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Stabilization of Periodic Discrete-Time Nonlinear Systems

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Abstract

This paper provides sufficient conditions for stabilization of periodic discrete systems. These conditions are derived from a discrete version of a Theorem due to Krasovski for continuous-time systems. This tool allows to give a systematic design of time-varying stabilizing control for autonomous discrete systems that can not be stabilized by time-invariant feedback.

Keywords: Lyapunov function, stability, stabilization, time-varying feedback, periodic discrete systems.

1 Introduction

The aim of this article is to design stabilizer feedback for a class of periodic discrete-time nonlinear control systems. More precisely, we consider systems that can be written $x_{k+1} = F(x_k, u_k, k)$, F being time-periodic and the unforced system being Lyapunov stable. It is also assumed that there exists a positive definite periodic function $V(x, k)$ in such a way $V(F(x, 0, k), k+1) - V(x, k) \leq 0$. The paper is organized as follows. In Section 2, we give a discrete version of Krasovski theorem [6] and a sufficient condition under which the attractivity of the equilibrium point is guaranteed. In Section 3, this will be used to derive a stabilization result of periodic systems. As an application, we derive a sufficient condition for time-varying feedback stabilization for a class of autonomous systems that can not be stabilized by time-invariant feedback [1, 2, 3, 5]. Due to the lack of space, the proofs are not reproduced here, but can be found in [4].

2 Stability of periodic systems

Let us consider a discrete nonlinear system

$$x_{k+1} = f(x_k, k), \quad f(0, k) = 0, \quad x(k) \in \mathbb{R}^n, \quad k \in \mathbb{N} \quad (1)$$

We assume that f is continuous and periodic with respect to k , i.e. there exists $T \in \mathbb{N}$ such that $f(x, k+T) = f(x, k)$, $\forall (x, k) \in \mathbb{R}^n \times \mathbb{N}$. We shall denote by $x(k, x_{k_0}, k_0)$ the solution of (1) with initial state x_{k_0} at initial time k_0 . Let $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function. We shall say that a belongs to \mathcal{K} if it is strictly increasing, vanishes at zero and $\lim_{r \rightarrow +\infty} a(r) = +\infty$.

Theorem 1 *Suppose there exists a continuous function $V : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathbb{R}$, periodic in k with period $T \geq 2$ such that for some $a \in \mathcal{K}$, $\forall x \in \mathbb{R}^n, \forall k \in \mathbb{N}$:*

$$(1) \quad V(x, k) \geq a(\|x\|), \quad V(0, k) = 0.$$

$$(2) \quad \Delta V(x, k) = V(f(x, k), k+1) - V(x, k) \leq 0.$$

(3) *The set $\{(x, k) \in \mathbb{R}^n \times \mathbb{N} : \Delta V(x, k) = 0\}$ contains no complete solution of (1) except the trivial one.*

Then the origin is globally uniformly asymptotically stable.

For the sequel, we define for all $0 \leq k \leq T$ a function \hat{f}_k in the following way : $\hat{f}_k^k(x) = x$ and by induction $\hat{f}_k^{p+1}(x) = f(\hat{f}_k^p(x), p)$ for all $p \geq k$. Actually $\hat{f}_k^p(x)$ is nothing but the value at time p of the solution of (1) with initial state x at initial time k . With this notation we can prove the following result which gives a sufficient condition to get the condition (3) of the above theorem.

Proposition 1 *Suppose (1) and (2) hold. If the set $\{x \in \mathbb{R}^n : V(\hat{f}_k^{p+1}(x), p+1) - V(\hat{f}_k^p(x), p) = 0, p \geq k\}$ is reduced to the origin for all $k = 0, \dots, T$, then assumption (3) of Theorem (1) is satisfied and so the origin is globally uniformly asymptotically stable.*

3 Stabilization

Consider a discrete-time nonlinear control system

$$x_{k+1} = F(x_k, u_k, k), \quad F(0, 0, k) = 0, \quad k \in \mathbb{N} \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the state at time k and $u \in \mathbb{R}^m$ is the control. $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \rightarrow \mathbb{R}^n$ is assumed to be continuous in (x, u) and periodic with respect to time k with period $T \geq 2$. The problem addressed here is the following: does a periodic feedback $u(x, k)$, $u(0, k) = 0$, exists such that the origin is a globally asymptotically stable equilibrium point for the closed-loop system

$$x_{k+1} = F(x_k, u(x_k, k), k)$$

We will give a sufficient condition under which such a feedback exists. To do this we need the following notations. For a Lyapunov function $V(x, k)$ which is C^2 in x and T -periodic in k let $\tilde{V} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \rightarrow \mathbb{R}$ and $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{N} \rightarrow \mathbb{R}$ be defined by

$$\tilde{V}(x, u, k) = V(F(x, u, k), k+1) \quad (3)$$

$$\varphi(x, u, v, k) = \int_0^1 (1-t)v^T \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu, k) v dt \quad (4)$$

Set $F_0(x, k) = F(x, 0, k)$. For a fixed $\eta > 0$, let $K_1(x, k)$ and $K_2(x, k)$ be any T -periodic and continuous real valued functions satisfying for all $(x, k) \in \mathbb{R}^n \times \mathbb{N}$, $K_1(x, k) + K_2(x, k) \neq 0$ and

$$K_1(x, k) \geq \sup_{\|u\| \leq \eta, \|v\|=1} |\varphi(x, u, v, k)| \quad (5)$$

$$K_2(x, k) \geq \left\| \frac{\partial V}{\partial x}(F_0(x, k), k+1) \frac{\partial F}{\partial u}(x, 0, k) \right\| \quad (6)$$

and set

$$K(x, k) = \frac{\eta}{\eta K_1(x) + K_2(x)} \quad (7)$$

One can notice that K is T -periodic with respect to k . Now, we can state the main result of this section :

Theorem 2 *Assume that $x_{k+1} = F_0(x_k, k)$ is stable and there exists a C^2 Lyapunov function $V(x, k)$ T -periodic in k such that V and F_0 satisfy the assumptions (1) and (2) of Theorem (1). If for all $k \in \{0, \dots, T\}$ the set*

$$\{x \in \mathbb{R}^n : V(\hat{F}_{0k}^{p+1}(x), p+1) = V(\hat{F}_{0k}^p(x), p), \\ \frac{\partial V}{\partial x}(\hat{F}_{0k}^{p+1}(x), p+1) \frac{\partial F}{\partial u}(\hat{F}_{0k}^p(x), 0, p) = 0, p \geq k\}$$

is reduced to the origin then, for any positive constant η , system (2) is globally asymptotically stabilizable by means of the continuous feedback law

$$u = -K(x, k) \left(\frac{\partial V}{\partial x}(F_0(x, k), k+1) \frac{\partial F}{\partial u}(x, 0, k) \right)^T$$

which satisfies

$$\begin{cases} \|u(x, k)\| \leq \eta & \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}, \\ u(0, k) = 0 & \forall k \in \mathbb{N}, \\ u(x, k+T) = u(x, k) & \forall (x, k) \in \mathbb{R}^n \times \mathbb{N}. \end{cases}$$

Now, we apply the above theorem to the following nonlinear control system

$$x_{k+1} = x_k + \Phi(x_k, u_k) \quad (8)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $\Phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a C^2 function satisfying $\Phi(x, 0) = 0$, $\forall x \in \mathbb{R}^n$. It is known that if the map Φ fails to be locally onto then (8) cannot be stabilized by means of continuous state feedback $u_k = u(x_k)$. In what follows we give a sufficient condition for system (8) to be stabilizable by a time-varying feedback $u_k = u(x_k, k)$.

Theorem 3 *Assume that for all $1 \leq i, j \leq m$*

$$\frac{\partial^2 \Phi_i}{\partial u_1 \partial u_j}(x, u) = 0, \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \quad (9)$$

and that there exist a C^2 Lyapunov function $V(x, k)$ and a continuous real valued function $\alpha(x, k)$ which are T -periodic with respect to time k such that V and the function F_0 defined by $F_0(x, k) = x + \alpha(x, k)g_1(x)$ satisfy the conditions of theorem 2. Then, for any positive constant η , system (8) is globally asymptotically stabilizable by means of the continuous feedback

$$u(x, k) = \nu(x, k) + \tilde{u}(x, k) \quad (10)$$

$$\nu(x, k) = [\alpha(x, k), 0, \dots, 0]^T$$

$$\tilde{u}(x, k) = -K(x, k) \left[\frac{\partial V}{\partial x}(F_0(x, k), k+1) \frac{\partial \Phi}{\partial u}(x, \nu(x, k)) \right]^T$$

where $K(x, k)$ is got from (3)-(7) with

$$\tilde{V}(x, u, k) = V(x + \Phi(x, \nu(x, k) + u), k+1)$$

Since the conditions and the design of the control laws in theorem 3 make use of the functions V and α , it is natural to look for particular systems of the form (8) for which V and α can be explicitated. Setting

$$g(x) = (g_1(x), \dots, g_m(x)) = \frac{\partial \Phi}{\partial u}(x, 0)$$

it turns out, as in continuous-time (see [7]), that if

$$g_1 = \frac{\partial \Phi}{\partial u_1}(x, 0) = \frac{\partial}{\partial x_1} \quad (11)$$

it is actually possible to give an explicit design of V and α , and by the way of the control laws.

Hereafter, for all $x \in \mathbb{R}^n$, set $\bar{x} = (x_2, \dots, x_n)^T$ and for a fixed integer $T \geq 2$ let

$$V(x, k) = \frac{1}{2} ([x_1 + h(\bar{x}, k)]^2 + \|\bar{x}\|^2)$$

$$\alpha(x, k) = -\frac{1}{2} [x_1 - h(\bar{x}, k)] - h(\bar{x}, k+1)$$

where h is a C^2 function satisfying $h(\bar{x}, k+T) = h(\bar{x}, k)$, $\forall (\bar{x}, k) \in \mathbb{R}^{n-1} \times \mathbb{N}$ and $h(0, k) = 0$, $\forall k \in \mathbb{N}$. A possible choice for h is $h(\bar{x}, k) = \psi(\bar{x})H(k)$ where ψ is a definite function with respect to \bar{x} , and $H(k)$ is a time periodic function with period T . So V and α are T -periodic with respect to time k , and V vanishes if and only if $x = 0$. With these notations we have:

Proposition 2 *Assume that (9) and (11) hold. If there exists a function h as specified above such that for all $\bar{x} \in \mathbb{R}^{n-1}$,*

$$\text{rank}\{\tilde{g}_i(h(\bar{x}, k), \bar{x}), 2 \leq i \leq m, 0 \leq k \leq T-1\} = n-1$$

where $\tilde{g}_i(x) = (0, g_i^2(x), \dots, g_i^n(x))$, then, for any positive constant η , system (8) is globally asymptotically stabilizable by means of the feedback (10).

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