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# Gomory Hu Tree and Pendant Pairs of a Symmetric Submodular System

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**Abstract.** Let  $\mathcal{S} = (V, f)$ , be a symmetric submodular system. For two distinct elements  $s$  and  $l$  of  $V$ , let  $\Gamma(s, l)$  denote the set of all subsets of  $V$  which separate  $s$  from  $l$ . By using every Gomory Hu tree of  $\mathcal{S}$  we can obtain an element of  $\Gamma(s, l)$  which has minimum value among all the elements of  $\Gamma(s, l)$ . This tree can be constructed iteratively by solving  $|V| - 1$  minimum  $sl$ -separator problem. An ordered pair  $(s, l)$  is called a pendant pair of  $\mathcal{S}$  if  $\{l\}$  is a minimum  $sl$ -separator. Pendant pairs of a symmetric submodular system play a key role in finding a minimizer of this system. In this paper, we obtain a Gomory Hu tree of a contraction of  $\mathcal{S}$  with respect to some subsets of  $V$  only by using contraction in Gomory Hu tree of  $\mathcal{S}$ . Furthermore, we obtain some pendant pairs of  $\mathcal{S}$  and its contractions by using a Gomory Hu tree of  $\mathcal{S}$ .

**Keywords:** Symmetric submodular system, Contraction of a system, Pendant pair, Maximum adjacency ordering, Gomory-Hu tree.

## 1 Introduction

Let  $V$  be a finite set. A set function  $f : 2^V \mapsto \mathbb{R}$  is called a submodular function if and only if for all  $X, Y \in 2^V$ , we have

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \quad (1)$$

Submodular functions play a key role in combinatorial optimization, see [3] for further discussion. Rank functions of matroids, cut capacity functions and entropy functions are some well known examples of submodular functions. For a given system  $\mathcal{S} = (V, f)$ , let  $f : 2^V \mapsto \mathbb{R}$  be a submodular function. The problem in which we want to find a subset  $X \subseteq V$ , for which  $f(X) \leq f(Y)$  for all  $Y \subseteq V$  is called submodular system minimization problem. Minimizing a submodular system is one of the most important problems in combinatorial optimization. Many problems in combinatorial optimization, such as finding minimum cut and minimum  $st$ -cut in graphs, or finding the largest common independent set in two matroids can be modeled as a submodular function minimization. Image segmentation [1, 8, 9], speech analysis [11, 12], wireless and power networks

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[20] are only a small part of applications of minimizing submodular functions. Grotchel, Lovasz and Scherijver have developed the first weakly and strongly polynomial time algorithm for minimizing submodular systems in [6] and [13], respectively. Each of them is designed based on the ellipsoid method. Then, nearly simultaneously Scherijver [18] and Iwata, Fleischer, and Fujishige [7] gave a combinatorial strongly polynomial time algorithms for this problem. Later, a faster algorithm for minimizing submodular system was proposed by Orlin [16]. To the best of our knowledge, the fastest algorithm to find a minimizer of a submodular system  $\mathcal{S} = (V, f)$  is due to Lee et al [10]. Their algorithm runs in  $O(|V|^3 \log^2 |V| \tau + |V|^4 \log^{O(1)} |V|)$  time, where  $\tau$  is the time taken to evaluate the function.

Stoer and Wagner [19] and Frank [2] independently have presented an algorithm that finds a minimum cut of a graph  $G = (V, E)$  in  $O(|E||V| + |V|^2 \log |V|)$  time. Their algorithms are based on Nagamochi and Ibaraki's algorithm [15] which finds a minimum cut of an undirected graph. Queyranne [17] developed a faster algorithm to find a minimizer in a special case of a submodular system. This algorithm was proposed to find a minimizer of a symmetric submodular system  $\mathcal{S} = (V, f)$ . It is a generalization of Stoer and Wagner's algorithm [19] and runs in  $O(|V|^3)$  time. This algorithm, similar to Stoer and Wagner's algorithm uses pendant pairs to obtain a minimizer of a symmetric submodular system.

For a given weighted undirected graph  $G = (V, E)$ , Gomory and Hu constructed a weighted tree, named as Gomory Hu tree [5]. By using a Gomory Hu tree of the graph  $G$ , one can solve the all pairs minimum  $st$ -cut problem with  $|V| - 1$  calls to the maximum flow subroutine instead of the  $\binom{|V|}{2}$  calls. Goemans and Ramakrishnan [4] illustrated that for every symmetric submodular system, there exists a Gomory-Hu tree. It is worth to mention that, there is neither an algorithm to construct a Gomory Hu tree of a symmetric submodular system by using pendant pairs nor any method to obtain pendant pairs of a symmetric submodular system by using a Gomory Hu tree of it.

In this paper, we obtain a Gomory Hu tree of a contraction of a symmetric submodular system  $\mathcal{S} = (V, f)$ , under some subsets of  $V$  only by using a Gomory Hu tree of  $\mathcal{S}$ . In other words, without solving any minimum  $st$ -separator problem of the contracted system, we obtain a Gomory Hu tree of it only by contracting the Gomory Hu tree of the original system. Furthermore, we obtain some pendant pairs of a symmetric submodular system  $\mathcal{S}$  and its contractions by using a Gomory Hu tree of  $\mathcal{S}$ .

The outline of this paper is as follows. Section 2 provides preliminaries and basic definitions. In Section 3, we obtain some pendant pairs of a symmetric submodular system by using a Gomory Hu tree of the system. In Section 4, we construct a Gomory Hu tree of a contracted system by using a Gomory Hu tree of the original system.

## 2 Preliminaries

Let  $V$  be a finite nonempty set. A function  $f : 2^V \mapsto \mathbb{R}$  is called a set function on  $V$ . For every  $X \subseteq V$ ,  $x \in X$  and  $y \in V \setminus X$ , we use  $X + y$  and  $X - x$  instead of  $X \cup \{y\}$  and  $X \setminus \{x\}$ , respectively. Also, for  $x \in V$  we use  $f(x)$  instead of  $f(\{x\})$ .

A pair  $\mathcal{S} = (V, f)$ , is called a system if  $f$  is a set function on  $V$ . A system  $\mathcal{S} = (V, f)$  is called a submodular system if for all  $X, Y \subseteq V$ , we have

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \quad (2)$$

Furthermore, it is called a symmetric system if for every  $X \subseteq V$ , we have

$$f(X) = f(V \setminus X). \quad (3)$$

Consider a symmetric submodular system  $\mathcal{S} = (V, f)$ . Suppose that  $A$  and  $B$  are two disjoint subsets of  $V$ . A subset  $X \subseteq V$  is called an  $AB$ -separator in  $\mathcal{S}$ , if  $X \cap (A \cup B) = A$  or  $X \cap (A \cup B) = B$ . Let  $\Gamma(A, B)$  denote the set of all  $AB$ -separators in  $\mathcal{S}$ . A subset  $X \in \Gamma(A, B)$  is called a minimum  $AB$ -separator in  $\mathcal{S}$  if  $f(X) = \min_{Y \in \Gamma(A, B)} f(Y)$ . If  $A$  and  $B$  are singletons  $\{a\}$  and  $\{b\}$ , then we use  $ab$ -separator and  $\Gamma(a, b)$  instead of  $\{a\}\{b\}$ -separator and  $\Gamma(\{a\}, \{b\})$ , respectively.

Let  $G = (V, E)$  be a weighted undirected graph with the weight function  $w : E \mapsto \mathbb{R}^+ \cup \{0\}$ . Suppose that  $X$  is a nonempty proper subset of  $V$ . The set of all edges connecting  $X$  to  $V \setminus X$ , is called the cut associated with  $X$  and is denoted by  $\delta(X)$ . The capacity of  $\delta(X)$  is denoted by  $C(X)$  and defined by

$$C(X) = \sum_{e \in \delta(X)} w(e). \quad (4)$$

By setting  $C(\emptyset) = C(V) = 0$ ,  $(V, C)$  is a symmetric submodular system [14]. For two distinct vertices  $u$  and  $v$  of  $G$ , every minimum  $uv$ -separator of  $(V, C)$  is a minimum  $uv$ -cut of  $G$ .

Let  $T = (V, F)$  be a tree and  $uv$  be an arbitrary edge of it. By  $T - uv$ , we mean the forest obtained from  $T$  by removing  $uv$ . The set of vertices of two components of  $T - uv$  which respectively contain  $u$  and  $v$ , is denoted by  $V_u(T - uv)$  and  $V_v(T - uv)$ . Also, for  $uv \in F$ , define  $\mathcal{F}_u(T - uv) = \{X | u \in X \subseteq V_u(T - uv)\}$ ,  $\mathcal{F}_v(T - uv) = \{X | v \in X \subseteq V_v(T - uv)\}$ .

Suppose that  $X$  is a nonempty subset of vertices of a given graph  $G = (V, E)$ . We denote by  $G_{>X<}$  the graph obtained from  $G$  by contracting all the vertices in  $X$  into a single vertex.

Let  $\mathcal{S} = (V, f)$  be a symmetric submodular system. Suppose that  $T = (V, F)$  is a weighted tree with the weight function  $w : E \mapsto \mathbb{R}^+$ . If for all  $u, v \in V$ , the minimum weight of the edges on the path between  $u$  and  $v$  in  $T$  is equal to minimum  $uv$ -separator in  $\mathcal{S}$ , then  $T$  is called a flow equivalent tree of  $\mathcal{S}$ . Also, we say that  $T$  has the cut property with respect to  $\mathcal{S}$  if  $w(e) = f(V_u(T - uv)) = f(V_v(T - uv))$  for every  $e = uv \in F$ . A flow equivalent tree of  $\mathcal{S}$  is called a Gomory Hu tree of  $\mathcal{S}$  if it has cut property with respect to  $\mathcal{S}$ .

Consider a system  $\mathcal{S} = (V, f)$ . A pair of elements  $(x, y)$  of  $V$  is called a pendant pair for  $\mathcal{S}$ , if  $\{y\}$  is a minimum  $xy$ -separator in  $\mathcal{S}$ .

Let  $\mathcal{S} = (V, f)$  be a symmetric submodular system. Suppose that  $\rho = (v_1, v_2, \dots, v_{|V|})$  is an ording of the elements of  $V$ , where  $v_1$  can be chosen arbitrarily. If for all  $2 \leq i \leq j \leq |V|$ , we have

$$f(V_{i-1} + v_i) - f(v_i) \leq f(V_{i-1} + v_j) - f(v_j), \quad (5)$$

where  $V_i = \{v_1, v_2, \dots, v_i\}$ , then  $\rho$  is called a maximum adjacency ordering (MA-ordering) of  $\mathcal{S}$ .

For a symmetric submodular system  $\mathcal{S} = (V, f)$ , Queyranne [17] showed that the last two elements  $(v_{|V|-1}, v_{|V|})$  of an MA-ordering of  $\mathcal{S}$ , is a pendant pair of this system.

Let  $\mathcal{S} = (V, f)$  be a system and  $X$  be an arbitrary subset of  $V$ . By  $\varphi(X)$ , we mean a single element obtained by unifying all elements of  $X$ . The contraction of  $\mathcal{S}$  with respect to a subset  $A \subseteq V$  is denoted by  $\mathcal{S}_A = (V_A, f_A)$  and defined by  $V_A = (V \setminus A) + \varphi(A)$  and

$$f_A(X) = \begin{cases} f(X) & \text{if } \varphi(A) \notin X \\ f((X - \varphi(A)) \cup A) & \text{if } \varphi(A) \in X. \end{cases} \quad (6)$$

Suppose that  $A$  and  $B$  are two nonempty disjoint subsets of  $V$ . We denote by  $(\mathcal{S}_A)_B$ , the contraction of  $\mathcal{S}_A$  with respect to  $B$ .

### 3 Obtaining Pendant Pairs from a Gomory Hu tree

Stoer and Wagner [19] obtained a pendant pair of a weighted undirected graph  $G = (V, E)$  by using MA-ordering in  $O(|E| + |V| \log |V|)$  time. By generalizing their algorithm to a symmetric submodular system  $\mathcal{S} = (V, f)$ , Queyranne [17] obtained a pendant pair of this system in  $O(|V|^2)$  time.

In this section, by using the fact that there exists a Gomory Hu tree for a symmetric submodular system  $\mathcal{S} = (V, f)$ , we obtain some pendant pairs of it from a Gomory Hu tree of this system. Also, we show that a Gomory Hu tree of a symmetric submodular system can be constructed by pendant pairs. Firstly, we prove the following lemma.

**Lemma 1.** *Let  $T = (V, F)$  be a flow equivalent tree of a symmetric submodular system  $\mathcal{S} = (V, f)$  with the weight function  $w : E \mapsto \mathbb{R}^+$ . If  $e = uv$  is an arbitrary edge of  $T$ , then for every  $A \in \mathcal{F}_u(T - uv)$  and  $B \in \mathcal{F}_v(T - uv)$ , we have  $w(e) \leq \min\{f(X) | X \in \Gamma(A, B)\}$ .*

*Proof.* Since  $T$  is a flow equivalent tree of  $\mathcal{S}$ , then the value of a minimum  $uv$ -separator in  $\mathcal{S}$  is equal to  $w(e)$ . In other words  $w(e) = \min\{f(X) | X \in \Gamma(u, v)\}$ . Since  $\mathcal{F}_u(T - uv)$  and  $\mathcal{F}_v(T - uv)$  are two subsets of  $\Gamma(u, v)$ , then  $w(e) \leq \min\{f(X) | X \in \Gamma(A, B)\}$ .

**Theorem 1.** *Let  $T = (V, F)$  be a Gomory Hu tree of a symmetric submodular system  $\mathcal{S} = (V, f)$  with the weight function  $w : E \mapsto \mathbb{R}^+$ . If  $e = uv$  is an arbitrary edge of  $T$ , then for every  $A \in \mathcal{F}_u(T - uv)$  and  $B \in \mathcal{F}_v(T - uv)$ ,  $V_v(T - uv)$  is a minimum  $AB$ -separator in  $\mathcal{S}$ .*

*Proof.* Since  $T$  has the cut property, then  $w(e) = f(V_v(T - uv))$ . Now, according to Lemma 1 we have  $f(V_v(T - uv)) \leq \min\{f(X) | X \in \Gamma(A, B)\}$ . On the other hand,  $V_v(T - uv)$  is one of the elements of  $\Gamma(A, B)$  then  $f(V_v(T - uv)) = \min\{f(X) | X \in \Gamma(A, B)\}$ , and the proof is completed.

The following theorem is immediate from Theorem 1.

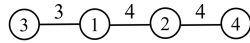
**Theorem 2.** *Let  $T = (V, F)$  be a Gomory Hu tree of a symmetric submodular system  $\mathcal{S} = (V, f)$ . If  $e = uv$  is an arbitrary edge of  $T$ , then  $(\varphi(A), \varphi(V_v(T - uv)))$  for every  $A \in \mathcal{F}_u(T - uv)$  is a pendant pair of  $(\mathcal{S}_A)_{V_v(T - uv)}$ .*

Theorem 2 shows that every Gomory Hu tree of a symmetric submodular system can be obtained by using pendant pairs. We know that every Gomory Hu tree of a symmetric submodular system  $\mathcal{S}$  is a flow equivalent tree having the cut property. We show by an example that Theorem 2 is not necessarily true for every flow equivalent tree of  $\mathcal{S}$ . Let  $V = \{1, 2, 3, 4\}$ . Consider the symmetric submodular system  $\mathcal{S} = (V, f)$  presented in Table 1.

**Table 1.** A symmetric submodular system  $\mathcal{S} = (V, f)$ .

$A$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	$V$
$f(A) = f(V \setminus A)$	4	6	3	5	4	5	9	0

It can be shown that the tree  $T = (V, f)$  depicted in Figure 1 is a flow equivalent tree of  $\mathcal{S}$ . Suppose that  $w : E \mapsto \mathbb{R}^+$  is the weight function of  $T$ . By considering the edge  $e = 24$  of  $T$ , we have  $V_4(T - 24) = \{4\}$ . Now, choose the element  $\{2\}$  from  $\mathcal{F}_2(T - 24)$ . According to Figure 1,  $\{4\}$  is a minimum  $24$ -cut and  $w(24) = 4$ . Since  $T$  is a flow equivalent tree of  $\mathcal{S}$ , then the value of minimum  $24$ -separator in  $\mathcal{S}$  is equal to 4. However, in the given system we have  $f(\{4\}) = 5$ . Therefore,  $(2, 4)$  cannot be a pendant pair of  $\mathcal{S}$ .



**Fig. 1.** Flow Equivalent tree of  $\mathcal{S}$ .

**Theorem 3.** *Let  $T = (V, F)$  be a flow equivalent tree of a symmetric submodular system  $\mathcal{S} = (V, f)$  with the weight function  $w : F \mapsto \mathbb{R}^+$ . If for every edge  $e = uv$*

of  $T$ , there exists a set  $A \in \mathcal{F}_u(T - uv)$  such that  $(\varphi(A), \varphi(V_v(T - uv)))$  is a pendant pair of  $(\mathcal{S}_A)_{V_v(T - uv)}$ , then  $T$  is a Gomory Hu tree.

*Proof.* Since for every  $e = uv$  of  $T$ , there exists a subset  $A$  in  $\mathcal{F}_u(T - uv)$ , that  $(\varphi(A), \varphi(V_v(T - uv)))$  is a pendant pair of  $(\mathcal{S}_A)_{V_v(T - uv)}$  then  $w(e) = f(V_v(T - uv))$ . Thus,  $T$  has the cut property. Therefore,  $T$  is a Gomory Hu tree of  $\mathcal{S}$ .

In the rest of this section, we will prove some properties of pendant pairs of a system.

**Theorem 4.** *Let  $(s, l)$  be a pendant pair of a system  $\mathcal{S} = (V, f)$ . For every  $A \subseteq V \setminus \{s, l\}$ ,  $(s, l)$  is a pendant pair of  $\mathcal{S}_A$ .*

*Proof.* Since  $(s, l)$  is a pendant pair of  $\mathcal{S}$ , then  $f(l) = \min\{f(X) | X \in \Gamma(s, l)\}$ . From (6) we have  $f_A(l) = \min\{f_A(X) | X \in \Gamma(s, l)\}$ .

Thus,  $(s, l)$  is a pendant pair of  $\mathcal{S}_A$ . The proof is completed.

Note that, the converse of Theorem 4 is not generally true. Consider the given system in Table 1. Table 2 contains MA-orderings of  $\mathcal{S}$  and  $\mathcal{S}_{\{1,3\}}$  and also pendant pairs, obtained from these MA-orderings.

**Table 2.** MA-orderings of  $\mathcal{S}$  and  $\mathcal{S}_{\{1,3\}}$ .

system	MA-ordering	Pendant Pair
$\mathcal{S}$	4, 2, 1, 3	(1, 3)
$\mathcal{S}_{\{1,3\}}$	13, 2, 4	(2, 4)

It can be observed that (2, 4) is a pendant pair of  $\mathcal{S}_{\{1,3\}}$ ; however, it is not a pendant pair of  $\mathcal{S}$ .

**Proposition 1.** *If  $(s, l)$  is a pendant pair of a system  $\mathcal{S} = (V, f)$ , then  $(l, s)$  is a pendant pair of  $\mathcal{S}$  iff  $f(l) = f(s)$ .*

*Proof.* Let  $(l, s)$  be a pendant pair of  $\mathcal{S}$ . Thus,  $f(l) = \min\{f(X) | X \in \Gamma(s, l)\}$ . Since  $(s, l)$  is also a pendant pair of  $\mathcal{S}$ , then  $f(s) = f(l)$ . Now, suppose that  $f(s) = f(l)$ . Since  $(s, l)$  is a pendant pair of  $\mathcal{S}$ , then  $(l, s)$  is also a pendant pair of  $\mathcal{S}$ .

## 4 Gomoru Hu Tree of the Contraction of a System

Let  $T = (V, F)$  be a tree. A subset  $X \subseteq V$  is called a  $T$ -connected subset of  $V$ , if the graph induced by  $X$  in  $T$  is a subtree. The following theorem shows that by having a flow equivalent tree  $T$  of a symmetric submodular system  $\mathcal{S} = (V, f)$ , we can easily obtain a flow equivalent tree of  $\mathcal{S}_X$  for every  $T$ -connected subset of  $V$ .

**Theorem 5.** *Let  $T = (V, F)$  be a flow equivalent tree of a symmetric submodular system  $\mathcal{S} = (V, f)$ . If  $X$  is a  $T$ -connected subset of  $V$ , then  $T_{>X<}$  is a flow equivalent tree of  $\mathcal{S}_X = (V_X, f_X)$ .*

*Proof.* Let  $u$  and  $v$  be two distinct elements of  $V_X$ . Consider the path  $P_{uv}$  connecting  $u$  and  $v$  in  $T$ . Suppose that  $T' = (V', F')$  is the induced subtree by  $X$  in  $T$ . Let  $E'$  be the set of edges in  $P_{uv}$  with the minimum weight. If  $E' \not\subseteq F'$ , then there is nothing to prove. Now, suppose that  $E'$  is a subset of  $F'$ . Assume that  $m_1 = \min\{f_X(A) | A \in \Gamma(u, \varphi(X))\}$ ,  $m_2 = \min\{f_X(A) | A \in \Gamma(\varphi(X), v)\}$  and  $m^* = \min\{f_X(A) | A \in \Gamma(u, v)\}$ . Obviously, the values of the minimum  $u\varphi(X)$ -cut, the minimum  $\varphi(X)v$ -cut and the minimum  $uv$ -cut in tree  $T_{>X<}$  are equal to  $m_1$ ,  $m_2$  and  $\min\{m_1, m_2\}$ , respectively. Thus, to show that  $T_{>X<}$  is a flow equivalent tree of  $\mathcal{S}_X$  it suffices to prove that  $m^* = \min\{m_1, m_2\}$ . Let  $T'$  be a flow equivalent tree of  $\mathcal{S}_X$  and  $P'_{uv}$  be a path connecting  $u$  and  $v$  in  $T'$ . Now, we have two cases: case (i),  $\varphi(X)$  is appeared in  $P'_{uv}$ . Therefore, the value of  $uv$ -separator in  $\mathcal{S}_X$  is equal to  $\min\{m_1, m_2\}$  which is equal to the value of minimum  $uv$ -cut in  $T_{>X<}$ . Case (ii), if  $\varphi(X)$  is not appeared in  $P'_{uv}$ , then we can easily conclude that the value of minimum  $uv$ -cut in  $T'$  and  $T_{>X<}$  are equal. The proof is completed.

**Theorem 6.** *Let  $T = (V, F)$  be a Gomory Hu tree of a symmetric submodular system  $\mathcal{S} = (V, f)$ . If  $X$  is a  $T$ -connected subset of  $V$ , then  $T_{>X<}$  is a Gomory Hu tree of  $\mathcal{S}_X = (V_X, f_X)$ .*

*Proof.* According to Theorem 5,  $T_{>X<}$  is a flow equivalent of  $\mathcal{S}_X$ . Furthermore, from (6),  $T_{>X<}$  has the cut property. Then,  $T_{>X<}$  is a Gomory Hu tree of  $\mathcal{S}_X$ .

Then, by having a Gomory Hu tree of a symmetric submodular system  $\mathcal{S} = (V, f)$ , we can find a Gomory Hu tree of the contracted system, with respect to a connected set  $X$ , without finding any minimum  $st$ -separators in  $\mathcal{S}_X$ . Also, we can deduce that  $T_{>V_v(T-uv)<}$  is a Gomory Hu tree of  $\mathcal{S}_{V_v(T-uv)}$ .

**Corollary 1.** *Let  $T = (V, f)$  be a Gomory Hu tree of a symmetric submodular system  $\mathcal{S} = (V, f)$  and  $uv$  be an arbitrary edge of  $T$ . For every  $A \in \mathcal{F}_u(T-uv)$ , for which  $A$  is a connecting set in  $T$ , the tree  $T_{>A \cup V_v(T-uv)<}$  is a Gomory Hu tree of  $(\mathcal{S}_A)_{V_v(T-uv)}$ .*

## 5 Conclusion

In this paper, we obtained some pendant pairs of a symmetric submodular system by using its Gomory Hu tree. Furthermore, for a contraction of  $\mathcal{S}$  with respect to a connected set, we constructed a Gomory Hu tree only by contracting the connected set in Gomory Hu tree of  $\mathcal{S}$ .

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