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► **To cite this version:**

Frédéric Havet, Dorian Mazauric, Viet-Ha Nguyen, Rémi Watrigant. Overlaying a hypergraph with a graph with bounded maximum degree. *Discrete Applied Mathematics*, In press, 319, pp.394-406. 10.1007/978-3-030-39219-2_32 . hal-02025469v3

HAL Id: hal-02025469

<https://inria.hal.science/hal-02025469v3>

Submitted on 26 Jul 2023

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1 Overlying a hypergraph with a graph with
2 bounded maximum degree

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7 **Abstract**

8 Let G and H be respectively a graph and a hypergraph defined on a same set
9 of vertices, and let F be a fixed graph. We say that G F -overlays a hyper-
10 edge S of H if F is a spanning subgraph of the subgraph of G induced by S ,
11 and that G F -overlays H if it F -overlays every hyperedge of H . Motivated
12 by structural biology, we study the computational complexity of two prob-
13 lems. The first problem, $(\Delta \leq k)$ F -OVERLAY, consists in deciding whether
14 there is a graph with maximum degree at most k that F -overlays a given
15 hypergraph H . It is a particular case of the second problem MAX $(\Delta \leq k)$
16 F -OVERLAY, which takes a hypergraph H and an integer s as input, and
17 consists in deciding whether there is a graph with maximum degree at most
18 k that F -overlays at least s hyperedges of H .

We give a complete polynomial/ \mathcal{NP} -complete dichotomy for the MAX $(\Delta \leq k)$ - F -OVERLAY problems depending on the pairs (F, k) , and establish the complexity of $(\Delta \leq k)$ F -OVERLAY for many pairs (F, k) .

19 *Keywords:* graphs, hypergraphs, complexity, algorithm, computational
20 structural biology

21 **1. Introduction**

22 A major problem in structural biology is the characterization of low res-
23 olution structures of macro-molecular assemblies [? ?]. To attack this very
24 difficult question, one has to determine the plausible contacts between the
25 subunits (e.g. proteins) of an assembly, given the lists of subunits involved
26 in some complexes. We assume that the composition, in terms of individual
27 subunits, of selected complexes is known. Indeed, a given assembly can be

28 chemically split into complexes by manipulating chemical conditions. This
 29 problem can be conveniently modelled by graphs and hypergraphs. We con-
 30 sider the hypergraph H whose vertices represent the subunits and whose
 31 hyperedges are the complexes. We are then looking for a graph G with the
 32 same vertex set as H whose edges represent the contacts between subunits,
 33 and satisfying (i) some local properties for every complex (*i.e.* hyperedge),
 34 and (ii) some other global properties.

35 For example, consider the following properties: (i) the subset of vertices
 36 of every hyperedge must form a connected subgraph of G and (ii) the number
 37 of edges of G is at most the number of vertices. The first property comes from
 38 the fact that the complexes obtained (e.g. by native mass spectrometry) are
 39 connected components of a given assembly. The second property is deduced
 40 from the constant number of subunits that are in contact to any subunit of an
 41 assembly. It means that the total number of edges is linear in the number of
 42 vertices (in our example, we assume that the number of edges is at most the
 43 number of vertices). Let us now consider a hypergraph H whose hyperedges
 44 are $\{v_1, v_2, v_3\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_5\}$, and $\{v_4, v_5, v_6, v_7\}$. These hyperedges
 45 are depicted in Figure 1 (left) in which a vertex labeled i represents v_i . The
 46 graph G described in Figure 1 (right) satisfies the previous properties: the
 47 set of vertices $\{v_4, v_5, v_6, v_7\}$ induces a connected subgraph (same observation
 48 for the three other hyperedges) and the number of edges of G is exactly the
 49 number of vertices.

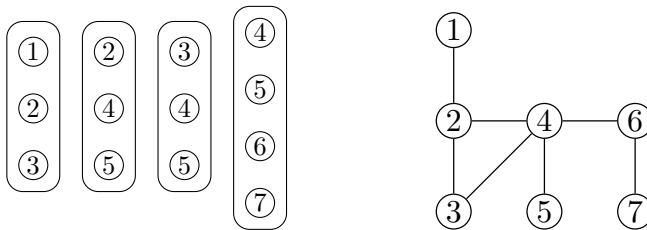


Figure 1: A hypergraph composed of seven vertices and four hyperedges (right). A graph G satisfying local properties (every hyperedge induces a connected subgraph in G) and a global property (the total number of edges is at most the number of vertices).

50 We first focus on the local properties. They are usually modelled by a
 51 (possibly infinite) family \mathcal{F} of admissible graphs to which each complex must
 52 belong: to this end, we define the notion of *enforcement* of a hyperedge and
 53 a hypergraph. A graph G \mathcal{F} -enforces a hyperedge $S \in E(H)$ if the subgraph
 54 $G[S]$ of G induced by S belongs to \mathcal{F} , and G \mathcal{F} -enforces H if G \mathcal{F} -enforces

55 all hyperedges of H . Very often, the considered family \mathcal{F} is closed on taking
 56 edge supergraphs [? ?]: if $F \in \mathcal{F}$, then every graph obtained from G by
 57 adding edges is also in \mathcal{F} . Such a family is completely defined by its set
 58 $\mathcal{M} = \mathcal{M}(\mathcal{F})$ of minimal graphs that are the elements of \mathcal{F} which are not
 59 edge supergraphs of any other. In this case, a graph G \mathcal{F} -enforcing S is such
 60 that there is an element of \mathcal{M} which is a spanning subgraph of $G[S]$. This
 61 leads to the following notion of *overlayment* when considering minimal graph
 62 families.

63 **Definition 1.** A graph G \mathcal{F} -overlays a hyperedge S if there exists $F \in \mathcal{F}$
 64 such that F is a spanning subgraph of $G[S]$, and G \mathcal{F} -overlays H if G \mathcal{F} -
 65 overlays every hyperedge of H .

66 As said previously, the graph sought will also have to satisfy some global
 67 constraints. Since in a macro-molecular assembly the number of contacts is
 68 small, the first natural idea is to look for a graph G with the minimum number
 69 of edges. This leads to the MIN- \mathcal{F} -OVERLAY problem: given a hypergraph
 70 H and an integer m , decide if there exists a graph G \mathcal{F} -overlying H such
 71 that $|E(G)| \leq m$.

72 A typical example of a family \mathcal{F} is the set of all connected graphs, in
 73 which case $\mathcal{M}(\mathcal{F})$ is the set of all trees. Agarwal et al. [?] focused on MIN-
 74 $\mathcal{M}(\mathcal{F})$ -OVERLAY for this particular family in the aforementioned context of
 75 structural biology. However, this problem was previously studied by several
 76 communities in other domains, as pointed out by Chen *et al.*[?]. Indeed,
 77 it is also known as SUBSET INTERCONNECTION DESIGN, MINIMUM TOPIC-
 78 CONNECTED OVERLAY or INTERCONNECTION GRAPH PROBLEM, and was
 79 considered (among others) in the design of vacuum systems [? ?], scalable
 80 overlay networks [? ?], and reconfigurable interconnection networks [? ?].
 81 Some variants have also been considered in the contexts of inferring a most
 82 likely social network [?], determining winners of combinatorial auctions [?
 83], as well as drawing hypergraphs [? ?].

84 Cohen et al. [?] presented a dichotomy regarding the polynomial vs.
 85 \mathcal{NP} -hard status of the problem MIN- \mathcal{F} -OVERLAY with respect to the con-
 86 sidered family \mathcal{F} . Roughly speaking, they showed that the easy cases one
 87 can think of (e.g. when edgeless graphs of the right sizes are in \mathcal{F} , or if \mathcal{F}
 88 contains only cliques) are the only families giving rise to a polynomial-time
 89 solvable problem: all others are \mathcal{NP} -complete. They also considered the
 90 FPT/W[1]-hard dichotomy for several families \mathcal{F} .

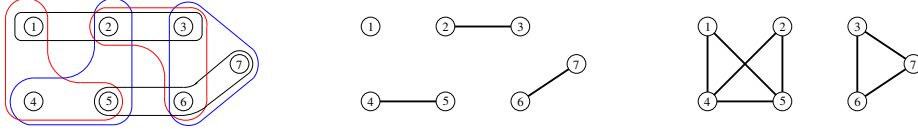


Figure 2: Example of $(\Delta \leq k)$ - \mathcal{F} -OVERLAY and MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY. In the figure, an instance H (left), a graph G with $\Delta(G) \leq 1$ that O_3 -overlays H (with O_3 being the graph with three vertices and one edge) (center), and a solution to MAX $(\Delta \leq 3)$ - C_3 -OVERLAY (with C_3 being the cycle on three vertices) (right).

91 In this paper, we consider the variant in which the additional constraint is
 92 that G must have a bounded maximum degree: this constraint is motivated
 93 by the context of structural biology, since a subunit (e.g. a protein) cannot
 94 be connected to many other subunits. This yields the following problem for
 95 any family \mathcal{F} of graphs and an integer k .

($\Delta \leq k$)- \mathcal{F} -OVERLAY
 96 Input: A hypergraph H .
 Question: Does there exist a graph G \mathcal{F} -overlying H such that $\Delta(G) \leq k$?

97 We denote by $over_{\mathcal{F}}(H, G)$ the number of hyperedges of H that are \mathcal{F} -
 98 overlaid by G . A natural generalization is to find $over_{\mathcal{F}}(H, k)$, the maximum
 99 number of hyperedges \mathcal{F} -overlaid by a graph with maximum degree at most
 100 k .
 101 k .

MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY
 Input: A hypergraph H and a positive integer s .
 102 Question: Does there exist a graph G such that $\Delta(G) \leq k$ and
 103 $over_{\mathcal{F}}(H, G) \geq s$?

104 Observe that there is an obvious reduction from $(\Delta \leq k)$ - \mathcal{F} -OVERLAY to
 105 MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY (by setting $s = |E(H)|$).

106 In this paper, we mainly consider the case when the family \mathcal{F} contains a
 107 unique graph F . We abbreviate $(\Delta \leq k)$ - $\{F\}$ -OVERLAY and MAX $(\Delta \leq k)$ -
 108 $\{F\}$ -OVERLAY to $(\Delta \leq k)$ - F -OVERLAY and MAX $(\Delta \leq k)$ - F -OVERLAY,
 109 respectively. By definition those two problems really make sense only for
 110 $|F|$ -uniform hypergraphs *i.e.* hypergraphs whose hyperedges are of size $|F|$.
 111 Therefore, we always assume the hypergraph to be $|F|$ -uniform.

112 If F is a graph with maximum degree greater than k , then solving $(\Delta \leq k)$ -
 113 F -OVERLAY or MAX $(\Delta \leq k)$ - F -OVERLAY is trivial as the answer is always
 114 ‘No’. So we only study the problems when $\Delta(F) \leq k$.

115 If F is an anticomplete graph (i.e. with no edges), then $\text{MAX } (\Delta \leq k)$ -
 116 F - OVERLAY is also trivial, because for any hypergraph H , the anticomplete
 117 graph on $V(H)$ vertices F -overlays H . Hence the first natural interesting
 118 cases are the graphs with one edge. For every integer $p \geq 2$, we denote by O_p
 119 the graph with p vertices and one edge. In Section 2, we prove the following
 120 dichotomy theorem.

121 **Theorem 1.** *Let $k \geq 1$ and $p \geq 2$ be integers. If $p = 2$ or if $k = 1$ and*
 122 *$p = 3$, then $\text{MAX } (\Delta \leq k)$ - O_p - OVERLAY and $(\Delta \leq k)$ - O_p - OVERLAY are*
 123 *polynomial-time solvable. Otherwise, they are \mathcal{NP} -complete.*

124 Then, in Section 3, we give a complete polynomial/ \mathcal{NP} -complete di-
 125 chotomy for the $\text{MAX } (\Delta \leq k)$ - F - OVERLAY problems.

126 **Theorem 2.** *$\text{MAX } (\Delta \leq k)$ - F - OVERLAY is polynomial-time solvable if ei-*
 127 *ther $\Delta(F) > k$, or F is an anticomplete graph, or $F = O_2$, or $k = 1$ and*
 128 *$F = O_3$. Otherwise it is \mathcal{NP} -complete.*

129 In Section 4, we investigate the complexity of $(\Delta \leq k)$ - F - OVERLAY and
 130 $(\Delta \leq k)$ - \mathcal{F} - OVERLAY problems. We believe that each such problem is either
 131 polynomial-time solvable or \mathcal{NP} -complete. However the dichotomy seems
 132 to be more complicated than the one for $\text{MAX } (\Delta \leq k)$ - F - OVERLAY . We
 133 exhibit several pairs (F, k) such that $(\Delta \leq k)$ - F - OVERLAY is polynomial-
 134 time solvable, while $\text{MAX } (\Delta \leq k)$ - F - OVERLAY is \mathcal{NP} -complete. This is
 135 in particular the case when F is a complete graph (Proposition 3), F is
 136 connected k -regular (Proposition 4), F is a path and $k = 2$ (Theorem 7),
 137 and when F is the cycle on 4 vertices and $k \leq 3$ (Theorem 6).

138 Most notations of this paper are standard. We now recall some of them,
 139 and we refer the reader to [?] for any undefined terminology. The degree
 140 of a vertex v in a graph is denoted by $d(v)$. For a positive integer p , let
 141 $[p] = \{1, \dots, p\}$. In some proofs of the paper we might use the modulo
 142 operation to keep indices in bounds.

143 Given $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S , that is
 144 the subgraph with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$. We
 145 denote by \overline{K}_k the *anticomplete* graph on k vertices, that is the graph with k
 146 vertices and no edges. The *disjoint union* of two graphs F and G is denoted
 147 by $F + G$.

148 Let H be a hypergraph. Two hyperedges are *adjacent* if their intersection
 149 has size at least 2. A hypergraph is *neat* if any two distinct hyperedges

150 intersect in at most one vertex. In other words, a hypergraph is neat if
 151 there is no pair of adjacent hyperedges. We denote by $K(H)$, the graph
 152 obtained by replacing each hyperedge by a complete graph. In other words,
 153 $V(K(H)) = V(H)$ and $E(K(H)) = \{xy \mid \exists S \in E(H), \{x, y\} \subseteq S\}$. The
 154 *edge-weight function induced by H on $K(H)$* , denoted by w_H , is defined by
 155 $w_H(e) = |\{S \in E(H) \mid e \subseteq S\}|$. In words, $w_H(e)$ is the number of hyperedges
 156 of H containing e . A hypergraph H is *connected* if $K(H)$ is connected, and
 157 the *connected components* of a hypergraph H are the connected components
 158 of $K(H)$. Finally, a graph G F -overlying H with maximum degree at most
 159 k is called an (F, H, k) -*graph* (resp. (\mathcal{F}, H, k) -*graph*) for F is a graph (resp.
 160 graph family).

161 2. The graphs with one edge

162 In this section, we establish Theorem 1. Let $p \geq 2$, and H be a p -
 163 uniform hypergraph. Consider the edge-weighted graph $(K(H), w_H)$. For
 164 every matching M with at least one edge of this graph, let $G_M = (V(H), M)$.
 165 Every hyperedge O_p -overlaid by G_M is a superset of at least one edge of M
 166 and at most $\lfloor \frac{p}{2} \rfloor$ edges of M . We thus have the following:

167 **Observation 1.** *For every matching M of $K(H)$, we have:*

$$\frac{1}{\lfloor \frac{p}{2} \rfloor} w_H(M) \leq \text{over}_{O_p}(H, G_M) \leq w_H(M), \quad (1)$$

168 where $w_H(M) = \sum_{e \in M} w_H(e)$.

169 Consider first the case when $p = 2$. Let H be a 2-uniform hypergraph.
 170 Every hyperedge is an edge, so $K(H) = H$. Moreover, a (hyper)edge of H
 171 is O_2 -overlaid by G if and only if it is in $E(G)$. Hence MAX $(\Delta \leq k)$ - O_2 -
 172 OVERLAY is equivalent to finding a maximum k -matching (that is a subgraph
 173 with maximum degree at most k) in $K(H)$. This problem is polynomial-time
 174 solvable, see [? , Chap. 31], hence:

175 **Proposition 1.** MAX $(\Delta \leq k)$ - O_2 -OVERLAY is polynomial-time solvable for
 176 all positive integer k .

177 If $p = 3$, Inequalities (1) are equivalent to $\text{over}_{O_3}(H, G_M) = w_H(M)$.
 178 Since the edge set of a graph with maximum degree 1 is a matching, MAX
 179 $(\Delta \leq 1)$ - O_3 -OVERLAY is equivalent to finding a maximum-weight matching
 180 in the edge-weighted graph $(K(H), w_H)$. This can be done in polynomial-
 181 time, see [? , Chap. 14].

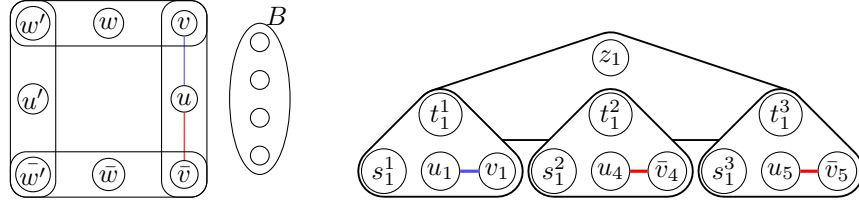


Figure 3: The construction of a variable gadget (left) which means that a blue (resp. red) edge in this gadget corresponds to that the variable is *true* (resp. *false*). And an example of clause gadget of $C_1 = (x_1 \vee \bar{x}_4 \vee \bar{x}_5)$ (right) with three edges $u_1v_1, u_4\bar{v}_4, u_5\bar{v}_5$ which corresponds to an assignment ϕ such that $\phi(x_1) = \phi(\bar{x}_4) = \phi(\bar{x}_5) = \text{true}$.

182 **Proposition 2.** $\text{MAX } (\Delta \leq 1)\text{-}O_3\text{-OVERLAY}$ is polynomial-time solvable.

183 We shall now prove that if $p \geq 4$, or $p = 3$ and $k \geq 2$, then $\text{MAX } (\Delta \leq k)\text{-}$
 184 $O_p\text{-OVERLAY}$ is \mathcal{NP} -complete. We prove it by a double induction on k and
 185 p . Theorems 3 and 4 first prove the base cases of the induction and Lemma 1
 186 corresponds to the inductive steps.

187 **Theorem 3.** $(\Delta \leq 1)\text{-}O_4\text{-OVERLAY}$ is \mathcal{NP} -complete.

188 *Proof.* We reduce 3-SAT to $(\Delta \leq 1)\text{-}O_4\text{-OVERLAY}$.

189 We shall need the following gadget hypergraph $J = J(A, B)$.
 190 $V(J) = A \cup B$ with $|A| = 3$ and $|B| = 4$ and $E(J) = \{A \cup \{b\} \mid b \in B\}$.

191 **Claim 1.** Every $(O_4, J, 1)$ -graph contains exactly one edge with both endver-
 192 tices in A .

193 *Proof of claim:* Let G be an $(O_4, J, 1)$ -graph. Each hyperedge of J contains
 194 an edge of G , which necessarily has an endvertex in A . Since $\Delta(G) \leq 1$, G
 195 has at most three edges with an endvertex in A . Since J has four hyperedges,
 196 there is an edge of G contained in two hyperedges of J . The two endvertices
 197 of this edge are in A . \triangleleft

198 Let Φ be an instance of 3-SAT with m clauses C_1, \dots, C_m on n variables
 199 x_1, \dots, x_n . Let $H = H(\Phi)$ be the 4-uniform hypergraph constructed as
 200 follows (see Figure 3).

- 201 • For each variable x_i , we create the variable gadget VG_i as follows.
 202 Its vertex set is $\{u_i, v_i, \bar{v}_i, w_i, \bar{w}_i, w'_i, \bar{w}'_i, u'_i\} \cup B_i$, with $|B_i| = 4$. We add
 203 the hyperedges of $J(\{u_i, v_i, \bar{v}_i\}, B_i)$, $J(\{v_i, w_i, w'_i\}, B_i)$, $J(\{\bar{v}_i, \bar{w}_i, \bar{w}'_i\}, B_i)$,
 204 and $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i)$.

- 205 • For each clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$, we create the clause hypergraph CG_j .
 206 Its vertex set is $\{z_j\} \cup \bigcup_{k=1}^3 \{s_j^k, t_j^k, u_j^k, v_j^k\}$. Set $Z_j = \{t_j^1, t_j^2, t_j^3, z_j\}$ and
 207 for $k \in \{1, 2, 3\}$, set $T_j^k = \{s_j^k, t_j^k, u_j^k, v_j^k\}$. The hyperedge set of CG_j
 208 is $\{Z_j, T_j^1, T_j^2, T_j^3\}$. Then for each $k \in \{1, 2, 3\}$, we identify u_j^k and v_j^k
 209 with u_i and v_i if $\ell_k = x_i$ and with u_i and \bar{v}_i if $\ell_k = \bar{x}_i$.

210 Let us prove that there is an $(O_4, H, 1)$ -graph if and only if Φ is satisfiable.
 211 The general idea of the proof is that the variable gadget enforces only one
 212 of edges $u_i v_i$, $u_i \bar{v}_i$ to belong to G and that $u_i v_i \in E(G)$ (resp. $u_i \bar{v}_i \in E(G)$)
 213 if and only if the corresponding variable x_i is *true* (resp. *false*), thus the
 214 appearance of such an edge in a clause gadget implies that there is a true
 215 literal in the clause.

216 Assume first that there is an $(O_4, H, 1)$ -graph G .

217 **Claim 2.** *For each $1 \leq i \leq n$, G contains either the edge $u_i v_i$ or the edge*
 218 *$u_i \bar{v}_i$. Moreover, for each $x \in \{u_i, v_i, \bar{v}_i\}$ there is some $x' \in VG_i$ such that x*
 219 *and x' are adjacent in G .*

220 *Proof of claim:* Since VG_i contains $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i)$, by Claim 1, G contains
 221 an edge e'' with both endvertices in $\{u'_i, w'_i, \bar{w}'_i\}$. Assume that e'' is incident
 222 to w'_i . Now VG_i contains $J(\{v_i, w_i, w'_i\}, B_i)$. So, by Claim 1, G contains
 223 an edge e' with both endvertices in $\{v_i, w_i, w'_i\}$. Since $\Delta(G) \leq 1$, e' is not
 224 incident to w'_i , so $e' = v_i w_i$. Now VG_i contains $J(\{u_i, v_i, \bar{v}_i\}, B_i)$. So, by
 225 Claim 1, G contains an edge e with both endvertices in $\{u_i, v_i, \bar{v}_i\}$. This edge
 226 is not incident to v_i , so $e = u_i \bar{v}_i$. Similarly, we get that G contains $\bar{v}_i \bar{w}_i$ and
 227 $u_i v_i$ if e'' is incident to \bar{w}'_i . \triangleleft

228 In view of this claim, one defines the truth assignment ϕ by $\phi(x_i) = \text{true}$
 229 if $u_i v_i \in E(G)$ and $\phi(x_i) = \text{false}$ if $u_i \bar{v}_i \in E(G)$. Let us prove that it satisfies
 230 Φ . Consider a clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$ and its corresponding clause gadget
 231 CG_j . If there is none of $u_j^k v_j^k$ in $E(G)$ (for all $k \in \{1, 2, 3\}$), that means
 232 both u_j^k and v_j^k are incident to an edge which is incident to a vertex in the
 233 variable gadget it they belongs to, T_j^k must be O_4 -overlaid by the edge $s_j^k t_j^k$.
 234 So, t_j^k (for all $k \in \{1, 2, 3\}$) has degree one in G . However, the hyperedge
 235 Z_j is O_4 -overlaid by $z_j t_j^k$ or $t_j^k t_j^{k'}$ for $k' \neq k$ and $k, k' \in \{1, 2, 3\}$, which
 236 implies that there is one of vertices t_j^k which is of degree one in $G[Z_j]$. It is
 237 a contradiction, thus there is an edge $u_j^k v_j^k \in E(G)$ (for some $k \in \{1, 2, 3\}$).
 238 Hence $\phi(\ell_k) = \text{true}$. Hence ϕ satisfies Φ .

239 Conversely, assume ϕ is a truth assignment satisfying Φ . Let G be the
 240 graph with vertex set $V(H)$ and whose edge set is constructed as follows.
 241 For every $1 \leq i \leq n$, add $\{u_i v_i, w_i w'_i, u'_i \bar{w}'_i, \bar{w}_i \bar{v}_i\}$ to $E(G)$ if $\phi(x_i) = \text{true}$
 242 and $\{u_i \bar{v}_i, \bar{w}_i \bar{w}'_i, u'_i w'_i, w_i v_i\}$ to $E(G)$ if $\phi(x_i) = \text{false}$. For every clause $C_j =$
 243 $\ell_1 \vee \ell_2 \vee \ell_3$, there exists $k \in \{1, 2, 3\}$ such that $\phi(\ell_k) = \text{true}$. Add the edges
 244 $z_j t_j^k, s_j^{k+1} t_j^{k+1}$ and $s_j^{k+2} t_j^{k+2}$ (the superscript are modulo 3) to $E(G)$. One
 245 easily checks that G is an $(O_4, H, 1)$ -graph. \square

246 **Theorem 4.** $(\Delta \leq 2)$ - O_3 -OVERLAY is \mathcal{NP} -complete.

247 *Proof.* In order to simplify the proof, we will prove that the problem $(\Delta \leq 2)$ -
 248 O_3 -OVERLAY WITH HANDICAP is \mathcal{NP} -complete, where, in addition to a
 249 hypergraph H , we are given a subset of vertices $A \subseteq V(H)$ (called *handicap*
 250 *vertices*), and we ask for a graph G that O_3 -overlays H with the additional
 251 constraint that for every $v \in A$, the degree of v is at most 1. Observe
 252 that we can easily reduce this problem to $(\Delta \leq 2)$ - O_3 -OVERLAY: for every
 253 $v \in A$, add six vertices t_v^1, \dots, t_v^6 , and hyperedges $\{v, t_v^1, t_v^p\}$ for every $p \in$
 254 $\{2, 3, 4, 5, 6\}$. Let H' be the obtained hypergraph. Observe that an $(O_3, H, 2)$ -
 255 graph G in which every $v \in A$ has degree at most 1 can be extended into an
 256 $(O_3, H', 2)$ -graph by adding the edge vt_v^1 (and all vertices of $V(H') \setminus V(H)$).
 257 Conversely, $(O_3, H', 2)$ -graph G' must contain the edge vt_v^1 , and thus v is of
 258 degree at most 1 in $G'[V(H)]$.

259 We reduce 3-SAT to $(\Delta \leq 2)$ - O_3 -OVERLAY WITH HANDICAP. Let Φ
 260 be an instance of 3-SAT with m clauses C_1, \dots, C_m on n variables $x_1, \dots,$
 261 x_n . We construct a 3-uniform hypergraph H as follows (see Figure 4 for an
 262 example):

263 • for every variable x_i , we create a variable gadget VG_i composed of
 264 vertices $a_i^1, \dots, a_i^{p_i}, \alpha_i^1, \dots, \alpha_i^{p_i}, b_i^1, \dots, b_i^{p_i}, \beta_i^1, \dots, \beta_i^{p_i}, z_i^1, \dots, z_i^{p_i},$
 265 and $\zeta_i^1, \dots, \zeta_i^{p_i}$ where p_i is the number of clauses where x_i appears.
 266 Then, for every $j \in [p_i]$, we create the following hyperedges (additions
 267 in superscripts are modulo p_i , and we write p_i instead of 0): $h_{i,j}^1 =$
 268 $\{z_i^j, a_i^j, \alpha_i^j\}, h_{i,j}^2 = \{z_i^j, b_i^j, \beta_i^j\}, h_{i,j}^3 = \{a_i^j, z_i^j, b_i^j\}, h_{i,j}^4 = \{z_i^j, b_i^j, \zeta_i^j\},$
 269 $h_{i,j}^5 = \{b_i^j, \zeta_i^j, a_i^{j+1}\}, h_{i,j}^6 = \{\zeta_i^j, a_i^{j+1}, z_i^{j+1}\}.$

270 Moreover, all vertices $z_i^1, \dots, z_i^{p_i}$, and $\zeta_i^1, \dots, \zeta_i^{p_i}$ are handicap vertices.

271 • for every clause C_j , we create a clause gadget CG_j composed of vertices
 272 c_j^1, \dots, c_j^5 , and, for every $p \in [5]$, of the hyperedge $\{c_j^p, c_j^{p+1}, c_j^{p+3}\}$

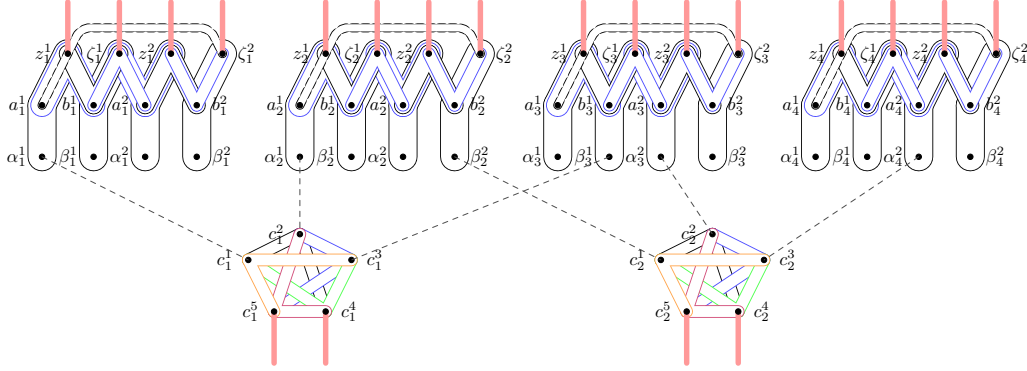


Figure 4: An example of the reduction for $\Phi = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3 \vee x_4)$. The dotted lines join two vertices that are identified. The thick red lines indicate the handicap vertices.

273 (additions are modulo 5, and we write 5 instead of 0). The vertices c_j^4
 274 and c_j^5 are handicap vertices.

- 275 • finally, for every clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$, if $\ell_p = x_i$ ($\ell_p = \bar{x}_i$) for some
 276 $p \in \{1, 2, 3\}$, then we identify the vertex c_j^p of CG_j with the vertex α_i^j
 277 (β_i^j) of VG_i , respectively.

278

279 The general idea of the proof is the following. There are only two possibil-
 280 ities to overlay a variable gadget VG_i (either by a set T_i or a set F_i of edges).
 281 See Claim 3. This two possibilities correspond to the two possible values
 282 (*true* or *false*) of the variable x_i . Then a clause gadget CG_j imposes that
 283 not all literals of the clauses are assigned *false* with this correspondence.

284 Assume first that there is an $(O_3, H, 2)$ -graph G such that all handicap
 285 vertices have degree at most 1.

286 For $i \in [n]$, let $T_i = \bigcup_{j=1}^{p_i} \{z_i^j \alpha_i^j, b_i^j \beta_i^j, c_i^j \zeta_i^j\}$ and $F_i = \bigcup_{j=1}^{p_i} \{a_i^j \alpha_i^j, z_i^j b_i^j, \zeta_i^j a_i^{j+1}\}$.

287 **Claim 3.** *Let $i \in [n]$. We have the following:*

- 288 1. $z_i^j \zeta_i^j \notin E(G)$ and $\zeta_i^j z_i^{j+1} \notin E(G)$, $j \in [m]$.
- 289 2. Either $T_i \subseteq E(G)$ or $F_i \subseteq E(G)$.

290 *Proof of claim:* 1. Assume for a contradiction that $z_i^j \zeta_i^j \in E(G)$. Since z_i^j
 291 and ζ_i^j are handicap vertices, the hyperedges $h_{i,j}^2, h_{i,j}^3, h_{i,j}^5$ must be O_3 -overlaid
 292 by the edges $b_i^j \beta_i^j, a_i^j b_i^j, b_i^j a_i^{j+1}$, respectively, contradicting the fact that b_i^j has

293 degree 2 in G .
 294 Similarly, if $\zeta_i^j z_i^{j+1} \in E(G)$, then the hyperedges $h_{i,j}^5, h_{i,j+1}^1, h_{i,j+1}^3$ must be O_3 -
 295 overlaid by the edges $b_i^j a_i^{j+1}, a_i^{j+1} \alpha_i^j, a_i^{j+1} b_i^{j+1}$, respectively, a contradiction.

296 2. By 1. the hyperedge $h_{i,j}^4$ is O_3 -overlaid by $z_i^j b_i^j$ or $\zeta_i^j b_i^j$.

297 In the first case $z_i^j b_i^j \in E(G)$, since z_i^j is a handicap vertex, the hyperedge
 298 $h_{i,j-1}^6$ must be O_3 -overlaid by $\zeta_i^{j-1} a_i^j$, and so hyperedge $h_{i,j-1}^3$ is O_3 -overlaid
 299 by $z_i^{j-1} b_i^{j-1}$. And by induction, we get that $\bigcup_{j=1}^m \{\zeta_i^j a_i^{j+1}, z_i^j b_i^j\} \subseteq E(G)$.
 300 Since the z_i^j are handicap vertices, the hyperedge $h_{i,j}^1$ is O_3 -overlaid by $a_i^j \alpha_i^j$.
 301 Hence $F_i \subseteq E(G)$.

302 In the second case $\zeta_i^j b_i^j \in E(G)$, a symmetrical argument shows $T_i \subseteq$
 303 $E(G)$. ◁

304 We define the truth assignment ϕ by $\phi(x_i) = true$ if $T_i \subseteq E(G)$ and
 305 $\phi(x_i) = false$ if $F_i \subseteq E(G)$. This is well-defined by Claim 3 and the fact
 306 that $E(G)$ cannot contain both T_i and F_i because it has maximum degree 2.

307 Now, for any $j \in [m]$, there exists $p \in \{1, 2, 3\}$ such that c_j^p has degree 2
 308 in $G[CG_j]$ (If not, then there are at most 2 edges. Hence there is a vertex
 309 c_j^p of degree 0, and thus there is a hyperedge containing c_j^p which is not
 310 overlaid). Therefore, it has degree 0 in the variable gadget c_j^p belongs to. By
 311 our definition of ϕ , we get that $\phi(\ell_p) = true$. Hence ϕ satisfies Φ .

312 Suppose now that $\phi : \{x_1, \dots, x_n\} \rightarrow \{true, false\}$ is a truth assignment
 313 satisfying Φ .

314 Let us first select a set of edges in the variable gadgets. For every $i \in [n]$, if
 315 $\phi(x_i) = true$, then select edges $\{z_i^j, a_i^j\}, \{b_i^j, \beta_i^j\}$ and $\{\zeta_i^j, b_i^j\}$ for all $j \in [p_i]$.
 316 If $\phi(x_i) = false$, then select edges $\{z_i^j, b_i^j\}, \{a_i^j, \alpha_i^j\}$ and $\{\zeta_i^j, a_i^{j+1}\}$ for all
 317 $j \in [p_i]$ (additions are modulo p_i). Let E_1 be the set of selected edges, and
 318 $G_1 = (V(H), E_1)$. Observe that G_1 O_3 -overlays the hyperedges of all variable
 319 gadgets: for every $i \in [n]$ and every $j \in [p_i]$,

- 320 • $h_{i,j}^1$ contains the edge $\{z_i^j, a_i^j\}$ if $\phi(x_i) = true$, and the edge $\{a_i^j, \alpha_i^j\}$
 321 otherwise.
- 322 • $h_{i,j}^2$ contains the edge $\{b_i^j, \beta_i^j\}$ if $\phi(x_i) = true$, and the edge $\{z_i^j, b_i^j\}$
 323 otherwise.
- 324 • $h_{i,j}^3$ contains the edge $\{z_i^j, a_i^j\}$ if $\phi(x_i) = true$, and the edge $\{z_i^j, b_i^j\}$
 325 otherwise.

- 326 • $h_{i,j}^4$ contains the edge $\{\zeta_i^j, b_i^j\}$ if $\phi(x_i) = true$, and the edge $\{z_i^j, b_i^j\}$
327 otherwise.
- 328 • $h_{i,j}^5$ contains the edge $\{\zeta_i^j, b_i^j\}$ if $\phi(x_i) = true$, and the edge $\{\zeta_i^j, a_i^{j+1}\}$
329 otherwise.
- 330 • $h_{i,j}^6$ contains the edge $\{z_i^{j+1}, a_i^{j+1}\}$ if $\phi(x_i) = true$, and the edge $\{\zeta_i^j, a_i^{j+1}\}$
331 otherwise.

332 Moreover the vertices $z_i^1, \dots, z_i^{p_i}$, and $\zeta_i^1, \dots, \zeta_i^{p_i}$ have degree 1 in G_1 and the
333 vertices α_i^j have degree 0 in G_1 if and only if $\phi(x_i) = true$ and the vertices
334 β_i^j have degree 0 in G_1 if and only if $\phi(x_i) = false$.

335 We now select a set of edges in the clause gadgets for every clause $C_j =$
336 $\ell_1 \vee \ell_2 \vee \ell_3$. Since ϕ satisfies Φ , there exists $p = p_j \in \{1, 2, 3\}$ such that
337 $\phi(\ell_p) = true$. Select the edges $\{c_j^p, c_j^{p+2}\}$, $\{c_j^p, c_j^{p+3}\}$ and $\{c_j^{p+1}, c_j^{p+4}\}$. Let
338 E_2 be the set of selected edges in clause gadgets, and $G_2 = (V(H), E_2)$.
339 Clearly, G_2 O_3 -overlays all hyperedges of the clause gadgets and the vertices
340 c_j^4 and c_j^5 , $j \in [m]$, have degree 1 in G_2 . Moreover, for all $j \in [m]$ the vertex
341 $c_j^{p_j}$ is identified with the vertex α_i^j or β_i^j , which has degree 0 in G_1 since
342 $\phi(\ell_p) = true$. Thus $G := (V(H), E_1 \cup E_2)$ has maximum degree 2, which
343 proves that it is an $(O_3, H, 2)$ -graph. \square

344 **Lemma 1.** *If $(\Delta \leq k)$ - O_p -OVERLAY is \mathcal{NP} -complete, then $(\Delta \leq k)$ - O_{p+1} -
345 OVERLAY and $(\Delta \leq k + 1)$ - O_p -OVERLAY are \mathcal{NP} -complete.*

346 Let k and p be two positive integers with $p \geq 2$, let $H_{k,p}$ be the p -uniform
347 hypergraph on $(k+1)(p-1)$ vertices whose hyperedge set contains all possible
348 p -subsets of vertices.

349 We first need the following claim in order to prove Lemma 1.

350 **Claim 4.** *Every $(O_p, H_{k,p}, k)$ -graph is k -regular. Moreover, there exists an
351 $(O_p, H_{k,p}, k)$ -graph.*

352 *Proof of claim:* Let G be an $(O_p, H_{k,p}, k)$ -graph. Assume for a contradic-
353 tion that it has a vertex v of degree at most $k - 1$. Then, observe that
354 $|V(G) \setminus N[v]| \geq (k+1)(p-2) + 1$. Now, by removing successively the closed
355 neighbourhood of a vertex of the graph (chosen arbitrarily) until the graph
356 is anticomplete, we can construct, together with v , an independent set S of
357 G of size p . This is impossible, since S is a hyperedge of $H_{k,p}$ which is not
358 O_p -overlaid by G .

359 Finally, observe that the disjoint union of $p - 1$ cliques of size $k + 1$ is a
 360 k -regular graph which O_p -overlays $H_{k,p}$. ◁

361 *Proof of Lemma 1.* First, we will reduce from $(\Delta \leq k)$ - O_p -OVERLAY to
 362 $(\Delta \leq k)$ - O_{p+1} -OVERLAY. Let H be a p -uniform hypergraph. We construct
 363 a $(p + 1)$ -uniform hypergraph H^* from a copy of H and replace, for every hy-
 364 peredge S , a hyperedge $S' = S \cup \{v_s\}$. We also add, for every hyperedge S , a
 365 set of $(k + 1)p - 1$ new vertices N_S , and form, together with v_S , a hypergraph
 366 H_S isomorphic to $H_{k,p+1}$.

367 Let G be an (O_p, H, k) -graph. Construct G^* from G by adding, for every
 368 hyperedge S of H , a disjoint union of p cliques of size $k + 1$ on the vertices
 369 $N_S \cup \{v_S\}$. Clearly, G^* is an (O_{p+1}, H^*, k) -graph.

370 Conversely, if G^* is an (O_{p+1}, H^*, k) -graph, then, by Claim 4, v_S must
 371 have degree k in N_S for every hyperedge S of H . Thus, the restriction of G^*
 372 to the vertices of H is an (O_p, H, k) -graph.

373 Now, we reduce $(\Delta \leq k)$ - O_p -OVERLAY to $(\Delta \leq k + 1)$ - O_p -OVERLAY.

374 Let H be a p -uniform hypergraph. Let H' be the p -uniform hypergraph
 375 obtained from H by doing the following for each vertex $v \in V(H)$:

- 376 • add the vertices $x^v, y_1^v, \dots, y_{p-2}^v$ and the hyperedge $S_v = \{v, x^v, y_1^v, \dots, y_{p-2}^v\}$.
- 377 • for any $i = \{1, \dots, p - 2\}$, add a copy H_i^v of $H_{k+1,p}$ whose vertices are
 378 y_i^v and $(p - 1)(k + 2) - 1$ new vertices.

379 Assume that G' is an $(O_p, H', k + 1)$ -graph. By Claim 4, the graph induced
 380 by G' on $V(H_i^v)$ is $(k + 1)$ -regular, for all $v \in V(H)$, $1 \leq i \leq p - 2$. Therefore
 381 each y_i^v has $k + 1$ -neighbours outside the hyperedge S_v . Hence G' must contain
 382 the edge vx^v because it O_p -overlays S_v . Thus the subgraph $G = G'[V(H)]$
 383 has maximum degree at most $\Delta(G') - 1 \leq k$ and G O_p -overlays H . In other
 384 words, G is an (O_p, H, k) -graph.

385 Conversely, assume G is an (O_p, H, k) -graph. Let G' be the graph with
 386 vertex set $V(H')$ and whose edge set is the union of $E(G)$, $\{vx^v \mid v \in V(H)\}$,
 387 and the edge set of an $(O_p, H_i^v, k + 1)$ -graph (which exists by Claim 4), for all
 388 $v \in V(H)$, $1 \leq i \leq p - 2$. One easily checks that G' is $(O_p, H', k + 1)$ -graph.

389 Therefore, there is an (O_p, H, k) -graph if and only if there is an
 390 $(O_p, H', k + 1)$ -graph.

391 □

392 Propositions 1 and 2, Theorems 3 and 4, and Lemma 1 imply Theorem 1.

393 **3. Complexity of Max ($\Delta \leq k$)- F -Overlay**

394 The aim of this section is to establish Theorem 2 that gives the \mathcal{P}/\mathcal{NP} -
395 complete dichotomy for the MAX ($\Delta \leq k$)- F -OVERLAY problems.

396 As noticed in the introduction, if $\Delta(F) > k$ or F is an anticomplete graph
397 then MAX ($\Delta \leq k$)- \mathcal{F} -OVERLAY is trivially polynomial-time solvable.
398 Moreover, by Propositions 1 and 2, MAX ($\Delta \leq 1$)- O_3 -OVERLAY as well as
399 MAX ($\Delta \leq k$)- O_2 -OVERLAY (for all positive integers k) are also polynomial-
400 time solvable.

401 We shall now prove that if we are not in one of the above cases, then
402 MAX ($\Delta \leq k$)- F -OVERLAY is \mathcal{NP} -complete. We first establish the \mathcal{NP} -
403 completeness when F has no isolated vertices.

404 **Theorem 5.** *Let F be a graph on at least three vertices with no isolated*
405 *vertices. If $k \geq \Delta(F)$, then MAX ($\Delta \leq k$)- F -OVERLAY is \mathcal{NP} -complete on*
406 *neat hypergraphs.*

407 *Proof.* Assume $k \geq \Delta(F)$. Let $n = |F|$, a_1, \dots, a_n be an ordering of the
408 vertices of F such that $\delta(F) = d(a_1) \leq d(a_2) \leq \dots \leq d(a_n) = \Delta(F)$.

409 Let $\gamma = \lfloor k/\delta(F) \rfloor - 1$, $\beta = k - \gamma\delta(F)$. Observe that $\delta(F) \leq \beta \leq 2\delta(F) - 1$.

410 We shall give a reduction from INDEPENDENT SET which is a well-known
411 \mathcal{NP} -complete problem even for cubic graphs (see [?].) We distinguish two
412 cases depending on whether $d(a_2) > \beta$ or not.

413 Let $\gamma_2 = \lfloor (k - d(a_2))/\delta(F) \rfloor$ and $\gamma_3 = \lfloor (k - d(a_3))/\delta(F) \rfloor$. And we define
414 the value γ_2 as follows.

- 415 - If $d(a_2) > \beta$, let $\gamma_1 = \gamma_2$.
- 416 - If $d(a_2) \leq \beta$, let $\gamma_1 = \lfloor (k - d(a_1))/\delta(F) \rfloor$.

417 Let Γ be a cubic graph. For each vertex $v \in V(\Gamma)$, let $(e_1(v), e_2(v), e_3(v))$
418 be an ordering of the edges incident to v . We shall construct the neat hyper-
419 graph $H = H(\Gamma)$ as follows.

- 420 • For each vertex $v \in \Gamma$, we create a hyperedge $S_v = \{a_1^v, \dots, a_n^v\}$. Then,
421 for $1 \leq i \leq 3$, we add γ_i a_i^v -leaves, that are hyperedges containing a_i^v
422 and $n - 1$ new vertices.
- 423 • For each edge $e = uv \in \Gamma$, let i and j be the indices such that $e =$
424 $e_i(u) = e_j(v)$. We create a new vertex z_e and hyperedges S_u^e (S_v^e)
425 containing z_e , a_i^u (a_j^v), and $n - 2$ new vertices, respectively. Then,

426 we add γ z_e -leaves, that are hyperedges containing z_e and $n - 1$ new
 427 vertices.

428 From the construction, observe that γ_i will be useful to enforce the degree
 429 of vertex a_i^v for $v \in V(G)$ which could have degree at most $d(a_3)$ in the
 430 hypergraph.

431 We shall prove that $over_F(H, k) = (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$,
 432 where $\alpha(\Gamma)$ denotes the cardinality of a maximum independent set in Γ .

The following claim shows that there are optimal solutions with specific structure. This leads to the inequality:

$$over_F(H, k) \leq (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$$

433 **Claim 5.** *There is a graph G with $\Delta(G) \leq k$ that F -overlays $over_F(H, k)$
 434 hyperedges of H such that:*

435 (a) *each x -leaf L is F -overlaid and x is incident to $\delta(F)$ edges in $G[L]$
 436 (with $x = a_i^v$ or $x = z_e$).*

437 (b) *for each edge $e = uv \in E(\Gamma)$, exactly one of the two hyperedges S_u^e and
 438 S_v^e is F -overlaid. Moreover if S_u^e (S_v^e) is F -overlaid, then a_i^u (a_j^v) is
 439 incident to at most $d(a_2)$ edges in S_u^e (S_v^e), respectively.*

440 (c) *the set of vertices v such that S_v is F -overlaid is an independent set
 441 in Γ .*

442 *Proof of claim:* Let G be a graph with $\Delta(G) \leq k$ that F -overlays $over_F(H, k)$
 443 hyperedges of H . Observe first that since H is a neat hypergraph, we may
 444 assume that, for every hyperedge S of H , $G[S] = F$ if S is F -overlaid and
 445 $G[S]$ has no edge otherwise.

446 Therefore if an x -leaf L is F -overlaid, then replacing $G[L]$ by a copy of F
 447 in which x has degree $\delta(F)$, we obtain another graph with maximum degree
 448 at most k that F -overlays $over_F(H, k)$ hyperedges of H . Henceforth we may
 449 assume that if an x -leaf is F -overlaid then x is incident to $\delta(F)$ edges included
 450 in it.

451 Now assume that the z_e -leaf L , for $e \in E(\Gamma)$, is not F -overlaid by G . We
 452 have $d_G(z_e) > k - \delta(F)$ for otherwise adding on L the edges of a copy of F
 453 in which z_e has degree $\delta(F)$, we obtain a graph F -overlying more hyperedges
 454 than G . But since there are γ z_e -leaves and z_e is incident to $\delta(F)$ edges in
 455 each of them, at least one of the hyperedges S_u^e and S_v^e is F -overlaid. Now

456 replacing the edges of G in this hyperedge by those of a copy of F in L , in
 457 which z_e has degree $\delta(F)$, we obtain another graph with maximum degree
 458 at most k that F -overlays $over_F(H, k)$ hyperedges of H . Henceforth we may
 459 assume that every z_e -leaf is F -overlaid.

460 A similar reasoning shows that every a_i^v -leaf is F -overlaid, for all $v \in V(\Gamma)$,
 461 $i \in \{1, 2, 3\}$. This proves (a).

462 Now by (a) each vertex z_e is incident to $\gamma\delta(F) = k - \beta > k - 2\delta(F)$ edges
 463 in the z_e -leaves. Thus z_e is incident to at most β edges in $S_u^e \cup S_v^e$. Since
 464 $\beta \leq 2\delta(F) - 1$, at most one of S_u^e and S_v^e is F -overlaid.

465 Moreover, if S_u^e is F -overlaid and $\beta < d(a_2)$ (resp. $d(a_2) \leq \beta$), then z_e must
 466 have degree $d(a_1)$ (resp. at most $d(a_2)$) in $G[S_u^e]$.

467 Assume now that none of S_u^e, S_v^e is F -overlaid, and let $e = e_i(u)$, a_i^u has γ_i
 468 a_i^u -leaves, so a_i^u is incident to $\gamma_i\delta(F) \leq k - d(a_i)$ edges in these leaves.

469 Thus, replacing the edges of G in S_u^e by those of a copy of F , in which

470 - if $d(a_2) > \beta$, we have $d_G(a_i^u) \leq k - d(a_2)$, then z_e has degree $\delta(F)$ and
 471 a_i^u has degree $d(a_2)$.

472 - if $d(a_2) \leq \beta$, since $d_G(a_1^u) \leq k - d(a_1)$, then for $i = 1$, a_i^u must have
 473 degree $d(a_1)$, so z_e has degree $d(a_2)$. And for $i = \{2, 3\}$, let $d(z_e) =$
 474 $d(a_1)$, $d(a_i^u) = d(a_2)$.

475 Hence, we obtain another graph with maximum degree at most k that F -
 476 overlays $over_F(H, k)$ hyperedges of H . Henceforth, we may assume that at
 477 least one of S_u^e and S_v^e is F -overlaid. This completes the proof of (b).

478 Consider now a vertex $v \in V(\Gamma)$ such that S_v is F -overlaid. We distin-
 479 guish the following cases.

480 - If $d(a_2) > \beta$, then by (a), vertex a_1^v is incident to $\gamma_1\delta(F)$ edges in the a_1^v -
 481 leaves and at least $\delta(F) \in S_v$. Therefore $S_v^{e_1(v)}$ cannot be F -overlaid for
 482 otherwise, by (b), a_1^v would have degree at least $\gamma_1\delta(F) + \delta(F) + d(a_2) >$
 483 k . Similarly, $S_v^{e_2(v)}$ cannot be F -overlaid.

484 Assume now for a contradiction that $S_v^{e_3(v)}$ is F -overlaid. By (a) and
 485 (b), a_3^v is incident to $\gamma_3\delta(F)$ edges in the a_3^v -leaves and at least $d(a_2) \in$
 486 $S_v^{e_3(v)}$. Since $\gamma_3\delta(F) + d(a_2) + d(a_3) > k$, a_3^v is incident to less than
 487 $d(a_3)$ edges in S_v . Thus, there is a vertex $a \in \{a_1^v, a_2^v\}$ which is incident
 488 to at least $d(a_3)$ edges in S_v and thus at least $\gamma_1\delta(F) + d(a_3)$ edges in
 489 G . But a_3^v is adjacent to at least $\delta(F)$ edges in S_v so at least $\gamma_3\delta(F) +$

490 $d(a_2) + \delta(F)$ edges in total. Thus $\gamma_3\delta(F) + d(a_2) + \delta(F) \leq k$, so
 491 $\gamma_3 \leq \gamma_1 - 1$. Thus a is adjacent to at least $(\gamma_3 + 1)\delta(F) + d(a_3) > k$
 492 edges, a contradiction.

493 - If $d(a_2) \leq \beta$, then by (a), vertex a_1^v is incident to $\gamma_1\delta(F)$ edges in
 494 the a_1^v -leaves and at least $\delta(F) \in S_v$. If $S_v^{e_1(v)}$ is F -overlaid, then
 495 (b) yields $d(a_i^v) = \gamma_i\delta(F) + \delta(F) + \delta(F) > k$, a contradiction. If
 496 $S_v^{e_2(v)}$ is F -overlaid, then $d(a_2^v) = \delta$ in $G[S_v]$. We obtain that $d(a_2^v) =$
 497 $\gamma_2\delta + d(a_2) + \delta \leq k$ which implies that $\gamma_2 \leq \gamma_2 - 1$, a contradiction.
 498 If $S_v^{e_3(v)}$ is F -overlaid, then we get a contradiction in the same way as
 499 that in the case $d(a_2) > \beta$.

500 To summarize, if S_v is F -overlaid, then none of $S_v^{e_1(v)}$, $S_v^{e_2(v)}$, $S_v^{e_3(v)}$ is
 501 F -overlaid. Together with (b), this implies (c). \triangleleft

502 Conversely, consider W a maximum independent set of Γ .

503 Let G be the graph with vertex $V(H)$ which is the union of the following
 504 subgraphs :

- 505 • for each x -leaf L , we add a copy of F on L in which x has degree $\delta(F)$;
- 506 • for each vertex $v \in W$, we add a copy of F on S_v in which a_i^v has degree
 507 $d(a_i)$ for all $1 \leq i \leq n$.
- 508 • for each edge $e \in E(\Gamma)$, we choose an endvertex u of e such that $u \notin W$,
 509 and add a copy of F in which z_e has degree $d(a_1)$ and a_i^u has degree
 510 $d(a_2)$ (with i the index such that $e_i(u) = e$) except the case when
 511 $d(a_2) \leq \beta$ and $i = 1$, we do that but $d(z_e) = d(a_2)$ and $d(a_1^u) = d(a_1)$
 512 in $G[S_u^{e_1(u)}]$.

513 It is simple matter to check that $\Delta(G) \leq k$ and that G F -overlays
 514 $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$ hyperedges of H . Thus $over_F(H, k) \geq$
 515 $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$.

516 \square

517 We then establish the following lemma, which allows to derive the \mathcal{NP} -
 518 completeness of $\text{MAX}(\Delta \leq k)\text{-}F\text{-OVERLAY}$ when F has isolated vertices.

519 **Lemma 2.** *Let k be a positive integer, let F be a graph with $\delta(F) \geq 1$, and*
 520 *let q be a non-negative integer. If $\text{MAX}(\Delta \leq k)\text{-}(F + \overline{K}_q)\text{-OVERLAY}$ is \mathcal{NP} -*
 521 *complete, then $\text{MAX}(\Delta \leq k)\text{-}(F + \overline{K}_{q+1})\text{-OVERLAY}$ is also \mathcal{NP} -complete.*

522 *Proof.* We shall reduce MAX $(\Delta \leq k)$ - $(F + \overline{K}_q)$ -OVERLAY to MAX $(\Delta \leq$
523 $k)$ - $(F + \overline{K}_{q+1})$ -OVERLAY. Set $p = |V(F)|$ and $r = \lfloor k/\delta(F) \rfloor$. Note that
524 $r \geq 1$, for otherwise MAX $(\Delta \leq k)$ - $(F + \overline{K}_q)$ -OVERLAY would trivially be
525 polynomial-time solvable.

526 Let H' be a $(p+q)$ -uniform hypergraph. Let H be the $(p+q+1)$ -uniform
527 hypergraph obtained from H' as follows. Its vertex set is partitioned into
528 $V(H')$, P_1, \dots, P_r , K , and $\{z\}$ with $|P_i| = p - 1$ for all $1 \leq i \leq r$ and
529 $|K| = (m+1)(k+1) + kp$ with $m = \max\{|E(H')|, q+1\}$. The hyperedge set
530 of H contains $Q \cup P_i \cup \{z\}$ for all $(q+1)$ -subsets Q of K and all $1 \leq i \leq r$,
531 and $S \cup \{z\}$ for all $S \in E(H')$.

532 The reduction can be performed in polynomial time, because for each
533 hyperedge in H' we construct a polynomial number of hyperedges in H (recall
534 that q is fixed), and each of them is straightforward to obtain. We shall now
535 prove that $over_{F+\overline{K}_{q+1}}(H, k) = over_{F+\overline{K}_q}(H', k) + r \binom{|K|}{q+1}$.

536 Let G'_0 be a graph with $V(G'_0) = V(H')$ and $\Delta(G'_0) \leq k$ that $(F + \overline{K}_q)$ -
537 overlays $over_{F+\overline{K}_q}(H', k)$ hyperedges of H' . For $1 \leq i \leq r$, let F_i be a copy
538 of F on $P_i \cup \{z\}$ in which z has degree $\delta(F)$. Let G_0 be the graph defined
539 by $V(G_0) = V(H)$ and $E(G'_0) \cup \bigcup_{i=1}^r E(F_i)$. Clearly, $\Delta(G_0) \leq k$ and G_0 F -
540 overlays all hyperedges of H except hyperedges $S \cup \{z\}$ in which $S \in E(H')$
541 are not overlaid by G'_0 . So $over_{F+\overline{K}_{q+1}}(H, k) \geq over_{F+\overline{K}_q}(H', k) + r \binom{|K|}{q+1}$.

542 Conversely, let G be a graph with $V(G) = V(H)$ and $\Delta(G) \leq k$ that
543 maximizes the number of $(F + \overline{K}_{q+1})$ -overlaid hyperedges.

544 **Claim 6.** *In G , vertex z has at least $\delta(F)$ neighbours in each P_i , $1 \leq i \leq r$.*

545 *Proof of claim:* Suppose for a contradiction that z has less than $\delta(F)$ neigh-
546 bours in P_i . Since $\Delta(G) \leq k$, at most kp vertices of K are incident to an edge
547 with an endvertex in $P_i \cup \{z\}$. Thus there is a subset R of K of $(m+1)(k+1)$
548 vertices non-adjacent to all vertices of $P_i \cup \{z\}$. Now $G[R]$ has maximum
549 degree k and is thus $(k+1)$ -colourable. Hence $G[R]$ has an independent set
550 S of size $m+1$. Observe that for each $(q+1)$ -subset Q of S , the subgraph
551 $G[Q \cup P_i \cup \{z\}]$ has $q+2$ vertices of degree less than $\delta(F)$ (namely the ones of
552 $Q \cup \{z\}$), and so the hyperedge $Q \cup P_i \cup \{z\}$ is not $(F + \overline{K}_{q+1})$ -overlaid. Thus
553 there are at least $m+1 \geq |E(H')| + 1$ hyperedges not $(F + \overline{K}_{q+1})$ -overlaid
554 by G . This contradicts the maximality of G as G_0 overlays more hyperedges
555 of H . ◁

556 Now z has degree at least $r\delta(F)$ in $\bigcup_{i=1}^r P_i$ and so less than $\delta(F)$ in each
557 hyperedge $S \cup \{z\}$ for $S \in E(H')$. Hence if G $(F + \overline{K}_{q+1})$ -overlays $S \cup z$, it

558 must also $(F + \overline{K}_q)$ -overlay S . Thus $over_{F+\overline{K}_{q+1}}(H, k) \leq over_{F+\overline{K}_q}(H', k) +$
 559 $r\binom{|K|}{q+1}$. \square

560 Now we can prove Theorem 2. As explained in the beginning of the
 561 section, it suffices to prove that MAX $(\Delta \leq k)$ - F -OVERLAY remains \mathcal{NP} -
 562 complete when $\Delta(F) \leq k$, $F \neq \overline{K}_{|F|}$, $|F| \geq 3$ and $(F, k) \neq (O_3, 1)$. Assume
 563 that the above conditions are satisfied. Let F' be the graph induced by the
 564 non-isolated vertices of F . Then $F = F' + \overline{K}_q$ with $q = |F| - |F'|$. If $|F'| = 2$,
 565 then $F = O_{|F|}$, and we have the result by Theorem 1. If $|F'| \geq 3$, then the
 566 result follows from Theorem 5, Lemma 2, and an immediate induction.

567 4. Complexity of $(\Delta \leq k)$ - F -Overlay and $(\Delta \leq k)$ - \mathcal{F} -Overlay

568 In this section, we investigate the complexity of $(\Delta \leq k)$ - F -OVERLAY
 569 and $(\Delta \leq k)$ - \mathcal{F} -OVERLAY problems. We believe that each such problem is
 570 either polynomial-time solvable or \mathcal{NP} -complete. However the dichotomy is
 571 more complicated than the one we proved in Theorem 2 for MAX $(\Delta \leq k)$ - F -
 572 OVERLAY. This dichotomy is simple because there are very few pairs (F, H)
 573 for which MAX $(\Delta \leq k)$ - F -OVERLAY is polynomial-time solvable. We now
 574 exhibit several pairs (F, k) such that $(\Delta \leq k)$ - F -OVERLAY is polynomial-
 575 time solvable, while MAX $(\Delta \leq k)$ - F -OVERLAY is \mathcal{NP} -complete.

576 4.1. Regular graphs

577 **Proposition 3.** *For every complete graph K and every positive integer k ,*
 578 *$(\Delta \leq k)$ - K -OVERLAY is polynomial-time solvable.*

579 *Proof.* Observe that a $|V(K)|$ -uniform hypergraph H is a positive instance
 580 of $(\Delta \leq k)$ - K -OVERLAY if and only if $K(H)$ is a (K, H, k) -graph. \square

581 **Proposition 4.** *For every connected k -regular graph F , $(\Delta \leq k)$ - F -OVERLAY*
 582 *is polynomial-time solvable.*

583 *Proof.* One easily sees that a $|V(F)|$ -uniform hypergraph H admits an (F, H, k) -
 584 graph if and only if the hyperedges of H are pairwise non-intersecting. \square

585 Let C_4 denote the cycle on 4 vertices. Proposition 4 implies that $(\Delta \leq 2)$ -
 586 C_4 -OVERLAY is polynomial-time solvable. We now show that $(\Delta \leq 3)$ - C_4 -
 587 OVERLAY is also polynomial-time solvable.

588 **Theorem 6.** *$(\Delta \leq 3)$ - C_4 -OVERLAY is polynomial-time solvable.*

589 *Proof.* Let H be a 4-uniform hypergraph.

590 Let us describe an algorithm to decide whether there is a $(C_4, H, 3)$ -graph.
591 It is sufficient to do it when H is connected since the disjoint union of the
592 $(C_4, K, 3)$ -graphs for connected components K of H is a $(C_4, H, 3)$ -graph.

593 Observe first that if two hyperedges of H intersect in exactly one vertex
594 u , then no such graph exists, since u must have degree 2 in each of the
595 hyperedges if they are C_4 -overlaid, and thus degree 4 in total. Therefore if
596 there are two such hyperedges, we return ‘No’. At this point we may assume
597 that $|E(H)| \geq 2$ for otherwise we return ‘Yes’.

598 From now on we may assume that two hyperedges either do not intersect,
599 or are adjacent (intersect on at least two vertices).

600 **Claim 7.** *If two hyperedges S_1 and S_2 intersect on three vertices and there*
601 *is a $(C_4, H, 3)$ -graph G , then $|V(H)| \leq 6$.*

602 *Proof of claim:* Assume $S_1 = \{a_1, b, c, d\}$ and $S_2 = \{a_2, b, c, d\}$. Let G be a
603 $(C_4, H, 3)$ -graph. Observe that a_1 and a_2 must have two common neighbours,
604 say b, c , in common in $\{b, c, d\}$, for otherwise their unique common vertex
605 would have degree 4 in G . This forces d to be also adjacent to b and c .

606 Consider a hyperedge S_3 intersecting $S_1 \cup S_2$. Since it is C_4 -overlaid by G ,
607 at least two edges connect $S_3 \cap (S_1 \cup S_2)$ to $S_3 \setminus (S_1 \cup S_2)$. The endvertices of
608 those edges in $S_1 \cup S_2$ must have degree 2 in $G[S_1 \cup S_2]$. Hence, without loss of
609 generality, there is a vertex e not in $S_1 \cup S_2$ such that either $S_3 = \{a_1, a_2, b, e\}$
610 and $\{a_1e, a_2e\} \subseteq E(G)$, or $S_3 = \{a_1, b, d, e\}$ and $\{a_1e, de\} \subseteq E(G)$. Now
611 no hyperedge can both intersect $S_1 \cup S_2 \cup S_3$ and contain a vertex not in
612 $S_1 \cup S_2 \cup S_3$, for such a hyperedge must contain either the vertices d, e or
613 a_2, e which are at distance 3 in $G[S_1 \cup S_2 \cup S_3]$. (However there can be more
614 hyperedges contained in $S_1 \cup S_2 \cup S_3$.) Hence $|V(H)| \leq 6$. \square

615 In view of Claim 7, if there are two hyperedges with three vertices in
616 common, either we return ‘No’ if $|V(H)| > 6$, or we check all possibilities (or
617 follow the proof of the above claim) to return the correct answer otherwise.
618 Henceforth, we may assume that any two adjacent hyperedges intersect in
619 exactly two vertices.

620 Let S_1 and S_2 be two adjacent hyperedges, say $S_1 = \{a, b, c, d\}$ and
621 $S_2 = \{c, d, e, f\}$. Note that every $(C_4, H, 3)$ -graph contains the edges ab ,
622 cd and ef , and that $N(c) \cup N(d) = S_1 \cup S_2$ in this $(C_4, H, 3)$ -graph.

623 **Claim 8.** *If there is another hyperedge than S_1 and S_2 containing c or d ,*
624 *and there is a $(C_4, H, 3)$ -graph G , then $|V(H)| \leq 8$.*

625 *Proof of claim:* Without loss of generality, we may assume that G contains
626 the cycle (a, b, d, f, e, c, a) and the edge cd . Hence the only possible hyper-
627 edges containing c or d and a vertex not in $S_1 \cup S_2$ are $S_3 = \{a, c, e, g\}$ for
628 some $g \notin S_1 \cup S_2$ and $S_4 = \{b, d, f, h\}$ for some $h \notin S_1 \cup S_2$.

629 If H contains both S_3 and S_4 , then G contains the edges ag, eg, bh
630 and hf . If G contains also gh , then $G[S_1 \cup S_2 \cup S_3 \cup S_4]$ is 3-regular, so
631 $G = G[S_1 \cup S_2 \cup S_3 \cup S_4]$. If G does not contain gh , then the only vertices
632 of degree 2 in $G[S_1 \cup S_2 \cup S_3 \cup S_4]$ are g and h , and they are at distance at
633 least 3 in this graph. Thus every hyperedge intersecting $S_1 \cup S_2 \cup S_3 \cup S_4$ is
634 contained in this set, so $|V(H)| = 8$.

635 Assume now that H is a superset of only one of S_3, S_4 . Without loss
636 of generality, we may assume that this is S_3 . Hence G also contains the
637 edges ag and eg . If $V(G) \neq S_1 \cup S_2 \cup S_3$, then there is a hyperedge S that
638 intersects $S_1 \cup S_2 \cup S_3$ and that is not contained in $S_1 \cup S_2 \cup S_3$. It does not
639 contain c and d . Hence it must contain one of the vertices a or e , because it
640 intersects each S_i along an edge of G or not at all. Without loss of generality,
641 $a \in S$. Hence $S = \{a, b, i, g\}$ for some vertex i not in $S_1 \cup S_2 \cup S_3$, and G
642 contains the edges bi and ig . Now, as previously, either i and f are adjacent
643 and $G = G[S_1 \cup S_2 \cup S_3 \cup S]$ or they are not adjacent, and every hyperedge
644 intersecting $S_1 \cup S_2 \cup S_3 \cup S$ is contained in this set. In both cases, $|V(H)| = 8$.

645 ◁

646 We now summarize the algorithm: if $|V(G)| \leq 8$, then we solve the
647 instance by brute force. Otherwise, for every pair of hyperedges S_1, S_2 , if
648 their intersection is of size 1 or 3, we answer ‘No’. In the remaining cases,
649 if S_1 and S_2 have non-empty intersection, then, they must intersect on two
650 vertices c and d , and these vertices do not belong to any other hyperedges
651 but S_1 and S_2 . Hence, if a vertex is in at least 3 hyperedges, then we return
652 ‘No’. If not, then every hyperedge intersects at most two hyperedges on two
653 disjoint pairs of vertices. We can then return the $(C_4, H, 3)$ -graph obtained as
654 follows. If a hyperedge $S = \{a, b, c, d\}$ intersects either two other hyperedges
655 S_1 and S_2 is $\{a, b\}$ and $\{c, d\}$, or exactly one hyperedge on $\{a, b\}$, then add
656 the edges ab, bc, cd, da to $E(G)$.

657 Clearly, the above-described algorithm runs in polynomial time. □

658 4.2. Paths

659 Let \mathcal{P} be the set of all paths. We have the following:

660 **Theorem 7.** $(\Delta \leq 2)$ - \mathcal{P} -OVERLAY is linear-time solvable.

661 *Proof.* Clearly, if H is not connected, it suffices to solve the problem on each
662 of the components and to return ‘No’ if the answer is negative for at least one
663 of the components, and ‘Yes’ otherwise. Henceforth, we shall now assume
664 that H is connected. In such a case, a $(\mathcal{P}, H, 2)$ -graph is either a path or a
665 cycle. However, if H is \mathcal{P} -overlaid by a path P , then it is also \mathcal{P} -overlaid by
666 the cycle obtained from P by adding an edge between its two endvertices.
667 Thus, we focus on the case where G is a cycle.

668 Let \mathcal{S} be a family of sets. The *intersection graph* of a family \mathcal{S} is the
669 graph $IG(\mathcal{S})$ whose vertices are the sets of \mathcal{S} , and in which two vertices are
670 adjacent if the corresponding sets in \mathcal{S} intersect.

The *intersection graph* of a hypergraph H , denoted by $IG(H)$, is the
intersection graph of its hyperedge set. We define two functions l_H and s_H
as follows:

$$l_H(S) = |S| - 1 \text{ for all } S \in E(H) \text{ and } s_H(S, S') = |S \cap S'| - 1 \text{ for all } S, S' \in E(H).$$

671 Let \mathbb{C}_ℓ be the circle of circumference ℓ . We identify the points of \mathbb{C}_ℓ with
672 the integer numbers (points) of the segment $[0, \ell]$, (with 0 identified with ℓ).
673 A *circular-arc graph* is the intersection graph of a set of arcs on \mathbb{C}_ℓ . A set
674 \mathcal{A} of arcs such that $IG(\mathcal{A}) = G$ is called an *arc representation* of G . We
675 denote by A_v the arc corresponding to v in \mathcal{A} . Let G be a graph and let
676 $l : V(G) \rightarrow \mathbb{N}$ and $s : E(G) \rightarrow \mathbb{N}$ be two functions. An arc representation \mathcal{A}
677 of G is *l -respecting* if A_v has length $l(v)$ for any $v \in V(G)$, *s -respecting* if
678 $A_v \cap A_u$ has length $s(u, v)$ for all $uv \in E(G)$, and *(l, s) -respecting* if it is both
679 l -respecting and s -respecting. One can easily adapt the algorithm given by
680 Köbler et al. [?] for (l, s) -respecting interval representations to decide in
681 linear time whether a graph admits an (l, s) -respecting arc representation in
682 \mathbb{C}_n for every integer n .

683 **Claim 9.** *Let H be a connected hypergraph on n vertices. There is a cy-*
684 *cle \mathcal{P} -overlying H if and only if $IG(H)$ admits an (l_H, s_H) -respecting arc*
685 *representation into \mathbb{C}_n .*

686 *Proof of claim:* Assume that H is \mathcal{P} -overlaid by a cycle $C = (v_0, v_1, \dots, v_{n-1}, v_0)$.
687 There is a canonical embedding of C to \mathbb{C}_n in which every vertex v_i is mapped
688 to i and every edge $v_i v_{i+1}$ to the circular arc $[i, i + 1]$. For every hyperedge
689 $S \in E(H)$, $P[S]$ is a subpath, which is mapped to the circular arc A_S of \mathbb{C}_n
690 that is the union of the circular arcs to which its edges are mapped. Clearly,
691 $\mathcal{A} = \{A_S \mid S \in E(H)\}$ is an (l_H, s_H) -respecting interval representation of
692 $IG(H)$.

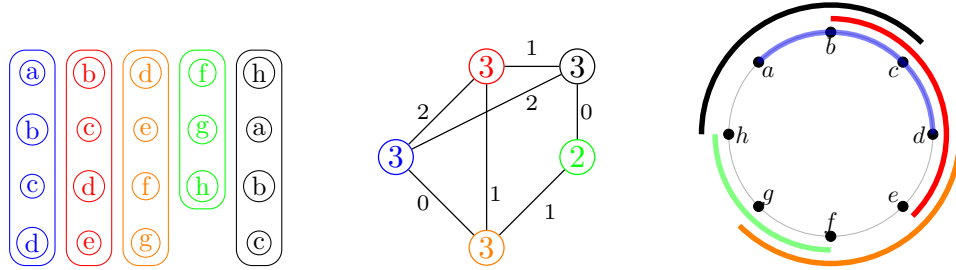


Figure 5: Given a hypergraph H (left), we determine its graph $IG(H)$ (middle) with each vertex represents the a hyperedge of the same color in which the value of any vertex $S \in V(H)$ is $l_H(S)$ and the weight of an edge between S, S' is $s_H(S, S')$. And a circular arc representation (each arc is associated to a hyperedge of the same color) is shown in the right which is (l, s) -respecting.

693 Conversely, assume that $IG(H)$ admits an (l_H, s_H) -respecting interval
 694 representation $\mathcal{A} = \{A_S \mid S \in E(H)\}$ into \mathbb{C}_n . Let S_0 be a hyperedge
 695 of minimum size. Free to rotate all intervals, we may assume that A_{S_0} is
 696 $[1, |S_0|]$. Now since \mathcal{A} is (l_H, s_H) -respecting and H is connected, we deduce
 697 that the extremities of A_S are integers for all $S \in E(H)$. Let v_1 be a vertex
 698 of H that belongs to the hyperedges whose corresponding arcs of \mathcal{A}
 699 contain 1. Then for all $i = 2$ to $n = |V(H)|$, denote by v_i an arbitrary vertex not in
 700 $\{v_1, \dots, v_{i-1}\}$ that belongs to the hyperedges whose corresponding arcs of \mathcal{A}
 701 contain i . Such a vertex exists because \mathcal{A} is (l_H, s_H) -respecting. Observe that
 702 such a construction yields $S = \{v_i \mid i \in A_S\}$ for all $S \in E(H)$. Furthermore,
 703 the cycle $C = (v_1, \dots, v_n, v_1)$ \mathcal{P} -overlays H . Indeed, for each $S \in E(H)$,
 704 $C[S]$ is the subpath corresponding to A_S , that is $V(C[S]) = \{v_i \mid i \in A_S\}$
 705 and $E(C[S]) = \{v_i v_{i+1} \mid [i, i+1] \subseteq A_S\}$. \triangleleft

706 The algorithm to solve $(\Delta \leq 2)$ - \mathcal{P} -OVERLAY for a connected hypergraph
 707 H in linear time is thus the following:

- 708 1. Construct the intersection graph $IG(H)$ and compute the associated
 709 functions l_H and s_H .
- 710 2. Check whether graph $IG(H)$ has an (l_H, s_H) -respecting interval repre-
 711 sentation. If it is the case, return ‘Yes’. If not return ‘No’.

712 □

713 **Remark 1.** We can also detect in polynomial time whether a connected
 714 hypergraph H is \mathcal{P} -overlaid by a path. Indeed, similarly to Claim 9, one can

715 show that there is a path \mathcal{P} -overlying H if and only if $IG(H)$ admits an
 716 (l_H, s_H) -respecting interval representation.

717 *4.3. On neat hypergraphs*

718 Let $F = (V, E)$ be a graph. We denote by V_i the set of vertices having
 719 degree i in F . For every increasing sequence of nonnegative integers $i_1 <$
 720 $i_2 < \dots < i_p$, an (i_1, i_2, \dots, i_p) -graph is a graph F such that $V_i = \emptyset$ if
 721 $i \notin \{i_1, i_2, \dots, i_p\}$. In particular, an (r) -graph is a r -regular graph.

722 Given a hypergraph H , we denote by $d_H(v)$ the number of hyperedges
 723 containing v for any $v \in V(H)$.

724 **Theorem 8.** *Let i_1, i_2, k be three non-negative integers with $i_1 < i_2$. For*
 725 *every (i_1, i_2) -graph F , one can solve $(\Delta \leq k)$ - F -OVERLAY in polynomial*
 726 *time on neat hypergraphs.*

727 *Proof.* Let H be a neat $|F|$ -uniform hypergraph.

728 If there is a vertex v such that $i_1 \cdot d_H(v) > k$, then we answer ‘No’. Indeed
 729 in an (H, F, k) -graph G , the vertex v has degree at least i_1 in each subgraph
 730 of G induced by a hyperedge to which it belongs, and so degree at least
 731 $i_1 \cdot d_H(v)$ in G because H is neat. Hence, we assume that $i_1 \cdot d_H(v) \leq k$ for
 732 every vertex.

733 Let $B(H)$ be the incidence graph of H : its vertex set is $E(H) \cup V(H)$
 734 and there is an edge between a hyperedge $S \in E(H)$ and a vertex $v \in V(H)$
 735 if and only if $v \in S$.

736 **Claim 10.** *There is an (H, F, k) -graph if and only if $B(H)$ has a subgraph*
 737 *in which every hyperedge vertex has degree $|V_{i_2}|$ and every vertex has degree*
 738 *at most $\lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$.*

739 *Proof of claim:* Assume that there is an (H, F, k) -graph G . Observe that
 740 since H is a neat hypergraph, free to remove some edges, we may assume
 741 that $G[S]$ is isomorphic to F for every hyperedge S of H . Let B' be the
 742 subgraph of $B(H)$ in which a vertex v is adjacent to a hyperedge S if and
 743 only if v has degree i_2 in $G[S]$. By definition every hyperedge has degree $|V_{i_2}|$
 744 in B' . Consider now a vertex v . Its degree in G is the sum of its degree in
 745 the subgraphs induced by the hyperedges S to which it belongs because H is
 746 neat. Hence $d_G(v) = i_2 \cdot d_{B'}(v) + i_1(d_H(v) - d_{B'}(v)) = i_1 \cdot d_H(v) + (i_2 - i_1)d_{B'}(v)$.
 747 But G is an (H, F, k) -graph, so $d_G(v) \leq k$. With the previous equality, this
 748 yields $d_{B'}(v) \leq \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1}$, and so $d_{B'}(v) \leq \lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$.

749 Conversely, assume $B(H)$ has a subgraph B' in which every $S \in E(H)$
750 has degree $|V_{i_2}|$ and any $v \in V(H)$ has degree at most $\lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$. We
751 construct G as follows: for each hyperedge S , the induced subgraph $G[S]$
752 is a copy of F such that any vertex v where $(S, v) \in E(B')$ is of degree
753 i_2 (there are $|V_{i_2}|$ such vertices) and the remaining vertices have degree i_1 .
754 Clearly, G F -overlays H . Moreover every vertex $v \in V(H)$ has degree i_2
755 in at most $\lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$ subgraphs and degree i_1 in the remaining subgraphs
756 induced by hyperedges to which it belongs. Hence v has degree at most k in
757 G . Therefore G is an (H, F, k) -graph. \triangleleft

758 Now the existence of a subgraph of $B(H)$ as shown in Claim 10 can be
759 done in polynomial time (see [?] Chap. 10). \square

760 **Theorem 9.** *Let i_1, i_2, i_3, k be three non-negative integers with $i_1 < i_2 < i_3$.
761 If F is a (i_1, i_2, i_3) -graph F such that all vertices of $V_{i_1} \cup V_{i_2}$ have the same
762 number of neighbours in V_{i_3} and $k < i_3 + i_1$, then one can solve $(\Delta \leq k)$ - \mathcal{F} -
763 OVERLAY in polynomial time on neat hypergraphs.*

764 *Proof.* Let j be the number of neighbours in V_{i_3} that each vertex of $V_{i_1} \cup V_{i_2}$
765 has.

766 Let H be an $|F|$ -uniform hypergraph. Observe that if a vertex has degree
767 i_3 in the subgraph induced by a hyperedge, then it cannot be in any other
768 hyperedge, for otherwise it would have degree at least $i_3 + i_1 > k$. Therefore,
769 we first check whether every hyperedge S contains a set $I(S)$ of $|V_{i_3}|$ vertices
770 that are uniquely in S . If not, then we return ‘No’. If it is the case, let $F' =$
771 $F - V_{i_3}$ and H' be the hypergraph whose vertex set is $V(H) \setminus \bigcup_{S \in E(H)} I(S)$
772 and whose hyperedge set is $\{S \setminus I(S) \mid S \in E(H)\}$, and let $k'(v) = k - j \cdot d_H(v)$
773 for every $v \in V(H')$. Since all vertices of $V_{i_1} \cup V_{i_2}$ have exactly j neighbours
774 in V_{i_3} , it is then easy to prove the following claim:

775 **Claim 11.** *There is an (H, F, k) -graph G if and only if there is a graph G'
776 F' -overlying H' such that $d_G(v) \leq k'(v)$ for all $v \in V(H')$.*

777 Deciding whether such a graph G' exists can be done similarly as the
778 proof of Theorem 8 replacing k by $k'(v)$. \square

779 5. Further research

780 Theorem 2 characterizes the complexity of MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY
781 when \mathcal{F} contains a unique graph. It would be nice to extend this character-
782 ization to families \mathcal{F} of arbitrary size.

783 **Problem 1.** *Characterize the pairs (\mathcal{F}, k) for which $\text{MAX } (\Delta \leq k)\text{-}\mathcal{F}$ -*
784 *OVERLAY is polynomial-time solvable and those for which it is \mathcal{NP} -complete.*

785 Theorem 1 and the results obtained in Section 4 give a first view of the
786 complexity of $(\Delta \leq k)\text{-}F\text{-OVERLAY}$. A natural problem is to close the
787 dichotomy:

788 **Problem 2.** *Characterize the pairs (F, k) for which $(\Delta \leq k)\text{-}F\text{-OVERLAY}$*
789 *is polynomial-time solvable and those for which it is \mathcal{NP} -complete.*

790 It would be interesting to consider the complexity of this problem when
791 F is k -regular but non-connected, and when F is a cycle. In order to attack
792 Problem 2, it would be helpful to prove the following conjecture.

793 **Conjecture 1.** *If $(\Delta \leq k)\text{-}F\text{-OVERLAY}$ is \mathcal{NP} -complete, then $(\Delta \leq k+1)\text{-}$*
794 *$F\text{-OVERLAY}$ is also \mathcal{NP} -complete.*

795 Furthermore, for each pair (\mathcal{F}, k) such that $\text{MAX } (\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$
796 is \mathcal{NP} -complete and $(\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$ is polynomial-time solvable, it
797 is natural to consider the parameterized complexity of $\text{MAX } (\Delta \leq k)\text{-}\mathcal{F}$ -
798 OVERLAY when parameterized by $|E(H)| - s$, because $(\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$
799 is the case $s = 0$.

800 It would also be interesting to obtain approximation algorithms for MAX
801 $(\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$ when this problem is \mathcal{NP} -complete.

802 As shown in Subsection 4.3, we proved that $(\Delta \leq k)\text{-}F\text{-OVERLAY}$ is some-
803 times easier when restricted to neat hypergraphs. It would then be interesting
804 to determine the complexity of $(\Delta \leq k)\text{-}F\text{-OVERLAY}$ on neat hypergraphs.
805 So far, we only proved some of these problems to be polynomial-time solvable,
806 but we believe some are \mathcal{NP} -complete.

807 **Problem 3.** *Characterize the pairs (F, k) for which $(\Delta \leq k)\text{-}F\text{-OVERLAY}$*
808 *on neat hypergraphs is polynomial-time solvable and those for which it is*
809 *\mathcal{NP} -complete.*