ &\quad + \frac{h^2}{2} f^\top(y_i) \frac{\partial^2 V(\xi)}{\partial \xi^2} \Big|_{\xi=y_i + \theta h f(y_i)} f(y_i), \end{aligned}$$

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*Email addresses:* [denis.efimov@inria.fr](mailto:denis.efimov@inria.fr) (Denis Efimov), [andrey.polyakov@inria.fr](mailto:andrey.polyakov@inria.fr) (Andrey Polyakov), [a.u.aleksandrov@spbu.ru](mailto:a.u.aleksandrov@spbu.ru) (Alexander Aleksandrov).

then

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= \|x_i\|_r^\mu \frac{\partial V(\xi)}{\partial \xi} \Big|_{\xi=y_i} h f(y_i) \\ &\quad + \|x_i\|_r^\mu \frac{h^2}{2} f^\top(y_i) \frac{\partial^2 V(\xi)}{\partial \xi^2} \Big|_{\xi=y_i + \theta h f(y_i)} f(y_i). \end{aligned}$$

Note that from (7),

$$\frac{\partial V(\xi)}{\partial \xi} \Big|_{\xi=y_i} h f(y_i) \leq -ha,$$

and there is  $v \in (0, +\infty)$  such that

$$\sup_{y \in S_r(1)} \sup_{\theta \in (0,1)} f^\top(y) \frac{\partial^2 V(\xi)}{\partial \xi^2} \Big|_{\xi=y + \theta f(y)} f(y) \leq v.$$

Therefore,

$$V(x_{i+1}) - V(x_i) \leq h \|x_i\|_r^\mu \left( \frac{h}{2} v - a \right)$$

and for all  $h \in (0, h_0]$  with  $h_0 = \min\{1, \frac{a}{v}\}$  we obtain:

$$V(x_{i+1}) - V(x_i) \leq -\frac{a}{2} h \|x_i\|_r^\mu \leq -\alpha h V(x_i),$$

where  $\alpha = \frac{a}{2c_2}$  and the property given in (7) was used on the last step. Assume that actually  $h_0 = \min\{1, \frac{a}{v}, 2\frac{c_2}{a}\}$ ,

then

$$V(x_{i+1}) \leq (1 - \alpha h)V(x_i)$$

for all  $i = 0, 1, \dots$  and the values of  $V(x_i)$  are monotonously decreasing, hence the sequence  $\{x_i\}_{i=0}^{\infty}$  is converging (the property (b) is substantiated). Global boundedness of  $\{x_i\}_{i=0}^{\infty}$  (i.e. the property (a)) follows from the properties of the function  $V$  for  $\gamma = (c_1^{-1}c_2)^{1/\mu}$ .

Let us evaluate the rate of convergence for different signs of the degree of homogeneity  $\nu$  of (1) stated in the property (c). Let  $\nu = 0$ , then trivially  $t_{i+1} = t_i + h$  and an exponential rate of convergence is recovered. From (2) we have:

$$t_{i+1} - t_i = \frac{h}{\|x_i\|_r^\nu} \leq \frac{h}{V^{\frac{\nu}{\mu}}(x_i)} \begin{cases} c_1^{\nu/\mu} & \nu < 0 \\ c_2^{\nu/\mu} & \nu > 0 \end{cases}.$$

As it has been proven above

$$V(x_i) \leq (1 - \alpha h)^i V(x_0)$$

for all  $i = 0, 1, \dots$ , then for  $\nu < 0$  and for any  $k \geq 0$  an estimate can be derived:

$$t_{k+1} \leq \sum_{i=0}^k \frac{c_1^{\nu/\mu} h}{V^{\frac{\nu}{\mu}}(x_i)} \leq \frac{c_1^{\nu/\mu} h}{V^{\frac{\nu}{\mu}}(x_0)} \sum_{i=0}^k (1 - \alpha h)^{-i \frac{\nu}{\mu}}.$$

Since  $|1 - \alpha h| < 1$  for the selected  $h_0$ , we get

$$t_{+\infty} \leq \frac{c_1^{\nu/\mu} h}{V^{\frac{\nu}{\mu}}(x_0)} \sum_{i=0}^{+\infty} (1 - \alpha h)^{-i \frac{\nu}{\mu}} \leq \frac{c_1^{\nu/\mu} h V^{-\frac{\nu}{\mu}}(x_0)}{1 - (1 - \alpha h)^{-\frac{\nu}{\mu}}} < +\infty$$

for all  $x_0 \neq 0$ , which implies a finite-time convergence to the origin of any sequence  $\{x_i\}_{i=0}^{\infty}$  in (2) but for  $i \rightarrow +\infty$ . For  $\nu > 0$  we are interesting in the time of convergence from an infinite initial condition to  $B_r(1)$ , then it is necessary to repeat all above arguments in the inverse time, which leads to exactly the same estimate of such a time.

## 2 Proof of Proposition 11

Let  $x_i \in S_r(1)$  be an arbitrary fixed vector, then the equation (3) can be rewritten as

$$\Delta = hz(x_i + \Delta)$$

for  $\Delta = x_{i+1} - x_i$ , where  $z : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function defined as  $z(x) = \frac{1}{\|x\|_r} f(x)$  with  $z(0) = 0$ . Indeed,  $z(\Lambda_r(\lambda)x) = \Lambda_r(\lambda)z(x)$  for any  $\lambda > 0$  and  $x \in \mathbb{R}^n$ , i.e.  $z$  is  $\mathbf{r}$ -homogeneous vector field of degree 0 that is also continuous on  $S_r(1)$ . Denote  $B(1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$  as the unit ball in  $\mathbb{R}^n$ , if

$$h < \inf_{x_i \in S_r(1), \Delta \in B(1)} \|z(x_i + \Delta)\|^{-1}$$

then the function  $hz(x_i + \cdot) : B(1) \rightarrow B(1)$  is continuous on the convex compact set  $B(1)$ . Hence, using the Brouwer fixed-point theorem [2] we conclude that the last equation has a solution with respect to  $\Delta \in B(1)$  for any  $x_i \in S_r(1)$ . The conclusion for any  $x_i \in \mathbb{R}^n$  follows from Corollary 6 or Proposition 5.

## 3 Proof of Theorem 12

Defining  $x_{i+1} = \Lambda_r(\|x_{i+1}\|_r)y_{i+1}$  with  $y_{i+1} \in S_r(1)$ , we obtain:

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= V(x_{i+1}) - V(x_{i+1} - \|x_{i+1}\|_r^{-\nu} h f(x_{i+1})) \\ &= \|x_{i+1}\|_r^\mu [V(y_{i+1}) - V(y_{i+1} - h f(y_{i+1}))] \\ &= \|x_{i+1}\|_r^\mu \frac{\partial V(\xi)}{\partial \xi} \Big|_{\xi=y_{i+1}} h f(y_{i+1}) \\ &\quad - \|x_{i+1}\|_r^\mu \frac{h^2}{2} f^\top(y_{i+1}) \frac{\partial^2 V(\xi)}{\partial \xi^2} \Big|_{\xi=y_{i+1} - \theta h f(y_{i+1})} f(y_{i+1}) \end{aligned}$$

with application of the Taylor expansion theorem with Lagrange remainder [1] on the last step. Next, similarly

$$\frac{\partial V(\xi)}{\partial \xi} \Big|_{\xi=y_{i+1}} h f(y_{i+1}) \leq -ha$$

from (7), and if the matrix  $\frac{\partial^2 V(\xi)}{\partial \xi^2}$  is nonnegative definite for all  $\xi \in \mathbb{R}^n$ , then

$$V(x_{i+1}) - V(x_i) \leq -ah \|x_{i+1}\|_r^\mu$$

for any  $h > 0$ . If this is not the case, then there is  $w \in (0, +\infty)$  such that

$$\sup_{y \in S_r(1)} \sup_{\theta \in (0,1)} f^\top(y) \frac{\partial^2 V(\xi)}{\partial \xi^2} \Big|_{\xi=y-\theta f(y)} f(y) \leq w,$$

which results in

$$V(x_{i+1}) - V(x_i) \leq h \|x_{i+1}\|_r^\mu \left( \frac{h}{2} w - a \right)$$

and for all  $h \in (0, h_0]$  with  $h_0 = \min\{1, \frac{a}{w}\}$  we obtain:

$$V(x_{i+1}) - V(x_i) \leq -\frac{a}{2} h \|x_{i+1}\|_r^\mu \leq -\alpha h V(x_{i+1}),$$

where again  $\alpha = \frac{a}{2c_2}$  and the property given in (7) was used on the last step. Finally,

$$V(x_{i+1}) \leq (1 + \alpha h)^{-1} V(x_i)$$

for all  $i = 0, 1, \dots$  and the values of  $V(x_i)$  are monotonously decreasing, hence the sequence  $\{x_i\}_{i=0}^{\infty}$  is converging (the property (b) is substantiated). Global boundedness of  $\{x_i\}_{i=0}^{\infty}$  (i.e. the property (a)) follows from the properties of the function  $V$  for  $\gamma = (c_1^{-1}c_2)^{1/\mu}$ .

The property (c) can be proven applying the same arguments as in the proof of Theorem 9.

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