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# Feedback stabilization of discrete-time nonlinear systems via the Control Lyapunov Functions

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## Abstract

In this paper we state sufficient conditions for the existence of feedback laws which render the equilibrium solution of a class of discrete-time nonlinear systems globally asymptotically stable.

**Keywords:** Control Lyapunov Functions, Nonlinear discrete-time systems, Feedback stabilization.

## 1 Introduction:

Feedback stabilization of various classes of discrete-time nonlinear systems have been studied in the past few years; see ([1, 2, 4, 5]) and references therein.

We consider the single-input discrete-time nonlinear systems of the form

$$x_{k+1} = f(x_k) + u_k g(x_k), \quad x_k \in \mathbb{R}^n, \quad u \in \mathbb{R} \quad (1)$$

where  $f$  and  $g$  are smooth on  $\mathbb{R}^n$  and  $f(0) = 0$ .

We say that (1) is globally asymptotically stable at the origin, if there exists a map  $x \mapsto K(x)$  such that the resulting system

$$x_{k+1} = f(x_k) + K(x_k)g(x_k) \quad (2)$$

is globally asymptotically stable at  $0 \in \mathbb{R}^n$ .

The main object of this paper is to provide explicit feedback  $K : \mathbb{R}^n \rightarrow \mathbb{R}$ , smooth on  $\mathbb{R}^n \setminus \{0\}$  and  $K(0) = 0$  such that  $u_k = K(x_k)$  globally asymptotically stabilizes the origin of the system (1), under the assumption that a 'quadratic Control Lyapunov Function' is known. This problem has been addressed for stochastic systems in [3].

This result represents a continuation of a line of work started in [7], in which, under the same condition, the author has studied the local stabilization for the multi-inputs discrete-time systems affine in the control, and the practical stabilization for the single-input systems.

## 2 Stabilization and clf's

In this section, we will state and prove an analogous of Sontag's result [6] for discrete-time systems.

**Definition 1** A smooth, proper and definite positive function  $V$  mapping  $\mathbb{R}^n$  into  $\mathbb{R}$  is said to be a control Lyapunov function (henceforth just 'clf') for the discrete systems (1) if and only if,

$$\inf_{u_k \in \mathbb{R}} (\Delta V(x_k, u_k) = V(x_{k+1}) - V(x_k)) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

In the following, we will assume that  $V$  is a quadratic clf for system (1) (i.e. there exists a positive definite matrix  $P$  such that  $V(x) = x^T P x = \|x\|_p^2 = \langle x, x \rangle_p$

**Remark:** In [1], the authors have studied the same problem and have given, under the same assumptions, the following feedback

$$u(x) = \begin{cases} 0 & \text{for } x \in \Omega_0 = \{x \in \mathbb{R}^n : g(x) = 0\} \\ -\frac{\langle f(x), g(x) \rangle_p}{\|g(x)\|_p^2} & \text{for } x \in \Omega_1 = \{x \in \mathbb{R}^n : g(x) \neq 0\} \end{cases}$$

but it is easy to show that  $\lim_{x \in \Omega_1 \rightarrow x_0} u(x) = \infty$  for all  $x_0 \in \partial\Omega_0 \setminus \{0\}$  such that  $f(x_0) \neq 0$  and so  $u(x)$  is not only discontinuous but is also unbounded in a neighbourhood of such a point.

Here, we will construct a stabilizing feedback which is smooth on  $\mathbb{R}^n \setminus \{0\}$ .

**Theorem 1** If there is a quadratic 'clf' for the discrete-time system (1), then there is a stabilizing feedback  $K : \mathbb{R}^n \rightarrow \mathbb{R}$ , smooth on  $\mathbb{R}^n \setminus \{0\}$  and  $K(0) = 0$ .

**Proof:** For every  $(x, u) \in \mathbb{R}^n \times \mathbb{R}$ , we let

$$h_x(u) = u^2 \|g(x)\|_p^2 + 2u \langle f(x), g(x) \rangle_p + \|f(x)\|_p^2 - \|x\|_p^2$$

and we introduce the sets  $\Omega_0 = \{x \in \mathbb{R}^n \setminus \{0\} \mid g(x) = 0\}$  and  $\Omega_1 = \{x \in \mathbb{R}^n \setminus \{0\} \mid g(x) \neq 0\}$ .

An easy computation shows that, for a given command law  $u$ , the difference  $\Delta V = V(x_{k+1}) - V(x_k)$  is equal to  $h_{x_k}(u_k)$ . If  $x \in \Omega_0$ , this difference does not depend on  $u$  ( $\Delta V = \|f(x_k)\|_p^2 - \|x_k\|_p^2$  in this case) and is therefore negative since  $V$  is a clf for system (1).

If  $x \in \Omega_1$ , for the same reason, there exists  $u \in \mathbb{R}$  such that  $h_x(u) < 0$ , this proves that  $h_x(u)$  regarded as a polynomial in  $u$  admits two separate roots  $\lambda_1(x)$  and  $\lambda_2(x)$ . In this case our task is to find a function  $K : x \mapsto K(x)$  such that  $\lambda_1(x) < K(x) < \lambda_2(x)$  for all  $x \in \Omega_1$ .

First we take two smooth functions  $\varphi$  and  $\psi$  defined by:

$$\varphi(t) = \begin{cases} 0 & \text{if } t \leq 2, \\ 1 & \text{if } t \geq 3. \end{cases} \quad \psi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t \geq 1. \end{cases}$$

and satisfying  $0 \leq \varphi(t), \psi(t) \leq 1$  for all  $t \in \mathbb{R}$ .

Next, for  $x \in \Omega_1$ , we introduce the following quantities,

$$\begin{aligned} A(x) &= (\lambda_1(x) + 1)\psi(\lambda_1(x) + 1) \\ &\quad + (\lambda_2(x) - 1)(1 - \psi(\lambda_2(x) - 1)) \\ B(x) &= \varphi(\lambda_2(x) - \lambda_1(x)) \\ C(x) &= \frac{\lambda_1(x) + \lambda_2(x)}{2}. \end{aligned}$$

Finally, we claim that the following mapping  $K$  defined by:

$$K(x) = \begin{cases} 0 & \text{if } x \in \Omega_0 \\ A(x)B(x) + (1 - B(x))C(x) & \text{if } x \in \Omega_1 \end{cases}$$

is smooth on  $\mathbb{R}^n \setminus \{0\}$  and globally asymptotically stabilizes system (1).

We first show that  $K(x) \in ]\lambda_1(x), \lambda_2(x)[$  if  $x \in \Omega_1$ :

- if  $\lambda_2(x) - \lambda_1(x) \leq 2$ , then  $B(x) = 0$  and so  $K(x) = C(x) \in ]\lambda_1(x), \lambda_2(x)[$ ;
- if  $\lambda_2(x) - \lambda_1(x) > 2$ , by considering all the possible placements of  $\lambda_1$  and  $\lambda_2$  regarding 0 and 1, we can see that  $A(x) \in ]\lambda_1(x), \lambda_2(x)[$  which in turn implies that  $K(x) \in ]\lambda_1(x), \lambda_2(x)[$ .

In order to show that  $K$  is smooth on  $\mathbb{R}^n \setminus \{0\}$ , we remark first that this is obviously the case in the interior of  $\Omega_0$  and in  $\Omega_1$ ; we will next prove that if  $x_0 \in \partial\Omega_0$ , there exists a neighborhood  $U$  of  $x_0$  on which  $K$  vanishes.

Let  $\varepsilon_0 = -(\|f(x_0)\|_p^2 - \|x_0\|_p^2)$ ,  $V$  is a clf implies  $\varepsilon_0 > 0$ . Let  $\varepsilon > 0$ , since  $g(x_0) = 0$  ( $x_0 \in \Omega_0$ ), by continuity there exists a neighborhood  $U$  of  $x_0$  such that  $\|f(x)\|_p^2 -$

$\|x\|_p^2 < -\varepsilon_0/2$ ,  $\|g(x)\|_p^2 < \varepsilon$  and  $2|\langle f(x), g(x) \rangle_p| < \varepsilon$ . From these inequalities, we deduce that

$$h_x(u) \leq \varepsilon u^2 + \varepsilon |u| - \varepsilon_0/2, \quad \forall (x, u) \in U \times \mathbb{R}. \quad (3)$$

Now, in order to have  $K(x) = 0$  for all  $x \in U \cap \Omega_1$ , it is sufficient that  $A(x) = 0$  and  $B(x) = 1$  for all  $x \in U \cap \Omega_1$ . To this end it is sufficient to have  $\lambda_1(x) \leq -1$  and  $\lambda_2(x) \geq 4$  which is equivalent to  $h_x(-1) \leq 0$  and  $h_x(4) \leq 0$ ; these last inequalities are satisfied in  $U$  if  $\varepsilon$  is chosen such that  $\varepsilon \leq \varepsilon_0/40$  and so the theorem is proven.

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