



HAL
open science

On global existence of strong and classical solutions of Navier-Stokes equations

Andrey Polyakov

► **To cite this version:**

Andrey Polyakov. On global existence of strong and classical solutions of Navier-Stokes equations. 2019. hal-02096215v3

HAL Id: hal-02096215

<https://inria.hal.science/hal-02096215v3>

Preprint submitted on 23 Apr 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On global existence of strong and classical solutions of Navier-Stokes equations

Andrey Polyakov

the date of receipt and acceptance should be inserted later

Abstract This short note studies the problem of a global expansion of local results on existence of strong and classical solutions of Navier-Stokes equations in \mathbb{R}^3 .

Keywords Navier-Stokes-Equations · Strong Solutions · Dilation Symetry

1 Introduction

In this short note we exploit a very simple idea of global expansion of regularity by means of dilation symmetry, which is well known for systems in \mathbb{R}^n .

Let a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be symmetric with respect to a dilation of the argument, i.e. $\exists \alpha \in \mathbb{R}$ such that

$$f(e^s u) = e^{(\alpha+1)s} f(u), \quad \forall u \in \mathbb{R}^n, \forall s \in \mathbb{R}$$

If $u(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$ is a classical solution of

$$\frac{du}{dt} = f(u), \quad t > 0$$

with the initial condition $u(0) = u_0$ then

$$u_s(t) := e^s u(e^{\alpha s} t)$$

is defined on $[0, +\infty)$ and, due to symmetry, we derive

$$\frac{du_s}{dt} = e^{(\alpha+1)s} f(u(e^{\alpha s} t)) = f(u_s(t)), \quad t > 0,$$

A. Polyakov
Inria Lille, Univ. Lille, CNRS, UMR 9189 - CRISTAL, (F-59000 Lille, France),
Tel.: +33-359577802
E-mail: andrey.polykov@inria.fr

i.e. u_s is a classical solution of the same differential equation with the initial condition $u_s(0) = e^s x_0$, where $s \in \mathbb{R}$.

Let $\exists \varepsilon > 0$ such that a classical solution of the differential equation exists on $[0, +\infty)$ for any initial value $u(0) = u_0 \in B_\varepsilon := \{u \in \mathbb{R}^n : |u| < \varepsilon\}$. To construct a solution for $u_0 \notin B_\varepsilon$ we first need to scale $u_0 \rightarrow e^{s_0} u_0$, where $s_0 \in \mathbb{R}$ is such that

$$|e^{s_0} u_0| < \varepsilon.$$

If $u(e^{s_0} u_0)$ is a solution with the initial condition $u(0) = \lambda_0 u_0$ then $\tilde{u}(t) = e^{-s_0} u(e^{-\alpha s_0}, e^{s_0} u_0)$ is also a solution of the considered system and, obviously,

$$\tilde{u}(0) = e^{s_0} u(0, e^{s_0} u_0) = e^{-s_0} e^{s_0} u_0 = u_0.$$

In this paper we use the dilation symmetry (see, [1, formula (1.5)]) of the Navier-Stokes equations in \mathbb{R}^3

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla) u - \nabla p, \\ \mathbf{0} &= \operatorname{div} u \end{aligned}$$

where u denotes the velocity of a fluid, p denotes the scalar pressure and $\nu > 0$ denotes viscosity of the fluid, in order to understand when a global-in-time existence of strong or classical solutions of the Navier-Stokes equations for *small initial data* implies the existence of global-in-time strong or classical solutions for *large initial data*. We refer the reader, for example, to [2] and [4], for more details about global-in-time existence of strong solutions for small initial data.

Mainly, the standard *notation* is utilized through the paper, e.g. \mathbb{R} is the field of real numbers; $L_{loc}^1((0, T) \times \mathbb{R}^n, \mathbb{R})$ denotes the space of locally integrable functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}$; $L^p(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p \leq +\infty$ is a Lebesgue space of function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ with the norm $\|\cdot\|_p$; $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ with compact support and $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions which vanish at infinity, where $0 < T \leq \infty$. For composition of operators A, B we also use the notation $A \circ B$.

Let $L_\mu^p(\mathbb{R}^n, \mathbb{R}^m)$ denotes the following normed vector space of functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$

$$L_\mu^p(\mathbb{R}^n, \mathbb{R}^m) := \{u : \|u\|_{p, \mu} < +\infty\}, \quad \mu \in \mathbb{R}$$

$$\|u\|_{p, \mu} := \left(\int_{\mathbb{R}^n} |x|^{\mu p} |u(x)|^p dx \right)^{1/p}, \quad 0 < p < \infty$$

$$\|u\|_{\infty, \mu} := \operatorname{ess\,sup}(|x|^\mu u(x)), \quad p = \infty,$$

which can be treated as a wighted L_p .

2 Preliminaries: Dilations in functional spaces

The following lemmas deal with the most common dilation groups in functional spaces.

Lemma 1 *The operator $\mathbf{d}(s)$ given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\gamma s} t, e^{\beta s} x), \quad (1)$$

where $s \in \mathbb{R}$, z is a function $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $0 < T \leq +\infty$, $x \in \mathbb{R}^n$ and $\alpha, \beta, \gamma \in \mathbb{R}$ are constant parameters,

- maps $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ onto $C_c^\infty((0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$;
- maps $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ onto $C_0^\infty([0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$.

The inverse operator is given by $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

Proof 1) Since the linear function $(t, x) \rightarrow (e^{\gamma s} t, e^{\beta s} x)$ maps a compact in $(0, T) \times \mathbb{R}^n$ to a compact in $(0, e^{-\gamma s} T) \times \mathbb{R}^n$ then $\mathbf{d}(s)$ defined on whole $C_c^\infty((0, T), \mathbb{R}^n)$ and if $z \in C_c^\infty((0, T), \mathbb{R}^n)$ (i.e. z is smooth and has a compact support in $(0, T) \times \mathbb{R}^n$) then $\mathbf{d}(s)z \in C_c^\infty((0, e^{-\gamma s} T), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth, but it has a compact support in $(0, e^{-\gamma s} T) \times \mathbb{R}^n$). Obviously, $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z = z$ for any $z \in C_c^\infty((0, T), \mathbb{R}^n)$ and any $s \in \mathbb{R}$.

Let us show that $\mathbf{d}(s)$ maps $C_c^\infty((0, T), \mathbb{R}^n)$ onto $C_c^\infty((0, e^{-\gamma s}), \mathbb{R}^n)$. Suppose the opposite: $\exists z^* \in C_c^\infty((0, e^{-\gamma s} T), \mathbb{R}^n)$ such that $z^* \neq \mathbf{d}(s)y$, $\forall y \in C_c^\infty((0, T), \mathbb{R}^n)$. This is impossible, since $\mathbf{d}(-s)z^* \in C_c^\infty((0, T), \mathbb{R}^n)$ and $z^* = (\mathbf{d}(s) \circ \mathbf{d}(-s))z^* \in \mathbf{d}(s)C_c^\infty((0, T), \mathbb{R}^n)$.

2) The proof for C_0^∞ is almost identical. Obviously, if $z \in C_0^\infty([0, T), \mathbb{R}^n)$ (i.e. z is smooth and vanishing at infinity) then $\mathbf{d}(s)z \in C_0^\infty([0, e^{-\gamma s} T), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth and vanishing at infinity) and $\mathbf{d}(s) \circ \mathbf{d}(-s)z = \mathbf{d}(-s) \circ \mathbf{d}(s)z = z$ for any $z \in C_0^\infty([0, T), \mathbb{R}^n)$.

Let us show that $\mathbf{d}(s)$ maps $C_0^\infty([0, T), \mathbb{R}^n)$ onto $C_0^\infty([0, e^{-\gamma s} T), \mathbb{R}^n)$. Suppose the opposite: there exists $z^* \in C_0^\infty([0, e^{-\gamma s} T), \mathbb{R}^n)$ such that $\mathbf{d}(s)y \neq z^*$, $\forall y \in C_0^\infty([0, T), \mathbb{R}^n)$. This is impossible since $\mathbf{d}(-s)z^* \in C_0^\infty([0, T), \mathbb{R}^n)$ and $z^* = \mathbf{d}(s) \circ \mathbf{d}(-s)z^* \in \mathbf{d}(s)C_0^\infty([0, T), \mathbb{R}^n)$.

Lemma 2 *The operator $\mathbf{d}(s)$ given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x), \quad (2)$$

where $s \in \mathbb{R}$, z is a function $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ are constant parameters, is

– a linear bounded invertible operator on $L^p(\mathbb{R}^n, \mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-n\beta/p)s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R},$$

– a linear bounded invertible operator on $L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-\beta(\mu+n/p))s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R},$$

where $0 < p \leq \infty$. The inverse operator is given by $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

Proof Notice that $L^p = L^p_0$.

Let $1 \leq p < \infty$. If $z \in L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n} |x|^{\mu p} |z(x)|^p dx < +\infty$$

and

$$\int_{\mathbb{R}^n} |x|^{\mu p} |z(x)|^p dx = e^{n\beta s} \int_{\mathbb{R}^n} |e^{\beta s} x|^{\mu p} |z(e^{\beta s} x)|^p dx =$$

$$e^{((n+\mu p)\beta - \alpha p)s} \int_{\mathbb{R}^n} |x|^{\mu p} (\mathbf{d}(s)z)(x)|^p dx < +\infty.$$

Since $e^{((n+\mu p)\beta - \alpha p)s} > 0$ for any $\alpha, \beta, p, s \in \mathbb{R}$ then $\mathbf{d}(s)z \in L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$ for any $s \in \mathbb{R}$. Obviously, $\mathbf{d}(s)$ is a linear operator on L^p_μ , i.e. $\mathbf{d}(s)(\mu_1 z_1 + \mu_2 z_2) = \mu_1 \mathbf{d}(s)z_1 + \mu_2 \mathbf{d}(s)z_2$, for any $\mu_1, \mu_2 \in \mathbb{R}$ and $z_1, z_2 \in L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$. Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_{p,\mu} = e^{(\alpha - (n/p + \mu)\beta)s} \|z\|_p, \quad \|z\|_{p,\mu} := \left(\int_{\mathbb{R}^n} |x|^{\mu p} |z(x)|^p dx \right)^{1/p}.$$

Hence, the operator $\mathbf{d}(s) : L^p(\mathbb{R}^n, \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^m)$ is bounded for any $s \in \mathbb{R}$.

Let $p = \infty$. If $z \in L^\infty_\mu(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\text{ess sup}|z(x)| = \text{ess sup}(|e^{\beta s} x|^\mu |z(e^{\beta s} x)|) < +\infty$$

for any $\beta, s, \mu \in \mathbb{R}$ and $\|\mathbf{d}(s)z\|_\infty = e^{(\alpha - \beta\mu)s} \|z\|_\infty$ for any $s \in \mathbb{R}$. Therefore, $\mathbf{d}(s)$ is also a linear bounded operator on $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Obviously, $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z$ for any $z : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and any $s \in \mathbb{R}$ and we derive $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

3 Global Existence of Strong Solutions of Navier-Stokes Equations

Below for shortness we omit \mathbb{R}^3 in the notations of $\int_{\mathbb{R}^3}$, L^p , C_0^∞ and C_c^∞ if the context is clear. Without loss of generality (see e.g. [4, page 4]) we also assume that $\nu = 1$, where ν is a viscosity coefficient.

Let us consider the weak form of the Navier-Stokes equations

$$\int_{\mathbb{R}^3} u(0) \cdot \xi(0) + \int_0^T \int_{\mathbb{R}^3} u \cdot (\partial_t \xi + \Delta \xi) + p \operatorname{div} \xi = \int_0^T \int_{\mathbb{R}^3} u \cdot (u \cdot \nabla) \xi, \quad \forall \xi \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3) \quad (3)$$

with $u(t) \in V$ for $t \in (0, T)$, where V is a set of the so-called weakly divergence free velocity fields:

$$V := \left\{ u \in L^2 : \int_{\mathbb{R}^3} u \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R}) \right\}.$$

The classical idea of analysis is to prove existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this analysis has been initiated by Jean Leray in 1938 (see [2]). We use the recent review [4] of his results.

Definition 1 ([4], Definition 3.7) A pair (u, p) is said to be a strong solution of the Navier-Stokes equations on $[0, T)$ if $u(t) \in V$ for $t \in (0, T)$, $p \in L_{loc}^1((0, T) \times \mathbb{R}^3, \mathbb{R})$,

$$u \in C([0, T), L^2) \cap C((0, T), L^\infty),$$

$\|u(t)\|_\infty$ is bounded as $t \rightarrow 0^+$ and (3) is satisfied.

A possible way for expansion of regularity to a larger set of initial conditions is to use the dilation symmetry as explained in the introduction. Several symmetries for Navier-Stokes equations are known (see e.g. [1] and references therein). Below we use just one of them (see [1, formula (1.5)]).

Lemma 3 If (u, p) is a strong solution of the Navier-Stokes equations with the initial data $u(0) = u_0 \in V \cap L^\infty$ defined on $[0, T)$ then for any $s \in \mathbb{R}$ the pair (u_s, p_s) given by

$$u_s(t, x) = e^s u(e^{2s}t, e^s x), \quad p_s(t, x) = e^{2s} p(e^{2s}t, e^s x), \quad t \geq 0, x \in \mathbb{R}^n, s \in \mathbb{R}.$$

is a strong solution of the Navier-Stokes equations defined on $[0, e^{-2s}T)$ with the initial value $u_s(0) = \mathbf{d}(s)u_0$.

Proof 1) Let $\mathbf{d}_1(s)$ be defined by the formula (2) with $\alpha = 1, \beta = 1$.

Since u is a strong solution then the function $t \rightarrow u(t)$ is continuous in L^2 and in L^∞ . According to Lemma 2, $\mathbf{d}_1(s)$ is a linear bounded invertible operator on L^2 and on L^∞ . Hence, for any fixed $s \in \mathbb{R}$ the function $t \rightarrow \mathbf{d}_1(s)u(t)$ is also continuous in L^2 and L^∞ , consequently, $u_s \in C([0, e^{-2s}T], L^2)$ and $u_s \in C((0, e^{-2s}T), L^\infty)$.

Notice $\|\mathbf{d}_1(s)z\|_\infty = e^s\|z\|_\infty$ for any $z \in L^\infty$ and any $s \in \mathbb{R}$ (see Lemma 2). Since $\|u(t)\|_\infty$ is bounded as $t \rightarrow 0^+$ then $\|u_s(t)\|_\infty = e^s\|u(e^st)\|_\infty$ is also bounded as $t \rightarrow 0^+$.

If $u_0 \in V$ then $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Indeed,

$$\int u_0 \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty$$

and using the change-of-variable theorem in the Lebesgue integral we derive

$$0 = \int_{\mathbb{R}^3} u_0(x) \cdot \nabla \phi(x) dx = e^{3s} \int_{\mathbb{R}^3} u_0(e^s x) \cdot (\nabla \phi)(e^s x) dx = e^{2s} \int_{\mathbb{R}^3} (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi},$$

where $\tilde{\phi} = \mathbf{d}_1(s)\phi$. Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ (see Lemma 1) then

$$0 = \int (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi}, \quad \forall \tilde{\phi} \in C_0^\infty,$$

i.e. $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Hence, the inclusion $u(t) \in V, t \in [0, T]$ implies $u_s(t) \in V, t \in [0, e^{-2s}T]$.

2) Let $\mathbf{d}_2(s)$ be given by the formula (1) with $\alpha = 2, \beta = 1$ and $\gamma = 2$. Since $p \in L_{loc}^1((0, T) \times \mathbb{R}^n, \mathbb{R})$ then

$$\int_0^T \int_{\mathbb{R}^3} |p(t, x)\xi(t, x)| dx dt < +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$$

hence using the change-of-variable theorem in the Lebesgue integral for the functions $t \rightarrow e^{2s}t$ and $x \rightarrow e^s x$ we derive

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} |p(t, x)\xi(t, x)| dx dt &= e^{5s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} |p(e^{2s}t, e^s x)\xi(e^{2s}t, e^s x)| dx dt = \\ &e^s \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} |p_s \cdot \mathbf{d}_2(s)\xi| < +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}). \end{aligned}$$

Since the operator $\mathbf{d}_2(s)$ maps $C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$ onto $C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$ then

$$\int_0^{e^{-2s}T} \int_{\mathbb{R}^3} |p_s \cdot \tilde{\xi}| < +\infty, \quad \forall \tilde{\xi} \in C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R}).$$

Therefore, $p_s \in L_{loc}^1((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$.

3) Let us show that (u_s, p_s) satisfies the equation (3). Since (u, p) is a solution defined on $[0, +\infty)$ then $\forall \xi \in C_0^\infty([0, T) \times \mathbb{R}^3, \mathbb{R}^3)$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} u(0, x) \cdot \xi(0, x) dx + \int_0^T \int_{\mathbb{R}^3} u(t, x) \cdot (\partial_t \xi(t, x) + (\Delta \xi)(t, x)) + p(t, x) (\operatorname{div} \xi)(t, x) dx dt = \\ \int_0^T \int_{\mathbb{R}^3} u(t, x) \cdot (u(t, x) \cdot \nabla) \xi(t, x) dx dt \end{aligned}$$

Using the change-of-variable theorem in the Lebesgue integral for the functions $t \rightarrow e^{2s}t$ and $x \rightarrow e^s x$ we derive

$$\begin{aligned} e^{3s} \int_{\mathbb{R}^3} u(0, e^s x) \cdot \xi(0, e^s x) dx + e^{5s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u(e^{2s}t, e^s x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + \\ (\Delta \xi)(e^{2s}t, e^s x)) + p(e^{2s}t, e^{2s}x) (\operatorname{div} \xi)(e^{2s}t, e^s x) dx dt = \\ e^{5s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u(e^{2s}t, e^s x) \cdot (u(e^{2s}t, e^s x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} e^{2s} \int_{\mathbb{R}^3} u_s(0, x) \cdot \xi(0, e^s x) dx + e^{2s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} e^{2s} u_s(t, x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + (\Delta \xi)(e^{2s}t, e^s x)) \\ + e^s p_s(t, x) (\operatorname{div} \xi)(e^{2s}t, e^s x) dx dt = \\ e^{3s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

Let us denote $\xi_s(t, x) = \xi(e^{2s}t, e^s x)$. Hence,

$$\begin{aligned} (\Delta \xi_s)(t, x) &= e^{2s} (\Delta \xi)(e^{2s}t, e^s x), \\ (\nabla \xi_s)(t, x) &= e^s (\nabla \xi)(e^{2s}t, e^s x), \\ (\partial_t \xi_s)(t, x) &= e^{2s} (\partial_t \xi)(e^{2s}t, e^s x), \\ (\operatorname{div} \xi_s)(t, x) &= e^s (\operatorname{div} \xi)(e^{2s}t, e^s x), \end{aligned}$$

and

$$e^{2s} \int_{\mathbb{R}^3} u_s(0) \cdot \xi_s(0) + e^{2s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s \cdot (\partial_t \xi_s + \Delta \xi_s + p_s (\operatorname{div} \xi_s)) = e^{2s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s \cdot (u_s \cdot \nabla) \xi_s,$$

where $\xi_s \in C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$. Since for any $s > 0$ the operator $\mathbf{d}_3(s)$ defined as

$$(\mathbf{d}_3(s)\xi)(t, x) = \xi(e^{2s}t, e^s x), \quad t > 0, x \in \mathbb{R}^3$$

maps $C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$ onto $C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$ (see Lemma 1), then

$$\int_{\mathbb{R}^3} u_s(0) \cdot \xi_s(0) + \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s(t, x) \cdot (\partial_t \xi_s + \Delta \xi_s + p_s(\operatorname{div} \xi_s)) = \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi_s.$$

holds for all $\xi_s \in C_0^\infty([0, e^{-\mu s}T] \times \mathbb{R}^3, \mathbb{R}^3)$. Therefore, (u_s, p_s) is a strong solution of the Navier-Stokes equations on $[0, e^{-2s}T)$ and $u_s(0) = \mathbf{d}(s)u_0$.

The proven lemma implies the following result, which describes the cases when global-in-time existence of strong solutions for small initial data is equivalent to global-in-time existence of strong solutions for large initial data.

Corollary 1 *Let $q_1, q_2 \in [1, \infty]$ and $r_1, r_2, \mu_1, \mu_2 \in \mathbb{R}$ and*

$$r_1(1 - \mu_1 - 3/q_1) + r_2(1 - \mu_2 - 3/q_2) \neq 0.$$

A strong solution of the Navier-Stokes equations with arbitrary the initial data $u(0) = u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2}$ exists on $[0, +\infty)$ if and only if there exist $\varepsilon > 0$ such that for any

$$u_0 \in \{u \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2} : \|u\|_{q_1, \mu_1}^{r_1} \|u\|_{q_2, \mu_2}^{r_2} < \varepsilon\}$$

a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$.

Proof Let \mathbf{d}_1 be defined as in the proof of Lemma 3. Assume that for any $u_0 \in \{u \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2} : \|u\|_{q_1, \mu_1}^{r_1} \|u\|_{q_2, \mu_2}^{r_2} < \varepsilon\}$ there exists a global in time strong solution with $u(0) = u_0$ and let us show that for any $u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2} : \|u_0\|_{q_1, \mu_1}^{r_1} \|u_0\|_{q_2, \mu_2}^{r_2} \geq \varepsilon$ then the Navier-Stokes equations also have a strong solution on $[0, +\infty)$.

In the proof of Lemma 3 we have shown that $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$, by Lemma 2 and $\mathbf{d}_1(s)u_0 \in L_{\mu_1}^{q_1}$ and for $\mathbf{d}_1(s)u_0 \in L_{\mu_1}^{q_2}$ any $s \in \mathbb{R}$. By Lemma 2 we also derive

$$\|\mathbf{d}_1(s)u_0\|_{q_1, \mu} = e^{s(1 - \mu_1 - 3/q_1)} \|u_0\|_{q_1, \mu_1},$$

and

$$\|\mathbf{d}_1(s)u_0\|_{q_2, \mu} = e^{s(1 - \mu_2 - 3/q_2)} \|u_0\|_{q_2, \mu_2}$$

and

$$\|\mathbf{d}_1(s)u_0\|_{q_1, \mu_1}^{r_1} \|\mathbf{d}_1(s)u_0\|_{q_2, \mu_2}^{r_2} = e^{s(r_1(1 - \mu_1 - \frac{3}{q_1}) + r_2(1 - \mu_2 - \frac{3}{q_2}))} \|u_0\|_{q_1, \mu_1}^{r_1} \|u_0\|_{q_2, \mu_2}^{r_2}.$$

Since by assumption $r_1(1 - \mu_1 - 3/q_1) + r_2(1 - \mu_2 - 3/q_2) \neq 0$ then for any $u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2}$ there exists $s_0 \in \mathbb{R}$ such that

$$\|\mathbf{d}_1(s)u_0\|_{q_1}^{r_1} \|\mathbf{d}_1(s)u_0\|_{q_2}^{r_2} < \varepsilon.$$

Hence, if a strong solution (u, p) with $u(0) = \mathbf{d}_1(s_0)u_0$ exists on $[0, +\infty)$ then by Lemma 3 the pair (\tilde{u}, \tilde{p}) given by

$$\tilde{u}(t, x) = e^{-s_0}u(e^{-2s_0}t, e^{-s_0}x), \tilde{p}(t, x) = e^{-2s_0}p(e^{-2s_0}t, e^{-s_0}x), t \in [0, +\infty), x \in \mathbb{R}^3$$

is also a strong solution of the Navier-Stokes equation. Since $u(0) = \mathbf{d}_1(s_0)u_0$ means that

$$u(0, x) = e^{s_0}u_0(e^{s_0}x), \quad x \in \mathbb{R}^3$$

then

$$\tilde{u}(0, x) = e^{-s_0}u(0, e^{-s_0}x) = u_0(x),$$

i.e. $\tilde{u}(0) = u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2}$ and the proof is complete.

Notice that taking $\mu_1 = \mu_2 = 0$ we derive the usual L^{q_1} and L^{q_2} spaces in the latter corollary.

Let us mention the following properties of the strong solutions (see Definition 1) proven before:

- *Global-in-time existence for small initial data* [4, Corollary 3.13 and Lemma 3.10]

There exist $\varepsilon > 0$ and $C > 0$ such that for any

$$u_0 \in \{u \in V \cap L^\infty \cap L^2 : \|u\|_2^2 \|u\|_\infty < \varepsilon\} \quad (4)$$

or

$$u_0 \in \{u \in V \cap L^\infty \cap L^2 : \|u\|_2 \|\nabla u\|_2 < \varepsilon\} \quad (5)$$

or

$$u_0 \in \{u \in V \cap L^\infty \cap L^2 : \|u\|_2^{2(q-3)} \|u\|_q^q < \varepsilon\}, q > 3 \quad (6)$$

a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$ and $\|u(t)\|_\infty \leq C\|u_0\|_\infty$.

- *Uniqueness of strong solutions* [4, Theorem 3.9]

For any $u_0 \in V \cap L^\infty$ a strong solution with $u(0) = u_0$ is unique.

- *Smoothness* [4, Corollary 3.3].

If (u, p) is a strong solution of the Navier-Stokes equation then

$$\partial_t^k \nabla^m u, \partial_t^k \nabla^m p \in C((0, T), L^2) \cap C((0, T), L^\infty), \quad \forall m, k \geq 0$$

and, in particular, $u, p \in C^\infty(\mathbb{R}^3 \times (0, T))$ constitute a classical solution of the Navier-Stokes equations on $(0, T) \times \mathbb{R}^3$.

None of conditions (4), (5), (6) satisfy Corollary 1.

To expand globally the regularity of the Navier-Stokes equation a global-in-time existence of strong solutions for small initial data has to be proven for the norms satisfying Corollary 1.

References

1. Fushchych, W., Popovych, R.: Symmetry reduction and exact solutions of the Navier-Stokes equations. *J. Nonlinear Math. Phys.* **1**, 75–113, 158–188 (1994)
2. Leray, J.: Sur le mouvement d'un liquide visqueux qui limite des parois. *Acta Math* **63**, 193–248 (1938)
3. Netuka, I.: The change-of-variables theorem for the Lebesgue integral. *ACTA UNIVERSITATIS MATTHIAE BELII*, series MATHEMATICS **19**, 37–42 (2011)
4. Ozanski, W., Pooley, B.: Leray's fundamental work on the Navier-Stokes equations: a modern review of "Sur le mouvement d'un liquide visqueux emplissant l'espace", pp. 11–203. *London Mathematical Society Lecture Note Series: Partial Differential Equations in Fluid Mechanics*, Series Number 452. Cambridge University Press (2018). URL (see also arXiv:1708.09787[math.AP])