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# An algorithmic characterization of $\mathbf{P}$ -matricity II: adjustments, refinements, and validation

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The paper “An algorithmic characterization of  $\mathbf{P}$ -matricity” (*SIAM Journal on Matrix Analysis and Applications*, 34:3 (2013) 904–916, by the same authors as here) implicitly assumes that the iterates generated by the Newton-min algorithm for solving a linear complementarity problem of dimension  $n$ , which reads  $0 \leq x \perp (Mx + q) \geq 0$ , are uniquely determined by some index subsets of  $\llbracket 1, n \rrbracket$ . Even if this is satisfied for a subset of vectors  $q$  that is dense in  $\mathbb{R}^n$ , this assumption is improper, in particular in the statements where the vector  $q$  is not subject to restrictions. The goal of the present contribution is to show that, despite this blunder, the main result of that paper is preserved. This one claims that a nondegenerate matrix  $M$  is a  $\mathbf{P}$ -matrix if and only if the Newton-min algorithm does not cycle between two distinct points, whatever is  $q$ . The proof is not more complex, requiring only some adjustments, which are essential however.

**Keywords:** linear complementarity problem,  $\mathbf{NM}$ -matrix, Newton-min algorithm,  $\mathbf{P}$ -matricity characterization,  $\mathbf{P}$ -matrix, semismooth Newton method.

**AMS Subject Classification (2010):** 15B99, 47B99, 49M15, 65K15, 90C33.

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## 1 Introduction

Four years after its official publication, we noticed that an error was made in a paper of ours [3; 2013]. Nevertheless, its main result is preserved. The error in the reasoning is a systematic confusion between an implication and an equivalence, the latter being thought to be true because it is linked to a definition.

The present contribution is therefore of a special nature; it has an unusual contents. Its goal is twofold. On the one hand, it is important to provide a correct proof of the main

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result, which, we think, is still interesting. On the other hand, since the publication [3] cannot be removed, it is also instructive to point the finger at what is wrong in some of its claims. Both goals will be pursued simultaneously, since the path to the final result proposed in [3] is still appropriate. As far as possible, we will try to make the paper self-contained, except when we recall some results whose proof is in extenso in [3].

The *linear complementarity problem* we consider here and in [3] has a standard form [12], which can be described as follows. Being given a positive integer  $n$ , a real matrix  $M \in \mathbb{R}^{n \times n}$ , and a real vector  $q \in \mathbb{R}^n$ , the problem consists in determining a real vector  $x \in \mathbb{R}^n$  such that one has in matrix notation

$$x \geq 0, \quad Mx + q \geq 0, \quad \text{and} \quad x^\top(Mx + q) = 0,$$

where the inequalities have to be understood componentwise, the notation  $v^\top$  is used to denote the transpose of the vector  $v$ , and  $(u, v) \mapsto u^\top v := \sum_i u_i v_i$  is the Euclidean scalar product. We will usually refer to the problem by its abbreviated form, namely

$$\text{LCP}(M, q) : \quad 0 \leq x \perp (Mx + q) \geq 0,$$

where  $\perp$  denotes perpendicularity with respect to the Euclidean scalar product. This framework can model many problems, including quadratic optimization problems [17, 16, 14, 4, 11].

Let us introduce some notation, definitions, and a general assumption. We denote by  $M_{IJ}$  the submatrix of the matrix  $M \in \mathbb{R}^{n \times n}$  formed of its rows with indices in  $I \subset \llbracket 1, n \rrbracket := \{1, \dots, n\}$  and its columns with indices in  $J \subset \llbracket 1, n \rrbracket$ . The matrix  $M$  is said to be a **P-matrix** if all its principal minors are positive (i.e.,  $\det M_{II} > 0$ , for all  $I \subset \llbracket 1, n \rrbracket$ ; by convention  $\det M_{\emptyset\emptyset} = 1$ ). It is known that problem  $\text{LCP}(M, q)$  has a unique solution, whatever is  $q$ , if and only if  $M$  is a **P-matrix** [32, 12; 1958]. In this paper, like in [3], it is always assumed that  $M$  is *nondegenerate*, meaning that the principal minors of  $M$  do not vanish (i.e.,  $\det M_{II} \neq 0$  for all  $I \subset \llbracket 1, n \rrbracket$ ).

The Newton-min algorithm is grounded on the following notion of node.

**Definition 1.1 (node)** For  $I \subset \llbracket 1, n \rrbracket$  and  $q \in \mathbb{R}^n$ , we denote by  $x^{(I, q)} \in \mathbb{R}^n$  the point defined by

$$x_{I^c}^{(I, q)} = 0 \quad \text{and} \quad (Mx^{(I, q)} + q)_I = 0, \quad (1.1a)$$

where  $I^c := \llbracket 1, n \rrbracket \setminus I$  denotes the complement of  $I$  in  $\llbracket 1, n \rrbracket$ . Such a point is called a *node* of problem  $\text{LCP}(M, q)$ . A node also depends on the matrix  $M$  but, since this one may be considered to be fixed in all the paper, this dependence is not mentioned. In contrast,  $I$  and  $q$  vary in some proposition claims. Since  $M$  is supposed to be nondegenerate, the system (1.1a) defines  $x^{(I, q)}$  unambiguously; it has indeed for unique solution

$$x_{I^c}^{(I, q)} = 0 \quad \text{and} \quad x_I^{(I, q)} = -M_{II}^{-1}q_I, \quad (1.1b)$$

where  $M_{II}^{-1}$  is a compact notation for  $(M_{II})^{-1}$ . □

Since there are  $2^n$  distinct subsets of  $\llbracket 1, n \rrbracket$ , there are at most  $2^n$  nodes, for given  $M$  and  $q$ . Actually, this number of nodes depends on the vector  $q$ . For example, when  $q = 0$ , we see by (1.1b) that there is a single node: the zero vector. It is shown in section 5 that there are  $2^n$  nodes for a set  $\mathcal{Q}(M)$  of  $q$ 's that is dense in  $\mathbb{R}^n$ .

The *Newton-min algorithm* is designed to find a solution to  $\text{LCP}(M, q)$ . It computes the next iterate  $x^+ \in \mathbb{R}^n$  from the current iterate  $x \in \mathbb{R}^n$  by

$$x^+ := x^{(\mathcal{S}(x, q), q)}, \quad (1.2)$$

where the *index selector*  $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^n \multimap \llbracket 1, n \rrbracket$  is the multifunction defined at  $(x, q) \in \mathbb{R}^n \times \mathbb{R}^n$  by

$$\mathcal{S}(x, q) := \{i \in \llbracket 1, n \rrbracket : x_i > (Mx + q)_i\}. \quad (1.3)$$

The multifunction  $\mathcal{S}$  was not used in [3], but it helps to clear up some ambiguities. Therefore, even if the first iterate is not a node, the next iterates are nodes. By (1.1b), each iteration requires computing the solution to a linear system of order  $|\mathcal{S}(x, q)|$ . We see that the Newton-min algorithm visits some of the potentially  $2^n$  nodes of the problem, in the hope of finding a solution node, if any. We recall that, when  $M$  is a  $\mathbf{P}$ -matrix, the Newton-min algorithm may cycle when  $n \geq 3$  but not for  $n \in \{1, 2\}$  [1, 2]. This algorithm is best viewed today as a semi-smooth Newton algorithm [29, 30] applied on the equation form of  $\text{LCP}(M, q)$  that reads  $\min(x, Mx + q) = 0$  (the “min” operator also acts componentwise). We refer the reader to the paragraph 7 of the introduction of [2] for a discussion on the origin of the algorithm and to [10, 26, 20, 19, 6, 5, 27, 21, 25, 9, 13, 23, 24, 22, 15] for other related contributions. For basic notions in optimization, the reader is referred to [7, 8].

The paper is organized as follows. Section 2 presents the common source of the errors made in [3; 2013], as well as the strategy used here to adapt its results and to prove them adequately. Section 3 is dedicated to finding a valid version of the equivalence in the proposition 3.2 of [3]; it is obtained by weakening one of its claims. Section 4 focuses on the proof of the main result, which remains correct and claims that a nondegenerate matrix  $M$  is a  $\mathbf{P}$ -matrix if and only if the Newton-min algorithm does not cycle between two distinct nodes, whatever is  $q$  (but it may make cycles having 3 or more nodes).

The references to the original paper [3] are specified here with the prefix [3]. Hence, “proposition [3].x.y” means proposition x.y of [3], “[3].a.b)” means formula (a.b) of [3], and section [3]. $\alpha$  means section  $\alpha$  of [3]. Table 1.1 summarizes the change from [3], by comparing the results of [3] with those of the present paper, and by specifying the modifications, if any.

In the paper [3]	In this paper	Comments
$x^{(I)}$	$x^{(I, q)}$	Emphasis on the dependence of $x^{(I)}$ on $q$
—	Sections 1 and 2	New sections
Proposition [3].3.2	Proposition 3.1	Weakening of (ii)
Lemma [3].4.1	Lemma 4.1	Reformulation
Theorem [3].4.2	Theorem 4.2	Weakening of (iii)
Corollary [3].4.3	Corollary 4.3	Unchanged
Proposition [3].4.4	Proposition 4.4	Two additional properties in point 2
—	Section 5	New section

Table 1.1: Comparison of the results of [3] and of the present paper

## 2 Common source of the errors in [3]

Even though it is not expressed in that way, the following wrong equivalence is implicitly used several times in [3]:

$$x^+ = x^{(I,q)} \stackrel{(\text{wrong})}{\iff} I = \mathcal{S}(x, q), \quad (2.1)$$

where  $x^+$  is supposed to be the node computed from  $x \in \mathbb{R}^n$  by the Newton-min algorithm,  $x^{(I,q)}$  is the node defined by (1.1), and  $\mathcal{S}$  is the index selector defined around (1.3).

For example, at the beginning of the proof of proposition [3].3.2, it is essentially written: “*by definition*, the Newton-min algorithm (1.2) generates the node  $x^{(J,q)}$  from the node  $x^{(I,q)}$ , for some given distinct index sets  $I$  and  $J \subset \llbracket 1, n \rrbracket$ , if *and only if* ([3].3.4) holds”. Let us clarify this claim. Formula ([3].3.4) reads

$$\begin{aligned} -(M_{II}^{-1}qI)_{J^c} \leq 0, \quad (M_{II}^{-1}qI)_J < 0, \\ q_{I^c \cap J} < M_{(I^c \cap J)I} M_{II}^{-1} qI, \quad \text{and} \quad M_{(I \cup J)^c I} M_{II}^{-1} qI \leq q_{(I \cup J)^c}. \end{aligned} \quad (2.2)$$

Since, from (1.1),  $x^{(I,q)}$  satisfies

$$\begin{cases} x_I^{(I,q)} = -M_{II}^{-1}qI \\ x_{I^c}^{(I,q)} = 0 \end{cases} \quad \text{and} \quad \begin{cases} (Mx^{(I,q)} + q)_I = 0 \\ (Mx^{(I,q)} + q)_{I^c} = q_{I^c} - M_{I^c I} M_{II}^{-1} qI, \end{cases} \quad (2.3)$$

we see that

$$\begin{aligned} (2.2) &\iff x_{I \cap J^c}^{(I,q)} \leq 0, \quad x_{I \cap J}^{(I,q)} > 0, \quad 0 > (Mx^{(I,q)} + q)_{I^c \cap J}, \quad 0 \leq (Mx^{(I,q)} + q)_{I^c \cap J^c} \\ &\iff x_{I \cap J^c}^{(I,q)} \leq (Mx^{(I,q)} + q)_{I \cap J^c}, \quad x_{I \cap J}^{(I,q)} > (Mx^{(I,q)} + q)_{I \cap J}, \\ &\quad x_{I^c \cap J}^{(I,q)} > (Mx^{(I,q)} + q)_{I^c \cap J}, \quad x_{I^c \cap J^c}^{(I,q)} \leq (Mx^{(I,q)} + q)_{I^c \cap J^c} \\ &\iff x_{J^c}^{(I,q)} \leq (Mx^{(I,q)} + q)_{J^c}, \quad x_J^{(I,q)} > (Mx^{(I,q)} + q)_J \\ &\iff J = \mathcal{S}(x^{(I,q)}, q), \end{aligned}$$

where we have used the definition (1.3) of the index selector  $\mathcal{S}$ . We quote this equivalence below for a future reference:

$$(2.2) \iff J = \mathcal{S}(x^{(I,q)}, q). \quad (2.4)$$

The sentence highlighted above, found at the beginning of the proof of proposition [3].3.2, therefore claims that the Newton-min computes  $x^{(J,q)}$  from  $x^{(I,q)}$  if and only if  $J = \mathcal{S}(x^{(I,q)}, q)$ . After the change in notation  $x^{(I,q)} \rightsquigarrow x$  and  $x^{(J,q)} \rightsquigarrow x^+$ , this corresponds to the alleged equivalence (2.1).

The right-to-left implication “ $\Leftarrow$ ” in (2.1) certainly holds by the very definition (1.2) of the Newton-min algorithm, but the left-to-right implication “ $\Rightarrow$ ” may fail, because the node  $x^+$  may also be defined by an index set  $I' \subset \llbracket 1, n \rrbracket$  different from the given index set  $I$ :  $x^+ = x^{(I,q)} = x^{(I',q)}$ . We stress this observation with a counter-example that will help becoming acquainted with the problem and the Newton-min algorithm.

**Counter-example 2.1 (left-to-right implication in (2.1) may fail)** Consider problem  $\text{LCP}(M, q)$  with  $n = 2$ ,  $M$  is the identity, and  $q = e^1 := (1 \ 0)^\top$ . Then, the problem has only two distinct nodes, namely  $x^{\{\{1\}, q\}} = x^{\{\{1,2\}, q\}} = -e^1$  and  $x^{\{\emptyset, q\}} = x^{\{\{2\}, q\}} = 0$ , the latter being the solution to the problem. If one takes  $I = \{1\}$  and  $J = \{2\}$ , the Newton-min algorithm goes indeed from the node  $x^{\{\{1\}, q\}}$  to the solution  $x^{\{\{2\}, q\}}$ , but it is not true that  $\{2\} = \mathcal{S}(x^{\{\{1\}, q\}}, q)$ . To see this, write  $x^{\{\{1\}, q\}} = -e^1$  and  $(Mx^{\{\{1\}, q\}} + q) = 0$ , from which and the definition (1.3) of  $\mathcal{S}$ , one concludes that  $\mathcal{S}(x^{\{\{1\}, q\}}, q) = \emptyset$ .  $\square$

In other words, in (1.2),  $\mathcal{S}(x, q)$  is just one of the index sets  $I'$  that defines the new iterate  $x^+$  as a node  $x^{(I', q)}$ , not necessarily the one that is fixed in the context where this wrong equivalence is used (in proposition [3].3.2 the index sets are fixed outside its claims (i) and (ii)). From this point of view, it is convenient to introduce the following definition.

**Definition 2.2 (uniquely determined node)** A node  $x$  of  $\text{LCP}(M, q)$  is said to be *uniquely determined* if there is a unique index set  $I \subset \llbracket 1, n \rrbracket$  such that  $x = x^{(I, q)}$ .  $\square$

It is precisely because some reasonings in [3] neglect the fact that the considered nodes may not be uniquely determined that corrections and refinements are necessary. The fact that all the nodes are uniquely determined depends on  $q$  and one can show that the  $q$ 's for which that property occurs is dense in  $\mathbb{R}^n$  (see section 5). Nevertheless, this density property seems to us useless when Motzkin's theorem of the alternative plays a key role in the analysis, like in [3]. Therefore our strategy to amend or to validate the results of [3] does not consist in using that density property.

Despite the misinterpretation (2.1) of the meaning of the definition (1.2) of the Newton-min iteration, the proofs of [3] are not meaningless. Our approach consists therefore to give a precise statement of what these proofs provide and next to give complements to enrich these results in order to make the outcomes as close as possible to the results claimed in [3] (these are sometimes erroneous). This approach is actually mainly used for proposition [3].3.2, whose role is prominent. Occasionally, these complements are even not necessary. In particular, the main result of [3] is valid: a nondegenerate matrix  $M$  is a  $\mathbf{P}$ -matrix if and only if the Newton-min algorithm does not cycle between two different nodes, whatever is  $q$ .

### 3 On proposition [3].3.2

Proposition 3.1 below gives the correct expression of the outcome of the reasoning used in the proof of proposition [3].3.2. The conditions (i) of proposition [3].3.2 and proposition 3.1 are identical, but their conditions (ii) are a little different. In particular, condition (ii) below is compatible with a cycle  $x^{(I, q)} \rightarrow x^{(J, q)} \rightarrow x^{(I, q)}$  that would be made by the Newton-min algorithm for some  $q$ , while condition (ii) in proposition [3].3.2 claims that such a cycle does not occur. The latter claim is wrong! To see this, take  $q = 0$  and arbitrary distinct index sets  $I$  and  $J \subset \llbracket 1, n \rrbracket$ ; then,  $x^{(I, q)} = x^{(J, q)} = 0$  and  $Mx^{(I, q)} + q = Mx^{(J, q)} + q = 0$ , so that the Newton-min algorithm makes the cycle  $0 \rightarrow 0 \rightarrow 0$  (0 is actually a solution to the problem in this case). In contrast, the conclusion in (ii) below is correct when  $q = 0$ , without having to use (i), since one cannot have  $J = \mathcal{S}(x^{(I, q)}, q)$  and  $I = \mathcal{S}(x^{(J, q)}, q)$  because

$\mathcal{S}(x^{(I,q)}, q)$  and  $\mathcal{S}(x^{(J,q)}, q)$  are both empty and  $I \neq J$  by assumption. It is important to require  $I \neq J$  in the assumption, otherwise (i) does not provide any information.

The following proposition gives necessary and sufficient conditions (NSC) for avoiding to have both  $J = \mathcal{S}(x^{(I,q)}, q)$  and  $I = \mathcal{S}(x^{(J,q)}, q)$ , whatever is  $q$ . Recall that the *symmetric difference* of the two index sets  $I$  and  $J \subset \llbracket 1, n \rrbracket$  is defined and denoted by

$$I \Delta J := (I \cap J^c) \cup (I^c \cap J) = (I \cup J) \setminus (I \cap J).$$

**Proposition 3.1 (NSC for  $J \neq \mathcal{S}(x^{(I,q)}, q)$  or  $I \neq \mathcal{S}(x^{(J,q)}, q)$ )** Suppose that  $M \in \mathbb{R}^{n \times n}$  is nondegenerate and let  $I$  and  $J \subset \llbracket 1, n \rrbracket$  be two different index sets. Then, the following conditions are equivalent:

(i) there is an  $\alpha \in \mathbb{R}_+^{|I \Delta J|} \setminus \{0\}$  such that

$$\begin{aligned} & \begin{pmatrix} M_{(I \cap J^c)(I \cap J^c)} & -M_{(I \cap J^c)(I^c \cap J)} \\ -M_{(I^c \cap J)(I \cap J^c)} & M_{(I^c \cap J)(I^c \cap J)} \end{pmatrix}^\top \alpha \\ & \geq \begin{pmatrix} -M_{(I \cap J)(I \cap J^c)} & M_{(I \cap J)(I^c \cap J)} \end{pmatrix}^\top M_{(I \cap J)(I \cap J)}^{-\top} \begin{pmatrix} -M_{(I \cap J^c)(I \cap J)} \\ M_{(I^c \cap J)(I \cap J)} \end{pmatrix}^\top \alpha, \end{aligned} \quad (3.1)$$

where the right-hand side is zero when  $I \cap J = \emptyset$ ,

(ii) whatever is  $q$ , one cannot have  $J = \mathcal{S}(x^{(I,q)}, q)$  and  $I = \mathcal{S}(x^{(J,q)}, q)$ .

PROOF. We only sketch the proof, since it is very similar to the one of proposition [3].3.2. Only the equivalence (3.2) below differs, since it takes into account the fact that the equivalence (2.1) may fail. Actually, instead of reading ([3].3.4) and ([3].3.5) as properties equivalent to the presence of a cycle  $x^{(I,q)} \rightarrow x^{(J,q)} \rightarrow x^{(I,q)}$  for the considered  $q$ , which is not correct, we express ([3].3.4) and ([3].3.5) by their meaning derived from (2.4).

From (2.4), ([3].3.4) reads  $J = \mathcal{S}(x^{(I,q)}, q)$ , as shown in the second paragraph of section 2. Since ([3].3.5) can be obtained from ([3].3.4) by switching  $I$  and  $J$ , it reads  $I = \mathcal{S}(x^{(J,q)}, q)$ . Therefore, for a fixed  $q \in \mathbb{R}^n$ , the following holds:

$$([3].3.4) \text{ and } ([3].3.5) \iff J = \mathcal{S}(x^{(I,q)}, q) \text{ and } I = \mathcal{S}(x^{(J,q)}, q). \quad (3.2)$$

Next, it is shown in the proof of proposition [3].3.2, using Motzkin's theorem of the alternative, that

$$\begin{aligned} & \exists q \in \mathbb{R}^n \text{ satisfying } ([3].3.4) \text{ and } ([3].3.5) \\ \iff & \nexists (\alpha, \alpha', \alpha'', \beta) \in \mathbb{R}_+^{|I \Delta J|} \times \mathbb{R}_+^{|I \cap J|} \times \mathbb{R}_+^{|I \cap J|} \times \mathbb{R}_+^{|I \Delta J|} \\ & \text{that satisfies } (\alpha, \alpha', \alpha'') \neq 0 \text{ and } ([3].3.6). \end{aligned} \quad (3.3)$$

Finally, it is shown in points 2 and 3 of the proof of proposition [3].3.2 that (3.3) simplifies to

$$\begin{aligned} & \exists q \in \mathbb{R}^n \text{ satisfying } ([3].3.4) \text{ and } ([3].3.5) \\ \iff & \nexists \alpha \in \mathbb{R}_+^{|I \Delta J|} \setminus \{0\} \text{ that satisfies } (3.1). \end{aligned} \quad (3.4)$$

One can now show the equivalence between (i) and (ii). Indeed, by the contrapositive of (3.4), (i) holds if and only if there is no  $q \in \mathbb{R}^n$  satisfying ([3].3.4) and ([3].3.5) or, by (3.2), if and only if there is no  $q \in \mathbb{R}^n$  satisfying both  $J = \mathcal{S}(x^{(I,q)}, q)$  and  $I = \mathcal{S}(x^{(J,q)}, q)$ .  $\square$

## 4 Revision of section [3].4

The next lemma reformulates the contrapositive of Lemma [3].4.1 in terms of the index selector  $\mathcal{S}$  defined in (1.3). It is now viewed as a condition ensuring that the Newton-min algorithm computes a nonzero displacement from  $x^{(I,q)}$ . This lemma is no longer used in the proof of theorem 4.2, like this was the case in the proof of theorem [3].4.2, but in the proof of proposition 4.4.

**Lemma 4.1 (nonzero displacement)** *Suppose that  $M$  is nondegenerate and let be given  $q \in \mathbb{R}^n$  and  $I \subset \llbracket 1, n \rrbracket$ . Then,*

$$\mathcal{S}(x^{(I,q)}, q) \setminus I \neq \emptyset \quad \implies \quad x^{(\mathcal{S}(x^{(I,q)}, q), q)} \neq x^{(I,q)}.$$

PROOF. Let  $J := \mathcal{S}(x^{(I,q)}, q)$ . On the one hand, since by assumption  $I^c \cap J = J \setminus I$  is nonempty, the following holds

$$\begin{aligned} (Mx^{(I,q)} + q)_{I^c \cap J} &< x_{I^c \cap J}^{(I,q)} \quad [\text{definition of } J = \mathcal{S}(x^{(I,q)}, q) \text{ and (1.3)}] \\ &= 0 \quad [(1.1b)]. \end{aligned}$$

On the other hand,

$$(Mx^{(J,q)} + q)_J = 0,$$

by the definition 1.1 of the node  $x^{(J,q)}$ . Therefore  $(Mx^{(I,q)} + q)_{I^c \cap J} \neq (Mx^{(J,q)} + q)_{I^c \cap J}$ , since the first vector is negative and the second vanishes. Since  $I^c \cap J \neq \emptyset$ , this certainly implies that  $x^{(J,q)} \neq x^{(I,q)}$ .  $\square$

Let us now consider the revision of theorem [3].4.2, which is given in theorem 4.2 below. It is the main result of the paper. The statement of the latter theorem is almost identical to the former, except that in (iii) the considered cycles are between *distinct nodes being described by the index sets determined by the index selector  $\mathcal{S}$* . The changes in the proof are the following.

- The proof of the implication (i)  $\Rightarrow$  (ii) has been changed to take into account the fact that the equivalence (2.1) does not necessarily hold. Nevertheless, the argument is essentially the same after a redefinition of the index sets associated with the nodes of the considered cycle.
- With the changes in the statement of (iii), the implication (ii)  $\Rightarrow$  (iii) becomes straightforward and no longer uses lemma 4.1.
- The implication (iii)  $\Rightarrow$  (iv) is proved similarly, but with the updated version of proposition 3.1, whose condition (ii) is weaker. This is why we have weakened the condition (iii) of the theorem.
- We have taken the opportunity of this new proof to be a little more explicit in the proof of the implication (iv)  $\Rightarrow$  (i).

For the reader's convenience, we have reproduced in full the parts of proof of theorem [3].4.2 that need no modification.



Let us recall some notation and associated properties. We denote by  $\text{cof}(M)$  the *cofactor matrix* of a matrix  $M \in \mathbb{R}^{n \times n}$ , whose element  $[\text{cof}(M)]_{ij}$  is the *cofactor*  $\text{cof}(M_{ij})$  of the element  $M_{ij}$  of  $M$ , that is

$$\text{cof}(M_{ij}) := (-1)^{i+j} \det M_{(\llbracket 1, n \rrbracket \setminus \{i\})(\llbracket 1, n \rrbracket \setminus \{j\})}. \quad (4.1)$$

We use the notation  $\text{cof}_{II}(M_{ij})$  for the cofactor of the element  $M_{ij}$  in  $M_{II}$ , assuming that both  $i$  and  $j \in I$ . Recall [28; 1987, chapter VI] that for any index  $i$  and  $j$ :

$$\det M = \sum_{i'} M_{i'j} \text{cof}(M_{i'j}) = \sum_{j'} M_{ij'} \text{cof}(M_{ij'}) \quad (4.2)$$

and that

$$M^{-1} = (\det M)^{-1} \text{cof}(M^T). \quad (4.3)$$

We also recall the following characterization of  $\mathbf{P}$ -matricity [18, 12; 1962]:

$$M \in \mathbf{P} \quad \iff \quad \text{any } x \text{ verifying } x \cdot (Mx) \leq 0 \text{ vanishes,} \quad (4.4)$$

where we have denoted by  $u \cdot v$  the *Hadamard product* of the vectors  $u$  and  $v$ , which is the vector whose  $i$ th component is  $u_i v_i$ .

**Theorem 4.2 (a characterization of  $\mathbf{P}$ -matricity)** *Suppose that  $M \in \mathbb{R}^{n \times n}$  is nondegenerate. Then, the following conditions are equivalent:*

- (i)  $M \in \mathbf{P}$ ,
- (ii) for any  $q$ , the Newton-min algorithm does not cycle between two distinct nodes when it is used to solve  $\text{LCP}(M, q)$ ,
- (iii) for any  $q$  and any index sets  $I$  et  $J \subset \llbracket 1, n \rrbracket$  such that  $I = J \cup \{i\}$  for some  $i \in \llbracket 1, n \rrbracket \setminus J$ , one cannot have  $I = \mathcal{S}(x^{(J,q)}, q)$  and  $J = \mathcal{S}(x^{(I,q)}, q)$ ,
- (iv) for any index sets  $I$  et  $J \subset \llbracket 1, n \rrbracket$  such that  $I = J \cup \{i\}$  for some  $i \in \llbracket 1, n \rrbracket \setminus J$ , the following holds

$$M_{ii} \geq M_{\{i\}J} M_{JJ}^{-1} M_{J\{i\}}, \quad (4.5)$$

where the right-hand side is zero when  $J = \emptyset$ .

PROOF. [(i)  $\Rightarrow$  (ii)] We prove the contrapositive, assuming that the algorithm visits in order the following nodes  $x^{(I,q)} \rightarrow x^{(J,q)} \rightarrow x^{(I,q)}$ , for some  $I$  and  $J \subset \llbracket 1, n \rrbracket$  and some  $q \in \mathbb{R}^n$  such that  $x^{(I,q)} \neq x^{(J,q)}$ . We simplify the notation by setting  $x^1 := x^{(I,q)}$  and  $x^2 := x^{(J,q)}$ .

By renaming the index sets  $I$  and  $J$ , we can assume that  $I = \mathcal{S}(x^2, q)$  and  $J = \mathcal{S}(x^1, q)$ . Indeed, set  $I' := \mathcal{S}(x^{(J,q)}, q)$  and  $J' := \mathcal{S}(x^{(I,q)}, q)$ . By the definition 1.3 of the index selector  $\mathcal{S}$ , the Newton-min algorithm goes from  $x^{(I,q)}$  to  $x^{(J',q)}$  and from  $x^{(J,q)}$  to  $x^{(I',q)}$ . By the existence of the cycle  $x^{(I,q)} \rightarrow x^{(J,q)} \rightarrow x^{(I,q)}$ , there must hold  $x^{(I',q)} = x^{(I,q)}$  and  $x^{(J',q)} = x^{(J,q)}$ . Therefore, one has the cycle  $x^{(I',q)} \rightarrow x^{(J',q)} \rightarrow x^{(I',q)}$  with the desired properties  $I' = \mathcal{S}(x^{(J',q)}, q)$  and  $J' = \mathcal{S}(x^{(I',q)}, q)$ . Now rename  $I' \rightsquigarrow I$  and  $J' \rightsquigarrow J$ .

Then, by the definition (1.3) of the index selector  $\mathcal{S}$ , there hold

$$x_{J^c}^1 \leq (Mx^1 + q)_{J^c} \quad \text{and} \quad x_J^1 > (Mx^1 + q)_J, \quad (4.6a)$$

$$x_{I^c}^2 \leq (Mx^2 + q)_{I^c} \quad \text{and} \quad x_I^2 > (Mx^2 + q)_I. \quad (4.6b)$$

We now express the fact that  $x^1$  is the node  $x^{(I,q)}$  and  $x^2$  is the node  $x^{(J,q)}$ . By the definition 1.1 of a node:

$$x_{I^c}^1 = 0 \quad \text{and} \quad (Mx^1 + q)_I = 0, \quad (4.7a)$$

$$x_{J^c}^2 = 0 \quad \text{and} \quad (Mx^2 + q)_J = 0. \quad (4.7b)$$

Using (4.7a)<sub>1</sub> and (4.7b)<sub>1</sub>, we get, after a possible rearrangement of the component order

$$x^2 - x^1 = \begin{pmatrix} x_{I \cap J}^2 \\ 0_{I \cap J^c} \\ x_{I^c \cap J}^2 \\ 0_{I^c \cap J^c} \end{pmatrix} - \begin{pmatrix} x_{I \cap J}^1 \\ x_{I \cap J^c}^1 \\ 0_{I^c \cap J} \\ 0_{I^c \cap J^c} \end{pmatrix} = \begin{pmatrix} (x^2 - x^1)_{I \cap J} \\ -x_{I \cap J^c}^1 \\ x_{I^c \cap J}^2 \\ 0_{I^c \cap J^c} \end{pmatrix}. \quad \begin{matrix} [?] \\ [+] \\ [-] \\ [0] \end{matrix}$$

The extra column on the right gives the sign of each component, when this is possible: the components of  $x^2 - x^1$  with indices in  $I \cap J^c$  are nonnegative since  $-x_{I \cap J^c}^1 \geq -(Mx^1 + q)_{I \cap J^c}$  [by (4.6a)<sub>1</sub>] = 0 [by (4.7a)<sub>2</sub>] and the components of  $x^2 - x^1$  with indices in  $I^c \cap J$  are nonpositive since  $x_{I^c \cap J}^2 \leq (Mx^2 + q)_{I^c \cap J}$  [by (4.6b)<sub>1</sub>] = 0 [by (4.7b)<sub>2</sub>]. Furthermore, by (4.7a)<sub>2</sub> and (4.7b)<sub>2</sub>, the following holds

$$M(x^2 - x^1) = \begin{pmatrix} -q_{I \cap J} \\ (Mx^2)_{I \cap J^c} \\ -q_{I^c \cap J} \\ (Mx^2)_{I^c \cap J^c} \end{pmatrix} - \begin{pmatrix} -q_{I \cap J} \\ -q_{I \cap J^c} \\ (Mx^1)_{I^c \cap J} \\ (Mx^1)_{I^c \cap J^c} \end{pmatrix} = \begin{pmatrix} 0_{I \cap J} \\ (Mx^2 + q)_{I \cap J^c} \\ -(Mx^1 + q)_{I^c \cap J} \\ (M(x^2 - x^1))_{I^c \cap J^c} \end{pmatrix}. \quad \begin{matrix} [0] \\ [-] \\ [+] \\ [?] \end{matrix}$$

The extra column on the right gives the sign of each component, when this is possible: the components of  $M(x^2 - x^1)$  with indices in  $I \cap J^c$  are nonpositive since  $(Mx^2 + q)_{I \cap J^c} < x_{I \cap J^c}^2$  [by (4.6b)<sub>2</sub>] = 0 [by (4.7b)<sub>1</sub>] and the components of  $M(x^2 - x^1)$  with indices in  $I^c \cap J$  are nonnegative since  $-(Mx^1 + q)_{I^c \cap J} > -x_{I^c \cap J}^1$  [by (4.6a)<sub>2</sub>] = 0 [by (4.7a)<sub>1</sub>]. Therefore

$$(x^2 - x^1) \cdot M(x^2 - x^1) \leq 0.$$

Since  $x^1 \neq x^2$ ,  $M$  cannot be a **P**-matrix (see (4.4)).

[(ii)  $\Rightarrow$  (iii)] Let  $q$ ,  $I$ , and  $J$  be given like in (iii). We consider two cases. If  $x^{(I,q)} = x^{(J,q)}$ , one cannot have  $I = \mathcal{S}(x^{(J,q)}, q)$  and  $J = \mathcal{S}(x^{(I,q)}, q)$ , since this would imply  $I = J$ , in contradiction with the definition of  $I$  and  $J$ . If  $x^{(I,q)} \neq x^{(J,q)}$ , one cannot have  $I = \mathcal{S}(x^{(J,q)}, q)$  and  $J = \mathcal{S}(x^{(I,q)}, q)$  since, otherwise, by the definition (1.2) of the Newton-min algorithm, there would be the cycle  $x^{(I,q)} \rightarrow x^{(J,q)} \rightarrow x^{(I,q)}$  with  $x^{(I,q)} \neq x^{(J,q)}$ , which would contradict (ii).

[(iii)  $\Rightarrow$  (iv)] Let  $I, J \subset \llbracket 1, n \rrbracket$  and  $i \in \llbracket 1, n \rrbracket$  be like in (iv). By (iii), whatever is  $q$ , one cannot have  $I = \mathcal{S}(x^{(J,q)}, q)$  and  $J = \mathcal{S}(x^{(I,q)}, q)$ . Then, the implication (ii)  $\Rightarrow$  (i) of proposition 3.1 shows that there is a scalar  $\alpha > 0$  such that (3.1) holds. Since  $I \cap J^c = \{i\}$ ,  $I^c \cap J = \emptyset$ ,  $I \cap J = J$ ,  $I \triangle J = \{i\}$ , this inequality (3.1) simplifies to

$$M_{ii}\alpha \geq (-M_{J\{i\}})^\top M_{JJ}^{-\top} (-M_{\{i\}J})^\top \alpha.$$

Now  $\alpha$  is a positive scalar that can be eliminated and the right-hand side is a scalar (hence equal to its transpose), so that the above inequality becomes (4.5). In case  $J = \emptyset$ , the inequality (3.1) simply yields  $M_{ii} \geq 0$ .

[(iv)  $\Rightarrow$  (i)] We prove by induction that  $\det M_{II} > 0$  for any  $I \subset \llbracket 1, n \rrbracket$ , which is equivalent to  $M \in \mathbf{P}$ . By applying (iv) with  $J = \emptyset$ , we obtain  $M_{ii} > 0$  for a nondegenerate matrix, so that  $\det M_{II} > 0$  when  $|I| = 1$ . Now, assume that  $J$  and  $i$  are chosen like in (iv), that  $I = J \cup \{i\}$ , that  $\det M_{JJ} > 0$  (induction assumption), and let us show that  $\det M_{II} > 0$ , which will conclude the proof of (iv)  $\Rightarrow$  (i).

Let us denote the indices in  $J$  by  $j_k$ ,  $k \in \llbracket 1, |J| \rrbracket$ , and let us label the elements of  $M_{JJ}$  by their indices in  $J$ . Using the cofactor matrix of  $M_{JJ}$  in (4.5) and the induction assumption  $\det M_{JJ} > 0$ , one gets (see the explanation of (4.8) and (4.9) below)

$$\begin{aligned}
0 &\leq M_{ii} \det M_{JJ} - M_{\{i\}J} \operatorname{cof}(M_{JJ}^\top) M_{J\{i\}} && [(4.5), (4.3), \det M_{JJ} > 0] \\
&= M_{ii} \det M_{JJ} - \sum_{k=1}^{|J|} \sum_{l=1}^{|J|} M_{ij_k} \operatorname{cof}_{JJ}([M_{JJ}]_{j_l j_k}) M_{j_l i} \\
&= M_{ii} \det M_{JJ} - \sum_{k=1}^{|J|} \sum_{l=1}^{|J|} M_{ij_k} (-1)^{l+k} \det M_{(J \setminus \{j_l\})(J \setminus \{j_k\})} M_{j_l i} && [(4.1)] \\
&= M_{ii} \det M_{JJ} + \sum_{k=1}^{|J|} (-1)^{k+|J|+1} M_{ij_k} \sum_{l=1}^{|J|} M_{j_l i} (-1)^{l+|J|} \det M_{(J \setminus \{j_l\})(J \setminus \{j_k\})} \\
&= M_{ii} \det M_{JJ} + \sum_{k=1}^{|J|} (-1)^{k+|J|+1} M_{ij_k} \det \begin{pmatrix} M_{J(J \setminus \{j_k\})} & M_{J\{i\}} \end{pmatrix} && [(4.2)_1] \quad (4.8) \\
&= \det M_{II} && [(4.2)_2]. \quad (4.9)
\end{aligned}$$

Formula (4.8) comes from the computation of the determinant of the  $|J| \times |J|$  matrix

$$\begin{pmatrix} M_{J(J \setminus \{j_k\})} & M_{J\{i\}} \end{pmatrix}$$

using the first identity in (4.2) on its last column. Formula (4.9) computes the determinant of the  $|I| \times |I|$  matrix

$$\begin{pmatrix} M_{JJ} & M_{J\{i\}} \\ M_{\{i\}J} & M_{ii} \end{pmatrix}$$

using the second identity in (4.2) on its last row. This one is indeed the determinant of  $M_{II}$  after permutations of two rows and two columns to put the row  $i$  and column  $i$  at the right place in  $I$  (this does not affect the sign of the determinant). Finally, using the nondegeneracy of  $M$ , we get  $\det M_{II} > 0$ .  $\square$

Let  $\mathbf{NM}$  be the class of nondegenerate matrices  $M \in \mathbb{R}^{n \times n}$  such that the Newton-min algorithm converges, when it is used to solve  $\text{LCP}(M, q)$ , whatever is  $q$  and the initial point. The corollary [3].4.3 is still valid and reads as follows. We omit its proof, which needs no change.

**Corollary 4.3 (NM is included in P)** *The set of nondegenerate matrices  $M$  ensuring the convergence of the Newton-min algorithm when it is used to solve  $\text{LCP}(M, q)$ ,*

whatever are the vector  $q$  and the initial point, is included in  $\mathbf{P}$ . More compactly

$$\mathbf{NM} \subset \mathbf{P}. \quad (4.10)$$

To be complete, we reproduce proposition [3].4.4 with one additional property, which is that the points  $x^{(I,q)}$  and  $x^{(J,q)}$  introduced in the proposition are different. The interest of that property is that the proposition can then be used to prove the contrapositive of the implication (iii)  $\Rightarrow$  (i) of theorem 4.2; see the comment after the proposition.

**Proposition 4.4 (2-cycle for  $M \notin \mathbf{P}$ )** Suppose that the nondegenerate matrix  $M$  is not a  $\mathbf{P}$ -matrix. Then,

- 1) there are two index sets  $I$  and  $J \subset \llbracket 1, n \rrbracket$  and an index  $i \in \llbracket 1, n \rrbracket \setminus J$  such that  $I = J \cup \{i\}$ ,  $\det M_{II} < 0$ , and  $\det M_{JJ} > 0$ ,
- 2) for any two index sets  $I$  and  $J \subset \llbracket 1, n \rrbracket$  and an index  $i \in \llbracket 1, n \rrbracket \setminus J$  having the properties given in point 1, the Newton-min algorithm cycles between the two distinct nodes  $x^{(I,q)}$  and  $x^{(J,q)}$  when the components of  $q$  are determined in order as follows

$$q_J = -M_{JJ}e^J, \quad (4.11)$$

$$q_i = -M_{iJ}e^J - \varepsilon, \quad \text{with } 0 < \varepsilon < \frac{|\det M_{II}|}{\max_{j \in J} [\text{cof}_{II}(M_{ij})]^+}, \quad (4.12)$$

$$q_{I^c} \geq \max(M_{I^cJ}M_{JJ}^{-1}q_J, M_{I^cI}M_{II}^{-1}q_I), \quad (4.13)$$

where  $e^J$  is the vector of all ones in  $\mathbb{R}^{|J|}$ ; in addition,  $I = \mathcal{S}(x^{(J,q)}, q)$  and  $J = \mathcal{S}(x^{(I,q)}, q)$ .

PROOF. The proof given in [3] is still valid, so that we only have to show that  $x^{(I,q)} \neq x^{(J,q)}$  and that  $I = \mathcal{S}(x^{(J,q)}, q)$  and  $J = \mathcal{S}(x^{(I,q)}, q)$  in point 2, which is straightforward. Indeed, the proof in [3] shows that the Newton-min algorithm makes the cycle  $x^{(I,q)} \rightarrow x^{(J,q)} \rightarrow x^{(I,q)}$ , with  $J = \mathcal{S}(x^{(I,q)}, q)$  and  $I = \mathcal{S}(x^{(J,q)}, q)$ . Since  $\mathcal{S}(x^{(J,q)}, q) \setminus J = I \setminus J = \{i\} \neq \emptyset$ , lemma 4.1 implies that  $x^{(I,q)} \neq x^{(J,q)}$ .  $\square$

To conclude this section, let us show how proposition 4.4 can be used to prove the implication (iii)  $\Rightarrow$  (i) of theorem 4.2. With this approach, the proof of our main result, the equivalence (i)  $\Leftrightarrow$  (ii) of theorem 4.2, no longer needs proposition 3.1.

**Remark 4.5 (another proof of (iii)  $\Rightarrow$  (i) in theorem 4.2)** We prove the contrapositive. Suppose that  $M$  is not a  $\mathbf{P}$ -matrix (but  $M$  is nondegenerate by assumption). By proposition 4.4, one can find a vector  $q$  and two index sets  $I$  and  $J \subset \llbracket 1, n \rrbracket$  with  $I = J \cup \{i\}$  for some  $i \in \llbracket 1, n \rrbracket$ , such that  $I = \mathcal{S}(x^{(J,q)}, q)$  and  $J = \mathcal{S}(x^{(I,q)}, q)$ . Hence (iii) does not hold.  $\square$

## 5 Uniquely determined nodes

This section highlights conditions under which the nodes of a particular instance of problem  $\text{LCP}(M, q)$  are *uniquely determined*, a concept introduced by definition 2.2. In particular, it is shown that, given a nondegenerate matrix  $M$ , the set  $\mathcal{Q}(M)$  of  $q$ 's in  $\mathbb{R}^n$  such that all the nodes are uniquely determined is dense in  $\mathbb{R}^n$  (proposition 5.5). We start with a proposition giving, in particular, conditions ensuring that the equivalence (2.1) holds. We have denoted by  $\mathcal{N}(M, q)$  the set of nodes of the problem  $\text{LCP}(M, q)$ :

$$\mathcal{N}(M, q) := \{x^{(I, q)} \in \mathbb{R}^n : I \subset \llbracket 1, n \rrbracket\}.$$

**Lemma 5.1 (2<sup>n</sup> nodes)** *Consider problem  $\text{LCP}(M, q)$  with a nondegenerate  $M \in \mathbb{R}^{n \times n}$  and  $q \in \mathbb{R}^n$ . Then, the following properties are equivalent*

- (i) *problem  $\text{LCP}(M, q)$  has  $2^n$  distinct nodes,*
  - (ii) *the map  $\mathcal{X} : I \subset \llbracket 1, n \rrbracket \mapsto x^{(I, q)} \in \mathcal{N}(M, q)$  is bijective,*
- and these properties imply that*
- (iii) *whatever is  $x \in \mathbb{R}^n$ , the equivalence (2.1) holds, with  $x^+$  being the point generated by the Newton-min algorithm from  $x$ .*

PROOF. [(i)  $\Rightarrow$  (ii)] By construction, the map  $\mathcal{X}$  is surjective. Since there are  $2^n$  distinct intervals  $I \subset \llbracket 1, n \rrbracket$ , the map  $\mathcal{X}$  must also be injective if there are  $2^n$  distinct nodes  $x^{(I, q)}$ .

[(ii)  $\Rightarrow$  (i)] If the map  $\mathcal{X}$  is bijective, the number of distinct nodes  $x^{(I, q)}$  is equal to the number of elements in the power set  $\mathcal{P}(\llbracket 1, n \rrbracket)$ , which is  $2^n$ .

[(ii)  $\Rightarrow$  (iii)] The implication “ $\Leftarrow$ ” in (2.1) holds by definition of the Newton-min algorithm. For the reverse implication “ $\Rightarrow$ ” in (2.1), suppose that  $x^+ = x^{(I, q)}$  for some  $I$ . By definition of the algorithm,  $x^+ = x^{(\mathcal{S}(x, q), q)}$ , so that  $x^{(I, q)} = x^{(\mathcal{S}(x, q), q)}$ . By (ii), one has  $I = \mathcal{S}(x, q)$ .  $\square$

For the purpose of clarification, let us introduce the following notion. For a given pair  $(M, q) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n$ , we say that a node  $x^{(I, q)}$  is *reachable* or *can be reached* if there is an  $x \in \mathbb{R}^n$  such that  $x^{(I, q)} = x^{(\mathcal{S}(x, q), q)}$  or, equivalently, if there is an  $x \in \mathbb{R}^n$  such that the Newton-min algorithm starting at  $x$  computes  $x^{(I, q)}$  as its next iterate. Let us denote by  $\mathcal{N}_r \equiv \mathcal{N}_r(M, q)$  the set of reachable nodes. This one is the range space of  $\mathcal{X} \circ \mathcal{S}(\cdot, q)$ :

$$\mathcal{N}_r = (\mathcal{X} \circ \mathcal{S})(\mathbb{R}^n, q).$$

**Remark 5.2 (reachable nodes)** *Not all the nodes of problem  $\text{LCP}(M, q)$  are reachable. For example, if  $M = I_n$  and  $q > 0$ , whatever is  $x \in \mathbb{R}^n$ , one has  $\mathcal{S}(x, q) = \emptyset$ , so that  $x^+ = x^{(\emptyset, q)} = 0$ . Now there are  $2^n$  distinct nodes for that problem, so that  $2^n - 1$  nodes are not reachable (this is clear, but point 2 of proposition 5.5 can also be invoked).*  $\square$

**Remark 5.3** *It is not true that the implication (iii)  $\Rightarrow$  (i) or (ii) holds in the previous lemma. Consider for example the case when*

$$n = 2, \quad M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (5.1)$$

The nodes of the problem are

$$x^{(\emptyset,q)} = 0, \quad x^{\{\{1\},q\}} = x^{\{\{1,2\},q\}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad x^{\{\{2\},q\}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Only two nodes can be reached by the Newton-min algorithm, the nodes  $x^{(\emptyset,q)}$  and  $x^{\{\{2\},q\}}$ . Indeed, the first node is reached when  $x_1 \geq -1$  (in this case  $\mathcal{S}(x,q)$  is indeed empty), while the second is reached when  $x_1 < -1$  (in this case  $\mathcal{S}(x,q)$  is indeed  $\{2\}$ ). These reachable nodes are uniquely identified by an index set:

$$\begin{aligned} x^+ = x^{(\emptyset,q)} = x^{(I,q)} &\implies I = \emptyset, \\ x^+ = x^{\{\{2\},q\}} = x^{(I,q)} &\implies I = \{2\}. \end{aligned}$$

Therefore the equivalence (2.1) holds, although there are less than  $2^n$  nodes.  $\square$

The observation made in the previous remark is formalized in the next lemma.

**Lemma 5.4 (uniquely determined reachable nodes)** *Consider problem LCP  $(M,q)$  with a nondegenerate  $M \in \mathbb{R}^{n \times n}$  and  $q \in \mathbb{R}^n$ . Then, the following properties are equivalent*

- (i) *for any  $x^+ \in \mathcal{N}_r$ , there is a unique  $I \subset \llbracket 1, n \rrbracket$  such that  $x^+ = x^{(I,q)}$ ,*
- (ii) *whatever is  $x \in \mathbb{R}^n$ , the equivalence (2.1) holds, with  $x^+$  being the point generated by the Newton-min algorithm from  $x$ .*

PROOF. [(i)  $\Rightarrow$  (ii)] The implication “ $\Leftarrow$ ” in (2.1) holds by definition of the Newton-min algorithm. For the reverse implication “ $\Rightarrow$ ” in (2.1), suppose that  $x^+ = x^{(I,q)}$  for some  $I$ . By definition of the algorithm,  $x^+ = x^{(\mathcal{S}(x,q),q)}$ , so that  $x^{(I,q)} = x^{(\mathcal{S}(x,q),q)}$ . By (i) and the fact that  $x^{(I,q)}$  is reachable node, one has  $I = \mathcal{S}(x,q)$ .

[(ii)  $\Rightarrow$  (i)] Let  $x^+ \in \mathcal{N}_r$ . Then, there is some  $x \in \mathbb{R}^n$  such that  $x^+$  is the iterate computed from  $x$  by the Newton-min algorithm, which reads  $x^+ = x^{(\mathcal{S}(x,q),q)}$ . Now, if  $x^+ = x^{(I,q)}$  for some  $I \subset \llbracket 1, n \rrbracket$ , one has  $I = \mathcal{S}(x,q)$  by (ii), implying the uniqueness of the set  $I$  such that  $x^+ = x^{(I,q)}$ .  $\square$

Let us introduce the set

$$\mathcal{Q}(M) := \{q \in \mathbb{R}^n : \text{LCP}(M,q) \text{ has } 2^n \text{ distinct nodes}\}.$$

**Proposition 5.5 (properties of  $\mathcal{Q}(M)$ )** *Suppose that  $M \in \mathbb{R}^{n \times n}$  is nondegenerate. Then,*

- 1)  $\mathcal{Q}(M) \neq \mathbb{R}^n$ ,
- 2)  $q \in \mathcal{Q}(M)$  if and only  $\forall I \subset \llbracket 1, n \rrbracket$ ,  $x_I^{(I,q)}$  has no zero components,
- 3)  $\mathcal{Q}(M)$  is open and dense in  $\mathbb{R}^n$ .

PROOF. Denote  $\mathcal{Q} := \mathcal{Q}(M)$ .

1) When  $q = 0$ , problem  $\text{LCP}(M, q)$  has 0 as single node. Therefore  $0 \notin \mathcal{Q}$  and  $\mathcal{Q}$  always differs from  $\mathbb{R}^n$ .

2) [ $\Rightarrow$ ] We prove the contrapositive. If, for some  $J \subsetneq I$ ,  $x_{I \setminus J}^{(I, q)} = 0$ , then,  $x^{(I, q)}$  satisfies

$$x_{J^c}^{(I, q)} = 0 \quad \text{and} \quad (Mx^{(I, q)} + q)_J = 0.$$

This system has for unique solution  $x^{(J, q)}$ , so that  $x^{(I, q)} = x^{(J, q)}$ . Since  $I \neq J$ ,  $\mathcal{X}$  is not injective and the implication (i)  $\Rightarrow$  (ii) of lemma 5.1 shows that problem  $\text{LCP}(M, q)$  has not  $2^n$  distinct nodes, meaning that  $q \notin \mathcal{Q}$ .

[ $\Leftarrow$ ] If, for all  $I \subset \llbracket 1, n \rrbracket$ , the components of  $x_I^{(I, q)}$  are nonzero, all the nodes are distinct (they differ by their zero components), so that problem  $\text{LCP}(M, q)$  has  $2^n$  distinct nodes, meaning that  $q \in \mathcal{Q}$ .

3) [ $\mathcal{Q}$  is open] We prefer showing that the complementary set  $\mathcal{Q}^c := \mathbb{R}^n \setminus \mathcal{Q}$  is closed, since the description of  $\mathcal{Q}^c$  given below intervenes in the proof of the density of  $\mathcal{Q}$ . Fix  $I \subset \llbracket 1, n \rrbracket$  and consider the map

$$\xi_I : q \in \mathbb{R}^n \mapsto x^{(I, q)} = (0_{I^c}, -M_{II}^{-1}q_I) \in \mathbb{R}^n,$$

which provides the node  $x^{(I, q)}$  as a function of  $q$ . We have seen in point 2 that  $q \notin \mathcal{Q}$  if and only if, for some  $I \subset \llbracket 1, n \rrbracket$  and some nonempty subset  $I_0$  of  $I$ ,  $(\xi_I(q))_{I_0} = 0$ . Since  $\xi_I$  is linear,  $(\xi_I(q))_{I_0} = 0$  if and only if  $q$  belongs to a proper subspace  $S_{I, I_0}$  of  $\mathbb{R}^n$  (“proper” means here “different from  $\mathbb{R}^n$ ”, and is justified by the fact that  $I_0 \neq \emptyset$  and the rows of  $M_{II}^{-1}$  do not vanish). Therefore, one can write

$$\mathcal{Q}^c = \bigcup \{S_{I, I_0} : I \subset \llbracket 1, n \rrbracket, I_0 \text{ non empty subset of } I\}.$$

Since  $S_{I, I_0}$  is a closed set,  $\mathcal{Q}^c$  is closed as a finite union of closed sets.

[ $\mathcal{Q}$  is dense] Let  $q_0 \in \mathcal{Q}^c$  and  $\varepsilon > 0$ . It suffices to show that the ball  $B(q_0, \varepsilon)$  centered at  $q_0$  with radius  $\varepsilon$  intersects  $\mathcal{Q}$ . This is clear, since otherwise  $\mathcal{Q}^c$  would contain the ball  $B(q_0, \varepsilon)$ , which is not possible for the finite union of proper subspaces of  $\mathbb{R}^n$  like  $\mathcal{Q}^c$  (this claim can be proved in a manner similar to the one of the fact that *a real vector space cannot be written as a finite union of proper subspaces*; for a proof of this last claim, see for example [31; theorem 1.2]).  $\square$

## Conclusion

This contribution brings some adjustments and complements to the paper [3] by the same authors. Some adjustments are necessary because it was implicitly assumed in [3] that the nodes  $x^{(I, q)}$  generated by the Newton-min algorithm were uniquely determined by their index sets  $I$ . This implicit assumption was not compatible with some claims, without bringing appropriate nuances, which is what is done in the present paper. The main result, according to which a nondegenerate matrix  $M$  is a  $\mathbf{P}$ -matrix if and only if the Newton-min algorithm does not cycle between two distinct points, whatever is  $q$ , is preserved. Because this fact is part of the discussion, it is also shown that the set of vectors  $q$  ensuring the existence of  $2^n$  distinct nodes for problem  $\text{LCP}(M, q)$  is dense in  $\mathbb{R}^n$ .



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