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# EXPONENTIAL STABILIZATION OF NONLINEAR SYSTEMS BY AN ESTIMATED STATE FEEDBACK

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**Abstract:** In this paper we investigate the stabilizability problem of a class of multi-input multi-output nonlinear systems which linearization at the origin is controllable and observable. Under assumptions on the nonlinear part we prove : (a) the system is globally exponentially stabilizable (G.E.S) by means of linear feedback law. (b) the system can be G.E.S using a state estimation given by an observer.

**Keywords:** exponential stabilization, feedback, nonlinear systems, Lyapunov functions, observer.

## 1 Introduction

We consider a nonlinear system of the form :

$$\begin{cases} \dot{x} = Ax + Bu + g(x, u) \\ x \in \mathbb{R}^n, u \in \mathbb{R}^r, A \in M_{n,n}(\mathbb{R}), B \in M_{n,r}(\mathbb{R}) \end{cases} \quad (1)$$

where  $M_{n,m}(\mathbb{R})$  is the set of matrices with n rows and m columns and the map

$$f = (f_1, \dots, f_n)^T : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is Lipschitz continuous such that  $f(0, 0) = 0$ . We assume that the pair  $(A, B)$  is controllable and is in Brunovsky canonical form, i.e.

- A is a block-diagonal matrix of the form

$$A = \begin{pmatrix} A_{k_1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & A_{k_r} \end{pmatrix}$$

where  $A_{k_i}, 1 \leq i \leq r$ , is a matrix in  $M_{k_i, k_i}(\mathbb{R})$  given by

$$A_{k_i} = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

- B is a block-diagonal matrix of the form

$$B = \begin{pmatrix} b_{k_1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & b_{k_r} \end{pmatrix}$$

where  $b_{k_i}, 1 \leq i \leq r$ , is a column-vector in  $\mathbb{R}^{k_i}$  given by

$$b_{k_i} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

In this paper, we study nonlinear systems of the form (1) which have the following property :

**(H1)** *There exists a positive constant K such that for any  $i = 1, \dots, n$  the following holds :*

$$\begin{cases} |g_i(x, u)| \leq K \|(x_1, \dots, x_i, 0, \dots, 0)\| \\ \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n, \forall u \in \mathbb{R}^r \end{cases} \quad (2)$$

where  $\| \cdot \|$  is the usual Euclidean norm on  $\mathbb{R}^n$ .

In section 2, we shall see that if condition **(H1)** is satisfied, then system (1) is globally exponentially stabilizable

(G.E.S.) at the origin by means of a linear feedback. This is a generalization of a result of Tsiniias [3] who studied the single input systems, our proof is different from its one and it is based on an idea from [1].

In section 3, we suppose in addition

**(H2)**  $g_i(x, u) = g_i(x_1, \dots, x_i, 0, \dots, 0, u)$

so we can construct an observer for system (1) with the output

$$y = Cx \quad (3)$$

where

$$C = \begin{pmatrix} C_{k_1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C_{k_p} \end{pmatrix}$$

$$C_{k_i} = (1 \quad 0 \quad \dots \quad 0)$$

This observer is of the form :

$$\dot{\hat{x}} = A\hat{x} + Bu + g(\hat{x}, u) - \tilde{E}(C\hat{x} - y) \quad (4)$$

We show that the error will tend to zero with an exponential rate of convergence and we construct a dynamic feedback which globally exponentially stabilizes system (1-3).

## 2 Stabilization

Before given our main result we recall some basic notions about exponential stability. Let us consider a system of ordinary differential equations

$$\dot{x} = X(x), \quad x \in \mathbb{R}^n \quad (5)$$

We suppose that the origin is an equilibrium point for the vector field  $X$  i.e,  $X(0) = 0$  and we denote  $X_t(x)$  the solution of (5) starting from the point  $x$  at  $t = 0$  ( $X_0(x) = x$ ). We say that (5) is *globally exponentially stable* at the origin if there exist positive constants  $M$  and  $\alpha$  such that

$$\|X_t(x)\| < M\|x\|e^{-\alpha t}$$

for any initial condition  $x$  in  $\mathbb{R}^n$  and any  $t > 0$ . To prove exponential stability we shall use the following Lyapunov theorem [2] :

**Theorem** *The solution  $x_t \equiv 0$  of the equation (5) is globally exponentially stable if there exist a Lyapunov function  $V$  and three positive constants  $k_1, k_2, k_3$  such that for any  $x \in \mathbb{R}^n$  one has :*

$$k_1\|x\|^2 \leq V(x) \leq k_2\|x\|^2$$

and

$$\dot{V}(x) = X.V(x) = \langle \nabla V(x), X(x) \rangle \leq -k_3\|x\|^2$$

We shall say that a control system

$$\begin{cases} \dot{x} = f(x, u), & x \in \mathbb{R}^n, u \in \mathbb{R}^r \\ f(0, 0) = 0 \end{cases}$$

is globally exponentially stabilizable (G.E.S) at the origin of  $\mathbb{R}^n$  if there exists a continuous feedback

$$\begin{aligned} u &: \mathbb{R}^n \rightarrow \mathbb{R}^r \\ x &\mapsto u(x) \end{aligned}$$

such that the closed loop system  $\dot{x} = f(x, u(x))$  is globally exponentially stable at the origin.

The aim of this section is to prove that system (1) is G.E.S and to give explicitly the stabilizing feedback provided that assumption **(H1)** is satisfied. To this end let  $\alpha \in \mathbb{R}$ ,  $\alpha > 1$  and introduce the following matrix :

$$\Phi = \begin{pmatrix} \alpha^{-1} & 0 & \dots & 0 \\ 0 & \alpha^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha^{-n} \end{pmatrix}$$

One can write

$$\Phi = \begin{pmatrix} \Phi_{k_1} & 0 & \dots & 0 \\ 0 & \Phi_{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \Phi_{k_r} \end{pmatrix}$$

where  $r$  is the number of the blocs of matrix  $A$  and

$$\Phi_{k_i} = \begin{pmatrix} \alpha^{-(k_{i-1}+1)} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha^{-(k_{i-1}+k_i)} \end{pmatrix}$$

with  $k_0 = 0$

Using the above decomposition, a simple computation can prove the following :

**lemma 1** *Matrices  $A, B$  and  $\Phi$  defined above satisfy :*

$$i) \alpha \Phi^{-1} A \Phi = A$$

ii)  $\forall F \in M_{r,n}(\mathbb{R})$  there exists  $\tilde{F} \in M_{r,n}(\mathbb{R})$  such that

$$B\tilde{F} = \alpha \Phi^{-1} B F \Phi$$

and  $\tilde{F}$  is given by the following formula :

$$\tilde{F} = \alpha B^T \Phi^{-1} B F \Phi$$

$$iii) \forall x \in \mathbb{R}^n : \alpha^{-n} \|x\| \leq \|\Phi x\| \leq \alpha^{-1} \|x\|$$

We can now prove the main result of this section.

**Theorem 1** *If the assumption **(H1)** holds then the system (1) is G.E.S. at the origin by means of a linear feedback*

$$u = \tilde{F} x$$

where  $\tilde{F}$  is defined by (ii) and  $F$  is such that  $(A + BF)$  has all its eigenvalues with negative real part.

**Proof**

Since the pair  $(A, B)$  is controllable there exists a matrix  $F \in M_{r,n}(\mathbb{R})$  such that  $(A + BF)$  has all its eigenvalues with negative real part. Let  $M = A + BF$ , there exists a symmetric positive definite matrix  $S$  such that  $M^T S + SM = -Q$ ,  $Q$  symmetric positive definite .

Now consider the function :

$$V(x) = x^T \Phi S \Phi x$$

$V$  is positive definite and proper ( $V$  is a quadratic Lyapunov function). Let us evaluate its derivative along the trajectories of the closed-loop system

$$\dot{x} = (A + B\tilde{F})x + g(x, \tilde{F}x) = \tilde{M}x + g(x, \tilde{F}x) \quad (6)$$

$$\dot{V}(x) = x^T (\Phi S \Phi \tilde{M} + \tilde{M}^T \Phi S \Phi)x + 2x^T \Phi S \Phi g(x, \tilde{F}x)$$

where  $\tilde{M} = A + B\tilde{F}$

Taking into account (i) and (ii) we have

$$\tilde{M} = A + B\tilde{F} = \alpha\Phi^{-1}A\Phi + \alpha\Phi^{-1}BF\Phi$$

$$\tilde{M} = \alpha\Phi^{-1}(A + BF)\Phi = \alpha\Phi^{-1}M\Phi$$

Therefore :

$$\begin{aligned} \dot{V}(x) = & \alpha x^T (\Phi S M \Phi + \Phi M^T S \Phi) x \\ & + 2x^T \Phi S \Phi g(x, \tilde{F}x) \end{aligned}$$

$$\dot{V}(x) = -\alpha x^T \Phi Q \Phi x + 2x^T \Phi S \Phi g(x, \tilde{F}x)$$

The matrix  $Q$  is symmetric definite positive so there exists a positive constant  $a$  such that :

$$x^T \Phi Q \Phi x \geq a \|\Phi x\|^2$$

where

$$a = \inf \{ z^T Q z / z \in S^{n-1} \text{ the unit sphere in } \mathbb{R}^n \}$$

And then :

$$\dot{V}(x) \leq -\alpha a \|\Phi x\|^2 + 2 \|\Phi x\| \|\Phi S\| \|\Phi g(x, \tilde{F}x)\|$$

According to **(H1)** we have

$$\begin{aligned} \|\Phi g(x, \tilde{F}x)\|^2 &= \sum_{i=1}^n \frac{1}{\alpha^{2i}} g_i^2(x, \tilde{F}x) \\ &\leq K^2 \sum_{i=1}^n \frac{1}{\alpha^{2i}} (x_1^2 + \dots + x_i^2) \\ &\leq K^2 \sum_{j=1}^n \left( \frac{x_j}{\alpha^j} \right)^2 \left( 1 + \frac{1}{\alpha^2} + \dots + \frac{1}{\alpha^{2(n-j)}} \right) \\ &\leq nK^2 \|\Phi x\|^2 \end{aligned}$$

So

$$\dot{V}(x) \leq (-\alpha a + 2\sqrt{n} K \|\Phi S\|) \|\Phi x\|^2$$

If we choose  $\alpha > \text{Max} \left( 1, \frac{2\sqrt{n} K \|\Phi S\|}{a} \right)$  then

$$\dot{V} \leq -c \|\Phi x\|^2$$

where  $c$  is a positive constant.

Since  $\|\Phi x\| \geq \frac{1}{\alpha^n} \|x\|$  it follows that :

$$\dot{V}(x) \leq -c' \|x\|^2$$

where  $c'$  is a positive constant , and this completes the proof of theorem 1.  $\blacksquare$

**Example 1.** The three dimensional system

$$\begin{cases} \dot{x}_1 = x_2 + x_1 \cos(u(x_2^2 + x_1 + x_3)) \\ \dot{x}_2 = x_3 + \frac{x_1 + x_2}{1 + x_3^2} \\ \dot{x}_3 = \sqrt{x_1^2 + x_2^2 + x_3^2} e^{-u^2} + u \end{cases} \quad (7)$$

has the form (1) with

$$g(x, u) = \begin{pmatrix} x_1 \cos(u(x_2^2 + x_1 + x_3)) \\ \frac{x_1 + x_2}{1 + x_3^2} \\ \sqrt{x_1^2 + x_2^2 + x_3^2} e^{-u^2} \end{pmatrix}$$

which satisfies condition **(H1)**, so system (7) is G.E.S and a stabilizing feedback can be computed according to theorem 1 as follows : We choose  $F = (-1, -3, -3)$ ,  $Q = -Id_{\mathbb{R}^3}$  and we solve  $M^T S + SM = Q$  :

$$S = \begin{pmatrix} \frac{37}{16} & \frac{31}{16} & \frac{1}{2} \\ \frac{31}{16} & \frac{13}{4} & \frac{13}{16} \\ \frac{1}{2} & \frac{13}{16} & \frac{7}{16} \end{pmatrix}$$

We compute  $\tilde{F} = \alpha B^T \Phi^{-1} B F \Phi$ . The feedback law

$$u = -\alpha^3 x_1 - 3\alpha^2 x_2 - 3\alpha x_3$$

globally exponentially stabilizes system (7) if we choose

$$\alpha > \text{Max} \left( 1, \frac{2\sqrt{n} K \|\Phi S\|}{a} \right) \simeq 10\sqrt{3}$$

### 3 Construction of the observer and stabilization using a state estimation

**Theorem 2** *If the condition **(H1)** and **(H2)** are satisfied then there exists  $\tilde{E} \in M_{n,p}(\mathbb{R})$  such that the system (4) is an exponential observer for (1-3).*

**Proof**

The error  $e = \hat{x} - x$  satisfies the following equation :

$$\dot{e} = (A - \tilde{E}C)e + g(\hat{x}, u) - g(x, u) \quad (8)$$

since  $(A, C)$  is observable, there exists  $E \in M_{n,p}(\mathbb{R})$  such that  $(A - EC)$  has all its eigenvalues with negative real part.

Remark that :

$$|g_i(x, u) - g_i(\hat{x}, u)| \leq K \|p_i(x - \hat{x})\| = K \|(e_1, \dots, e_i)\|$$

where  $p_i : \mathbb{R}^n \rightarrow \mathbb{R}^i$  is the canonical projection.

So if we take

$$\tilde{E} = \alpha \Phi^{-1} E C \Phi C^T$$

then , according to the proof of theorem 1, there exists a quadratic Lyapunov function  $W$  such that for  $\alpha$  large enough :

$$\dot{W}(e) \leq -b \|e\|^2, b > 0$$

This proves that  $\|e(t)\| \leq M \|e_0\| e^{-\alpha t}$  so (4) is an exponential observer for (1-3). ■

Now we use the above results to achieve the stabilization of system (1) with the state estimation given by the observer (4). Consider the following system defined on  $\mathbb{R}^n \times \mathbb{R}^n$  :

$$\begin{cases} \dot{x} = Ax + B\tilde{F}\hat{x} + g(x, \tilde{F}\hat{x}) \\ \dot{e} = (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \end{cases} \quad (9)$$

**Theorem 3** *If the assumption (H1) and (H2) hold then (9) is globally exponentially stable i.e. the closed-loop system (1-3), with the state estimation given by the observer (4) is G.E.S.*

**Proof**

Since  $e = \hat{x} - x$ , system (9) becomes :

$$\begin{cases} \dot{x} = (A + B\tilde{F})x + B\tilde{F}e + g(x, \tilde{F}\hat{x}) \\ \dot{e} = (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \end{cases} \quad (10)$$

According to the proofs of theorem 1 and theorem 2, there exist two quadratic Lyapunov functions  $V$  and  $W$  such that :

$$\langle \nabla V(x), (A + B\tilde{F})x + g(x, \tilde{F}\hat{x}) \rangle \leq -c' \|x\|^2$$

$$\langle \nabla W(e), (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \rangle \leq -b \|e\|^2$$

where  $\langle \cdot, \cdot \rangle$  is the usual Euclidean inner product on  $\mathbb{R}^n$ .

Let  $U$  be the function defined on  $\mathbb{R}^n \times \mathbb{R}^n$  by :

$$U(x, e) = \beta V(x) + W(e), \beta > 0$$

$U$  is a quadratic positive definite function and we have :

$$\begin{aligned} \dot{U}(x, e) = & \beta \langle \nabla V(x), (A + B\tilde{F})x + g(x, \tilde{F}\hat{x}) \rangle \\ & + \beta \langle \nabla V(x), B\tilde{F}e \rangle \\ & + \langle \nabla W(e), (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \rangle \end{aligned}$$

So

$$\dot{U}(x, e) \leq -\beta c' \|x\|^2 + 2\beta K' \|x\| \|e\| - b \|e\|^2$$

where  $K'$  is a positive constant defined by :

$$\langle \nabla V(x), B\tilde{F}e \rangle \leq 2K' \|x\| \|e\|$$

If we choose  $\beta$  such that  $0 < \beta < \frac{c' b}{K'^2}$  then  $\dot{U}(x, e)$  is negative definite on  $\mathbb{R}^n \times \mathbb{R}^n$  and so theorem 3 is proved.

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