



**HAL**  
open science

## On the Decision Problem for MELL

Lutz Strassburger

► **To cite this version:**

Lutz Strassburger. On the Decision Problem for MELL. [Research Report] 9203, Inria Saclay Ile de France. 2018. hal-01870148

**HAL Id: hal-01870148**

**<https://inria.hal.science/hal-01870148>**

Submitted on 7 Sep 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



# On the Decision Problem for MELL

Lutz Straßburger

**RESEARCH  
REPORT**

**N° 9203**

September 2018

Project-Team Parsifal





## On the Decision Problem for MELL

Lutz Straßburger

Project-Team Parsifal

Research Report n° 9203 — September 2018 — 11 pages

**Abstract:** In this short paper I will exhibit several mistakes in the recent attempt by Bimbó [1] to prove the decidability of the multiplicative exponential fragment of linear logic (MELL). In fact, the main mistake is so serious that there is no obvious fix, and therefore the decidability of MELL remains to be an open problem. As a side effect, this paper contains a complete (syntactic) proof of the decidability of the relevant version of MELL (called RMELL in this paper), that is the logic obtained from MELL by replacing the linear logic contraction rule by a general unrestricted version of the contraction rule. This proof can also be found (with a small error) in [1], and a semantic proof can be found in [33].

**Key-words:** linear logic, MELL, decidability, relevant logic

**RESEARCH CENTRE  
SACLAY – ÎLE-DE-FRANCE**

1 rue Honoré d'Estienne d'Orves  
Bâtiment Alan Turing  
Campus de l'École Polytechnique  
91120 Palaiseau

## Sur le problème de la décision pour MELL

**Résumé :** Dans cet article, je présenterai plusieurs erreurs dans l'article récent de Bimbó [1] sur la décidabilité de la fragment multiplicatif exponentiel de la logique linéaire (MELL). En fait, l'erreur principale est si grave qu'il n'y a pas une solution évidente, et par conséquent, la décidabilité de MELL reste un problème ouvert. En plus, cet article contient un complet preuve (syntaxique) de la décidabilité de la version pertinente de MELL (appelé RMELL dans cet article), c'est la logique obtenue à partir de MELL en remplaçant la règle de contraction de la logique linéaire par un version générale sans restriction. Cette preuve peut également être trouvé (avec une petite erreur) dans [1], et une preuve sémantique peut être trouvée dans [33].

**Mots-clés :** logique linéaire, MELL, décidabilité

## 1 Introduction

Since the beginning of linear logic [7], the complexity of the decision problem of its various fragments has been studied by many researchers. For example, its multiplicative fragment (MLL) is NP-complete [16], and its multiplicative-additive fragment (MALL) is PSPACE-complete [25]. But the complexity of its multiplicative-exponential fragment (MELL) is still unknown. In fact, it is an open problem whether that logic is decidable. Finally, the multiplicative-additive-exponential fragment, i.e., full propositional linear logic (LL) is undecidable [25]. However, if we add second-order propositional quantifiers, already the multiplicative fragment (MLL2) is undecidable [20]. On the other hand, if we add a self-dual non-commutative multiplicative connective, the multiplicative fragment stays NP-complete [15] (this logic is called *pomset logic* [36] or BV [11, 8]), but the multiplicative exponential fragment is undecidable [41] (this logic is called NEL [12, 42, 13]).

It was observed early on that the reachability problem for Petri nets [35] can be encoded into MELL [30]. That problem has been shown to be decidable [27, 18, 28], but its precise complexity is still unknown. It is EXPSPACE-hard [26] and the known algorithms have runtimes that are not primitive recursive [24]. It has been known for a long time that the reachability problem for Petri nets is equivalent to the reachability problem of vector addition systems with states (VASS) [37]. Furthermore, it has been shown recently that the decidability problem of MELL is equivalent to the reachability problem for branching VASS [6], for which very recently a non-elementary lower bound has been found [22]. More precisely, if MELL turns out to be decidable it will be at least TOWER-hard [22, Corollary 22].

Since all known proofs [27, 18, 28, 37, 23] of the decidability of the reachability for Petri nets and VASS are very involved—in fact, the complete proof fills a textbook [37]—there is interest in the community in an alternative proof, which could be provided by a proof-theoretical proof of the decidability of MELL. The recent proposal by Bimbó [1] gives such a proof which is surprisingly simple. All the details could be given in less than ten pages.<sup>1</sup>

However, an inspection of the proof by Bimbó [1] shows immediately that the same argument also works for the subexponential variant of MELL, denoted by MSELL, that is obtained by allowing not one but several pairs of the modalities  $?$  and  $!$ , which are subject to a partial order relation. But that logic has recently been shown to be undecidable for the case of three pairs of  $?$  and  $!$  by Chaudhuri in [4]. Furthermore, a minor variation of the argument in [1] (using results from [33]) would also allow to prove decidability of LL, which also known to be undecidable [25], as mentioned above.

This, of course, is enough to dismiss [1] as being erroneous. However, that is not helpful, neither for the author of [1], nor anyone else who would like to understand what is going on. For this reason, I will in Section 4 of this paper explain in more detail the technical mistakes of [1]. After all, the main gap in the proof is a rather subtle mistake that could easily be repeated in other contexts.

Before coming to that, let me finish this introduction with the observation that the decision problem for MELL is—no matter how it will turn out—very close to the border between the decidable and the undecidable: adding just a little bit, i.e., a third self-dual (non-commutative) multiplicative connective or the additive connectives or subexponentials, renders the problem undecidable, and removing just a little bit, i.e., stepping down from branching VASS to non-branching VASS, puts the problem in the realm of the decidable.

This might be one of the reasons why the problem receives so much attention and is, after more than 3 decades, still open.

## 2 MELL and RMELL

To make this paper self-contained, I give the sequent calculus for MELL (called CLL<sub>int</sub> in [1]) and its relevant version RMELL (called RLL<sub>int</sub> in [1]) below. *Formulas* (denoted by capital Roman letters  $A, B, C, \dots$ ) are generated from propositional variables  $\{a, b, c, \dots\}$  and their duals  $\{a^\perp, b^\perp, c^\perp, \dots\}$  via the grammar:

$$A, B ::= a \mid a^\perp \mid A \wp B \mid A \otimes B \mid ?A \mid !A$$

As in [1], I work here in the unit-free fragment. *Sequents* (denoted by capital Greek letters  $\Gamma, \Delta, \dots$ ) are (possibly empty) finite multisets of formulas, written as list separated by commas, with a preceding turnstile:  $\vdash A_1, A_2, \dots, A_n$ . The inference rules for MELL are the following<sup>2</sup>:

$$\begin{array}{c} \text{id} \frac{}{\vdash a, a^\perp} \quad \wp \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \quad \otimes \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \\ \\ \! \frac{\vdash A, ?\Delta}{\vdash !A, ?\Delta} \quad ?\text{d} \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \quad ?\text{w} \frac{\vdash \Gamma}{\vdash \Gamma, ?A} \quad ?\text{c} \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} \end{array} \quad (1)$$

<sup>1</sup>Bimbó [1] does not speak about complexity, but previous work by Urquhart [43, 44] gives an EXPSPACE lower bound, and more recently Weiermann and Bunder [45] have shown an ACKERMANN lower bound.

<sup>2</sup>I am using here the standard names from the linear logic literature.

where  $?\Delta$  stands for a sequent in which every formula is of shape  $?B$  for some  $B$ .

The relevant version RMELL, is obtained from MELL by adding a general contraction rule

$$\text{c} \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \quad (2)$$

Of course, the standard linear logic contraction rule  $?c$  is a special case of this and can therefore be omitted from the system.

Note that in order to save paper, ink, and the patience of the reader, I use here the one-sided presentation of the sequent calculus, whereas [1] uses a two-sided presentation. Everything I present here also works in the two-sided systems, we only have to replace the one-sided axiom by a negation rule and a two-sided identity axiom:

$$\text{id} \frac{}{\vdash a, a^\perp} \quad \rightarrow \quad (\cdot)^\perp \frac{\text{id} \frac{}{a \vdash a}}{\vdash a, a^\perp}$$

### 3 MSELL and RMSELL

For making some arguments in this paper clearer, I will also introduce the subexponential version of MELL, also called *multiplicative subexponential linear logic* [31, 32], or MSELL. However, for the main points of this paper, subexponentials are not needed, and the reader can safely skip over this section on first reading. Formulas of MSELL are generated from propositional variables and their duals (see previous section) via the grammar:

$$A, B ::= a \mid a^\perp \mid A \wp B \mid A \otimes B \mid ?^v A \mid !^v A$$

where (as before) we omit the units, and where exponentials are indexed by elements from a (countable) set  $V$  of *labels* which comes equipped with a partial order  $\leq \subseteq V \times V$  and a subset  $U \subseteq V$  of *unbounded labels*. The inference rules for MSELL are the ones in the first line of (1) above together with

$$\begin{aligned} & !^v \frac{\vdash !^v A, ?^{w_1} B_1, \dots, ?^{w_n} B_n}{\vdash !^v A, ?^{w_1} B_1, \dots, ?^{w_n} B_n} \quad v \leq w_i \text{ for all } i \in \{1, \dots, n\} \\ & ?^v \text{d} \frac{\vdash \Gamma, A}{\vdash \Gamma, ?^v A} \quad ?^u \text{w} \frac{\vdash \Gamma}{\vdash \Gamma, ?^u A} \quad u \in U \quad ?^u \text{c} \frac{\vdash \Gamma, ?^u A, ?^u A}{\vdash \Gamma, ?^u A} \quad u \in U \end{aligned} \quad (3)$$

Note that contraction and weakening are only allowed if the label of the  $?$  is among the unbounded labels, and promotion on a  $!$  is only allowed if its label is smaller or equal than all the labels of the  $?$  in the context. Only the dereliction rule  $?^v \text{d}$  can be applied without restriction.

In [4] it has been shown that provability in MSELL is undecidable for the case  $V = \{\alpha, \beta, \gamma\}$  with  $\leq$  being the reflexive closure of  $\leq_0$  with  $\alpha \leq_0 \gamma$  and  $\beta \leq_0 \gamma$  and  $U = \{\gamma\}$ , by a straightforward encoding of two-counter machines [29, 21].

Clearly, we can also define a relevant version of MSELL, that I call here RMSELL, and that is obtained from MSELL by adding the general contraction rule  $\text{c}$  in (2). As before in that case, the restricted contraction rule  $?^u \text{c}$  can be omitted.

## 4 Technical flaws in Bimbó's decidability proof

There are three main step in Bimbó's proof [1] of the decidability of MELL:

1. Cut admissibility for MELL and RMELL and modified systems that restrict the application of contraction.
2. Decidability for RMELL by showing that proof search in the modified system terminates.
3. Decidability of MELL by deriving an upper bound for the proof search trees from the decision procedure for RMELL.

In the paper [1] all three steps have technical flaws. For the first and the second, these are easily fixable, but for the third this is not the case, and therefore the decision problem for MELL remains open. Below I explain the mistakes in each of the three steps.

## 4.1 Cut admissibility

The cut rule for all logical systems presented so far can be given as follows:

$$\text{cut} \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \quad (4)$$

where  $A^\perp$  is the De Morgan dual of  $A$ . The cut admissibility result for a logical system  $S$  says that if a formula or sequent can be derived in  $S + \text{cut}$  then it can also be derived in  $S$  without the cut-rule. All four logical systems presented here have this property. For MELL and MSELL this has been proved in [7] and [31], respectively. For their relevant version, it can be shown by similar methods. But since we defined the logics without cut, the cut admissibility theorem is technically not needed for the decidability proof (for RMELL and RMSELL).

Nonetheless, [1] provides a cut admissibility proof for MELL and claims that this is a new proof. However, the proof method in [1] is standard: permute the cut upwards, start with the topmost one, and make sure that some measure decreases at each step. The measure used in [1] consists of the tuple  $\langle \rho, \mu, \delta \rangle$  where  $\rho$  is the number of rule instances above the cut having the cut formula in the conclusion,  $\mu$  is the number of  $?c$ -instances that are applied to ancestors of the cut formula, and  $\delta$  is the size of the cut formula, and where the order is lexicographic.<sup>3</sup> But one case in [1] is problematic, namely when

$$\text{cut} \frac{! \frac{\vdash ?\Delta, A}{\vdash ?\Delta, !A} \quad ?c \frac{\vdash ?A^\perp, ?A^\perp, \Gamma}{\vdash ?A^\perp, \Gamma}}{\vdash ?\Delta, \Gamma} \quad \text{is reduced to} \quad \text{cut} \frac{! \frac{\vdash ?\Delta, A}{\vdash ?\Delta, !A} \quad ! \frac{\vdash ?\Delta, A}{\vdash ?\Delta, !A} \quad \text{cut} \frac{\vdash ?\Delta, !A \quad \vdash ?A^\perp, ?A^\perp, \Gamma}{\vdash ?\Delta, ?A^\perp, \Gamma}}{?c \frac{\vdash ?\Delta, ?\Delta, \Gamma}{\vdash ?\Delta, \Gamma}}$$

where the dotted line stands for several instances of the  $?c$ -rule. On first sight one might be tempted to agree with the argument that the two new cuts have a lower  $\mu$  than the original one, and therefore the induction hypothesis can be applied to both cuts. However, the elimination process for the first cut can increase the  $\mu$ -value for the second cut, and thus, the induction hypothesis cannot be applied. In order to make this kind of argument work, it is not enough to just count how often the cut formula is duplicated in a contraction. One also has to find a way to take into account the instances of  $?c$  that are newly created in the cut elimination process. This is best achieved through a notion “flow graph”, as done for classical logic in [3] using Buss’ *logical flowgraphs* or in [9, 10] using *atomic flows*. For MELL, these flow graphs are studied in [40] and [42] in the setting of the calculus of structures. Then, for ensuring termination, it has to be shown that there no cycle in the flow graph, or that the cycles can be eliminated.

In any case, cut elimination for MELL is a well-established result with several different published proofs, so that there is no need to go into further details here.

## 4.2 Decidability of RMELL

Okada and Terui have shown in [33] via a semantic argument that the relevant version of full propositional linear logic, denoted by RLL<sup>4</sup>, is decidable. Therefore, RMELL is also decidable.<sup>5</sup>

On the other hand, for a reader familiar with the syntactic proof of the decidability of various fragments of *relevant logic*, attributed to Kripke [19] and first written up in all detail by Belnap and Wallace in [14] for the logic of *entailment with negation*<sup>6</sup>, it should be clear that the decidability of RMELL can also be shown by almost literally the same proof. This proof is presented in [1], but contains a mistake.

More precisely, Kripke’s theorem (Theorem 18) is stated wrongly in [1]. That theorem is needed for cognate sequents in general, and not just modally cognate sequents. I will explain this in further detail now, because for understanding the problem with the decidability proof for MELL, one needs to understand how the decidability proof for RMELL works. So, in order to make this paper self-contained, I will give this proof.

For this, we need some definitions: Two sequents  $\Gamma$  and  $\Delta$  are *cognate* if they contain the same formulas, i.e., they only differ in the number of occurrences of the formulas in the sequent. They are *modally cognate*, if additionally every formula that is not of the shape  $?A$  has the same number of occurrences in both sequents.

<sup>3</sup>In [1], the value  $\rho$  is called the *rank* of the cut, whereas usually in the literature the value  $\delta$  is called the *rank* of the cut.

<sup>4</sup>That is LL extended with the general contraction rule (2). This logic is called CLL in [33], but the C is more often used for “classical”. For this reason I use the R for “relevant” throughout this paper.

<sup>5</sup>More recently, it has been shown that RLL is ACKERMANN-complete [22, Corollary 25], and that RMELL is in 2EXP [39, Theorem 6.1].

<sup>6</sup>This logic is equivalent to the one obtained from MLL (the three rules in the first line of (1)) together with the contraction rule (2).



A set of sequents that are cognate to each other are called a *cognition class*. In the example below, all four sequents are in the same cognition class, but only the first two are modally cognate:

$$\vdash a, b, b, ?a \quad \vdash a, b, b, ?a, ?a, ?a \quad \vdash a, a, a, b, ?a, ?a \quad \vdash a, a, a, a, a, b, b, ?a, ?a, ?a \quad (5)$$

In the following, we use the notation  $\Gamma_1 \succ \Gamma_2$  if there is a derivation with premise  $\vdash \Gamma_1$  and conclusion  $\vdash \Gamma_2$ , using only the *c*-rule, and we write  $\Gamma_1 \succcurlyeq \Gamma_2$  (or equivalently  $\Gamma_2 \preccurlyeq \Gamma_1$ ) iff  $\Gamma_1 \succ \Gamma_2$  or  $\Gamma_1 = \Gamma_2$  (where  $=$  stands for multiset equality).

A (finite or infinite) sequence  $\Gamma_1, \Gamma_2, \dots$  of sequents is *irredundant* if for all  $i < j$ , we have  $\Gamma_i \not\preccurlyeq \Gamma_j$ . We can now state Kripke's lemma:

**Theorem 4.1** (Kripke [19]). *If a sequence of cognate sequents is irredundant, then it is finite.*

A proof can be found in [14, p.289]. Note that for this theorem it is irrelevant what the inference rules are and what the language of the formulas is.<sup>7</sup>

We can now use Theorem 4.1 to exhibit a terminating complete proof search procedure for RMELL, from which decidability follows. This is done by using the following variant of the proof system, called  $\langle\langle$ RMELL $\rangle\rangle$ :<sup>8</sup>

$$\begin{array}{ccc} \text{id} \frac{}{\vdash a, a^\perp} & \langle\langle \wp \rangle\rangle \frac{\vdash \Gamma, A, B}{\vdash \langle\langle \Gamma, A \wp B \rangle\rangle} & \langle\langle \otimes \rangle\rangle \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \langle\langle \Gamma, A \otimes B, \Delta \rangle\rangle} \\ & ! \frac{\vdash A, ?\Delta}{\vdash !A, ?\Delta} & \langle\langle ?d \rangle\rangle \frac{\vdash \Gamma, A}{\vdash \langle\langle \Gamma, ?A \rangle\rangle} \quad ?w \frac{\vdash \Gamma}{\vdash \Gamma, ?A} \end{array} \quad (6)$$

where the rules  $\langle\langle \wp \rangle\rangle$ ,  $\langle\langle \otimes \rangle\rangle$ , and  $\langle\langle ?d \rangle\rangle$  are variants of the rules  $\wp$ ,  $\otimes$ , and  $?d$ , respectively, which have the contraction rule built in, i.e., a sequent  $\vdash \langle\langle \Sigma \rangle\rangle$  is obtained from  $\vdash \Sigma$  by application of the *c*-rule in (2), such that the following additional conditions are satisfied:<sup>9</sup>

- $\langle\langle \wp \rangle\rangle$ : If the formula  $A \wp B$  occurs  $n$  times in  $\Gamma$ , then it occurs at least  $\max(n, 1)$  times in  $\langle\langle \Gamma, A \wp B \rangle\rangle$ . Any other formula occurs as often in  $\Gamma$  as in  $\langle\langle \Gamma, A \wp B \rangle\rangle$ .
- $\langle\langle \otimes \rangle\rangle$ : If the formula  $A \otimes B$  occurs  $n$  times in  $\Gamma$  and  $m$  times in  $\Delta$ , then it occurs at least  $\max(n, m, 1)$  times in  $\langle\langle \Gamma, A \otimes B, \Delta \rangle\rangle$ . If another formula occurs  $n$  times in  $\Gamma$  and  $m$  times in  $\Delta$ , then it occurs at least  $\max(n, m)$  times in  $\langle\langle \Gamma, A \otimes B, \Delta \rangle\rangle$ .
- $\langle\langle ?d \rangle\rangle$ : If the formula  $?A$  occurs  $n$  times in  $\Gamma$ , then it occurs at least  $\max(n, 1)$  times in  $\langle\langle \Gamma, ?A \rangle\rangle$ . Any other formula occurs as often in  $\Gamma$  as in  $\langle\langle \Gamma, ?A \rangle\rangle$ .

We can now prove the following two theorems about  $\langle\langle$ RMELL $\rangle\rangle$ . The first one states the height-preserving admissibility of contraction, and is in the literature often (e.g. in [1]) attributed to Curry.

**Theorem 4.2** (Curry). *If  $\vdash \Gamma$  has a  $\langle\langle$ RMELL $\rangle\rangle$  proof  $\pi$  with height  $h$ , and  $\Gamma \succ \Gamma'$ , then  $\vdash \Gamma'$  has a  $\langle\langle$ RMELL $\rangle\rangle$  proof  $\pi'$  with height  $h' \leq h$ .*

*Proof.* This is proved by a straightforward induction on  $h$ . □

**Theorem 4.3.** *A sequent  $\vdash \Gamma$  is provable in  $\langle\langle$ RMELL $\rangle\rangle$  if and only if it is provable in RMELL.*

*Proof.* Any proof in  $\langle\langle$ RMELL $\rangle\rangle$  can be expanded to a proof in RMELL by adding the necessary instances of the *c*-rule. Conversely, every rule in RMELL, except for *c*, is an instance of a rule in  $\langle\langle$ RMELL $\rangle\rangle$ . Hence, a proof in RMELL is already a proof in  $\langle\langle$ RMELL $\rangle\rangle + c$ , and the result follows from Theorem 4.2.<sup>10</sup> □

With these ingredients, we can give the full proof of decidability for RMELL.

**Theorem 4.4.** *Provability in RMELL is decidable.*

*Proof.* By Theorem 4.3 we can restrict proof search to  $\langle\langle$ RMELL $\rangle\rangle$ . By Theorem 4.2, we can stop the search when we reach a sequent  $\Gamma$  such that on the current branch of the proof search tree there is an ancestor  $\Gamma'$  with  $\Gamma' \preccurlyeq \Gamma$ .

Then, we observe that any formula that occurs in an  $\langle\langle$ RMELL $\rangle\rangle$  is a subformula of the endsequent and there are only finitely many such formulas. There are infinitely many sequents that can be formed from these formulas,

<sup>7</sup>In fact, it is equivalent to Dickson's Lemma in number theory which states that every set of  $n$ -tuples of natural numbers has finitely many minimal elements (see also [38]).

<sup>8</sup>The  $\langle\langle \cdot \rangle\rangle$  notation is taken from [1].

<sup>9</sup>In [1], the formulation of these conditions is slightly ambiguous. For this reason, I took here the formulation of [14], adapted to the case of RMELL.

<sup>10</sup>In [1] it is stated that also cut admissibility is necessary for obtaining this theorem, but as we have shown here, it is not needed.

but there are only finitely many cognation classes. Furthermore, from the previous paragraph and Theorem 4.1 it follows that from each cognation class only finitely many sequents need to be visited in a single branch in the proof search tree. Since the proof search tree is finitely branching (each inference rule has only finitely many premises and at each step there are only finitely many choices for applying an inference rule), we can conclude by König's lemma that the proof search tree is finite.  $\square$

It is important to observe that the whole argument breaks down if we assume that Theorem 4.1 only holds for sequences of modally cognate sequents. In that case it is still true that in each branch only finitely many sequents of the same cognation class are visited, but now there are infinitely many cognation classes (see example in (5)), i.e., the sequents visited in a single branch of the proof search tree can become arbitrarily large. And this is exactly the reason why the decidability proof for MELL in [1] is not correct, as we will see in the next section.

Finally, it is easy to see that we can define a system  $(\text{RMSELL})$  in the same way as  $(\text{RMELL})$ , and that we can prove the decidability of provability in RMSELL with almost literally the same proof as for RMELL.

**Theorem 4.5.** *Provability in RMSELL is decidable.*

### 4.3 Concerning the Decidability Proof for MELL

The proof of the decidability of MELL in [1] is based on the observation that every MELL-proof is also a RMELL-proof of the same endsequent. Thus, given a sequent  $\Gamma$ , if the RMELL proof search for  $\Gamma$  comes back with a failure, then we know that  $\Gamma$  is not provable in MELL. On the other hand, if there is an RMELL-proof of  $\Gamma$ , then we can count for each subformula occurrence  $?A$  of  $\Gamma$ , how often the contraction rule is applied to it in any RMELL-proof of  $\Gamma$ .

Even though there is no bound on the proof search for RMELL, we have that  $(\text{RMELL})$  proof search is bounded. The argument of [1] is now that it is enough to count for each occurrence of a subformula  $?A$  of a sequent  $\Gamma$ , how often the subformula occurrence  $A$  and its ancestors<sup>11</sup> are contracted in any  $(\text{RMELL})$  proof of  $\Gamma$ . This is called the *heap number of  $?A$*  [1, Def. 22], which is then used to restrict proof search in the system  $[\text{MELL}]$  shown below:

$$\begin{array}{c} \text{id} \frac{}{\vdash a, a^\perp} \quad \wp \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \quad [\otimes] \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash [\Gamma, A \otimes B, \Delta]} \\ \\ \text{!} \frac{\vdash A, ?\Delta}{\vdash !A, ?\Delta} \quad [?d] \frac{\vdash \Gamma, A}{\vdash [\Gamma, ?A]} \quad ?w \frac{\vdash \Gamma}{\vdash \Gamma, ?A} \end{array} \quad (7)$$

where  $[\otimes]$  and  $[?d]$  are variants of the rules  $\otimes$  and  $?d$ , respectively, which have the  $?c$ -rule built in, i.e., a sequent  $\vdash [\Sigma]$  is obtained from  $\vdash \Sigma$  by application of the  $?c$ -rule, such that the following additional conditions are satisfied:

$[\otimes]$ : If a formula  $?C$  occurs  $n$  times in  $\Gamma$  and  $m$  times in  $\Delta$ , then it occurs at least  $\max(n, m)$  times in  $[\Gamma, A \otimes B, \Delta]$ .

$[?d]$ : If the formula  $?A$  occurs  $n$  times in  $\Gamma$ , then it occurs at least  $\max(n, 1)$  times in  $[\Gamma, ?A]$ . Any other formula occurs as often in  $\Gamma$  as in  $[\Gamma, ?A]$ .

We have then for  $[\text{MELL}]$  analogous results as stated in Theorems 4.2 and 4.3 for  $(\text{RMELL})$ , namely the  $?c$  is height-preserving admissible for  $[\text{MELL}]$ , and a sequent is provable in  $[\text{MELL}]$  if and only if it is provable in MELL.

The decidability proof for MELL in [1, Theorem 23] now simply bounds the number of application of the  $[?d]$ -rule to (the ancestors of) an occurrence of a  $?A$ -formula by its heap-number.

However, the same argument would apply to prove decidability of MSELL from the decidability of RMSELL, by going via the system  $[\text{MSELL}]$  that is defined in the same way as  $[\text{MELL}]$ . But MSELL has been shown undecidable in [4].

Therefore, something must be wrong with Bimbó's argument in [1]. First, it is obvious that Bimbó's decision procedure terminates. This follows from the termination argument for  $(\text{RMELL})$ . However, there is no argument explaining why her decision procedure should be complete.

To understand the problem, consider for example the following MELL sequent:

$$\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, ?(a^\perp \otimes (a \wp a)) \quad (8)$$

<sup>11</sup>I.e., the ancestors of the auxiliary formulas of the instances of the  $[?d]$ -rule in which an ancestors of that  $?A$  are principal.

The obvious proof in (RMELL) removes the  $?$ -formula with a weakening, applies the  $(\otimes)$ -rule three times, and puts a copy of  $a$  into each branch:

$$\begin{array}{c}
 \text{id} \frac{}{a^\perp, a} \quad \text{id} \frac{}{a^\perp, a} \quad \text{id} \frac{}{a^\perp, a} \quad \text{id} \frac{}{a^\perp, a} \\
 (\otimes) \frac{}{\vdash a^\perp \otimes a^\perp, a} \quad (\otimes) \frac{}{\vdash a^\perp \otimes a^\perp, a} \\
 (\otimes) \frac{}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a} \\
 ?w \frac{}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, ?(a^\perp \otimes (a \wp a))}
 \end{array} \tag{9}$$

For proving the sequent (8) in [MELL] we need to apply the  $[?d]$ -rule three times in order to create the necessary four copies of  $a$ :

$$\begin{array}{c}
 \text{id} \frac{}{a^\perp, a} \quad \text{id} \frac{}{a^\perp, a} \quad \text{id} \frac{}{a^\perp, a} \quad \text{id} \frac{}{a^\perp, a} \\
 (\otimes) \frac{}{\vdash a^\perp \otimes a^\perp, a, a} \quad (\otimes) \frac{}{\vdash a^\perp \otimes a^\perp, a, a} \\
 \text{id} \frac{}{a, a^\perp} \quad \wp \frac{}{\vdash a^\perp \otimes a^\perp, a \wp a} \quad \text{id} \frac{}{a, a^\perp} \quad \wp \frac{}{\vdash a^\perp \otimes a^\perp, a \wp a} \\
 (\otimes) \frac{}{\vdash a^\perp \otimes a^\perp, a, a^\perp \otimes (a \wp a)} \quad (\otimes) \frac{}{\vdash a^\perp \otimes a^\perp, a, a^\perp \otimes (a \wp a)} \\
 [?d] \frac{}{\vdash a^\perp \otimes a^\perp, a, ?(a^\perp \otimes (a \wp a))} \quad [?d] \frac{}{\vdash a^\perp \otimes a^\perp, a, ?(a^\perp \otimes (a \wp a))} \\
 (\otimes) \frac{}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, a, ?(a^\perp \otimes (a \wp a))} * \\
 \text{id} \frac{}{a, a^\perp} \quad \wp \frac{}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a \wp a, ?(a^\perp \otimes (a \wp a))} \\
 (\otimes) \frac{}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, a^\perp \otimes (a \wp a), ?(a^\perp \otimes (a \wp a))} \\
 [?d] \frac{}{\vdash (a^\perp \otimes a^\perp) \otimes (a^\perp \otimes a^\perp), a, ?(a^\perp \otimes (a \wp a))}
 \end{array} \tag{10}$$

This proof is at the same time a correct (RMELL) proof, but it is not visited by the (RMELL) proof search described in the proof of Theorem 4.4, because the sequent marked with a  $*$  in (10) is in  $\succ$  relation with the conclusion, and therefore the (RMELL) proof search is aborted at that point. For this reason, we cannot assume, *a priori*, that the *heap numbers* determined by the (RMELL) decision procedure are high enough to ensure a complete [MELL] proof search, which leaves a large gap in the decidability proof for MELL in [1].

However, if we consider again the derivation in (10), we can see that there is a rule permutation variant that does not visit the sequent  $*$ : we can permute one instance of  $[?d]$  down below the  $(\otimes)$  and the  $\wp$ -rule instance and keep a copy of  $?(a^\perp \otimes (a \wp a))$ . Unfortunately, it is not at all clear whether such a rule permutation always exists. More generally, we can formulate the following conjecture:

**Conjecture 4.6.** *Let  $\pi$  be a proof in [MELL] of a sequent  $\Gamma$ . Then there is a proof  $\pi'$  in [MELL] with the same endsequent  $\Gamma$ , such that for any two sequents  $\Gamma_1$  and  $\Gamma_2$  occurring in  $\pi'$ , such that  $\Gamma_1$  is an ancestor of  $\Gamma_2$  in the tree of  $\pi'$ , we have  $\Gamma_1 \not\prec \Gamma_2$ .*

It follows immediately from [4] that Conjecture 4.6 does not hold if we replace [MELL] by [MSELL], but it might well be that it holds for the special case where the label set  $V$  is a singleton. A proof of Conjecture 4.6 would indeed close the gap in Bimbó's proof of the decidability of MELL, and would provide an alternative proof for the decidability of the reachability problem for VASS and Petri nets.

Note that a counterexample to Conjecture 4.6 would not show that MELL is undecidable. But it would show that the proof idea of [1] cannot work.

## 5 Conclusion

Whenever a paper is published whose main proof is faulty, it is always easy to blame the author or the editor or the referee or some other victim. However, in this case this would be too short-sighted. I think that the publication of [1] is the consequence of a systemic problem in the field of structural proof theory. Namely, that it is studied by two different communities: one coming from a computer science background and the other coming from a philosophical background. These two communities use different notation and terminology and do not talk to each other. Furthermore, they also consider each other to be less skilled and take this as a justification for not taking each other's papers seriously.

On the one hand, the philosophers rightfully accuse the computer scientists of ignoring the vast amount of literature on substructural logic that existed *before linear logic*, and the computer scientists rightfully accuse the philosophers of considering linear logic as just another substructural logic, ignoring the semantic consideration that gave rise to it and the vast amount of literature that came *after linear logic*, exhibiting its enormous influence in many areas of theoretical computer science that makes linear logic very special among the zoo of substructural logics.

The decision problem for MELL is open for three decades now, and many people in the computer science community worked on it, but nobody has observed its containment in RMELL, even though decision problems for relevant logics have been studied since the 1960's, and the relation between MELL and RMELL is obvious to anyone with a background in philosophical logic. This observation leads naturally to Bimbó's proof idea: Can we bound the proof search for MELL by using information that we can extract from the RMELL proofs?

On the one hand, it is quite embarrassing for the computer science community that nobody has explored this idea before Bimbó.<sup>12</sup> On the other hand, the gap in Bimbó's reasoning could be spotted immediately by anybody familiar with the peculiarities of MELL. However, Bimbó's error is very easy to overlook for someone not familiar with linear logic, and this led to the unfortunate publication of [1].

Interestingly, there is a recent proof by Bimbó and Dunn (presented in [2]) of the decidability of the logic of ticket entailment<sup>13</sup>, using similar methods as discussed here. In [5] Dawson and Goré tried to formally verify that proof using the interactive theorem prover Isabelle/HOL. Their work uncovered a gap in the proof of [2] which seems to be of similar nature as the gap uncovered here for MELL.

We should take this as a lesson to take each other more seriously in the future. It seems that with the decidability of MELL we have a problem that needs for its solution both communities, philosophy and computer science.

## Acknowledgments

Many thanks to Sylvain Schmitz, Dale Miller, Willem Heijltjes, and Revantha Ramanayake for helpful comments and discussions improving the readability and comprehensiveness of this short paper.

## References

- [1] Katalin Bimbó. The decidability of the intensional fragment of classical linear logic. *Theor. Comput. Sci.*, 597:1–17, 2015.
- [2] Katalin Bimbó and J. Michael Dunn. On the decidability of implicational ticket entailment. *J. Symb. Log.*, 78(1):214–236, 2013.
- [3] Alessandra Carbone. Turning cycles into spirals. *Ann. Pure Appl. Logic*, 96(1-3):57–73, 1999.
- [4] Kaustuv Chaudhuri. Undecidability of multiplicative subexponential logic. In Sandra Alves and Iliano Cervesato, editors, *3rd International Workshop on Linearity (LINEARITY)*, volume 176 of *EPTCS*, pages 1–8, Vienna, Austria, July 2014.
- [5] Jeremy E. Dawson and Rajeev Goré. Issues in machine-checking the decidability of implicational ticket entailment. In Renate A. Schmidt and Cláudia Nalon, editors, *Automated Reasoning with Analytic Tableaux and Related Methods - 26th International Conference, TABLEAUX 2017, Brasília, Brazil, September 25-28, 2017, Proceedings*, volume 10501 of *Lecture Notes in Computer Science*, pages 347–363. Springer, 2017.
- [6] Philippe de Groote, Bruno Guillaume, and Sylvain Salvati. Vector addition tree automata. In *19th IEEE Symposium on Logic in Computer Science (LICS 2004), 14-17 July 2004, Turku, Finland, Proceedings*, pages 64–73, 2004.
- [7] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
- [8] Alessio Guglielmi. A system of interaction and structure. *ACM Transactions on Computational Logic*, 8(1):1–64, 2007.
- [9] Alessio Guglielmi and Tom Gundersen. Normalisation control in deep inference via atomic flows. *Logical Methods in Computer Science*, 4(1:9):1–36, 2008.
- [10] Alessio Guglielmi, Tom Gundersen, and Lutz Straßburger. Breaking paths in atomic flows for classical logic. In *LICS 2010*, 2010.
- [11] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In Laurent Fribourg, editor, *Computer Science Logic, CSL 2001*, volume 2142 of *LNCS*, pages 54–68. Springer-Verlag, 2001.

<sup>12</sup>At this point it is worth mentioning that the Karp-Miller trees [17], which are finite structures that are crucial for the decidability proof for VASS, play in that proof a similar role as the (RMELL) derivations in Bimbó's proof attempt for MELL. Given that it is still a big leap from Karp-Miller trees to the full decidability of the reachability problem for VASS, a proof of Conjecture 4.6 will very likely be much harder than it seems on first sight, if it exists at all.

<sup>13</sup>Another published proof of the decidability of ticket entailment can be found in [34].

- 
- [12] Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 231–246. Springer-Verlag, 2002.
- [13] Alessio Guglielmi and Lutz Straßburger. A system of interaction and structure V: the exponentials and splitting. *Mathematical Structures in Computer Science*, 21(3):563–584, 2011.
- [14] Nuel D. Belnap Jr. and John R. Wallace. A decision procedure for the system  $e_{\bar{i}}$  of entailment with negation. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 11:277–289, 1965.
- [15] Ozan Kahramanoğulları. System BV is NP-complete. *Annals of Pure and Applied Logic*, 2007. In press.
- [16] Max I. Kanovich. Horn programming in linear logic is NP-complete. In *Proceedings of the 7th Annual IEEE Symposium on Logic in Computer Science (LICS 92)*, 1992.
- [17] Richard M. Karp and Raymond E. Miller. Parallel program schemata. *J. Comput. Syst. Sci.*, 3(2):147–195, 1969.
- [18] S. Rao Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In Harry R. Lewis, Barbara B. Simons, Walter A. Burkhard, and Lawrence H. Landweber, editors, *Proceedings of the 14th Annual ACM Symposium on Theory of Computing, May 5-7, 1982, San Francisco, California, USA*, pages 267–281. ACM, 1982.
- [19] Saul A. Kripke. The problem of entailment (abstract). *The Journal of Symbolic Logic*, 24(4), 1959.
- [20] Yves Lafont and Andre Scedrov. The undecidability of second order multiplicative linear logic. *Information and Computation*, 125:46–51, 1996.
- [21] Joachim Lambek. How to program an infinite abacus. *Canad. Math. Bull.*, 4(3):295–302, 1961.
- [22] Ranko Lazic and Sylvain Schmitz. Nonelementary complexities for branching VASS, MELL, and extensions. *ACM Trans. Comput. Log.*, 16(3):20:1–20:30, 2015.
- [23] Jérôme Leroux. Vector addition systems reachability problem (A simpler solution). In Andrei Voronkov, editor, *Turing-100 - The Alan Turing Centenary, Manchester, UK, June 22-25, 2012*, volume 10 of *EPiC Series in Computing*, pages 214–228. EasyChair, 2012.
- [24] Jérôme Leroux and Sylvain Schmitz. Demystifying reachability in vector addition systems. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015*, pages 56–67, 2015.
- [25] P. Lincoln, J. Mitchell, A. Scedrov, and N. Shankar. Decision problems for propositional linear logic. *Annals of Pure and Applied Logic*, 56(1–3):239–311, 1992.
- [26] Richard J. Lipton. The reachability problem requires exponential space. Technical Report Technical Report 63, Yale University, 1976.
- [27] Ernst W. Mayr. An algorithm for the general petri net reachability problem. In *Proceedings of the 13th Annual ACM Symposium on Theory of Computing, May 11-13, 1981, Milwaukee, Wisconsin, USA*, pages 238–246. ACM, 1981.
- [28] Ernst W. Mayr. An algorithm for the general petri net reachability problem. *SIAM J. Comput.*, 13(3):441–460, 1984.
- [29] Marvin L. Minsky. Recursive unsolvability of Post’s problem of ”tag” and other topics in theory of Turing machines. *The Annals of Mathematics*, 74(3):437–455, 1961.
- [30] Virgile Mogbil. *Sémantique des phases, réseaux de preuve et divers problèmes de décision en logique linéaire. (Phase semantics, proof nets and some decision problems in linear logic)*. PhD thesis, University of the Mediterranean, Marseille, France, 2001.
- [31] Vivek Nigam. *Exploiting non-canonicity in the sequent calculus*. PhD thesis, Ecole Polytechnique, 2009.
- [32] Vivek Nigam and Dale Miller. Algorithmic specifications in linear logic with subexponentials. In *ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP)*, pages 129–140, 2009.
- [33] Mitsuhiro Okada and Kazushige Terui. The finite model property for various fragments of intuitionistic linear logic. *J. Symb. Log.*, 64(2):790–802, 1999.

- 
- [34] Vincent Padovani. Ticket entailment is decidable. *CoRR*, abs/1106.1875, 2011.
- [35] C. A. Petri. *Kommunikation mit Automaten*. Dissertation, Universität Bonn, 1962.
- [36] Christian Retoré. Pomset logic: A non-commutative extension of classical linear logic. In Ph. de Groot and J. R. Hindley, editors, *Typed Lambda Calculus and Applications, TLCA '97*, volume 1210 of *LNCS*, pages 300–318. Springer, 1997.
- [37] Christophe Reutenauer. *Aspects mathématiques des réseaux de Petri*. Masson, 1989.
- [38] Jaques Riche and Robert K. Meyer. Belnap, Urquhart and Relevant Decidability & Complexity: "Das ist nicht Mathematik, das ist Theologie". In Georg Gottlob, Etienne Grandjean, and Katrin Seyr, editors, *Computer Science Logic, 12th International Workshop, CSL '98, Annual Conference of the EACSL, Brno, Czech Republic, August 24-28, 1998, Proceedings*, volume 1584 of *Lecture Notes in Computer Science*, pages 224–240. Springer, 1998.
- [39] Sylvain Schmitz. Implicational relevance logic is 2-exptime-complete. *J. Symb. Log.*, 81(2):641–661, 2016.
- [40] Lutz Straßburger. MELL in the Calculus of Structures. *Theoretical Computer Science*, 309(1–3):213–285, 2003.
- [41] Lutz Straßburger. System NEL is undecidable. In Ruy De Queiroz, Elaine Pimentel, and Lucília Figueiredo, editors, *10th Workshop on Logic, Language, Information and Computation (WoLLIC)*, volume 84 of *Electronic Notes in Theoretical Computer Science*, 2003.
- [42] Lutz Straßburger and Alessio Guglielmi. A system of interaction and structure IV: The exponentials and decomposition. *ACM Trans. Comput. Log.*, 12(4):23, 2011.
- [43] Alasdair Urquhart. The complexity of decision procedures in relevance logic. In J. Michael Dunn and Anil Gupta, editors, *Truth or Consequences: Essays in Honor of Nuel Belnap*, pages 61–76, Dordrecht, 1990. Springer Netherlands.
- [44] Alasdair Urquhart. The complexity of decision procedures in relevance logic II. *J. Symb. Log.*, 64(4):1774–1802, 1999.
- [45] Andreas Weiermann and Martin Bunder. Ackermannian lower bounds for a combinatorial problem related to the decidability of certain relevant logics. Unpublished note, 2006.



**RESEARCH CENTRE  
SACLAY – ÎLE-DE-FRANCE**

1 rue Honoré d'Estienne d'Orves  
Bâtiment Alan Turing  
Campus de l'École Polytechnique  
91120 Palaiseau

Publisher  
Inria  
Domaine de Voluceau - Rocquencourt  
BP 105 - 78153 Le Chesnay Cedex  
[inria.fr](http://inria.fr)

ISSN 0249-6399