The multiple electromagnetic wave scattering by small spheres

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The multiple electromagnetic wave scattering by small spheres

Justine Labat, Victor Péron, Sébastien Tordeux

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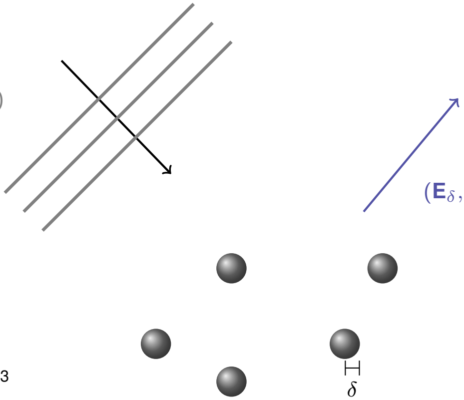
Seminar of RTWH Aachen University

Aachen, December 4th 2018

3D Scattering problem by small obstacles

Incident field $(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}})$

with wavelength λ

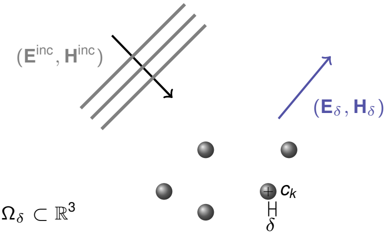


$(\mathbf{E}_\delta, \mathbf{H}_\delta)$ Scattered field

Exterior domain $\Omega_\delta \subset \mathbb{R}^3$

$\boxed{\text{Asymptotic assumption : } \delta \ll \lambda}$

Model problem



- Time-harmonic domain
- Homogeneous & isotropic medium Ω_δ
- Perfect conductors $\mathcal{B}(c_k, \delta)$, $k = 1 \dots N_{\text{obs}}$

Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl}\, \mathbf{E}_\delta - i\kappa\, \mathbf{H}_\delta = 0 & \text{in } \Omega_\delta \\ \mathbf{curl}\, \mathbf{H}_\delta + i\kappa\, \mathbf{E}_\delta = 0 & \text{in } \Omega_\delta \end{cases}$$

with $\kappa^2 = \omega^2 \mu(\varepsilon + \frac{i\sigma}{\omega})$, $\Im(\kappa) \geq 0$

Boundary condition

$$\mathbf{n} \times \mathbf{E}_\delta = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \qquad \text{on } \partial\Omega_\delta$$

Silver-Müller radiation condition

$$r\big(\mathbf{H}_\delta \times \widehat{x} - \mathbf{E}_\delta\big) \underset{r\to\infty}{\longrightarrow} 0 \quad \text{unif. in } \widehat{x} = \frac{\mathbf{x}}{r}$$

Mathematical well-posedness

For any $\mathbf{E}^{\text{inc}} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}\,, \Omega_\delta)$ there exist a unique solution $(\mathbf{E}_\delta, \mathbf{H}_\delta) \in \mathbf{H}_{\text{loc}}(\mathbf{curl}\,, \Omega_\delta)^2$ to the exterior Maxwell problem.

Motivations

Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

Motivations

Multiple scattering →

Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

Foldy-Lax model

- ✓ Interactions taken into account
- ✓ Low computational cost
- ✓ Meshless method

Superposition principle ↓

Born approximation

- ✗ No interaction between the obstacles
- ✓ Low computational cost
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Motivations

Multiple scattering $\longrightarrow$

Asymptotic models

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Superposition principle $\downarrow$

Born approximation

- ✗ No interaction between the obstacles
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High-order numerical solution

- ✗ High computational cost
  - ✗ volumical methods FEM, DG, . . .
  - ∼ boundary element methods
- ✗ Mesh refinement
- ✓ High-order spectral method

Outline

1. Single electromagnetic scattering by small spheres
   - Asymptotic models
   - Numerical results
   - Application: the Born approximation

2. Multiple electromagnetic scattering by small spheres
   - Foldy-Lax model
   - Spectral method
   - Numerical results

3. Conclusions and perspectives

Outline

1. Single electromagnetic scattering by small spheres
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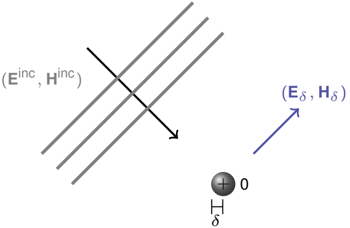
    - Numerical results

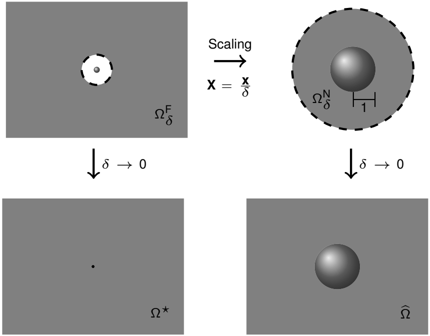
3. Conclusions and perspectives

Approximation of solution to single scattering

Method of matched asymptotic expansions

- Domain decomposition
- Local approximations
- Matching procedure





Approximation of solution to single scattering

Method of matched asymptotic expansions

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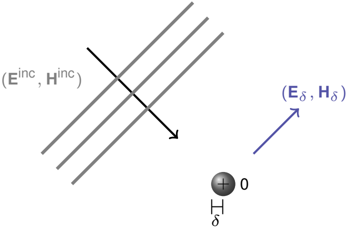
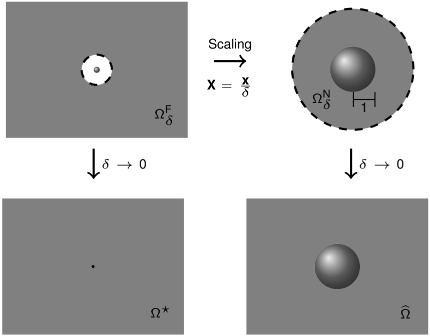


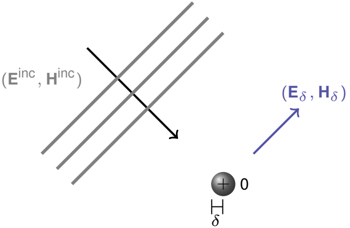


Asymptotic expansions

Far field expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$

Near field expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0,1)}$

Approximation of solution to single scattering



Method of matched asymptotic expansions

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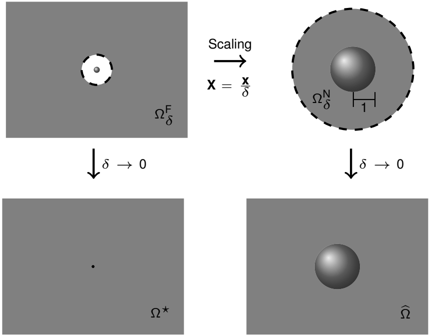

Asymptotic expansions

Far field expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$

Near field expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0,1)}$

Near field $\leftrightsquigarrow$ Far field

Asymptotic expansions

Near field expansion

$$\mathbf{E}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{E}}_0(\mathbf{X}) + \delta\widehat{\mathbf{E}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{E}}_2(\mathbf{X}) + \dots$$

$$\mathbf{H}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{H}}_0(\mathbf{X}) + \delta\widehat{\mathbf{H}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{H}}_2(\mathbf{X}) + \dots$$

Far field expansion

$$\mathbf{E}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{E}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{E}}_5(\mathbf{x}) + \dots$$

$$\mathbf{H}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{H}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{H}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{H}}_5(\mathbf{x}) + \dots$$

$\mathbf{X} = \frac{\mathbf{x}}{\delta}$: fast variable

- For an obstacle of arbitrary shape

  Numerical solution of elementary problems independent of δ
  - Near-field: stationnary problems
  - Far-field: time-harmonic problems + equivalent multipole distributions

- For a spherical obstacle

  - Analytical solutions of elementary problems
  - Equivalent multipole distributions

Equivalent multipole distributions

Boundary Value Problem	$\Longleftrightarrow$	Problem in the free-space

$$\begin{cases} \textbf{curl}\, \widehat{\mathbf{E}}_0 = 0 & \text{in } \widehat{\Omega} \\ \text{div}\, \widehat{\mathbf{E}}_0 = 0 & \text{in } \widehat{\Omega} \\ \mathbf{n} \times \widehat{\mathbf{E}}_0 = -\mathbf{n} \times \mathbf{E}^{\text{inc}}(0) & \text{on } \widehat{\Gamma} \\ \widehat{\mathbf{E}}_0 = O(\mathrm{R}^{-1}) & \mathrm{R} \to \infty \end{cases}$$

$$\widehat{\mathbf{E}}_0 = -\nabla V^{0,\text{stat}}_{1,\text{elec}} = -\nabla(\lim_{\epsilon \to 0} V^{\epsilon,\text{stat}}_{1\text{elec}})$$

$$\begin{cases} -\Delta V^{\epsilon,\text{stat}}_{1,\text{elec}} = \varrho^{\epsilon}_1 & \text{in } \mathbb{R}^3 \\ V^{\epsilon,\text{stat}}_{1,\text{elec}} = O(\mathrm{R}^{-1}) & \mathrm{R} \to \infty \end{cases}$$

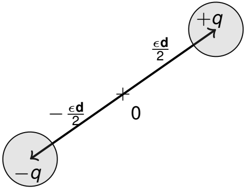

Figure: Electrostatic dipole

Distribution of two electric point-charges

$$\varrho^{\epsilon}_1 = \frac{1}{\epsilon}\left(\boldsymbol{\delta}_{\frac{\epsilon \mathbf{d}}{2}} - \boldsymbol{\delta}_{-\frac{\epsilon \mathbf{d}}{2}}\right)$$

Dipole moment **d** related to the BC

$$\mathbf{d} = 4\pi \mathbf{E}^{\text{inc}}(0)$$

Equivalent multipole distributions

Boundary Value Problem $\Longleftrightarrow$ Problem in the free-space

$$\begin{cases} \mathbf{curl}\, \widehat{\mathbf{E}}_0 = 0 & \text{in } \widehat{\Omega} \\ \operatorname{div} \widehat{\mathbf{E}}_0 = 0 & \text{in } \widehat{\Omega} \\ \mathbf{n} \times \widehat{\mathbf{E}}_0 = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}}(0) & \text{on } \widehat{\Gamma} \\ \widehat{\mathbf{E}}_0 = O(\mathrm{R}^{-1}) & \mathrm{R} \to \infty \end{cases}$$

$$\begin{cases} \operatorname{div} \widehat{\mathbf{E}}_0 = D\delta_0(\mathbf{d}) & \text{in } \mathbb{R}^3 \\ \mathbf{curl}\, \widehat{\mathbf{E}}_0 = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

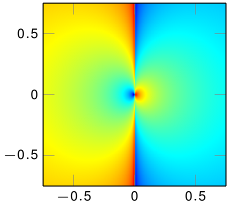

Figure: Electrostatic potential $\epsilon \to 0$

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Equivalent multipole distributions

Boundary Value Problem $\Longleftrightarrow$ Problem in the free-space

$$\begin{cases} \mathbf{curl}\,\widetilde{\mathbf{E}}_3 - i\kappa\,\widetilde{\mathbf{H}}_3 = 0 & \text{in } \Omega^\star \\ \mathbf{curl}\,\widetilde{\mathbf{H}}_3 + i\kappa\,\widetilde{\mathbf{E}}_3 = 0 & \text{in } \Omega^\star \\ \widetilde{\mathbf{E}}_3, \widetilde{\mathbf{H}}_3 = O(r^{-3}) & r \to 0 \\ +\ \text{Silver-Müller condition} & \text{at } \infty \end{cases}$$

$$\widetilde{\mathbf{E}}_3 = -\nabla\left(\lim_{\epsilon\to 0} V^\epsilon_{1,\mathrm{elec}}\right) + i\kappa \lim_{\epsilon\to 0} \mathbf{A}^\epsilon_{1,\mathrm{elec}} - \mathbf{curl} \lim_{\epsilon\to 0} \mathbf{A}^\epsilon_{1,\mathrm{mag}}$$

$$\widetilde{\mathbf{H}}_3 = -\nabla\left(\lim_{\epsilon\to 0} V^\epsilon_{1,\mathrm{mag}}\right) + i\kappa \lim_{\epsilon\to 0} \mathbf{A}^\epsilon_{1,\mathrm{mag}} + \mathbf{curl} \lim_{\epsilon\to 0} \mathbf{A}^\epsilon_{1,\mathrm{elec}}$$

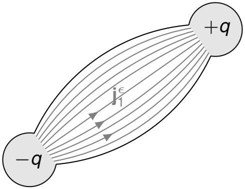


Figure: Time-harmonic electric dipole

Filiform current distribution

$$\mathbf{j}_1^\epsilon = \frac{i\omega}{\epsilon}\frac{\mathbf{d}_\mathrm{E}}{|\mathbf{d}_\mathrm{E}|}\boldsymbol{\delta}_{\mathcal{F}_\epsilon}$$

Dipole moment $\mathbf{d}_\mathrm{E}$ related to $\widehat{\mathbf{E}}_0$

$$\mathbf{d}_\mathrm{E} = 4\pi\mathbf{E}^{\mathrm{inc}}(0)$$

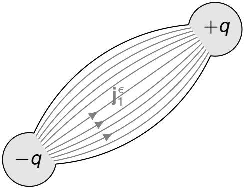
Equivalent multipole distributions

Boundary Value Problem $\Longleftrightarrow$ Problem in the free-space

$$\begin{cases} \textbf{curl}\, \widetilde{\mathbf{E}}_3 - i\kappa\, \widetilde{\mathbf{H}}_3 = 0 & \text{in } \Omega^\star \\ \textbf{curl}\, \widetilde{\mathbf{H}}_3 + i\kappa\, \widetilde{\mathbf{E}}_3 = 0 & \text{in } \Omega^\star \\ \widetilde{\mathbf{E}}_3, \widetilde{\mathbf{H}}_3 = O(r^{-3}) & r \to 0 \\ +\ \text{Silver-Müller condition} & \text{at } \infty \end{cases}$$

$$\begin{cases} -\Delta V^{\epsilon}_{1,\text{elec}} - \kappa^2 V^{\epsilon}_{1,\text{elec}} = \varrho^{\epsilon}_1 & \text{in } \mathbb{R}^3 \\ +\ \text{Sommerfield condition} & \text{R} \to \infty \end{cases}$$

$$\begin{cases} -\Delta \mathbf{A}^{\epsilon}_{1,\text{elec}} - \kappa^2 \mathbf{A}^{\epsilon}_{1,\text{elec}} = \dfrac{\mathbf{j}^{\epsilon}_1}{c} & \text{in } \mathbb{R}^3 \\ +\ \text{Sommerfield condition} & \text{R} \to \infty \end{cases}$$



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Equivalent multipole distributions

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$$\begin{cases} \textbf{curl}\, \widetilde{\mathbf{E}}_3 - i\kappa \widetilde{\mathbf{H}}_3 = -i\kappa \mathbf{d}_{\mathrm{H}} \delta_0 & \text{in } \mathbb{R}^3 \\ \textbf{curl}\, \widetilde{\mathbf{H}}_3 + i\kappa \widetilde{\mathbf{E}}_3 = i\kappa \mathbf{d}_{\mathrm{E}} \delta_0 & \text{in } \mathbb{R}^3 \\ \text{div}\, \widetilde{\mathbf{E}}_3 = D\delta_0(\mathbf{d}_{\mathrm{E}}) & \text{in } \mathbb{R}^3 \\ \text{div}\, \widetilde{\mathbf{H}}_3 = D\delta_0(\mathbf{d}_{\mathrm{H}}) & \text{in } \mathbb{R}^3 \\ +\ \text{Silver-Müller condition} & \text{at } \infty \end{cases}$$

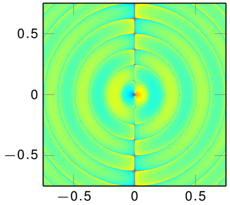

Figure: Electric potential $\epsilon \to 0$

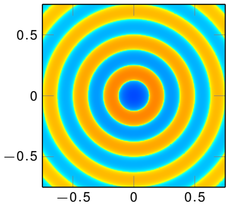

Figure: Magnetic potential $\epsilon \to 0$

Equivalent multipole distributions

$$\widehat{\mathbf{E}}_p = \sum_{\substack{n=1 \\ |n-p| \text{ odd}}}^{p+1} \mathcal{E}^{\text{stat}}_{n,\text{elec}}[\mathbf{u}^{p,\text{E}}_{1,n}, \dots, \mathbf{u}^{p,\text{E}}_{n,n}] + \sum_{\substack{n=1 \\ |n-p| \text{ even}}}^{p+1} \mathcal{E}^{\text{stat}}_{n,\text{mag}}[\mathbf{u}^{p,\text{H}}_{1,n}, \dots, \mathbf{u}^{p,\text{H}}_{n,n}]$$

$$\widetilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \left\{ \mathcal{E}_{n,\text{elec}}[\mathbf{v}^{p,\text{E}}_{1,n}, \dots, \mathbf{v}^{p,\text{E}}_{n,n}] + \mathcal{E}_{n,\text{mag}}[\mathbf{v}^{p,\text{H}}_{1,n}, \dots, \mathbf{v}^{p,\text{H}}_{n,n}] \right\}$$

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	●	●				
	1						
	2						
Far field	3	● ●	● ●				
	4						
	5						

● Electric field ● Magnetic field

+ Far field collected dipolar model

J. Labat, V. Péron, S. Tordeux (Inria Research Report n$^\circ$ 9169, Mars 2018)
Asymptotic Modeling of the Electromagnetic Scattering by Small Spheres Perfectly Conducting.

Equivalent multipole distributions

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	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	•	•				
	1	•	•	•	•		
	2						
Far field	3	• •	• •				
	4	• •	• •	• •	• •		
	5						

• Electric field • Magnetic field

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	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	electric	magnetic				
	1	magnetic	electric	electric	magnetic		
	2	electric	magnetic	magnetic	electric	electric	magnetic
Far field	3	electric, magnetic	electric, magnetic				
	4	electric, magnetic	electric, magnetic	electric, magnetic	electric, magnetic		
	5	electric, magnetic	electric, magnetic	electric, magnetic	electric, magnetic	electric, magnetic	electric, magnetic

• Electric field • Magnetic field

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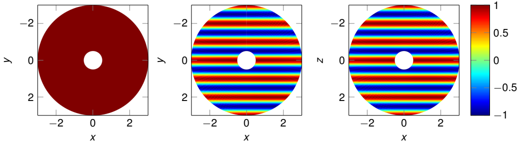
Numerical validation

Small parameter

$$\delta \in \left\{ \frac{1}{10^p},\ p = 0.5 : 0.1 : 4 \right\}$$

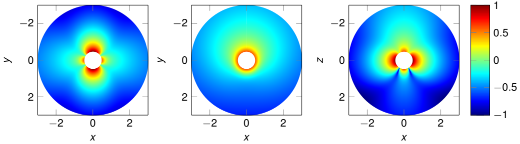
Physical parameters

- $\varepsilon = 1.0$
- $\mu = 1.0$
- $\sigma = 0$
- $\lambda = 5.0$



(a) $z = 0$ (b) $x = 0$ (c) $y = 0$

Figure: Incident plane wave (real part)



(a) $z = 0$ (b) $x = 0$ (c) $y = 0$

Figure: Reference scattered field (log)

Numerical validation

Small parameter

$$\delta \in \left\{ \frac{1}{10^p},\ p = 0.5 : 0.1 : 4 \right\}$$

Physical parameters

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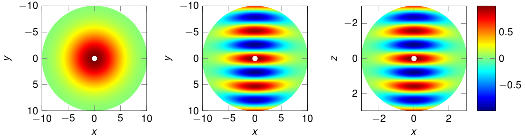
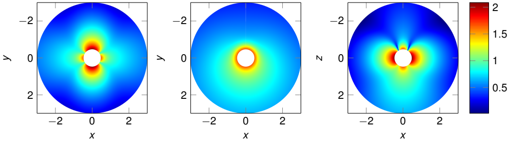


(a) $z = 0$ (b) $x = 0$ (c) $y = 0$

Figure: Incident gaussian beam (real part)



(a) $z = 0$ (b) $x = 0$ (c) $y = 0$

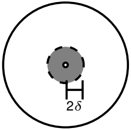
Figure: Reference scattered field (log)

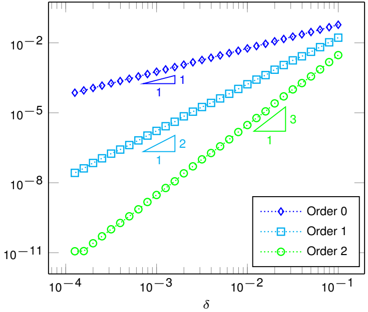
Numerical convergence of near field approximations

Order 0 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

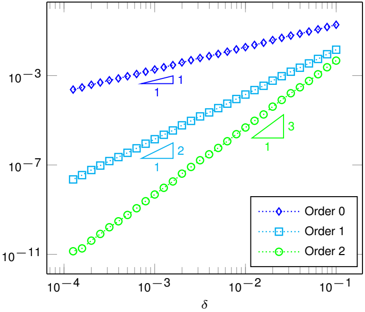
Order 1 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Order 2 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta}) - \delta^2\widehat{\mathbf{E}}_2(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$





(a) Electric field



(b) Magnetic Field

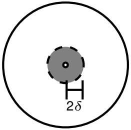
Figure: Relative $\mathbf{L}^2(\Omega_\delta^{2\delta})$-errors for near field approximations – Incident plane wave

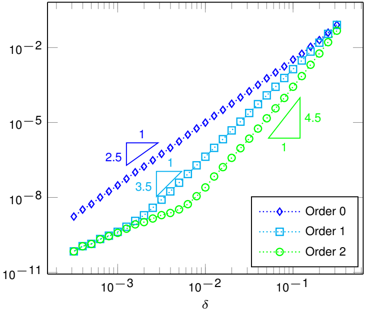
Numerical convergence of near field approximations

Order 0 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

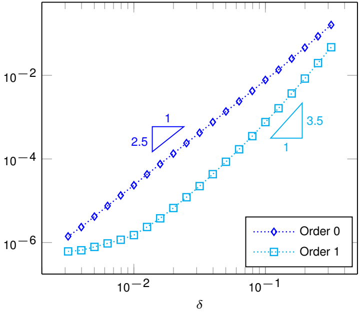
Order 1 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Order 2 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta}) - \delta^2\widehat{\mathbf{E}}_2(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$





(a) Electric field



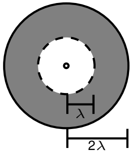
(b) Magnetic field

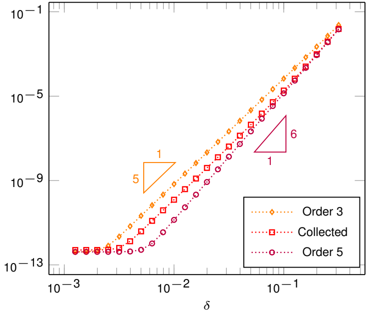
Figure: Relative $\mathbf{L}^2(\Omega_\delta^{2\delta})$-errors for near field approximations – Incident gaussian beam

Numerical convergence of far field approximations

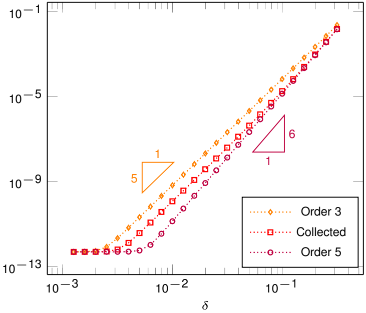
Order 3 : $\|\mathbf{E}_\delta - \delta^3 \widetilde{\mathbf{E}}_3\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

Order 5 : $\|\mathbf{E}_\delta - \delta^3 \widetilde{\mathbf{E}}_3 - \delta^5 \widetilde{\mathbf{E}}_5\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$





(a) Electric field



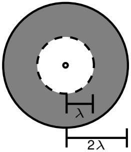
(b) Magnetic field

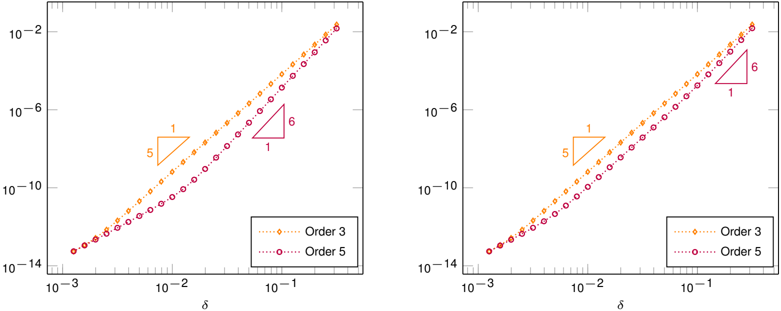
Figure: Absolute $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$-errors for far field approximations – Incident plane wave

Numerical convergence of far field approximations

Order 3 : $\|\mathbf{E}_\delta - \delta^3\widetilde{\mathbf{E}}_3\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

Order 5 : $\|\mathbf{E}_\delta - \delta^3\widetilde{\mathbf{E}}_3 - \delta^5\widetilde{\mathbf{E}}_5\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$





(a) Electric field

(b) Magnetic field

Figure: Absolute $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$-errors for far field approximations – Incident gaussian beam

Application: the Born approximation

- The electric field is approximated by the superposition principle

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a dipolar source around c_k

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \mathcal{E}_{1,\text{elec}}[\mathbf{d}^{\text{E}}_{\delta,k}](\mathbf{x} - c_k) + \mathcal{E}_{1,\text{mag}}[\mathbf{d}^{\text{H}}_{\delta,k}](\mathbf{x} - c_k)$$

- The dipole moments $\mathbf{d}^{\text{E}}_{\delta,k}$ and $\mathbf{d}^{\text{H}}_{\delta,k}$ depend on the nature of the obstacles

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For perfectly conducting spheres:

- Approximation of order 3

$$\mathbf{d}^{\text{E}}_{\delta,k} = 4\pi\delta^3\mathbf{E}^{\text{inc}}(c_k) \qquad \mathbf{d}^{\text{H}}_{\delta,k} = -2\pi\delta^3\mathbf{H}^{\text{inc}}(c_k)$$

- Collected dipolar approximation

$$\mathbf{d}^{\text{E}}_{\delta,k} = 4\pi\delta^3\left(1 + \frac{3(\kappa\delta)^2}{10}\right)\mathbf{E}^{\text{inc}}(c_k) \qquad \mathbf{d}^{\text{H}}_{\delta,k} = -2\pi\delta^3\left(1 - \frac{3(\kappa\delta)^2}{5}\right)\mathbf{H}^{\text{inc}}(c_k)$$

No interaction are taken into account

Outline

Foldy-Lax model

- The electromagnetic fields are approximated by the superposition principle

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \qquad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a dipolar source around c_k

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \mathcal{E}_{1,\text{elec}}[\mathbf{d}^{\text{E}}_{\delta,k}](\mathbf{x} - c_k) + \mathcal{E}_{1,\text{mag}}[\mathbf{d}^{\text{H}}_{\delta,k}](\mathbf{x} - c_k)$$
$$\mathbf{H}_{\delta,k}(\mathbf{x}) = \mathcal{H}_{1,\text{elec}}[\mathbf{d}^{\text{E}}_{\delta,k}](\mathbf{x} - c_k) + \mathcal{H}_{1,\text{mag}}[\mathbf{d}^{\text{H}}_{\delta,k}](\mathbf{x} - c_k)$$

For perfectly conducting spheres:

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$$\mathbf{d}^{\text{E}}_{\delta,k} = 4\pi\delta^3 \Big(\mathbf{E}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell\neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}(c_k) \Big) \qquad \mathbf{d}^{\text{H}}_{\delta,k} = -2\pi\delta^3 \Big(\mathbf{H}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell\neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}(c_k) \Big)$$

- Collected dipolar approximation

$$\mathbf{d}^{\text{E}}_{\delta,k} = 4\pi\delta^3 \Big(1 + \frac{3(\kappa\delta)^2}{10}\Big) \Big(\mathbf{E}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell\neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}(c_k) \Big) \qquad \mathbf{d}^{\text{H}}_{\delta,k} = -2\pi\delta^3 \Big(1 - \frac{3(\kappa\delta)^2}{5}\Big) \Big(\mathbf{H}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell\neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}(c_k) \Big)$$

Foldy-Lax model

- The electromagnetic fields are approximated by the superposition principle

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \qquad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

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$$\mathbf{H}_{\delta,k}(\mathbf{x}) = \mathcal{H}_{1,\text{elec}}[\mathbf{d}^{\text{E}}_{\delta,k}](\mathbf{x} - c_k) + \mathcal{H}_{1,\text{mag}}[\mathbf{d}^{\text{H}}_{\delta,k}](\mathbf{x} - c_k)$$

- Vectorial formulation (3-order approximation)

Find $\mathbf{d} = \left((\mathbf{d}^{\text{E}}_1), \dots, (\mathbf{d}^{\text{E}}_{N_{\text{obs}}}), (\mathbf{d}^{\text{H}}_1) \dots, (\mathbf{d}^{\text{H}}_{N_{\text{obs}}})\right)^\top \in \mathbb{C}^{6N_{\text{obs}}}$ such that

$$\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

with $\mathbf{f} = \begin{pmatrix} 4\pi\mathbf{E}^{\text{inc}}(c_1) \\ \vdots \\ 4\pi\mathbf{E}^{\text{inc}}(c_{N_{\text{obs}}}) \\ -2\pi\mathbf{H}^{\text{inc}}(c_1) \\ \vdots \\ -2\pi\mathbf{H}^{\text{inc}}(c_{N_{\text{obs}}}) \end{pmatrix}$ and $\mathbb{A}$ the "interaction" matrix

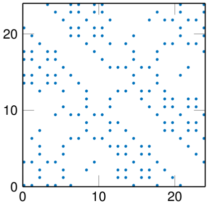

Figure: Skeleton of $\mathbb{A}$ for $N_{\text{obs}} = 4$

Spectral

Foldy-Lax approximations

- Approximation of order 3

$$\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

- Collected dipolar approximation

$$\mathbb{D}^{-1}\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\mathbb{D} = \mathrm{diag}(\underbrace{\alpha, \dots, \alpha}_{3N_{\mathrm{obs}}}, \underbrace{\beta, \dots, \beta}_{3N_{\mathrm{obs}}}) \qquad \alpha = 1 + \frac{3(\kappa\delta)^2}{10} \qquad \beta = 1 - \frac{6(\kappa\delta)^2}{10}$$

- Modified dipolar approximation

$$\widetilde{\mathbb{D}}^{-1}\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\widetilde{\mathbb{D}} = \mathrm{diag}(\underbrace{\widetilde{\alpha}, \dots, \widetilde{\alpha}}_{3N_{\mathrm{obs}}}, \underbrace{\widetilde{\beta}, \dots, \widetilde{\beta}}_{3N_{\mathrm{obs}}}) \qquad \widetilde{\alpha} = \frac{3i}{2(\kappa\delta)^3} \frac{j_1(\kappa\delta)}{h_1^{(1)}(\kappa\delta)} \qquad \widetilde{\beta} = -\frac{3i}{(\kappa\delta)^3} \frac{j_1(\kappa\delta) + \kappa\delta j_1'(\kappa\delta)}{h_1^{(1)}(\kappa\delta) + \kappa\delta h_1^{(1)\prime}(\kappa\delta)}$$

Foldy-Lax approximations

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$$\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

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$$\mathbb{D}^{-1}\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

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$$\mathbb{D} = \text{diag}(\underbrace{\alpha, \dots, \alpha}_{3N_{\text{obs}}}, \underbrace{\beta, \dots, \beta}_{3N_{\text{obs}}}) \qquad \alpha = 1 + \frac{3(\kappa\delta)^2}{10} \qquad \beta = 1 - \frac{6(\kappa\delta)^2}{10}$$

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Spectral method

- The electromagnetic fields are represented by the Stratton-Chu formula for $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_k(\mathbf{y}) \, ds_{\mathbf{y}}$$

where $\Phi(\mathbf{x}, \mathbf{y}) = \frac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$

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- Each $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ satisfies the boundary integral equation

$$\sum_{j=1}^{N_{\text{obs}}} \mathcal{M}_\Gamma^{kj} \mathbf{p}_j = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \qquad \text{on } \Gamma_\delta^k$$

where $\mathcal{M}_\Gamma^{kj} : \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\delta^j}, \Gamma_\delta^j) \longrightarrow \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ is the extension of

$$\mathcal{M}_\Gamma^{kj} \boldsymbol{\lambda}(\mathbf{x}_\Gamma) = \mathbf{n}(\mathbf{x}_\Gamma) \times \lim_{\mathbf{x} \to \mathbf{x}_\Gamma} \mathbf{curl} \int_{\Gamma_\delta^j} \Phi(\mathbf{x}, \mathbf{y}) \boldsymbol{\lambda}(\mathbf{y}) \, ds_{\mathbf{y}} \qquad \boldsymbol{\lambda} \in \mathscr{C}^\infty(\Gamma_\delta^j) \qquad \mathbf{x}_\Gamma \in \Gamma_\delta^k$$

Spectral method

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- Galerkin discretization of the BIE on local spectral basis with N_{mod} modes

$$\mathbf{p}_j(\mathbf{x}) = \sum_{n=1}^{N_{\text{mod}}} \sum_{m=-n}^{n} p_{n,m}^{j,\perp} \nabla_{\mathbb{S}^2} Y_{n,m}(\widehat{x}_j) + p_{n,m}^{j,\times} \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}(\widehat{x}_j) \qquad \mathbf{x} \in \Gamma_\delta^j$$

with $\widehat{x}_j = \frac{\mathbf{x} - c_j}{|\mathbf{x} - c_j|}$ and $\nabla_{\mathbb{S}^2} Y_{n,m}, \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}$: complex-valued vector spherical harmonics

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M. Ganesh, S.C. Hawkins (2009)
A high-order algorithm for multiple electromagnetic scattering in three dimensions

H. Barucq, J. Chabassier, H. Pham, S. Tordeux (2017)
Numerical robustness of single-layer method with Fourier basis for multiple obstacle acoustic scattering in homogeneous media

Vectorial formulation

- Variational formulation: Find $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ such that

$$\langle \mathcal{M}_\Gamma^{kj} \mathbf{p}_j, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\mathrm{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \qquad \forall\, \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

- Vectorial formulation: Find $\mathbf{p} = ((p_{n,m}^{1,\perp}), \dots, (p_{n,m}^{N_{\mathrm{obs}},\perp}), (p_{n,m}^{1,\times}), \dots, (p_{n,m}^{N_{\mathrm{obs}},\times}))^\top \in \mathbb{C}^{\boldsymbol{N}}$ s.t.

$$\mathbb{M}\, \mathbf{p} = \mathbf{f}$$

with $\boldsymbol{N} = 2N_{\mathrm{mod}}(N_{\mathrm{mod}} + 2)N_{\mathrm{obs}}$ and

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_{\perp\perp} & \mathbb{M}_{\perp\times} \\ \mathbb{M}_{\times\perp} & \mathbb{M}_{\times\times} \end{pmatrix} \quad \text{with} \quad \mathbb{M}_{\alpha\beta} = \begin{pmatrix} \mathbb{M}_{\alpha\beta}^{11} & \mathbb{M}_{\alpha\beta}^{12} & \dots & \mathbb{M}_{\alpha\beta}^{1N_{\mathrm{obs}}} \\ \mathbb{M}_{\alpha\beta}^{21} & \mathbb{M}_{\alpha\beta}^{22} & \dots & \mathbb{M}_{\alpha\beta}^{2N_{\mathrm{obs}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{M}_{\alpha\beta}^{N_{\mathrm{obs}}1} & \mathbb{M}_{\alpha\beta}^{N_{\mathrm{obs}}2} & \dots & \mathbb{M}_{\alpha\beta}^{N_{\mathrm{obs}}N_{\mathrm{obs}}} \end{pmatrix}$$

$$= \mathbb{M}_{\alpha\beta}(\delta, \mathbf{d}_{jk})$$

Y.-L. Xu (1995)
Electromagnetic scattering by an aggregate of small spheres

Vectorial formulation

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- Vectorial formulation: Find $\mathbf{p} = ((p_{n,m}^{1,\perp}), \ldots, (p_{n,m}^{N_{\mathrm{obs}},\perp}), (p_{n,m}^{1,\times}), \ldots, (p_{n,m}^{N_{\mathrm{obs}},\times}))^\top \in \mathbb{C}^{\boldsymbol{N}}$ s.t.

$$\mathbb{M}\, \mathbf{p} = \mathbf{f}$$

Computation time
Memory consumption
Ill-conditionned system

- Anti-diagonal coefficients
- Number of unknowns
- Dense matrix

Mex interface (C)
Preconditionning
Smart storage and assembling

- Sub-blocks $\mathbb{M}_{\alpha\beta}$ depend on $\mathbf{d}_{jk}$
- Linear algebra
- Sparse matrix

Linear transformations

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}) \qquad \text{with} \qquad \mathbf{p} = \mathbf{p}(\delta, \mathbf{d}_{jk}) \qquad \text{solves} \qquad \mathbb{M}\,\mathbf{p} = \mathbf{f}$$

- Step 1: Change of variable $\mathbb{T}_\delta$

$$\widetilde{\mathbf{p}} := \mathbb{T}_\delta \mathbf{p} \qquad \Longrightarrow \qquad \mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}})$$

- Step 2: Permutation matrix $\mathbb{P}$

$$(\mathrm{diag}\,\mathbb{P}\,\mathbb{M})_k \neq 0 \qquad \forall\, k = 1, \ldots, \boldsymbol{N}$$

- Step 3: Diagonal matrix $\mathbb{D}_\delta$

$$\widetilde{\mathbb{M}} := \mathbb{D}_\delta^{-1}\mathbb{P}\,\mathbb{M} \qquad \Longrightarrow \qquad (\mathrm{diag}\,\widetilde{\mathbb{M}})_k = 1 \qquad \forall\, k = 1, \ldots, \boldsymbol{N}$$

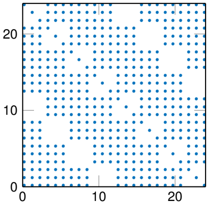
$$\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}) \qquad \text{with} \qquad \widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \qquad \text{solves} \qquad \widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}} = \mathbb{D}_\delta^{-1}\mathbb{P}\,\mathbf{f}$$

Preconditionning

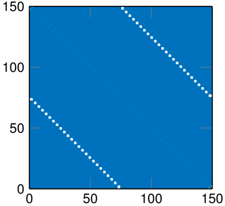
$$\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}) \qquad \text{with} \qquad \widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \qquad \text{solves} \qquad \widetilde{\mathbb{M}}\, \widetilde{\mathbf{p}} = \mathbb{D}_\delta^{-1} \mathbb{P}\, \mathbf{f}$$

Number of unknowns $\boldsymbol{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$
Dense matrix (approximation of coefficients)

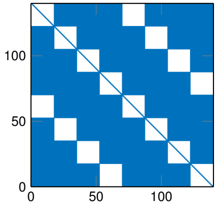
Memory consumption



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$



(b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$



(c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

Figure: Skeletton of $\widetilde{\mathbb{M}}$ Foldy

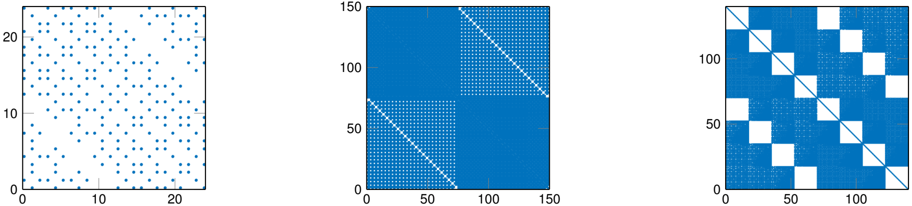
Preconditionning

$$\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}) \qquad \text{with} \qquad \widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \qquad \text{solves} \qquad \widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}} = \mathbb{D}_\delta^{-1}\mathbb{P}\,\mathbf{f}$$

Number of unknowns $\boldsymbol{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$
Dense matrix (approximation of coefficients)

Memory consumption

Error of approximation
Analytic preconditionning (dipole)
Algebraic preconditionning (small criterion)



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$ (b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$ (c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

Figure: Skeletton of $\widetilde{\mathbb{M}}$

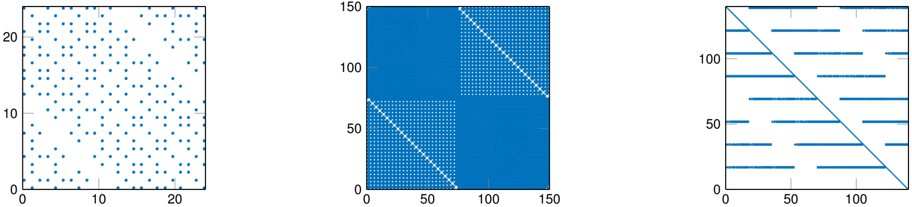
Preconditionning

$$\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}) \quad \text{with} \quad \widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}} = \mathbb{D}_\delta^{-1}\mathbb{P}\,\mathbf{f}$$

- Number of unknowns $\boldsymbol{N} = 2N_{\mathrm{mod}}(N_{\mathrm{mod}}+2)N_{\mathrm{obs}}$
  Dense matrix (approximation of coefficients)

Memory consumption

- Error of approximation
  Analytic preconditionning (dipole)
  Algebraic preconditionning (small criterion)



(a) $N_{\mathrm{obs}} = 4$ and $N_{\mathrm{mod}} = 1$ (b) $N_{\mathrm{obs}} = 25$ and $N_{\mathrm{mod}} = 1$ (c) $N_{\mathrm{obs}} = 4$ and $N_{\mathrm{mod}} = 5$

Figure: Skeletton of Precond($\widetilde{\mathbb{M}}$)

Preconditionning

$$\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}) \quad \text{with} \quad \widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}(\delta, \mathbf{d}_{jk}) \quad \text{solves} \quad \widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}} = \mathbb{D}_\delta^{-1}\mathbb{P}\,\mathbf{f}$$

Number of unknowns $\boldsymbol{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$
Dense matrix (approximation of coefficients)

Memory consumption

Error of approximation
Analytic preconditionning (dipole)
Algebraic preconditionning (small criterion)

Sparse matrix

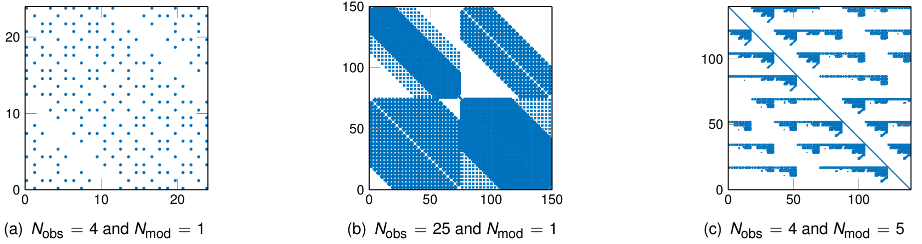


Figure: Skeletton of Precond($\widetilde{\mathbb{M}}$)

Assembling for axisymmetric configurations

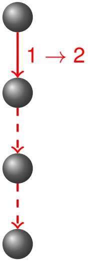
- Example: 4 aligned obstacles

$$\widetilde{\mathbb{M}} = \left[\begin{array}{cccc|cccc} \mathrm{Id} & & & & 0 & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & \mathrm{Id} & & & & 0 \\ \hline 0 & & & & \mathrm{Id} & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & \mathrm{Id} \end{array}\right]$$

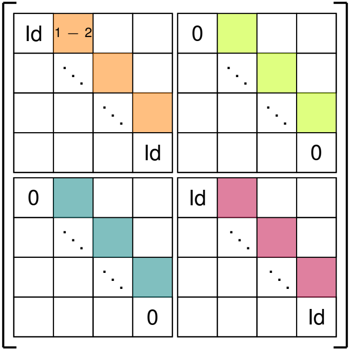
The sub-blocks $\mathbb{M}^{kj}_{\alpha\beta}$ depend only on δ and $\mathbf{d}_{jk}$

Assembling for axisymmetric configurations

- Example: 4 aligned obstacles

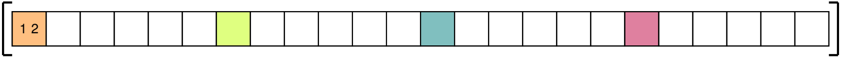


$\widetilde{\mathbb{M}} =$



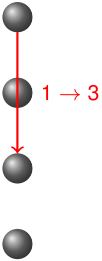
- Storage of sub-blocks

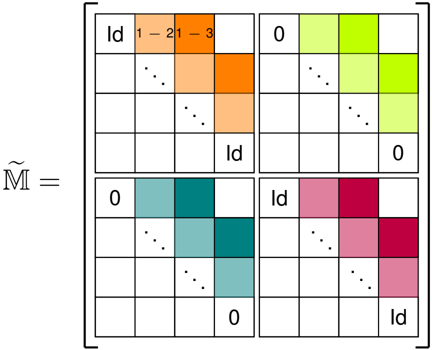
$\widetilde{\mathbb{M}}_{\text{Block}} =$



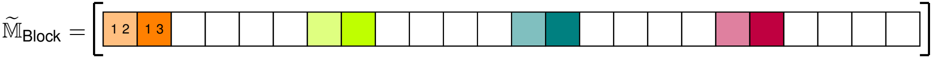
Assembling for axisymmetric configurations

- Example: 4 aligned obstacles



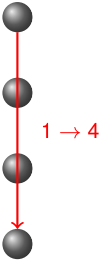


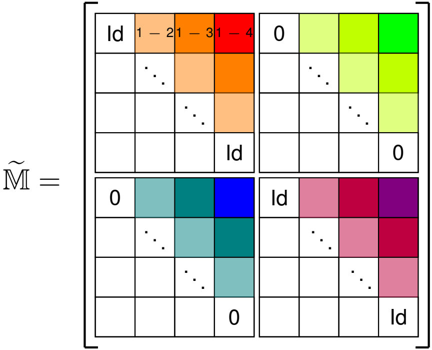
- Storage of sub-blocks



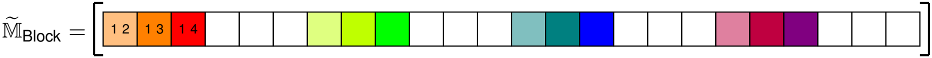
Assembling for axisymmetric configurations

- Example: 4 aligned obstacles



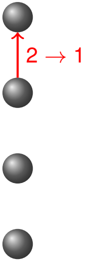


- Storage of sub-blocks

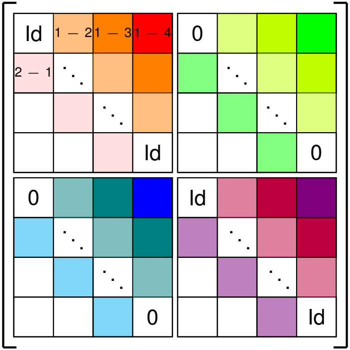


Assembling for axisymmetric configurations

- Example: 4 aligned obstacles

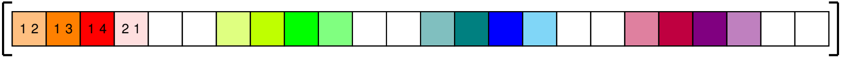


$\widetilde{\mathbb{M}} =$



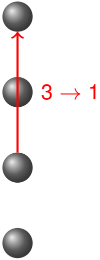
- Storage of sub-blocks

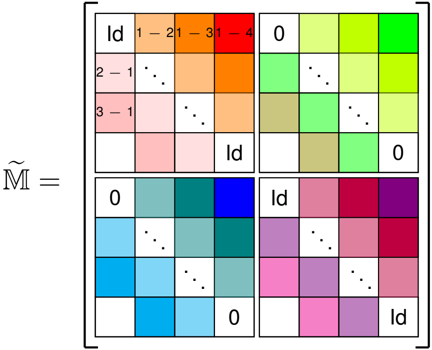
$\widetilde{\mathbb{M}}_{\text{Block}} =$



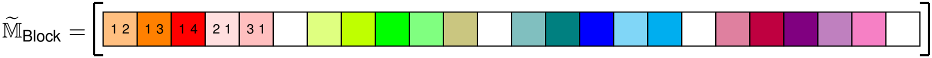
Assembling for axisymmetric configurations

- Example: 4 aligned obstacles



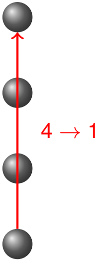


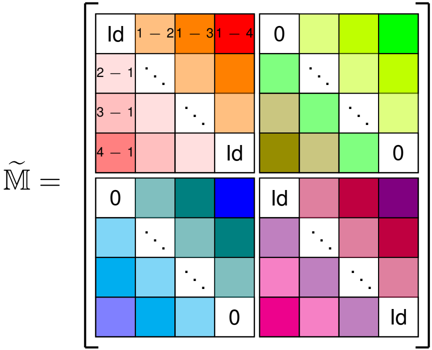
- Storage of sub-blocks



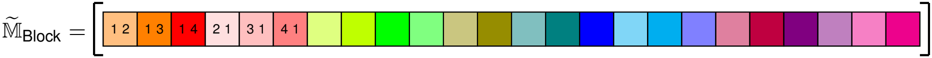
Assembling for axisymmetric configurations

- Example: 4 aligned obstacles





- Storage of sub-blocks



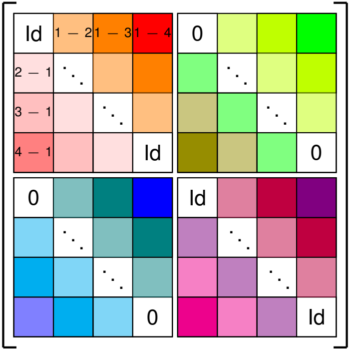
Assembling for axisymmetric configurations

- Example: 4 aligned obstacles

$8(N_{\text{obs}} - 1)$ blocks

instead of $4(N_{\text{obs}})^2$

$\widetilde{\mathbb{M}} =$



- Storage of sub-blocks

$\widetilde{\mathbb{M}}_{\text{Block}} =$

N_{obs}	$4(N_{\text{obs}})^2$	$8(N_{\text{obs}} - 1)$	Ratio (%)	$\widetilde{\mathbb{M}}$ (GB)	$\widetilde{\mathbb{M}}_{\text{Block}}$ (GB)
100	40 000	792	1.98	$5.76 \cdot 10^{-3}$	$1.15 \cdot 10^{-4}$
1 000	4 000 000	7 992	$1.998 \cdot 10^{-3}$	$5.76 \cdot 10^{-1}$	$1.15 \cdot 10^{-3}$
3 000	36 000 000	23 992	$6.664 \cdot 10^{-4}$	4.01	$3.45 \cdot 10^{-3}$
10 000	400 000 000	79 992	$1.9998 \cdot 10^{-4}$	- -	$1.15 \cdot 10^{-2}$

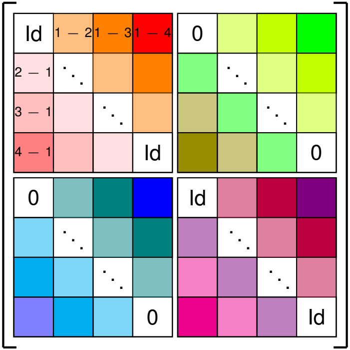
Assembling for axisymmetric configurations

- Example: 4 aligned obstacles

$8(N_{\text{obs}} - 1)$ blocks

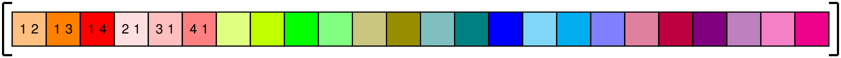
instead of $4(N_{\text{obs}})^2$

$\widetilde{\mathbb{M}} =$



- Storage of sub-blocks

$\widetilde{\mathbb{M}}_{\text{Block}} =$

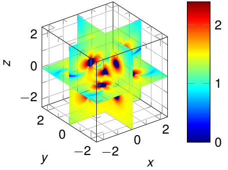


- Iterative solvers: define the action of $\widetilde{\mathbb{M}}_{\text{Block}}$ on $\mathbf{u}$

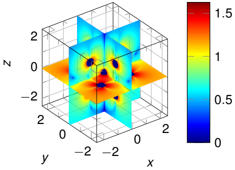
$$\mathcal{A}(\widetilde{\mathbb{M}}_{\text{Block}}, \mathbf{u}) := \widetilde{\mathbb{M}}\, \mathbf{u}$$

Numerical validation

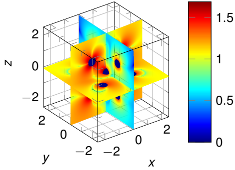
- 13 obstacles shared around the origin – Arbitrary configuration



(a) *x*-coordinates

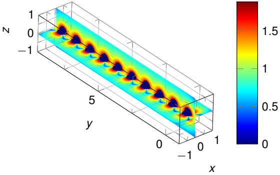


(b) *y*-coordinates

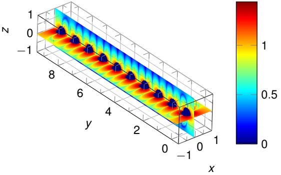


(c) *z*-coordinates

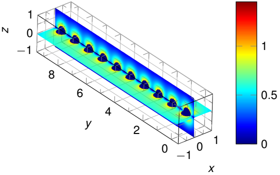
- 10 aligned obstacles – Axisymmetric configuration



(d) *x*-coordinates

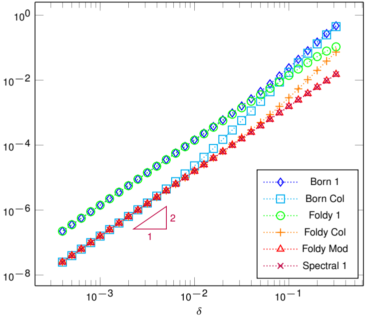


(e) *y*-coordinates

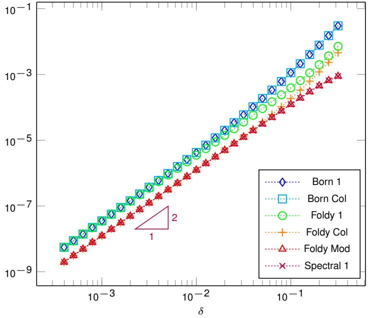


(f) *z*-coordinates

Numerical convergence of reduced models



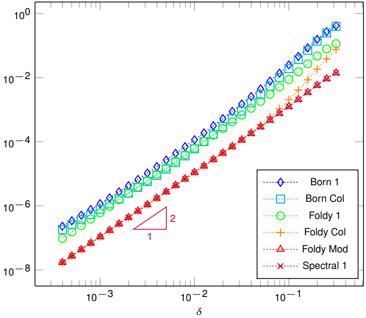
(g) Arbitrary configuration $N_{\text{obs}} = 13$



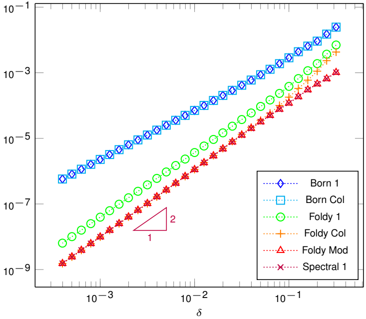
(h) Aligned obstacles $N_{\text{obs}} = 20$

Figure: Relative $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$-errors for electric field with $|\mathbf{d}_{jk}| \approx 1.0$

Numerical convergence of reduced models



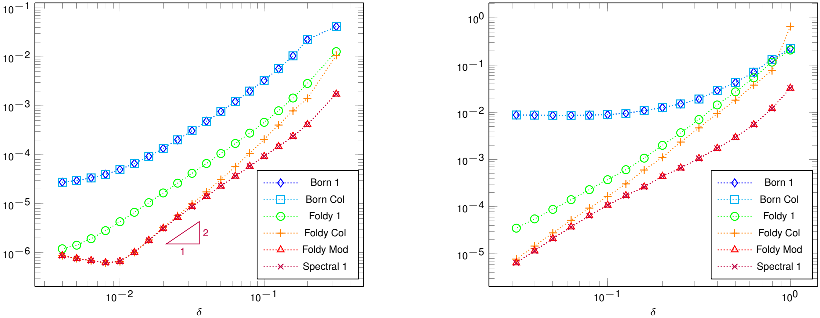
(a) Arbitrary configuration $N_{\text{obs}} = 13$



(b) Aligned obstacles $N_{\text{obs}} = 20$

Figure: Relative $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$-errors for electric field with $|\mathbf{d}_{jk}| \approx 2\sqrt{\delta}$

Numerical convergence of reduced models



(a) Arbitrary configuration $N_{\text{obs}} = 13$ (b) Aligned obstacles $N_{\text{obs}} = 20$

Figure: Relative $\mathbf{L}^2(\Omega_\lambda^{2\lambda})$-errors for electric field with $|\mathbf{d}_{jk}| \approx 4\delta$

Time comparison

In terms of accuracy

Modified Foldy $\Longleftrightarrow$ Spectral 1

	Modified Foldy			Spectral 1 (arbitrary configuration)		
N_{obs}	100	1000	2000	100	1000	2000
Matrix	0.5588	51.0912	190.6899	10.0850	722.7368	5252
Right hand-side	0.000525	0.000648	0.0232	0.0045	0.0343	0.0722
Inversion	0.0157	7.1581	72.2991	0.0948	10.6212	90.1667
Time linear system	0.5750	58.2499	263.0122	10.1843	733.3923	5342.2
Representation	4.3079	48.4771	105.1629	4.2486	46.5048	89.4991
Total time	4.8829	106.7270	368.1751	14.4329	779.8971	5431.7

Table: Time computation in seconds

Time comparison

In terms of accuracy

Modified Foldy $\Longleftrightarrow$ Spectral 1

	Modified Foldy			Spectral 1 (aligned configuration)		
N_{obs}	100	1000	2000	100	1000	2000
Matrix	0.5588	51.0912	190.6899	9.4903	105.2278	208.5052
Right hand-side	0.000525	0.000648	0.0232	0.0042	0.0320	0.0722
Inversion	0.0157	7.1581	72.2991	0.1656	10.8251	88.4188
Time linear system	0.5750	58.2499	263.0122	9.6001	116.0849	298.4607
Representation	4.3079	48.4771	105.1629	4.3087	44.6488	89.8913
Total time	4.8829	106.7270	368.1751	13.9088	160.7337	388.3520

Table: Time computation in seconds

Outline

Conclusions and Perspectives

Conclusion

- ✓ Numerical validation of the matched asymptotic expansions for single scattering
- ✓ Numerical validation of Born and Foldy-Lax models for multiple scattering
- ✓ Derivation and implementation of spectral method at any order

On-going work

- ∼ Smart assembling of spectral matrix for axisymmetric configurations
- ∼ Numerical validation of spectral method with finite element solutions
- ∼ Comparison of preconditionners and iterative solvers

Perspectives

- ✗ Justification of the matched asymptotic expansions
- ✗ High-order Foldy-Lax models for multiple scattering
- ✗ Extension to time-dependent domain
- ✗ Numerical extension to obstacles of arbitrary shape

Thank you for your attention :)