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Abstract

This work presents Lyapunov analysis conditions for fixed-time stability, a property where all the system's trajectories converge exactly to zero in a finite amount of time that is independent of the system's initial condition. Necessary and sufficient conditions for fixed-time stability without taking into account the regularity of the settling-time function are presented first. Next, a characterization for fixed-time stability with continuous settling-time function is introduced. A particular form of the characterizing functions follows, it allows to establish more constructive conditions and in order to obtain a converse result, the concept of *complete* fixed-time stability is introduced. A set of academic examples and an example of allocation of mobile agents illustrate the given concepts. Finally, a sufficient condition for fixed-time stabilization of nonlinear affine systems is obtained.

1. Introduction

In the theory of dynamical systems, the paramount Lyapunov's direct method allows to determine the stability of a system via the existence of a positive definite function V with a negative derivative, derived from the system equations [1]. Converse results obtained by Massera and Kurzweil [1] show that whenever a system is stable, there exists such a function V, called Lyapunov function. Further results show that a similar technique can be used to determine rates of convergence *e.g.* asymptotic or exponential stability [2]. These rates of convergence guarantee that an equilibrium point will be reached in an infinite amount of time. In contrast, finite-time stability, described early in the works of Roxin [3] and Zubov [4], provides a *faster than asymptotic* rate where the system's trajectories converge exactly to zero in a finite amount of time. The definition of this type of stability implies the existence of a function, often called *settling-time* function, that determines, given an initial condition, the minimum time at which the system's trajectories have converged exactly to zero and remain at zero for all future time. A counterpart of finite-time stable systems is that, as shown in [5], a finite-time stable system cannot be Lipschitz continuous at the origin, so that uniqueness of solutions in the backward time is lost. Another aspect to take into account is the regularity of the settling-time function since it might be discontinuous with respect to the initial conditions. The classical work of Bhat and Bernstein [5] provides an insightful study of finite-time stable systems and provides a sufficient condition for finite-time stability with continuous settling-time function using a Lyapunov-like formulation, *i.e.* a positive definite function that satisfies a differential inequality. It was later given in [6] necessary and sufficient conditions for finite-time stability for both cases of settling-time function (continuous and discontinuous) and in [7] analogous results for non-autonomous systems were given.

Yet another type of stability, reported in works such as [8, 9, 10, 11, 12], occurs when the settling-time function of a finite-time stable system is bounded on the whole domain by a finite value. This means that all the system's trajectories will converge exactly to zero before a specified, known in advance time regardless of the initial conditions. Being a particular case of finite-time stability, this type of stability, known as *fixed-time* stability, inherits many of the results derived for finite-time stable systems. Although some sufficient conditions for fixed-time stability can be found in [9, 11, 13, 14] and references therein, so far, necessary and sufficient conditions that take into account the regularity of the settling-time function haven't been presented. The aim of this article is to provide these conditions along with a characterization of the fixed-time stability property using Lyapunov theory. The theoretical tools employed to derive these conditions are founded in the works [5, 6] as well as in classic results on direct and converse Lyapunov theorems [15]. The main challenge consists in the analysis of the system's behavior *far away* from the origin since fixed-time stability, contrary to finite-time stability, is a property that preserves the finiteness of the settling-time over the whole domain of definition, even if this domain is unbounded.

In addition, based on Sontag's universal formula [16], we will present a sufficient condition for fixed-time stabilization of continuous autonomous systems that are affine in the control.

2. Preliminaries

2.1. Notation

- ⋄ $\mathbb{R}$ denotes the set of real numbers while $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.
- ⋄ a class-$\mathcal{K}$ function is a continuous, strictly increasing function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\gamma(0) = 0$ while a class-$\mathcal{K}_\infty$ function is a class $\mathcal{K}$ function γ such that $\gamma(s) \to \infty$ as $s \to \infty$.
- ⋄ throughout the article, the domain $\Gamma \subseteq \mathbb{R}^n$ will denote an open connected set containing the origin and $\partial\Gamma$ will denote its boundary. $\mathcal{B}^n$ denotes the unit open ball in $\mathbb{R}^n$.
- ⋄ a continuous function $V : \Gamma \to \mathbb{R}_{\geq 0}$ is said to be *radially unbounded* on Γ if $V(x) \to +\infty$ as $x \to \partial\Gamma$. If Γ is unbounded then, in addition, $V(x) \to +\infty$ as $\|x\| \to +\infty$.
- ⋄ a function $V : \Gamma \to \mathbb{R}$ is said to be $\mathcal{C}^0$ (or to belong to the $\mathcal{C}^0$ class), if it is continuous, $\mathcal{C}^1$ if it is continuously differentiable and if the derivative $V^{(n)}(x)$ exists and it is continuous for all $x \in \Gamma$ and for any integer n, then it is said to be $\mathcal{C}^\infty$.
- ⋄ The Lie derivative of $V : \Gamma \to \mathbb{R}$ along $g : \mathbb{R}^n \to \mathbb{R}^n$ is represented by $\mathcal{L}_g V : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{L}_g V = \langle \nabla V(x), f(x) \rangle$.
- ⋄ The brackets $\lceil \cdot \rfloor^\nu$ simplify the notation $|\cdot|^\nu \operatorname{sign}(\cdot)$, where $|\cdot|$ denotes the absolute value.
- ⋄ The upper-right Dini derivative of a function $g : [a, b) \to \mathbb{R}$, $b > a$, is the function $D^+ g : [a, b) \to \bar{\mathbb{R}}$ given by $D^+ g(t) = \limsup_{h \to 0^+} \frac{1}{h}[g(t+h) - g(t)]$, $t \in [a, b)$. If g is differentiable at t, then $D^+ g(t)$ is the ordinary derivative of g at t.

2.2. Theoretical Background

Consider the following autonomous system

$$\dot{x} = f(x), \quad x \in \Gamma, \tag{1}$$

where $f : \Gamma \to \mathbb{R}^n$ is a continuous function and $f(0) = 0$. Let us assume that f is such that (1) has the properties of existence and uniqueness of solutions in forward time outside the origin. Then $\Phi(t, x)$ denotes the solution to system (1) starting from $x \in \Gamma$ at $t = 0$.

Definition 1 ([17, 11]**).** The origin of the system (1) is said to be

- ***Lyapunov stable*** if for any $x_0 \in \Gamma$ the solution $\Phi(t, x_0)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x_0 \in \Gamma$, if $\|x_0\| \leq \delta$ then $\|\Phi(t, x_0)\| \leq \epsilon$ for all $t \geq 0$;
- ***asymptotically stable*** if it is Lyapunov stable and $\|\Phi(t, x_0)\| \to 0$ as $t \to +\infty$ for any $x_0 \in \Gamma$;
- ***finite-time stable*** if it is Lyapunov stable and finite-time converging from Γ, *i.e.* for any $x_0 \in \Gamma$ there exists $0 \leq T < +\infty$ such that $\Phi(t, x_0) = 0$ for all $t \geq T$. The function $T(x_0) = \inf\{T \geq 0 : \Phi(t, x_0) = 0 \ \forall t \geq T\}$ is called the ***settling-time*** function of system (1);
- ***fixed-time stable*** if it is finite-time stable and $\sup_{x_0 \in \Gamma} T(x_0) < +\infty$.

The set Γ is called the ***domain of attraction***. If $\Gamma = \mathbb{R}^n$, then the corresponding properties become ***global***.

2.3. Lyapunov Functions

For a continuous function $V : \Gamma \to \mathbb{R}$, the upper-right Dini derivative of V along the solutions of (1) is given by

$$\dot{V}(x) := D^{+}(V \circ \Phi)(0, x), \tag{2}$$

if V is continuously differentiable on $\Gamma \backslash \{0\}$, then

$$\dot{V}(x) = \frac{d(V \circ \Phi)}{dt}(0, x) = \frac{\partial V}{\partial x} f(x), \quad x \in \Gamma \backslash \{0\}. \tag{3}$$

Definition 2 ([15]). A function $V : \Gamma \to \mathbb{R}_{\geq 0}$ is called a ***strict Lyapunov function*** for system (1) if it fulfills the following properties:

L1 *Positive Definiteness.* $V(x) > 0$ for all $x \in \Gamma \setminus \{0\}$ and $V(0) = 0$.

L2 V is radially unbounded on Γ.

L3 V is continuous on Γ and
$$\dot{V}(x) < 0 \text{ for each } x \in \Gamma \backslash \{0\}.$$

Theorem 1 ([15, Theorems 5.2 and 5.3]). *The origin of* (1) *is asymptotically stable on* Γ *if and only if there exists a strict Lyapunov function for* (1).

The next result is a Corollary of Theorem 1 and will be used later in the development of the proofs of some results. Remark that it is stated in a converse form and that instead of a differential inequality, an equality is obtained.

Corollary 1. *Suppose that the origin of system* (1) *is asymptotically stable on* Γ*. Then there exist a strict Lyapunov function* $\tilde{V} : \Gamma \to \mathbb{R}_{\geq 0}$ *for* (1) *and a continuous positive definite function* $\tilde{W} : \Gamma \to \mathbb{R}_{\geq 0}$ *that satisfy*

M1 $$\dot{\tilde{V}}(x) = -\tilde{W}(x) \quad \forall x \in \Gamma.$$

M2 $$\alpha_1(\tilde{V}(x)) \leq \tilde{W}(x) \leq \alpha_2(\tilde{V}(x)) \quad \forall x \in \Gamma, \textit{ for some } \alpha_1, \alpha_2 \in \mathcal{K}_\infty.$$

Proof. Following Theorem 1, there exists a continuous positive definite and radially unbounded function V such that

$$W^{*}(x) := -\frac{\partial V}{\partial x} f(x) > 0 \quad \forall x \in \Gamma \backslash \{0\}.$$

Let us propose the function $\tilde{V}(x) = \int_0^{V(x)} \xi(p)dp$, where $\xi \in \mathcal{K}_\infty$ is defined later on. Note that from its definition, $\tilde{V}$ is continuous and radially unbounded (the integral of the radially unbounded function ξ over V is itself radially unbounded). Then

$$\dot{\tilde{V}}(x) = \xi(V(x))\dot{V}(x) = -\xi(V(x))W^{*}(x),$$

$$\xi(s) := s \begin{cases} 1 & \text{if } s \leq 1 \\ \max\{1, \frac{w(1)}{w(s)}\} & \text{if } s > 1 \end{cases}$$

and

$$w(s) := \inf_{V(x)=s} W^{*}(x)$$

By defining $\tilde{W}(x) := \xi(V(x))W^{*}(x)$ we obtain **M1** and it becomes clear that $\tilde{W}$ is continuous, $\tilde{W}(0) = 0$ and $\tilde{W}(x) > 0$ for all $x \in \Gamma \backslash \{0\}$ such that **M2** is satisfied with the above selected ξ and the class-$\mathcal{K}_\infty$ functions

$$\alpha_1(s) = \frac{s}{s+1} \inf_{x \in \Gamma : \tilde{V}(x) \geq s} \tilde{W}(x), \qquad \alpha_2(s) = s + \sup_{x \in \Gamma : \tilde{V}(x) \leq s} \tilde{W}(x).$$

Since $\tilde{V}$ satisfies all conditions of Definition 2, it constitutes a strict Lyapunov function for system (1). $\square$

2.4. Relevant Theorems and Lemmas on Finite-Time Stability for Autonomous Systems

The following theorem gives necessary and sufficient conditions for finite-time stability of autonomous systems.

Theorem 2 ([6]). *Let us consider system* (1) *with uniqueness of solutions in forward time outside the origin. The following properties are equivalent:*

1) the origin of system (1) *is finite-time stable on* Γ.

2) there exists a class-$\mathcal{C}^\infty$ *strict Lyapunov function* $V : \Gamma \to \mathbb{R}_{\geq 0}$ *for system* (1)*, satisfying for all* $x \in \Gamma$

$$\int_{V(x)}^{0} \frac{ds}{\dot{V}(\Phi(\theta_x(s), x))} < +\infty, \tag{4}$$

where the map $s \overset{\theta_x}{\mapsto} t$ *fulfills the identity* $s = V(\Phi(\theta_x(s), x))$.

Moreover if 1) or 2) are verified, all smooth Lyapunov functions $V : \Gamma \to \mathbb{R}_{\geq 0}$ *for the system* (1) *satisfy for all* $x \in \Gamma$

$$T(x) = \int_{V(x)}^{0} \frac{ds}{\dot{V}(\Phi(\theta_x(s), x))} < +\infty. \tag{5}$$

The following theorem, introduced initially in [5] provides a particular differential inequality from which finite-time stability with continuous settling-time function can be asserted.

Theorem 3 ([6, Proposition 12],[5]). *For the system* (1)*, the following properties are equivalent:*

1. *the origin is finite-time stable on* Γ *with a continuous settling-time function.*
2. *there exist real numbers* $c > 0$, $\alpha \in (0, 1)$ *and a strict Lyapunov function* $V : \Gamma \to \mathbb{R}_{\geq 0}$ *satisfying*

$$\dot{V}(x) \leq -cV(x)^\alpha \quad \forall x \in \Gamma.$$

The next lemma states relevant properties of the solutions of finite-time stable systems and is used later in the article.

Lemma 1 ([5, Proposition 2.4]). *Suppose that the origin of system* (1) *is finite-time stable on* Γ*. Let* $T : \Gamma \to \mathbb{R}_{\geq 0}$ *be the settling-time function. Then the following statements hold:*

i. *If* $x \in \Gamma$ *and* $t \in \mathbb{R}_{\geq 0}$*, then*

$$T(\Phi(t, x)) = \max\{T(x) - t, 0\}. \tag{6}$$

ii. T *is continuous on* Γ *if and only if* T *is continuous at* 0.

Since fixed-time stability implies finite-time stability, this lemma remains valid for the fixed-time case.

3. Necessary and Sufficient Conditions For Fixed-Time Stability

Let us introduce the following example that illustrates the behavior of a fixed-time stable system.

Example 1. Consider the scalar system

$$\dot{x} = -\lceil x \rfloor^{\frac{1}{2}} - \lceil x \rfloor^{\frac{3}{2}}. \tag{7}$$

The trajectories of this dynamics, for any initial condition $x_0 \in \mathbb{R}$ and any $t \geq 0$, can be obtained by direct integration and are given by

$$\Phi(t,x_0) = \begin{cases} \tan[\arctan(|x_0|^{\frac{1}{2}}) - \frac{1}{2}t]^2 \operatorname{sign}(x_0) & \text{if } 0 \le t \le T(x_0) \\ 0 & \text{if } t > T(x_0) \end{cases}.$$

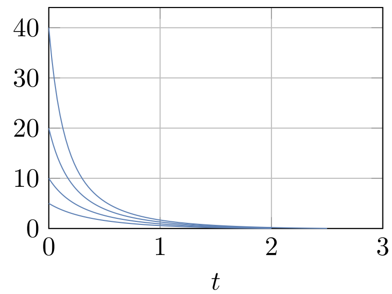

Figure 1: Solutions of system (7) for four different initial conditions.

It is clear from the system's solutions that for any $t \ge 2\arctan(|x_0|^{\frac{1}{2}})$, $\Phi(t,x_0) = 0$ so that the settling-time function is $T(x_0) = 2\arctan(|x_0|^{\frac{1}{2}})$. Moreover, this function is bounded on $\mathbb{R}$, indeed $\sup_{x_0 \in \mathbb{R}} T(x_0) = \pi$ so that all trajectories, regardless of where they start, converge exactly to zero in $t \le \pi$ seconds (see Figure 1).

System (7) possesses yet another property: for all initial conditions $x_0 \neq 0$ there exist well-defined solutions in the backward time. This property will be called *complete fixed-time stability* and it is defined as follows:

Definition 3. The system (1) is called *complete* fixed-time stable, if it is fixed-time stable, and for $x_0 \in \Gamma \setminus \{0\}$ there exist unique solutions in the backward time denoted as $\Phi(-t,x_0)$ for $t \in [0,\tau(x_0))$, where $\tau : \Gamma \setminus \{0\} \to \mathbb{R}_{\ge 0}$ is the *escape-time function* such that

$$\limsup_{t \to \tau(x_0)} \|\Phi(-t,x_0)\|_{\delta\Gamma} = 0 \quad \forall x_0 \in \Gamma \setminus \{0\}.$$

Note that by definition, if $T(x_0) \le T_{\max}$ for all $x_0 \in \Gamma$, then $\tau(x_0) \le T_{\max}$ also for all $x_0 \in \Gamma \setminus \{0\}$, since

$$T(\Phi(t,x_0)) + \tau(\Phi(t,x_0)) = \text{const}$$

for all $t \ge 0$ such that $\Phi(t,x_0) \neq 0$. Therefore, by Lemma 1:

$$\tau(\Phi(-t,x_0)) = \max\{0, \tau(x_0) - t\},$$

then, provided that τ is continuous,

$$\frac{\partial \tau(\Phi(-t,x_0))}{\partial t} = -1$$

for all $t \in [0,\tau(x_0))$ and all $x_0 \in \Gamma \setminus \{0\}$.

Analogously to Theorem 2 we first present a necessary and sufficient condition for an autonomous system to be fixed-time stable.

Theorem 4. *Consider system* (1). *The following properties are equivalent*

1) The origin is fixed-time stable on Γ.

2) There exists a strict Lyapunov function V *for system* (1) *satisfying for all* $x \in \Gamma$

$$\sup_{x \in \Gamma} \int_{V(x)}^{0} \frac{ds}{\dot{V}(\Phi(\theta_x(s),x))} < +\infty, \tag{8}$$

where the map $s \overset{\theta_x}{\mapsto} t$ *fulfills the identity* $s = V(\Phi(\theta_x(s),x))$.

Proof. 1) $\Rightarrow$ 2). If the system (1) is fixed-time stable, then its settling-time function is such that $T_* := \sup_{x \in \Gamma} T(x) < +\infty$ and $\Phi(t,x) = 0$ for all $t \ge T_*$ and for all $x \in \Gamma$. Since fixed-time stability implies asymptotic stability, according to Theorem 1, there exists a strict Lyapunov function V for (1) and therefore there exists a well defined mapping $[0,T(x)) \to (0,V(x)]$, $t \mapsto V(\Phi(t,x))$ strictly decreasing and differentiable for all $t \in [0, T(x))$. Hence, for any $x \in \Gamma$, there exists a differentiable inverse mapping $(0, V(x)] \to [0, T(x))$, $s \overset{\theta_x}{\mapsto} t$, also decreasing that satisfies for all $s \in (0, V(x)]$

$$\theta_x'(s) = \frac{1}{\dot{V}(\Phi(\theta_x(s), x))}.$$

The change of variables $s = V(\Phi(t, x))$ and the fact that $V(\Phi(T(x), x)) = 0 \; \forall x \in \Gamma$ lead to

$$T(x) = \int_0^{T(x)} dt = \int_{V(x)}^0 \theta_x'(s) ds = \int_{V(x)}^0 \frac{ds}{\dot{V}(\Phi(\theta_x(s), x))}. \tag{9}$$

Then we have that

$$+\infty > \sup_{x \in \Gamma} T(x) = \sup_{x \in \Gamma} \int_{V(x)}^0 \frac{ds}{\dot{V}(\Phi(\theta_x(s), x))}. \tag{10}$$

for all $x \in \Gamma$ and the conclusion readily follows.
2) $\Rightarrow$ 1). According to Theorem 1, because there exists a strict Lyapunov function V for system (1), its origin is asymptotically stable and therefore the inverse mapping θ_x exists. The equation (10) implies, furthermore, that the origin is fixed-time stable. □

Even though the conditions (4) and (8) are in general difficult to verify (they require explicit knowledge of the system's trajectories), they can be used to discard finite-time or fixed-time stability of certain systems. The next example, borrowed from [6], shows that linear systems cannot be finite-time stable.

Example 2. Consider the Cauchy problem

$$\begin{cases} \dot{x} = -x, \\ x_0 = 1, \end{cases} \tag{11}$$

and the Lyapunov function candidate $V(x) = x^2$. Then $\Phi(t, x_0) = e^{-t}$, $V(\Phi(t), x_0) = e^{-2t}$, $\theta(s) = -\frac{1}{2}\ln(s)$ and $\Phi(\theta(s), x_0) = \sqrt{s}$ lead to $\dot{V}(\Phi(\theta(s), x_0) = -2s$, where $s > 0$. Substituting in (5) yields

$$T(1) = \int_0^1 \frac{ds}{2s} = +\infty$$

and according to Theorem 2, the system (11) is not finite-time stable.

Example 3. The scalar system $\dot{x} = -\lceil x \rfloor^{\mu}$, $\mu \in (0, 1)$ has, for any initial condition $x_0 \in \mathbb{R}$, the explicit solution

$$\Phi(t, x_0) = \begin{cases} (|x_0|^{1-\mu} - (1-\mu)t)^{\frac{1}{1-\mu}} \operatorname{sign}(x_0) & \text{if } 0 \le t \le T(x_0) \\ 0 & \text{if } t \ge T(x_0) \end{cases}.$$

Although in this simple example it is possible to derive the settling-time function T from the general solution, as we did in Example 1, and immediately check that it is not globally bounded, here we will propose the $\mathcal{C}^1$ candidate function $V(x) = |x|^{1-\mu}$, which yields $\Phi(\theta_{x_0}(s), x_0) = s^{\frac{1}{1-\mu}} \operatorname{sign}(x_0)$, $s > 0$ and $\dot{V}(\Phi(\theta_{x_0}(s), x_0)) = -(1-\mu)$. Then we have that

$$\int_{V(x_0)}^0 \frac{ds}{\dot{V}(\Phi(\theta_{x_0}(s), x_0))} = \frac{1}{1-\mu} \int_0^{|x_0|^{1-\mu}} ds = \frac{1}{1-\mu} |x_0|^{1-\mu} \tag{12}$$

and it becomes clear that (4) is satisfied while (8) is not. Hence, the system $\dot{x} = -\lceil x \rfloor^{\mu}$ is finite-time stable but not fixed-time stable.

Note that in Theorem 4 nothing is said about the regularity of the settling-time function, therefore T might be discontinuous.

3.1. Sufficient Conditions for Fixed-Time Stability With Continuous Settling-Time Function

The next set of results presents more constructive conditions for fixed-time stability, they will no longer require explicit knowledge of the system's solutions and in the sufficiency case, they will guarantee continuity of the settling-time function.

Theorem 5. *Suppose that there exists a continuously differentiable strict Lyapunov function $V : \Gamma \to \mathbb{R}_{\geq 0}$ for system (1) such that*

S1 *there exists a continuous positive definite function $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that satisfies* $\int_0^{\sup_{x\in\Gamma} V(x)} \frac{dz}{r(z)} < +\infty$;

S2 *the inequality $\dot{V}(x) \leq -r(V(x))$ holds for all $x \in \Gamma$.*

Then the origin of (1) is fixed-time stable with continuous settling-time function $T : \Gamma \to \mathbb{R}_{\geq 0}$ and

$$T(x) \leq \int_0^{\sup_{x\in\Gamma} V(x)} \frac{dz}{r(z)} \quad \forall x \in \Gamma. \tag{13}$$

Proof. Let us define the inverse mapping $(0, V(x)] \to [0, T(x))$ as $s \overset{\varphi_x}{\mapsto} t$, where $T : \Gamma \to \mathbb{R}_{\geq 0}$ is the settling-time function *i.e.* $T(x) := \inf\{T \geq 0 : V(\Phi(T, x)) = 0\}$. Since V is, by assumption, differentiable on $x \in \Gamma$ and satisfies **S2**, $\varphi_x(s)$ is strictly decreasing and differentiable with

$$\varphi_x'(s) = \frac{1}{\dot{V}(\Phi(\varphi_x(s), x))}.$$

Then we have that for all $x \in \Gamma$

$$T(x) = \int_0^{T(x)} dt = -\int_{V(x)}^0 \frac{ds}{-\dot{V}(\Phi(\varphi_x(s), x))}.$$

From condition **S2** we have that $-\dot{V}(x) \geq r(V(x)) \; \forall x \in \Gamma$ so that

$$\begin{aligned} -\int_{V(x)}^0 \frac{ds}{-\dot{V}(\Phi(\varphi_x(s), x))} &\leq \int_0^{V(x)} \frac{ds}{r(V(\Phi(\varphi_x(s), x)))} \\ &\leq \int_0^{\sup_{x\in\Gamma} V(x)} \frac{ds}{r(V(\Phi(\varphi_x(s), x)))} \\ &= \int_0^{\sup_{x\in\Gamma} V(x)} \frac{ds}{r(s)} < +\infty, \end{aligned}$$

where in this last step, the condition **S1** was used. Therefore, $\sup_{x\in\Gamma} T(x) < +\infty$. Equivalently, the origin of (1) is fixed-time stable. Taking any $x_k \in \Gamma\backslash\{0\}$ converging to zero, we have that $T(x_k) \leq \int_0^{V(x_k)} \frac{ds}{r(s)}$, by continuity of V and since r is assumed to be positive definite we obtain

$$\lim_{x_k \to 0} \int_0^{V(x_k)} \frac{ds}{r(s)} = 0,$$

therefore T is continuous at the origin and due to Lemma 1.**ii**, T is continuous on Γ and, from the analysis above, bounded by (13). $\square$

As can be seen from this theorem, fixed-time stability with continuous T can be completely characterized by the pair of functions (V, r) satisfying conditions **S1** and **S2**. The following corollary gives more insight about some of the forms that the function r might take.

Corollary 2. *Suppose there exists a strict Lyapunov function $V : \Gamma \to \mathbb{R}_{\geq 0}$ for system (1) such that*

$$\dot{V}(x) \leq -c_1 V(x)^\alpha - c_2 V(x)^\beta, \quad \forall x \in \Gamma, \tag{14}$$

with $c_1, c_2 > 0$, $\alpha \in [0, 1)$ and $\beta > 1$. Then the origin of (1) is fixed-time stable with continuous settling-time function $T(x)$ satisfying

$$T(x) \leq \frac{1}{c_1(1-\alpha)} + \frac{1}{c_2(\beta-1)}.$$

Proof. Although this result was proven in [11], here we will use the characterization proposed in Theorem 5. Let us consider the ball $\mathcal{D} := \{x \in \mathbb{R}^n : V(x) \leq 1\}$ and denote $T_{\mathcal{D}} := \inf\{x \in \Gamma \backslash \mathcal{D}, T \geq 0 : V(\Phi(T,x)) \leq 1\}$, and $T_0 := \inf\{x \in \mathcal{D}, T \geq 0 : V(\Phi(T,x)) = 0\}$ as the time functions that provide the time that takes to any trajectory outside $\mathcal{D}$ to arrive to $\mathcal{D}$, and the time that takes to any trajectory in $\mathcal{D}$ to arrive to zero, respectively. Then we have that for all $x \in \Gamma\backslash\mathcal{D}$, $\dot{V}(x) \leq -r_1(V(x))$, where $r_1(z) = c_2 z^{\beta}$, and therefore

$$\begin{aligned} T_{\mathcal{D}}(x) \leq \int_1^{\sup_{x\in\mathbb{R}^n} V(x)} \frac{dz}{r_1(z)} &= \int_1^{+\infty} \frac{dz}{c_2 z^{\beta}} \\ &= \frac{z^{1-\beta}}{c_2(\beta-1)}\Big|_{+\infty}^{1} = \frac{1}{c_2(\beta-1)} < +\infty. \end{aligned}$$

For all $x \in \mathcal{D}$, we have that $\dot{V}(x) \leq -r_2(V(x))$, where $r_2(z) = c_1 z^{\alpha}$ and therefore

$$T_0(x) \leq \int_0^1 \frac{dz}{r_2(z)} = \int_0^1 \frac{dz}{c_1 z^{\alpha}} = \frac{z^{1-\alpha}}{c_1(1-\alpha)}\Big|_0^1 = \frac{1}{c_1(1-\alpha)} < +\infty.$$

Hence, **S1** and **S2** are satisfied with $r(z) = r_1(z) + r_2(z)$ and

$$T(x) = T_{\mathcal{D}} + T_0 \leq \frac{1}{c_1(1-\alpha)} + \frac{1}{c_2(\beta-1)} < +\infty \quad \forall x \in \Gamma.$$

□

Example 4 (*Two dimensional systems*)**.** Consider the systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -\lceil x_1 \rfloor^{\gamma} + x_2 \\ \dot{x}_2 = -\lceil x_2 \rfloor^{\gamma} - x_1 \end{cases}, \qquad \Sigma_2 : \begin{cases} \dot{x}_1 = -\lceil x_1 \rfloor^{\gamma} - x_1^3 + x_2 \\ \dot{x}_2 = -\lceil x_2 \rfloor^{\gamma} - x_2^3 - x_1 \end{cases}, \quad x \in \mathbb{R}^2, \gamma \in (0,1)$$

and the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. For Σ_1 we have

$$\dot{V}(x)\big|_{\Sigma_1} = -(|x_1|^{\gamma+1} + |x_2|^{\gamma+1}) \leq -V(x)^{\frac{\gamma+1}{2}},$$

while for Σ_2

$$\dot{V}(x)\big|_{\Sigma_2} = -(x_1^4 + x_2^4) - (|x_1|^{\gamma+1} + |x_2|^{\gamma+1}) \leq -V(x)^{\frac{\gamma+1}{2}} - V(x)^{\gamma+1},$$

where $\frac{\gamma+1}{2} < 1$ and $1 + \gamma > 1$. Then, according to Corollary 2, Σ_1 is finite-time stable while Σ_2 is fixed-time stable with $T(x) \leq \frac{1}{1-\gamma} + \frac{1}{\gamma}$. Figure 2 shows the trajectories of Σ_1 and Σ_2 under small initial conditions (left) and with slightly larger ones (right). The time $\bar{t}$ represents the instant at which $\|x\| \leq 10^{-3}$. It is possible to see that the settling time of Σ_1 increases significantly with larger initial conditions, that of Σ_2 remains in a close vicinity.

3.2. Necessary Conditions for Fixed-Time Stability.

The first theorem of this section presents a necessary condition for fixed-time stability using a similar characterization as the one employed in Theorem 5.

Theorem 6. *Consider system* (1) *and suppose that the origin is fixed-time stable on* Γ*. Then there exist a strict Lyapunov function* V *and a class-*$\mathcal{K}_\infty$ *function* q *that satisfies*

N1 $\displaystyle\int_0^{\sup_{x\in\Gamma} V(x)} \frac{dz}{q(z)} < +\infty;$

N2 $-q(V(x)) \leq \dot{V}(x) \qquad \forall x \in \Gamma.$

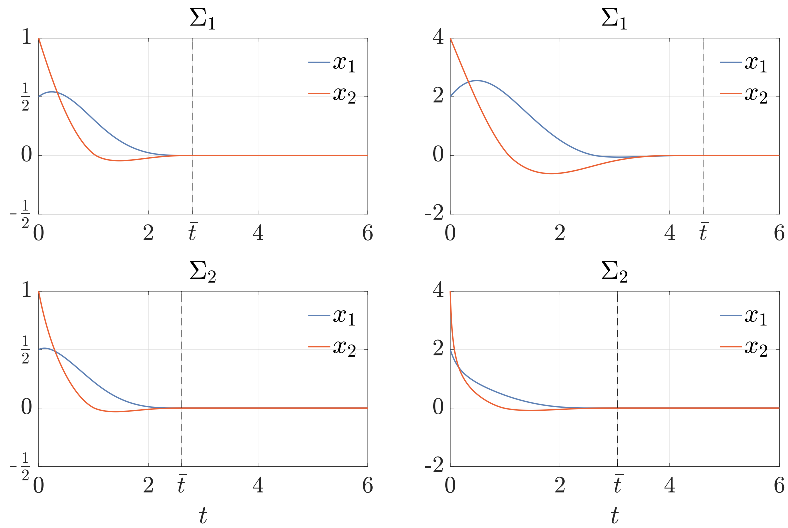

Figure 2: Trajectories of Σ_1 and Σ_2 for $\gamma = \frac{1}{2}$ with initial conditions $x_0 = (-1, 2)$ (left) and $x_0 = (-5, 10)$ (right).

Proof. Since fixed-time stability implies asymptotic stability we know, from Corollary 1, that there exist a strict Lyapunov function V and a continuous positive definite function W that satisfy the **M** conditions. Then we have that for all $x \in \Gamma$:

I there exist a decreasing differentiable mapping $[0, T(x)] \to (0, V(x_0)]$, $t \mapsto V(\Phi(t, x))$, with $T(x) = \inf\{T \geq 0 : V(\Phi(T, x)) = 0\}$ and its corresponding inverse mapping $(0, V(x)] \to [0, T(x)]$, $s \overset{\varphi_x}{\mapsto} t$, also decreasing and differentiable such that $\varphi_x'(s) = 1/\dot{V}(\Phi(\varphi_x(s), x))$.

II Since $V(x)$ and $W(x) := \dot{V}(x)$ satisfy **M**, there exists some $q \in \mathcal{K}_\infty$ such that $-q(V(x)) \leq \dot{V}(x)$ $\forall x \in \Gamma$ and **N2** is satisfied.

III Then

$$T(x) = \int_0^{T(x)} dt = -\int_{V(x)}^0 \frac{ds}{-\dot{V}(\Phi(\varphi_x(s), x))}.$$

From **II**, $-\dot{V}(x) \leq q(V(x))$ and therefore

$$-\int_{V(x)}^0 \frac{ds}{-\dot{V}(\Phi(\varphi_x(s), x))} \geq \int_0^{V(x)} \frac{ds}{q(V(\Phi(\varphi_x(s), x)))}.$$

Hence

$$+\infty > \sup_{x \in \Gamma} T(x) \geq \int_0^{\sup_{x\in\Gamma} V(x)} \frac{ds}{q(s)}$$

and **N1** is fulfilled. $\square$

It is difficult to obtain a converse result to Corollary 2 without adding some assumptions about the behavior of V when $x \to \partial\Gamma$. Nonetheless, for the case of complete fixed-time stability, the next necessary and sufficient condition can be obtained.

Theorem 7. *The system* (1) *is complete fixed-time stable on* Γ *with continuous settling-time* T *and escape-time* τ *functions if and only if there exists a strict Lyapunov function* $V : \Gamma \to \mathbb{R}_{\geq 0}$ *and some constants* $c_1 > 0$, $c_2 > 0$, $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\frac{\partial V(x)}{\partial x} f(x) \leq -c_1 V^{\frac{\alpha}{\alpha+1}}(x) - c_2 V^{\frac{\beta+1}{\beta}}(x)$$

for all $x \in \Gamma \setminus \{0\}$.

Proof. The system is fixed-time stable, and in complement to the maximal time of convergence to the origin:

$$T_{\max} = \limsup_{\|x\|_{\delta\Gamma}\to 0} T(x),$$

let us introduce also into consideration the minimal time of convergence from the boundary of Γ:

$$T_{\min} = \liminf_{\|x\|_{\delta\Gamma}\to 0} T(x),$$

which is well defined due to continuity of T and $T_{\min} \le T_{\max}$. Obviously, $T(0) = 0$ and $T(x_0) > 0$ for all $x_0 \in \Gamma \setminus \{0\}$, and

$$T(\Phi(t, x_0)) = \max\{0, T(x_0) - t\},$$

then

$$\frac{\partial T(\Phi(t, x_0))}{\partial t} = -1$$

for all $t \in [0, T(x_0))$ and all $x_0 \in \Gamma \setminus \{0\}$.

For any $a > A = \frac{T_{\max}}{T_{\min}} \ge 1$ introduce the set

$$\Omega_a = \{x \in \Gamma : T(x_0) \le \frac{1}{a} T_{\max}\},$$

which is then a closed set containing the origin, $\Omega_a \subset \Gamma$. Define a Lyapunov function candidate $V_1 : \Omega_a \to \mathbb{R}_{\ge 0}$ as $V_1(x) = \frac{\rho_1}{\alpha+1} T^{\alpha+1}(x)$ for some $\alpha > 0$ and $\rho_1 > 0$, then

$$\frac{\partial V_1(x)}{\partial x} f(x) = -\rho_1 T^{\alpha}(x) = -\rho_1 \left(\frac{\alpha+1}{\rho_1} V_1(x)\right)^{\frac{\alpha}{\alpha+1}}$$

for any $x \in \Omega_a \setminus \{0\}$. For further analysis assume that ρ_1 is taken small enough to guarantee that $V_1 : \Omega_a \to [0, 1)$.

Now consider an open set $\Xi = \Gamma - \Omega_{2a}$. On this set consider the system (1) in the backward time:

$$\dot{x}(t) = -f(x(t)),\ t \ge 0,$$

then for any $x_0 \in \Xi$ there is a unique solution $\Phi(-t, x_0)$ (to denote a solution of this system we use the solutions of (1) with negative argument since we assumed that they exist due to complete fixed-time stability of (1)) that is defined on some finite interval of time for $t \in [0, \tau(x_0))$, and

$$\lim_{t\to\tau(x_0)} \|\Phi(-t, x_0)\|_{\delta\Gamma} = 0.$$

Again,

$$\tau(\Phi(-t, x_0)) = \max\{0, \tau(x_0) - t\},$$

then for a continuous function τ:

$$\frac{\partial \tau(\Phi(-t, x_0))}{\partial t} = -1$$

for all $t \in [0, \tau(x_0))$ and all $x_0 \in \Xi$. Define another Lyapunov function candidate $V_2 : \Xi \to \mathbb{R}_{\ge 0}$ as $V_2(x) = \frac{\rho_2}{\beta} \tau^{-\beta}(x)$ for some $\beta > 0$ and $\rho_2 > 0$, then

$$\frac{\partial V_1(x)}{\partial x} f(x) = -\rho_2 \tau^{-\beta-1}(x) = -\rho_2 \left(\frac{\beta}{\rho_2} V_2(x)\right)^{\frac{\beta+1}{\beta}}$$

for any $x \in \Xi$. We can also select ρ_2 sufficiently large to ensure that $V_2 : \Xi \to (1, +\infty)$.

It is rest to unite these functions on the set $\Xi \cap \Omega_a = \{x \in \Gamma : \frac{1}{2a} T_{\max} < T(x) \le \frac{1}{a} T_{\max}\}$ to obtain a Lyapunov function $V : \Gamma \to \mathbb{R}_{\ge 0}$:

$$V(x) = \begin{cases} V_1(x) & x \in \Omega_{2a} \\ V_2(x) & x \notin \Omega_a \\ \lambda(x) V_1(x) + (1 - \lambda(x)) V_2(x) & x \in \Xi \cap \Omega_a \end{cases},$$

where

$$\lambda(x) = 2(1 - \frac{a}{T_{\max}} T(x)).$$

By definition,

$$V(0) = 0,\ V(x) > 0\ \forall x \in \Gamma \setminus \{0\},\ \lim_{\|x\|_{\delta\Gamma} \to 0} V(x) = +\infty.$$

Finally,

$$\frac{\partial V(x)}{\partial x} f(x) = -\begin{cases} \rho_1 \left(\frac{\alpha+1}{\rho_1} V(x)\right)^{\frac{\alpha}{\alpha+1}} & x \in \Omega_{2a} \setminus \{0\} \\ \rho_2 \left(\frac{\beta}{\rho_2} V(x)\right)^{\frac{\beta+1}{\beta}} & x \notin \Omega_a \\ \frac{2a}{T_{\max}}(V_2(x) - V_1(x)) + \rho_2(1-\lambda(x)) \left(\frac{\beta}{\rho_2} V_2(x)\right)^{\frac{\beta+1}{\beta}} & \\ +\rho_1 \lambda(x) \left(\frac{\alpha+1}{\rho_1} V_1(x)\right)^{\frac{\alpha}{\alpha+1}} & x \in \Xi \cap \Omega_a \end{cases}.$$

Then the condition to ensure is that

$$V_2(x) > V_1(x)$$

for $x \in \Xi \cap \Omega_a$, which is satisfied due to selection of the coefficients ρ_1, ρ_2 (*i.e.* $V_1 : \Omega_a \to [0, 1)$ and $V_2 : \Xi \to (1, +\infty)$).

Finally, for any $\alpha > 0$ and $\beta > 0$, it is straightforward to check the existence of $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{\partial V(x)}{\partial x} f(x) \le -c_1 V^{\frac{\alpha}{\alpha+1}}(x) - c_2 V^{\frac{\beta+1}{\beta}}(x)$$

for all $x \in \Gamma \setminus \{0\}$ as needed.

Now assume that the last inequality is satisfied, and let us show that the system (1) is complete fixed-time stable. Obviously, it is fixed-time stable with a continuous at the origin settling-time function by Corollary 2. Consider the system behavior in the backward time, then we obtain that

$$-\frac{\partial V(x)}{\partial x} f(x) \ge c_2 V^{\frac{\beta+1}{\beta}}(x),$$

from which we conclude that the escape-time function is upper bounded by a continuous function, and hence it is also continuous. □

4. Mobile Agents Allocation

In order to illustrate the most important features of fixed-time stability, let us consider the problem of equidistant allocation of n mobile agents along a line segment. The one dimensional position of each agent is given by $x_i(t) \in \mathbb{R}$, $i = 1, 2, ..., n$ and x_0 and x_{n+1} denote fixed endpoints on the segment where the agents will align equidistantly. The dynamical model of each agent is given by the integrator system

$$\dot{x}_i = v_i, \quad i = 1, 2, ..., n \tag{15}$$

where $x = [x_1, x_2, ..., x_n]^T$ is the state vector of the multi-agent system and $v_i \in \mathbb{R}$ is a continuous scalar function called ***allocation algorithm***. It was shown in [18] that the allocation algorithm

$$v_i = \tfrac{1}{2}(x_{i-1} - x_i) + \tfrac{1}{2}(x_{i+1} - x_i), \tag{16}$$

drives the states of system (15) from any initial condition to the equilibrium point $x_i^* \in \mathbb{R}^n$, $x^* = x_0 + \frac{i}{n+1}(x_{n+1} - x_0)$ exponentially. The equilibrium points x_i^* represent the state where all the agents are aligned sequentially and equidistantly along the interval $[x_0, x_{n+1}]$. Note that the allocation algorithm given above only uses the information of the current agent x_i and of the neighbor agents x_{i-1} and x_{i+1}.

The extension of system (15)-(16) to the 2D case is achieved by combining independently the systems

$$\begin{aligned} \dot{x}_i &= \tfrac{1}{2}(x_{i-1} - x_i) + \tfrac{1}{2}(x_{i+1} - x_i), \\ \dot{y}_i &= \tfrac{1}{2}(y_{i-1} - y_i) + \tfrac{1}{2}(y_{i+1} - y_i), \end{aligned} \tag{17}$$

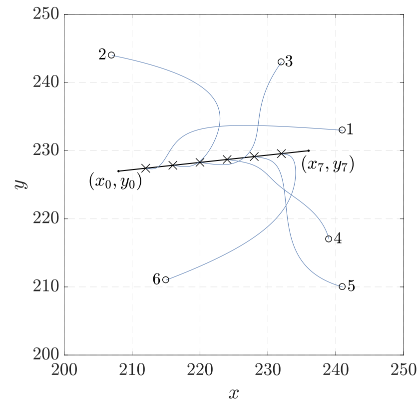

Figure 3: Position trajectories $(x_i(t), y_i(t))$ of the multi-agent system (17) with $n = 6$, initial positions $x(0) = [241, 207, 232, 239, 241, 215]$, $y(0) = [233, 244, 243, 217, 210, 211]$ and a target line segment given by the points $(x_0, y_0) = (208, 227)$ and $(x_7, y_7) = (236, 230)$. The symbol $\circ$ denotes the initial position of each agent and $\times$ denotes its target position along the line segment depicted in bold black. The number beside each initial point corresponds to the agent's index i.

where $x, y \in \mathbb{R}^n$ are the vectors of horizontal and vertical positions, respectively. Then, each agent position $(x_i(t), y_i(t)) \to (x_i^*, y_i^*)$ as $t \to \infty$, where x_i^* is as above and $y_i^* = y_0 + \frac{i}{n+1}(y_{n+1} - y_0)$. This corresponds to the allocation of the agents along the line segment that joins the points (x_0, y_0) and (x_{n+1}, y_{n+1}). Figure 3 shows a numerical simulation of (17) over a time interval $t \in [0, 40]$ seconds, with agents' initial position *close* to the target line segment.

Consider now the allocation algorithm

$$v_i = \phi\left(\tfrac{1}{2}(x_{i-1} - x_i) + \tfrac{1}{2}(x_{i+1} - x_i)\right) \tag{18}$$

where

$$\phi(s) := c_1 \lceil s \rfloor^{1-\frac{1}{\mu}} + c_2 \lceil s \rfloor^{1+\frac{1}{\mu}}, \quad c_1, c_2 > 0, \mu > 1.$$

In [19], it was shown using Corollary 2, that the system (15)-(18) is fixed-time stable and that its settling-time function satisfies the estimate

$$T(x) \leq \frac{\pi \mu n^{\frac{1}{4\mu}}}{2\bar{\lambda}\sqrt{c_1 c_2}}, \quad \bar{\lambda} = 2\sin^2\left(\tfrac{\pi}{2n+1}\right).$$

The extension of system (15) with the allocation algorithm (18) to the 2D case is given by

$$\begin{aligned} \dot{x}_i &= \phi(\tfrac{1}{2}(x_{i-1} - x_i) + \tfrac{1}{2}(x_{i+1} - x_i)), \\ \dot{y}_i &= \phi(\tfrac{1}{2}(y_{i-1} - y_i) + \tfrac{1}{2}(y_{i+1} - y_i)). \end{aligned} \tag{19}$$

Figure 4 presents a comparison between (17) and (19) with larger initial conditions than in Figure 3 and the same time interval $t \in [0, 40]$. It is possible to see in the inner plots that the fixed-time algorithm (18) manages to drive the agents to their target positions (Fig 4.b) whereas the exponential algorithm (16) falls short, *i.e.* the given time interval is not long enough to drive the agents to the equilibrium point. Moreover, the calculation of the settling-time estimate gives $T(x) \leq 26.21$, which means that the agents will converge exactly to their target positions in less than 26.21 seconds, irrespectively of their initial position. Figure 4.c shows a plot over time of the norm of the distance between the agents' position and the target points, *i.e.* $e_i(t) = \sqrt{(x_i(t) - x_i^*)^2 + (y_i(t) - y_i^*)^2}$ of each allocation algorithm e_{exp} and e_{fx}. It is noticeable that the fixed-time algorithm exhibits a much faster decay rate.

5. Fixed-Time Stabilization of Continuous Affine Systems

In this section, we will give a sufficient condition for fixed-time stabilization following the same structure of well known results on asymptotic stabilization of continuous autonomous systems.

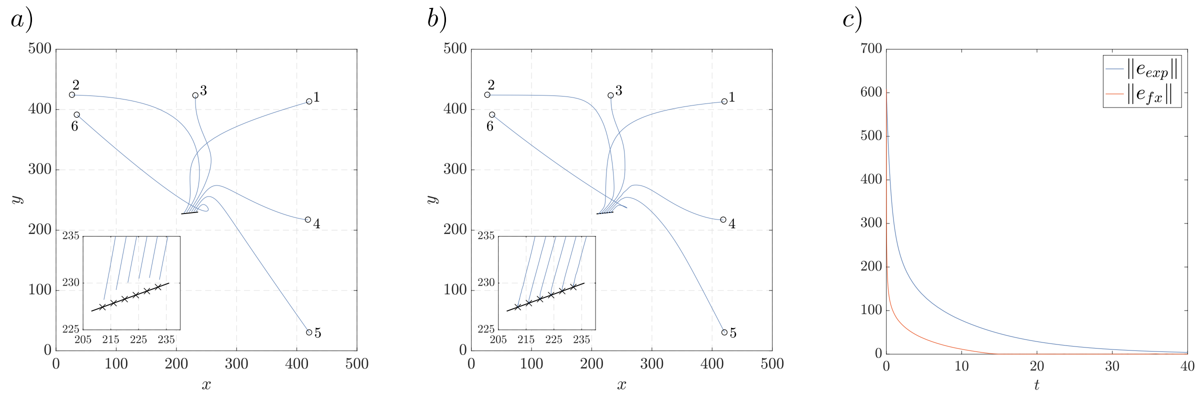

Figure 4: Comparison of systems (17) and (19) with initial conditions $x(0) = 208, 241+d, 207-d, 232, 239+d, 241+d, 215-d, 236$, $y = [227, 233+d, 244+d, 243+d, 217, 210-d, 211+d, 230]$, $d = 180$ and parameters $c_1 = c_2 = 1$, $\mu = 1.1$ for the allocation algorithm (18). a) and b) shows the trajectories $(x_i(t), y_i(t))$ of systems (17) and (19), respectively. c) depicts a plot of the norm of the distance between the initial points and the target points of each system, denoted $\|e_{exp}\|$ and $\|e_{fx}\|$, respectively.

Consider the following affine in the input u system:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i, \quad x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m, \tag{20}$$

where $f_0(0) = 0$, f_i is continuous for all $0 \leq i \leq m$ and such that (20) has uniqueness of solutions in forward time. Its closed-loop representation is given by

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i(x), \quad x \in \mathbb{R}^n. \tag{21}$$

Let us recall the definition of *stabilization* and propose a definition of *fixed-time stabilization*. In the latter, we will only consider fixed-time stabilization with continuous settling-time functions.

Definition 4. The control system (20) is ***stabilizable*** (respectively ***fixed-time stabilizable***) if there exists a $\mathcal{C}^0$ feedback control law $u : \Gamma \to \mathbb{R}^m$ such that:

1. $u(0) = 0$;
2. the origin of the system (21) is asymptotically stable (respectively fixed-time stable with a continuous settling-time function).

Such a feedback law $u(x)$ is called a ***stabilizer*** (respectively ***fixed-time stabilizer***) for system (20).

A radially unbounded, positive definite, $\mathcal{C}^1$ function $V : \Gamma \to \mathbb{R}_{\geq 0}$ is a ***control Lyapunov function*** (CLF) for the system (20) if for all $x \in \Gamma \backslash \{0\}$,

$$\inf_{u \in \mathbb{R}^m} (a(x) + \langle B(x), u \rangle) < 0, \tag{22}$$

where $a(x) = \mathcal{L}_{f_0} V(x)$, $B(x) = (B_1(x), \ldots, B_m(x))$ with $B_i(x) = \mathcal{L}_{f_i} V(x)$ for $1 \leq i \leq m$. Such a control Lyapunov function satisfies the ***small control property*** (SCP) if for each $\epsilon > 0$, there exists $\tau > 0$ such that, if $x \in \tau\mathcal{B}^n$, then there exists some $u \in \epsilon\mathcal{B}^m$ such that

$$a(x) + \langle B(x), u \rangle < 0. \tag{23}$$

The small control property is equivalent to

$$\limsup_{\|x\| \to 0} \frac{a(x)}{\|B(x)\|} \leq 0 \tag{24}$$

and this limit may very well be $-\infty$. For a CLF, $a(x) < 0$ whenever $B(x) = 0$, and one may think that this is a necessary condition for asymptotic stabilization. However, as shown in [16], it is also a sufficient one. Indeed, E. Sontag shows that if a radially unbounded and positive definite $\mathcal{C}^1$ function V satisfies $b(x) = 0 \Rightarrow a(x) < 0$ where $b(x) = \|B(x)\|^2$, then the feedback law $u = w(x)$,

$$w_i(x) := \begin{cases} -\dfrac{a(x) + \sqrt{a(x)^2 + b(x)^2}}{b(x)} B_i(x) & \text{if } x \in \Gamma \backslash \{0\} \\ 0 & \text{if } x = 0 \end{cases}, \tag{25}$$

known as *Sontag's universal formula*, is a stabilizer for (20). It was also shown in [16] that this feedback law is continuous on $\Gamma\backslash\{0\}$ and that if V further satisfies the SCP, then w is continuous on Γ (see, for instance, [20, Chapter 9]).

The following theorem presents an analogous formulation for fixed-time stabilization and in order to prove it, we will provide a continuous fixed-time stabilizer, akin to Sontag's universal formula.

Theorem 8. *For the system (20) there is a continuous fixed-time stabilizer $u = v(x)$ if there exists a radially unbounded and positive definite $\mathcal{C}^1$ control Lyapunov function $V : \Gamma \to \mathbb{R}_{\geq 0}$ that satisfies the SCP and*

$$a(x)^2 + b(x)^2 \rho(x)^2 \geq (c_1 V(x)^\alpha + c_2 V(x)^\beta)^2 \quad \forall x \in \Gamma, \tag{26}$$

for some $c_1, c_2 > 0$, $\alpha \in (0, 1)$ and $\beta > 1$, where $\rho : \Gamma \to \mathbb{R}$ is a continuous function on $\Gamma\backslash\{0\}$ such that

$$\limsup_{x \to 0} |\rho(x)| \sqrt{b(x)} = 0. \tag{27}$$

Proof. Let us introduce, for brevity in the notation, the function $\varphi(s) := c_1 s^\alpha + c_2 s^\beta$ and define the feedback law

$$v_i(x) := \begin{cases} -\dfrac{a(x) + \sqrt{a(x)^2 + b(x)^2 \tilde{\rho}(x)}}{b(x)} B_i(x) & \text{if } x \in \Gamma \backslash \{0\} \\ 0 & \text{if } x = 0 \end{cases}, \tag{28}$$

where $\tilde{\rho}(s) := 1 + \rho^2(x)$ and $1 \leq i \leq m$.

I The derivative of V along the trajectories of (20), (28) is given by

$$\dot{V}(x) = \left\langle \nabla V(x), f_0(x) + \sum_{i=1}^{m} f_i(x) v_i(x) \right\rangle = -\sqrt{a(x)^2 + b(x)^2 \tilde{\rho}(x)} < 0 \quad \forall x \in \Gamma \backslash \{0\}. \tag{29}$$

Thus the control (28) is a stabilizer for (20) and V is a strict Lyapunov function for the closed-loop system (21).

II Since $\tilde{\rho}(x) \geq 1$ for all $x \in \Gamma$, using the change of variables $\tilde{B}_i(x) = B_i(x) \sqrt[4]{\tilde{\rho}(x)}$, $\tilde{b}(x) = b(x)\sqrt{\tilde{\rho}(x)}$ in (28) we obtain

$$v_i(x) = \begin{cases} -\sqrt[4]{1 + \rho^2(x)} \dfrac{a(x) + \sqrt{a(x)^2 + \tilde{b}(x)^2}}{\tilde{b}(x)} \tilde{B}_i(x) & \text{if } x \in \Gamma \backslash \{0\} \\ 0 & \text{if } x = 0 \end{cases},$$

so that Sontag's universal formula is recovered multiplied by a gain $\sqrt[4]{1 + \rho^2(x)}$, which is continuous for all $x \in \Gamma\backslash\{0\}$. In addition, this observation also implies that $v(x) = 0$ whenever $b(x) = 0$. Therefore, for further analysis assume that $b(x) \neq 0$. Since by assumption V satisfies the SCP, $f_i : \Gamma \to \mathbb{R}^n$ for $0 \leq i \leq m$ are continuous and $\frac{dV}{dx}(0) = 0$, then for any $\epsilon > 0$ there exists $\tau > 0$ such that $\sqrt{b(x)} \leq \epsilon$ and

$$|a(x)| \leq \sqrt{b(x)}\epsilon \leq \epsilon^2$$

for all $0 < \|x\| \leq \tau$. Reducing τ and due to the restrictions imposed on the product $\rho(x)\sqrt{b(x)}$ we also have:

$$|\rho(x)|\sqrt{b(x)} \leq \epsilon$$

for all $0 < \|x\| \leq \tau$. Recall that

$$|v(x)| = \frac{|a(x) + \sqrt{a(x)^2 + b(x)^2 \tilde{\rho}(x)}|}{\sqrt{b(x)}},$$

then if $a(x) > 0$ we obtain:

$$\begin{aligned} |v(x)| &\leq \frac{2a(x) + b(x)\sqrt{\tilde{\rho}(x)}}{\sqrt{b(x)}} \\ &\leq 2\frac{a(x)}{\sqrt{b(x)}} + \sqrt{b(x)}(1 + |\rho(x)|) \leq 4\epsilon. \end{aligned}$$

For $a(x) \leq 0$,

$$a(x) + \sqrt{a(x)^2 + b(x)^2 \tilde{\rho}(x)} \leq b(x)\sqrt{\tilde{\rho}(x)}$$

and

$$|v(x)| \leq \sqrt{b(x)}\sqrt{\tilde{\rho}(x)} \leq \sqrt{b(x)}(1 + |\rho(x)|) \leq 2\epsilon.$$

Hence, due to an arbitrary selection of ϵ, (28) is a continuous feedback stabilizer for (20).

III Now let us show that the control (28) is a fixed-time stabilizer for (20). We have from (29) that

$$\begin{aligned} \dot{V}(x) &= -\left(\left(\frac{a(x)}{\varphi(V(x))}\right)^2 + (1 + \rho^2(x))\left(\frac{b(x)}{\varphi(V(x))}\right)^2\right)^{\frac{1}{2}} \varphi(V(x)) \\ &\leq -\left(\frac{a(x)^2 + \rho(x)^2 b(x)^2}{\varphi(V(x))^2}\right)^{\frac{1}{2}} \varphi(V(x)) \\ &\leq -\varphi(V(x)), \end{aligned}$$

which implies fixed-time stability for all $x \in \Gamma$ and from Corollary 2, (28) is moreover a continuous fixed-time stabilizer for (20). $\square$

Example 5. Consider the following system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + \lceil x_1 \rfloor^{\mu} u \\ \dot{x}_2 &= x_2 + \lceil x_2 \rfloor^{\mu} u \end{aligned} \tag{30}$$

where $\mu > 0$. For $u = 0$ the system (30) becomes a linear unstable system and our aim is to design a fixed-time stabilizer for it. Consider the $\mathcal{C}^1$ control Lyapunov function candidate $V(x) = \frac{1}{\eta+1}|x_1|^{\eta+1} + \frac{1}{\eta+1}|x_2|^{\eta+1}$, with $\eta > 0$. We have that

$$\begin{aligned} a(x) &= |x_1|^{\eta+1} + x_2\lceil x_1 \rfloor^{\eta} + |x_2|^{\eta+1}, \\ B(x) &= |x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}, \quad b(x) = (|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu})^2. \end{aligned}$$

Note that $b(x) = 0$ only at $x = 0$ and that $a(0) = 0$, such that $b(x) = 0 \Rightarrow a(x) = 0$. Selecting

$$\rho(x)^2 := \max\left\{0, \frac{\varphi(V(x))^2 - a(x)^2}{b(x)^2}\right\},$$

where $\varphi(s) := s^{\eta} + s^{\eta+1}$ yields

$$a(x)^2 + b(x)^2 \rho(x)^2 \leq (V(x)^{\eta} + V(x)^{\eta+1})^2,$$

such that (26) is satisfied. To verify that the SCP is satisfied we investigate the behavior of

$$\frac{a(x)}{|B(x)|} = \frac{|x_1|^{\eta+1} + x_2\lceil x_1 \rfloor^{\eta} + |x_2|^{\eta+1}}{|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}}.$$

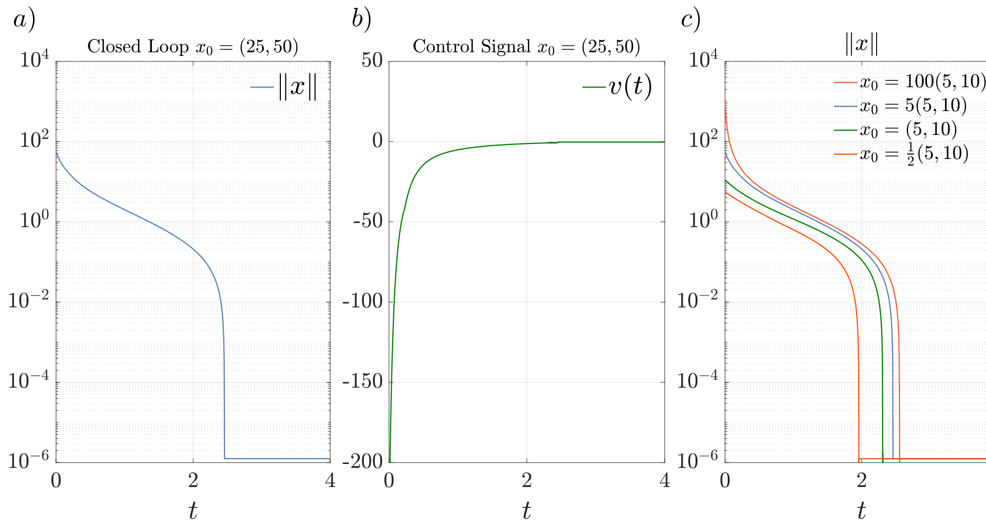

Figure 5: State norm $\|x\|$ in semi-logarithmic scale (a) and the control signal $v(t)$ (b) of the system (30) in closed loop with $x_0 = (25, 50)$, $\nu = \frac{1}{2}$ and $\mu = \frac{1}{5}$. The norm of the state in closed loop for four different initial conditions is depicted in c.

On the one hand

$$\limsup_{x\to 0} \frac{|x_1|^{\eta+1} + |x_2|^{\eta+1}}{|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}} = 0 \tag{31}$$

provided that $0 < \mu < 1$. On the other hand, for the case $x_1x_2 < 0$, $\limsup_{x\to 0} \frac{x_2\lceil x_1 \rfloor^{\eta}}{|x_1|^{\eta+\mu}+|x_2|^{\eta+\mu}} \leq 0$ is immediately satisfied. For $x_1x_2 > 0$ we apply Young's inequality to deduce that

$$\limsup_{x\to 0} \frac{x_2\lceil x_1 \rfloor^{\eta}}{|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}} \leq \limsup_{x\to 0} \frac{x_2^2}{|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}} + \frac{|x_1|^{2\eta}}{|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}} = 0, \tag{32}$$

provided that $0 < \eta + \mu < 2$ and that $\mu < \eta$. Hence, the SCP property holds. In order to check the condition (27), assume that the expression on the right hand side is greater that zero, then we obtain

$$|\rho(x)|\sqrt{b(x)} = \sqrt{\left(\frac{V(x)^{\eta} + V(x)^{\eta+1}}{|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}}\right)^2 - \left(\frac{a(x)}{|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu}}\right)^2}.$$

Similarly to the calculation above, it can be easily shown that $\limsup_{x\to 0} \left(\frac{a(x)}{B(x)}\right)^2 = 0$ under similar restrictions on μ and $\limsup_{x\to 0} \left(\frac{\varphi(V)}{B(x)}\right)^2 = 0$, provided that $\mu < \eta^2$. Finally, from the analysis above, it is straightforward to verify that $\inf_{u\in\mathbb{R}}(a(x) + B(x)u) < 0$ for all $x \in \mathbb{R}^2\backslash\{0\}$ and therefore $V(x)$ is a CLF.

Thus, all the conditions of Theorem 8 are satisfied and gathering all the estimates we conclude that for $\mu \in (0, 1)$, $\eta + \mu < 2$, $\mu < \eta$ and $\mu < \eta^2$ the control law (28) is a continuous fixed-time stabilizer for (30).

Figure 5.a shows the norm of the state of the closed loop system (30) with $x_0 = (25, 50)$, $\nu = \frac{1}{2}$ and $\mu = \frac{1}{5}$ in semi-logarithmic scale; Figure 5.b depicts the control signal $v(t)$. It can be seen that stability is achieved in a finite amount of time and that the control signal is indeed continuous. Figure 5.c shows the norm of the state of system (30) for four different initial conditions. The plot reveals the expected low sensitivity of the settling-time with respect to initial conditions.

6. Conclusions

Complete necessary and sufficient conditions for fixed-time stability of continuous autonomous systems have been presented. A characterization of this property using a simple Lyapunov dissipation inequality has been proposed and it allows, in the sufficiency case, to rule out the case of discontinuous settling-time function. It is worth noticing that in the sufficiency case, the characterizing function r is continuous and positive, and this assumption is enough to assert fixed-time stability with continuous T. In the necessary case, however, the characterizing function q is not only continuous and positive definite but also increasing and unbounded *i.e.* a class-$\mathcal{K}_\infty$ function and no assumptions on the regularity of T were made. A particular, more constructive and previously studied form of the characterizing function r has shown to be consistent with the framework here presented.

The concept of complete fixed-time stability has been introduced and a necessary and sufficient condition for this property has been obtained. Finally, a sufficient condition for fixed-time stabilization of continuous affine systems, analogous to previous results on asymptotic stabilization, has also been obtained. It is left as an open problems, to find a necessary condition for fixed-time stability with continuous settling-time function and to find a necessary condition for fixed-time stabilization of affine systems.

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