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# On global existence of strong and classical solutions of Navier-Stokes equations

Andrey Polyakov

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**Abstract** This short note studies the problem of a global expansion of local results on existence of strong and classical solutions of Navier-Stokes equations in  $\mathbb{R}^3$ .

**Keywords** Navier-Stokes-Equations · Strong Solutions · Dilation Symetry

## 1 Introduction

In this short note we exploit a very simple idea of global expansion of regularity by means of dilation symmetry, which is well known for systems in  $\mathbb{R}^n$ .

Let a vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be symmetric with respect to a dilation of the argument, i.e.  $\exists \alpha \in \mathbb{R}$  such that

$$f(e^s u) = e^{(\alpha+1)s} f(u), \quad \forall u \in \mathbb{R}^n, \forall s \in \mathbb{R}$$

If  $u(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$  is a classical solution of

$$\frac{du}{dt} = f(u), \quad t > 0$$

with the initial condition  $u(0) = u_0$  then

$$u_s(t) := e^s u(e^{\alpha s} t)$$

is defined on  $[0, +\infty)$  and, due to symmetry, we derive

$$\frac{du_s}{dt} = e^{(\alpha+1)s} f(u(e^{\alpha s} t)) = f(u_s(t)), \quad t > 0,$$

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i.e.  $u_s$  is a classical solution of the same differential equation with the initial condition  $u_s(0) = e^s x_0$ , where  $s \in \mathbb{R}$ .

Let  $\exists \varepsilon > 0$  such that a classical solution of the differential equation exists on  $[0, +\infty)$  for any initial value  $u(0) = u_0 \in B_\varepsilon := \{u \in \mathbb{R}^n : |u| < \varepsilon\}$ . To construct a solution for  $u_0 \notin B_\varepsilon$  we first need to scale  $u_0 \rightarrow e^{s_0} u_0$ , where  $s_0 \in \mathbb{R}$  is such that

$$|e^{s_0} u_0| < \varepsilon.$$

If  $u(e^{s_0} u_0)$  is a solution with the initial condition  $u(0) = \lambda_0 u_0$  then  $\tilde{u}(t) = e^{-s_0} u(e^{-\alpha s_0}, e^{s_0} u_0)$  is also a solution of the considered system and, obviously,

$$\tilde{u}(0) = e^{s_0} u(0, e^{s_0} u_0) = e^{-s_0} e^{s_0} u_0 = u_0.$$

In this paper we use the dilation symmetry (see, [1, formula (1.5)]) of the Navier-Stokes equations in  $\mathbb{R}^3$

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla) u - \nabla p, \\ \mathbf{0} &= \operatorname{div} u \end{aligned}$$

where  $u$  denotes the velocity of a fluid,  $p$  denotes the scalar pressure and  $\nu > 0$  denotes viscosity of the fluid, in order to understand when a global-in-time existence of strong or classical solutions of the Navier-Stokes equations for *small initial data* implies the existence of global-in-time strong or classical solutions for *large initial data*. We refer the reader, for example, to [2] and [4], for more details about global-in-time existence of strong solutions for small initial data.

Mainly, the standard *notation* is utilized through the paper, e.g.  $\mathbb{R}$  is the field of real numbers;  $L_{loc}^1((0, T) \times \mathbb{R}^n, \mathbb{R})$  denotes the space of locally integrable functions  $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}$ ;  $L^p(\mathbb{R}^n, \mathbb{R}^m)$ ,  $1 \leq p \leq +\infty$  is a Lebesgue space of function  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  with the norm  $\|\cdot\|_p$ ;  $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$  is a space of smooth functions  $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  with compact support and  $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$  is a space of smooth functions which vanish at infinity, where  $0 < T \leq \infty$ . For composition of operators  $A, B$  we also use the notation  $A \circ B$ .

Let  $L_\mu^p(\mathbb{R}^n, \mathbb{R}^m)$  denotes the following normed vector space of functions  $\mathbb{R}^n \rightarrow \mathbb{R}^m$

$$L_\mu^p(\mathbb{R}^n, \mathbb{R}^m) := \{u : \|u\|_{p, \mu} < +\infty\}, \quad \mu \in \mathbb{R}$$

$$\|u\|_{p, \mu} := \left( \int_{\mathbb{R}^n} |x|^{\mu p} |u(x)|^p dx \right)^{1/p}, \quad 0 < p < \infty$$

$$\|u\|_{\infty, \mu} := \operatorname{ess\,sup}(|x|^\mu u(x)), \quad p = \infty,$$

which can be treated as a wighted  $L_p$ .

## 2 Preliminaries: Dilations in functional spaces

The following lemmas deal with the most common dilation groups in functional spaces.

**Lemma 1** *The operator  $\mathbf{d}(s)$  given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\gamma s} t, e^{\beta s} x), \quad (1)$$

where  $s \in \mathbb{R}$ ,  $z$  is a function  $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $0 < T \leq +\infty$ ,  $x \in \mathbb{R}^n$  and  $\alpha, \beta, \gamma \in \mathbb{R}$  are constant parameters,

- maps  $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$  onto  $C_c^\infty((0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$ ;
- maps  $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$  onto  $C_0^\infty([0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$ .

The inverse operator is given by  $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$ .

*Proof 1)* Since the linear function  $(t, x) \rightarrow (e^{\gamma s} t, e^{\beta s} x)$  maps a compact in  $(0, T) \times \mathbb{R}^n$  to a compact in  $(0, e^{-\gamma s} T) \times \mathbb{R}^n$  then  $\mathbf{d}(s)$  defined on whole  $C_c^\infty((0, T), \mathbb{R}^n)$  and if  $z \in C_c^\infty((0, T), \mathbb{R}^n)$  (i.e.  $z$  is smooth and has a compact support in  $(0, T) \times \mathbb{R}^n$ ) then  $\mathbf{d}(s)z \in C_c^\infty((0, e^{-\gamma s} T), \mathbb{R}^n)$  (i.e.  $\mathbf{d}(s)z$  is also smooth, but it has a compact support in  $(0, e^{-\gamma s} T) \times \mathbb{R}^n$ ). Obviously,  $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z = z$  for any  $z \in C_c^\infty((0, T), \mathbb{R}^n)$  and any  $s \in \mathbb{R}$ .

Let us show that  $\mathbf{d}(s)$  maps  $C_c^\infty((0, T), \mathbb{R}^n)$  onto  $C_c^\infty((0, e^{-\gamma s}), \mathbb{R}^n)$ . Suppose the opposite:  $\exists z^* \in C_c^\infty((0, e^{-\gamma s} T), \mathbb{R}^n)$  such that  $z^* \neq \mathbf{d}(s)y$ ,  $\forall y \in C_c^\infty((0, T), \mathbb{R}^n)$ . This is impossible, since  $\mathbf{d}(-s)z^* \in C_c^\infty((0, T), \mathbb{R}^n)$  and  $z^* = (\mathbf{d}(s) \circ \mathbf{d}(-s))z^* \in \mathbf{d}(s)C_c^\infty((0, T), \mathbb{R}^n)$ .

2) The proof for  $C_0^\infty$  is almost identical. Obviously, if  $z \in C_0^\infty([0, T), \mathbb{R}^n)$  (i.e.  $z$  is smooth and vanishing at infinity) then  $\mathbf{d}(s)z \in C_0^\infty([0, e^{-\gamma s} T), \mathbb{R}^n)$  (i.e.  $\mathbf{d}(s)z$  is also smooth and vanishing at infinity) and  $\mathbf{d}(s) \circ \mathbf{d}(-s)z = \mathbf{d}(-s) \circ \mathbf{d}(s)z = z$  for any  $z \in C_0^\infty([0, T), \mathbb{R}^n)$ .

Let us show that  $\mathbf{d}(s)$  maps  $C_0^\infty([0, T), \mathbb{R}^n)$  onto  $C_0^\infty([0, e^{-\gamma s} T), \mathbb{R}^n)$ . Suppose the opposite: there exists  $z^* \in C_0^\infty([0, e^{-\gamma s} T), \mathbb{R}^n)$  such that  $\mathbf{d}(s)y \neq z^*$ ,  $\forall y \in C_0^\infty([0, T), \mathbb{R}^n)$ . This is impossible since  $\mathbf{d}(-s)z^* \in C_0^\infty([0, T), \mathbb{R}^n)$  and  $z^* = \mathbf{d}(s) \circ \mathbf{d}(-s)z^* \in \mathbf{d}(s)C_0^\infty([0, T), \mathbb{R}^n)$ .

**Lemma 2** *The operator  $\mathbf{d}(s)$  given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x), \quad (2)$$

where  $s \in \mathbb{R}$ ,  $z$  is a function  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $x \in \mathbb{R}^n$  and  $\alpha, \beta \in \mathbb{R}$  are constant parameters, is

– a linear bounded invertible operator on  $L^p(\mathbb{R}^n, \mathbb{R}^m)$ ,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-n\beta/p)s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R},$$

– a linear bounded invertible operator on  $L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$ ,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-\beta(\mu+n/p))s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R},$$

where  $0 < p \leq \infty$ . The inverse operator is given by  $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$ .

*Proof* Notice that  $L^p = L^p_0$ .

Let  $1 \leq p < \infty$ . If  $z \in L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$  then

$$\int_{\mathbb{R}^n} |x|^{\mu p} |z(x)|^p dx < +\infty$$

and

$$\int_{\mathbb{R}^n} |x|^{\mu p} |z(x)|^p dx = e^{n\beta s} \int_{\mathbb{R}^n} |e^{\beta s} x|^{\mu p} |z(e^{\beta s} x)|^p dx =$$

$$e^{((n+\mu p)\beta - \alpha p)s} \int_{\mathbb{R}^n} |x|^{\mu p} (\mathbf{d}(s)z)(x)|^p dx < +\infty.$$

Since  $e^{((n+\mu p)\beta - \alpha p)s} > 0$  for any  $\alpha, \beta, p, s \in \mathbb{R}$  then  $\mathbf{d}(s)z \in L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$  for any  $s \in \mathbb{R}$ . Obviously,  $\mathbf{d}(s)$  is a linear operator on  $L^p_\mu$ , i.e.  $\mathbf{d}(s)(\mu_1 z_1 + \mu_2 z_2) = \mu_1 \mathbf{d}(s)z_1 + \mu_2 \mathbf{d}(s)z_2$ , for any  $\mu_1, \mu_2 \in \mathbb{R}$  and  $z_1, z_2 \in L^p_\mu(\mathbb{R}^n, \mathbb{R}^m)$ . Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_{p,\mu} = e^{(\alpha - (n/p + \mu)\beta)s} \|z\|_p, \quad \|z\|_{p,\mu} := \left( \int_{\mathbb{R}^n} |x|^{\mu p} |z(x)|^p dx \right)^{1/p}.$$

Hence, the operator  $\mathbf{d}(s) : L^p(\mathbb{R}^n, \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^m)$  is bounded for any  $s \in \mathbb{R}$ .

Let  $p = \infty$ . If  $z \in L^\infty_\mu(\mathbb{R}^n, \mathbb{R}^m)$  then

$$\text{ess sup}|z(x)| = \text{ess sup}(|e^{\beta s} x|^\mu |z(e^{\beta s} x)|) < +\infty$$

for any  $\beta, s, \mu \in \mathbb{R}$  and  $\|\mathbf{d}(s)z\|_\infty = e^{(\alpha - \beta\mu)s} \|z\|_\infty$  for any  $s \in \mathbb{R}$ . Therefore,  $\mathbf{d}(s)$  is also a linear bounded operator on  $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ .

Obviously,  $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z$  for any  $z : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and any  $s \in \mathbb{R}$  and we derive  $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$ .

### 3 Global Existence of Strong Solutions of Navier-Stokes Equations

Below for shortness we omit  $\mathbb{R}^3$  in the notations of  $\int_{\mathbb{R}^3}$ ,  $L^p$ ,  $C_0^\infty$  and  $C_c^\infty$  if the context is clear. Without loss of generality (see e.g. [4, page 4]) we also assume that  $\nu = 1$ , where  $\nu$  is a viscosity coefficient.

Let us consider the weak form of the Navier-Stokes equations

$$\int_{\mathbb{R}^3} u(0) \cdot \xi(0) + \int_0^T \int_{\mathbb{R}^3} u \cdot (\partial_t \xi + \Delta \xi) + p \operatorname{div} \xi = \int_0^T \int_{\mathbb{R}^3} u \cdot (u \cdot \nabla) \xi, \quad \forall \xi \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3) \quad (3)$$

with  $u(t) \in V$  for  $t \in (0, T)$ , where  $V$  is a set of the so-called weakly divergence free velocity fields:

$$V := \left\{ u \in L^2 : \int_{\mathbb{R}^3} u \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^3, \mathbb{R}) \right\}.$$

The classical idea of analysis is to prove existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this analysis has been initiated by Jean Leray in 1938 (see [2]). We use the recent review [4] of his results.

**Definition 1** ([4], **Definition 3.7**) A pair  $(u, p)$  is said to be a strong solution of the Navier-Stokes equations on  $[0, T)$  if  $u(t) \in V$  for  $t \in (0, T)$ ,  $p \in L_{loc}^1((0, T) \times \mathbb{R}^3, \mathbb{R})$ ,

$$u \in C([0, T), L^2) \cap C((0, T), L^\infty),$$

$\|u(t)\|_\infty$  is bounded as  $t \rightarrow 0^+$  and (3) is satisfied.

A possible way for expansion of regularity to a larger set of initial conditions is to use the dilation symmetry as explained in the introduction. Several symmetries for Navier-Stokes equations are known (see e.g. [1] and references therein). Below we use just one of them (see [1, formula (1.5)]).

**Lemma 3** If  $(u, p)$  is a strong solution of the Navier-Stokes equations with the initial data  $u(0) = u_0 \in V \cap L^\infty$  defined on  $[0, T)$  then for any  $s \in \mathbb{R}$  the pair  $(u_s, p_s)$  given by

$$u_s(t, x) = e^s u(e^{2s}t, e^s x), \quad p_s(t, x) = e^{2s} p(e^{2s}t, e^s x), \quad t \geq 0, x \in \mathbb{R}^n, s \in \mathbb{R}.$$

is a strong solution of the Navier-Stokes equations defined on  $[0, e^{-2s}T)$  with the initial value  $u_s(0) = \mathbf{d}(s)u_0$ .

*Proof* 1) Let  $\mathbf{d}_1(s)$  be defined by the formula (2) with  $\alpha = 1, \beta = 1$ .

Since  $u$  is a strong solution then the function  $t \rightarrow u(t)$  is continuous in  $L^2$  and in  $L^\infty$ . According to Lemma 2,  $\mathbf{d}_1(s)$  is a linear bounded invertible operator on  $L^2$  and on  $L^\infty$ . Hence, for any fixed  $s \in \mathbb{R}$  the function  $t \rightarrow \mathbf{d}_1(s)u(t)$  is also continuous in  $L^2$  and  $L^\infty$ , consequently,  $u_s \in C([0, e^{-2s}T], L^2)$  and  $u_s \in C((0, e^{-2s}T), L^\infty)$ .

Notice  $\|\mathbf{d}_1(s)z\|_\infty = e^s\|z\|_\infty$  for any  $z \in L^\infty$  and any  $s \in \mathbb{R}$  (see Lemma 2). Since  $\|u(t)\|_\infty$  is bounded as  $t \rightarrow 0^+$  then  $\|u_s(t)\|_\infty = e^s\|u(e^st)\|_\infty$  is also bounded as  $t \rightarrow 0^+$ .

If  $u_0 \in V$  then  $\mathbf{d}_1(s)u_0 \in V$  for any  $s \in \mathbb{R}$ . Indeed,

$$\int u_0 \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty$$

and using the change-of-variable theorem in the Lebesgue integral we derive

$$0 = \int_{\mathbb{R}^3} u_0(x) \cdot \nabla \phi(x) dx = e^{3s} \int_{\mathbb{R}^3} u_0(e^s x) \cdot (\nabla \phi)(e^s x) dx = e^{2s} \int_{\mathbb{R}^3} (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi},$$

where  $\tilde{\phi} = \mathbf{d}_1(s)\phi$ . Since  $\mathbf{d}_1(s)$  maps  $C_0^\infty$  onto  $C_0^\infty$  (see Lemma 1) then

$$0 = \int (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi}, \quad \forall \tilde{\phi} \in C_0^\infty,$$

i.e.  $\mathbf{d}_1(s)u_0 \in V$  for any  $s \in \mathbb{R}$ . Hence, the inclusion  $u(t) \in V, t \in [0, T]$  implies  $u_s(t) \in V, t \in [0, e^{-2s}T]$ .

2) Let  $\mathbf{d}_2(s)$  be given by the formula (1) with  $\alpha = 2, \beta = 1$  and  $\gamma = 2$ . Since  $p \in L_{loc}^1((0, T) \times \mathbb{R}^n, \mathbb{R})$  then

$$\int_0^T \int_{\mathbb{R}^3} |p(t, x)\xi(t, x)| dx dt < +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$$

hence using the change-of-variable theorem in the Lebesgue integral for the functions  $t \rightarrow e^{2s}t$  and  $x \rightarrow e^s x$  we derive

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} |p(t, x)\xi(t, x)| dx dt &= e^{5s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} |p(e^{2s}t, e^s x)\xi(e^{2s}t, e^s x)| dx dt = \\ &e^s \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} |p_s \cdot \mathbf{d}_2(s)\xi| < +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}). \end{aligned}$$

Since the operator  $\mathbf{d}_2(s)$  maps  $C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$  onto  $C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$  then

$$\int_0^{e^{-2s}T} \int_{\mathbb{R}^3} |p_s \cdot \tilde{\xi}| < +\infty, \quad \forall \tilde{\xi} \in C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R}).$$

Therefore,  $p_s \in L_{loc}^1((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$ .

3) Let us show that  $(u_s, p_s)$  satisfies the equation (3). Since  $(u, p)$  is a solution defined on  $[0, +\infty)$  then  $\forall \xi \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$  we have

$$\begin{aligned} \int_{\mathbb{R}^3} u(0, x) \cdot \xi(0, x) dx + \int_0^T \int_{\mathbb{R}^3} u(t, x) \cdot (\partial_t \xi(t, x) + (\Delta \xi)(t, x)) + p(t, x) (\operatorname{div} \xi)(t, x) dx dt = \\ \int_0^T \int_{\mathbb{R}^3} u(t, x) \cdot (u(t, x) \cdot \nabla) \xi(t, x) dx dt \end{aligned}$$

Using the change-of-variable theorem in the Lebesgue integral for the functions  $t \rightarrow e^{2s}t$  and  $x \rightarrow e^s x$  we derive

$$\begin{aligned} e^{3s} \int_{\mathbb{R}^3} u(0, e^s x) \cdot \xi(0, e^s x) dx + e^{5s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u(e^{2s}t, e^s x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + \\ (\Delta \xi)(e^{2s}t, e^s x)) + p(e^{2s}t, e^{2s}x) (\operatorname{div} \xi)(e^{2s}t, e^s x) dx dt = \\ e^{5s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u(e^{2s}t, e^s x) \cdot (u(e^{2s}t, e^s x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} e^{2s} \int_{\mathbb{R}^3} u_s(0, x) \cdot \xi(0, e^s x) dx + e^{2s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} e^{2s} u_s(t, x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + (\Delta \xi)(e^{2s}t, e^s x)) \\ + e^s p_s(t, x) (\operatorname{div} \xi)(e^{2s}t, e^s x) dx dt = \\ e^{3s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

Let us denote  $\xi_s(t, x) = \xi(e^{2s}t, e^s x)$ . Hence,

$$\begin{aligned} (\Delta \xi_s)(t, x) &= e^{2s} (\Delta \xi)(e^{2s}t, e^s x), \\ (\nabla \xi_s)(t, x) &= e^s (\nabla \xi)(e^{2s}t, e^s x), \\ (\partial_t \xi_s)(t, x) &= e^{2s} (\partial_t \xi)(e^{2s}t, e^s x), \\ (\operatorname{div} \xi_s)(t, x) &= e^s (\operatorname{div} \xi)(e^{2s}t, e^s x), \end{aligned}$$

and

$$e^{2s} \int_{\mathbb{R}^3} u_s(0) \cdot \xi_s(0) + e^{2s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s \cdot (\partial_t \xi_s + \Delta \xi_s + p_s (\operatorname{div} \xi_s)) = e^{2s} \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s \cdot (u_s \cdot \nabla) \xi_s,$$



where  $\xi_s \in C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$ . Since for any  $s > 0$  the operator  $\mathbf{d}_3(s)$  defined as

$$(\mathbf{d}_3(s)\xi)(t, x) = \xi(e^{2s}t, e^s x), \quad t > 0, x \in \mathbb{R}^3$$

maps  $C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$  onto  $C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$  (see Lemma 1), then

$$\int_{\mathbb{R}^3} u_s(0) \cdot \xi_s(0) + \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s(t, x) \cdot (\partial_t \xi_s + \Delta \xi_s + p_s(\operatorname{div} \xi_s)) = \int_0^{e^{-2s}T} \int_{\mathbb{R}^3} u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi_s.$$

holds for all  $\xi_s \in C_0^\infty([0, e^{-\mu s}T] \times \mathbb{R}^3, \mathbb{R}^3)$ . Therefore,  $(u_s, p_s)$  is a strong solution of the Navier-Stokes equations on  $[0, e^{-2s}T)$  and  $u_s(0) = \mathbf{d}(s)u_0$ .

The proven lemma implies the following result, which describes the cases when global-in-time existence of strong solutions for small initial data is equivalent to global-in-time existence of strong solutions for large initial data.

**Corollary 1** *Let  $q_1, q_2 \in [1, \infty]$  and  $r_1, r_2, \mu_1, \mu_2 \in \mathbb{R}$  and*

$$r_1(1 - \mu_1 - 3/q_1) + r_2(1 - \mu_2 - 3/q_2) \neq 0.$$

*A strong solution of the Navier-Stokes equations with arbitrary the initial data  $u(0) = u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2}$  exists on  $[0, +\infty)$  if and only if there exist  $\varepsilon > 0$  such that for any*

$$u_0 \in \{u \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2} : \|u\|_{q_1, \mu_1}^{r_1} \|u\|_{q_2, \mu_2}^{r_2} < \varepsilon\}$$

*a strong solution  $(u, p)$  with the initial data  $u(0) = u_0$  exists on  $[0, +\infty)$ .*

*Proof* Let  $\mathbf{d}_1$  be defined as in the proof of Lemma 3. Assume that for any  $u_0 \in \{u \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2} : \|u\|_{q_1, \mu_1}^{r_1} \|u\|_{q_2, \mu_2}^{r_2} < \varepsilon\}$  there exists a global in time strong solution with  $u(0) = u_0$  and let us show that for any  $u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2} : \|u_0\|_{q_1, \mu_1}^{r_1} \|u_0\|_{q_2, \mu_2}^{r_2} \geq \varepsilon$  then the Navier-Stokes equations also have a strong solution on  $[0, +\infty)$ .

In the proof of Lemma 3 we have shown that  $\mathbf{d}_1(s)u_0 \in V$  for any  $s \in \mathbb{R}$ , by Lemma 2 and  $\mathbf{d}_1(s)u_0 \in L_{\mu_1}^{q_1}$  and for  $\mathbf{d}_1(s)u_0 \in L_{\mu_1}^{q_2}$  any  $s \in \mathbb{R}$ . By Lemma 2 we also derive

$$\|\mathbf{d}_1(s)u_0\|_{q_1, \mu} = e^{s(1 - \mu_1 - 3/q_1)} \|u_0\|_{q_1, \mu_1},$$

and

$$\|\mathbf{d}_1(s)u_0\|_{q_2, \mu} = e^{s(1 - \mu_2 - 3/q_2)} \|u_0\|_{q_2, \mu_2}$$

and

$$\|\mathbf{d}_1(s)u_0\|_{q_1, \mu_1}^{r_1} \|\mathbf{d}_1(s)u_0\|_{q_2, \mu_2}^{r_2} = e^{s(r_1(1 - \mu_1 - \frac{3}{q_1}) + r_2(1 - \mu_2 - \frac{3}{q_2}))} \|u_0\|_{q_1, \mu_1}^{r_1} \|u_0\|_{q_2, \mu_2}^{r_2}.$$

Since by assumption  $r_1(1 - \mu_1 - 3/q_1) + r_2(1 - \mu_2 - 3/q_2) \neq 0$  then for any  $u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2}$  there exists  $s_0 \in \mathbb{R}$  such that

$$\|\mathbf{d}_1(s)u_0\|_{q_1}^{r_1} \|\mathbf{d}_1(s)u_0\|_{q_2}^{r_2} < \varepsilon.$$

Hence, if a strong solution  $(u, p)$  with  $u(0) = \mathbf{d}_1(s_0)u_0$  exists on  $[0, +\infty)$  then by Lemma 3 the pair  $(\tilde{u}, \tilde{p})$  given by

$$\tilde{u}(t, x) = e^{-s_0}u(e^{-2s_0}t, e^{-s_0}x), \tilde{p}(t, x) = e^{-2s_0}p(e^{-2s_0}t, e^{-s_0}x), t \in [0, +\infty), x \in \mathbb{R}^3$$

is also a strong solution of the Navier-Stokes equation. Since  $u(0) = \mathbf{d}_1(s_0)u_0$  means that

$$u(0, x) = e^{s_0}u_0(e^{s_0}x), \quad x \in \mathbb{R}^3$$

then

$$\tilde{u}(0, x) = e^{-s_0}u(0, e^{-s_0}x) = u_0(x),$$

i.e.  $\tilde{u}(0) = u_0 \in V \cap L_{\mu_1}^{q_1} \cap L_{\mu_2}^{q_2}$  and the proof is complete.

Notice that taking  $\mu_1 = \mu_2 = 0$  we derive the usual  $L^{q_1}$  and  $L^{q_2}$  spaces in the latter corollary.

Let us mention the following properties of the strong solutions (see Definition 1) proven before:

- *Global-in-time existence for small initial data* [4, Corollary 3.13 and Lemma 3.10]

There exist  $\varepsilon > 0$  and  $C > 0$  such that for any

$$u_0 \in \{u \in V \cap L^\infty \cap L^2 : \|u\|_2^2 \|u\|_\infty < \varepsilon\} \quad (4)$$

or

$$u_0 \in \{u \in V \cap L^\infty \cap L^2 : \|u\|_2 \|\nabla u\|_2 < \varepsilon\} \quad (5)$$

or

$$u_0 \in \{u \in V \cap L^\infty \cap L^2 : \|u\|_2^{2(q-3)} \|u\|_q^q < \varepsilon\}, q > 3 \quad (6)$$

a strong solution  $(u, p)$  with the initial data  $u(0) = u_0$  exists on  $[0, +\infty)$  and  $\|u(t)\|_\infty \leq C\|u_0\|_\infty$ .

- *Uniqueness of strong solutions* [4, Theorem 3.9]

For any  $u_0 \in V \cap L^\infty$  a strong solution with  $u(0) = u_0$  is unique.

- *Smoothness* [4, Corollary 3.3].

If  $(u, p)$  is a strong solution of the Navier-Stokes equation then

$$\partial_t^k \nabla^m u, \partial_t^k \nabla^m p \in C((0, T), L^2) \cap C((0, T), L^\infty), \quad \forall m, k \geq 0$$

and, in particular,  $u, p \in C^\infty(\mathbb{R}^3 \times (0, T))$  constitute a classical solution of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$ .

None of conditions (4), (5), (6) satisfy Corollary 1.

To expand globally the regularity of the Navier-Stokes equation a global-in-time existence of strong solutions for small initial data has to be proven for the norms satisfying Corollary 1.

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