The Applied Pi Calculus: Mobile Values, New Names, and Secure Communication

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We study the interaction of the programming construct "new", which generates statically scoped names, with communication via messages on channels. This interaction is crucial in security protocols, which are the main motivating examples for our work; it also appears in other programming-language contexts.

We define the applied pi calculus, a simple, general extension of the pi calculus in which values can be formed from names via the application of built-in functions, subject to equations, and be sent as messages. (In contrast, the pure pi calculus lacks built-in functions; its only messages are atomic names.) We develop semantics and proof techniques for this extended language and apply them in reasoning about security protocols.

This paper essentially subsumes the conference paper that introduced the applied pi calculus in 2001. It fills gaps, incorporates improvements, and further explains and studies the applied pi calculus. Since 2001, the applied pi calculus has been the basis for much further work, described in many research publications and sometimes embodied in useful software, such as the tool ProVerif, which relies on the applied pi calculus to support the specification and automatic analysis of security protocols. Although this paper does not aim to be a complete review of the subject, it benefits from that further work and provides better foundations for some of it. In particular, the applied pi calculus has evolved through its implementation in ProVerif, and the present definition reflects that evolution.

CCS Concepts: • **Security and privacy → Formal methods and theory of security**; •**Theory of computation** → *Process calculi*

Additional Key Words and Phrases: Security protocols

ACM Reference format:
Martín Abadi, Bruno Blanchet, and Cédric Fournet. 2017. The Applied Pi Calculus: Mobile Values, New Names, and Secure Communication. *J. ACM* 65, 1, Article 1 (October 2017), 103 pages.
https://doi.org/10.1145/3127586

1 A CASE FOR IMPURITY

Purity often comes before convenience and even before faithfulness in the lambda calculus, the pi calculus, and other foundational programming languages. For example, in the standard pi calculus, the only messages are atomic names [Milner 1999]. This simplicity is extremely appealing from a foundational viewpoint, and helps in developing the theory of the pi calculus. Furthermore,

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0004-5411/2017/10-ART1
https://doi.org/10.1145/3127586

ingenious encodings demonstrate that it may not entail a loss of generality. In particular, integers, objects, and even higher-order processes can be represented in the pure pi calculus. Similarly, various encodings of cryptographic operations in the pi calculus have been considered [Abadi and Gordon 1999; Baldamus et al. 2004; Carbone and Maffeis 2003; Martinho and Ravara 2011].

On the other hand, this purity has a price. In applications, the encodings can be futile, cumbersome, and even misleading. For instance, in the study of programming languages based on the pi calculus (such as Pict [Pierce and Turner 2000], JoCaml [Conchon and Fessant 1999], or occam-pi [Welch and Barnes 2005]), there is little point in pretending that integers are not primitive. The encodings may also hinder careful reasoning about communication (for example, because they require extra messages), and they may complicate static analysis and proofs.

These difficulties are often circumvented through on-the-fly extensions. The extensions range from quick punts ("for the next example, let's pretend that we have a datatype of integers") to the laborious development of new calculi, such as the spi calculus [Abadi and Gordon 1999] (a calculus with cryptographic operations) and its variants. Generally, the extensions bring us closer to a realistic programming language or modeling language—that is not always a bad thing.

Although many of the resulting calculi are ad hoc and poorly understood, others are robust and uniform enough to have a rich theory and a variety of applications. In particular, impure extensions of the lambda calculus with function symbols and with equations among terms ("delta rules") have been developed systematically, with considerable success. Similarly, impure versions of CCS and CSP with value-passing are not always deep but often neat and convenient [Milner 1989].

In this paper, we introduce, study, and use an analogous uniform extension of the pi calculus, which we call the applied pi calculus (by analogy with "applied lambda calculus"). From the pure pi calculus, we inherit constructs for communication and concurrency, and for generating statically scoped new names ("new"). We add functions and equations, much as is done in the lambda calculus. Messages may then consist not only of atomic names but also of values constructed from names and functions. This embedding of names into the space of values gives rise to an important interaction between the "new" construct and value-passing communication, which appears in neither the pure pi calculus nor value-passing CCS and CSP. Further, we add an auxiliary substitution construct, roughly similar to a floating "let"; this construct is helpful in programming examples and especially in semantics and proofs, and serves to capture the partial knowledge that an environment may have of some values.

The applied pi calculus builds on the pure pi calculus and its substantial theory, but it shifts the focus away from encodings. In comparison with ad hoc approaches, it permits a general, systematic development of syntax, operational semantics, equivalences, and proof techniques.

Using the calculus, we can write and reason about programming examples where "new" and value-passing appear. First, we can easily treat standard datatypes (integers, pairs, arrays, etc.). We can also model unforgeable capabilities as new names, then model the application of certain functions to those capabilities. For instance, we may construct a pair of capabilities. More delicately, the capabilities may be pointers to composite structures, and then adding an offset to a pointer to a pair may yield a pointer to its second component (e.g., as in [Liblit and Aiken 2000]). Furthermore, we can study a variety of security protocols. For this purpose, we represent fresh channels, nonces, and keys as new names, and primitive cryptographic operations as functions, obtaining a simple but useful programming-language perspective on security protocols (much as in the spi calculus). A distinguishing characteristic of the present approach is that we need not craft a special calculus and develop its proof techniques for each choice of cryptographic operations. Thus, we can express and analyze fairly sophisticated protocols that combine several cryptographic primitives (encryptions, hashes, signatures, XORs, ...). We can also describe attacks against the protocols that rely on

(equational) properties of some of those primitives. In our work to date, security protocols are our main source of examples.

The next section defines the applied pi calculus. Section 3 introduces some small, informal examples. Section 4 defines semantic concepts, such as process equivalence, and develops proof techniques. Sections 5 and 6 treat larger, instructive examples; they concern a Diffie-Hellman key exchange, cryptographic hash functions, and message authentication codes. (The two sections are independent.) Many other examples now appear in the literature, as explained below. Section 7 discusses related work, and Section 8 concludes. The body of the paper contains some proofs and outlines others; many details of the proofs, however, are in appendices.

This paper essentially subsumes the conference paper that introduced the applied pi calculus in 2001. It fills gaps, incorporates various improvements, and further explains and studies the applied pi calculus. Specifically, it presents a revised language, with a revised semantics, as explained in Sections 2 and 4. It also includes precise definitions and proofs; these address gaps in the conference paper, discussed in further detail in Section 4. Finally, some of the examples in Sections 3, 5, and especially 6 are polished or entirely new.

Since 2001, the applied pi calculus has been the basis for much further work, described in many research publications (some of which are cited below) and tutorials [Abadi 2007; Cortier and Kremer 2014; Ryan and Smyth 2011]. This further work includes semantics, proof techniques, and applications in diverse contexts (key exchange, electronic voting, certified email, cryptographic file systems, encrypted Web storage, website authorization, zero-knowledge proofs, and more). It is sometimes embodied in useful software, such as the tool ProVerif [Blanchet 2001, 2004, 2016; Blanchet et al. 2008]. This tool, which supports the specification and automatic analysis of security protocols, relies on the applied pi calculus as input language. Other software that builds on ProVerif targets protocol implementations, Web-security mechanisms, or stateful systems such as hardware devices [Arapinis et al. 2011; Bansal et al. 2012; Bhargavan et al. 2008b]. Finally, the applied pi calculus has also been implemented in other settings, such as the prover Tamarin [Kremer and Künnemann 2014; Meier et al. 2013].

Although this paper does not aim to offer a complete review of the subject and its growth since 2001, it benefits from that further work and provides better foundations for some of it. In particular, the applied pi calculus has evolved through its implementation in ProVerif, and the present definition reflects that evolution.

2 THE APPLIED PI CALCULUS

In this section we define the applied pi calculus: its syntax and informal semantics (Section 2.1), then its operational semantics (Section 2.2). We also discuss a few variants and extensions of our definitions (Section 2.3).

2.1 Syntax and Informal Semantics

A *signature* Σ consists of a finite set of function symbols, such as f, encrypt, and pair, each with an arity. A function symbol with arity 0 is a constant symbol.

Given a signature Σ, an infinite set of names, and an infinite set of variables, the set of *terms* is defined by the grammar:

$L, M, N, T, U, V ::=$	terms
$a, b, c, \ldots, k, \ldots, m, n, \ldots, s$	name
x, y, z	variable
$f(M_1, \ldots, M_l)$	function application

where f ranges over the functions of Σ and l matches the arity of f.

Although names, variables, and constant symbols have similarities, we find it clearer to keep them separate. A term is ground when it does not contain variables (but it may contain names and constant symbols). We use meta-variables u, v, w to range over both names and variables. We abbreviate tuples $u_1, \ldots, u_l$ and $M_1, \ldots, M_l$ to $\widetilde{u}$ and $\widetilde{M}$, respectively.

The grammar for *processes* is similar to the one in the pi calculus, but here messages can contain terms (rather than only names) and names need not be just channel names:

$P, Q, R ::=$	processes (or plain processes)
$\mathbf{0}$	null process
$P \mid Q$	parallel composition
$!P$	replication
$\nu n.P$	name restriction ("new")
$\mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$	conditional
$N(x).P$	message input
$\overline{N}\langle M\rangle.P$	message output

The null process $\mathbf{0}$ does nothing; $P \mid Q$ is the parallel composition of P and Q; the replication $!P$ behaves as an infinite number of copies of P running in parallel. The process $\nu n.P$ makes a new, private name n then behaves as P. The conditional construct $\mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$ is standard, but we should stress that $M = N$ represents equality, rather than strict syntactic identity. We abbreviate it $\mathit{if}\ M = N\ \mathit{then}\ P$ when Q is $\mathbf{0}$. Finally, $N(x).P$ is ready to input from channel N, then to run P with the actual message replaced for the formal parameter x, while $\overline{N}\langle M\rangle.P$ is ready to output M on channel N, then to run P. In both of these, we may omit P when it is $\mathbf{0}$.

Further, we extend processes with *active substitutions*:

$A, B, C ::=$	extended processes
P	plain process
$A \mid B$	parallel composition
$\nu n.A$	name restriction
$\nu x.A$	variable restriction
$\{^{M}/_{x}\}$	active substitution

We write $\{^{M}/_{x}\}$ for the substitution that replaces the variable x with the term M. Considered as a process, $\{^{M}/_{x}\}$ is like $\mathit{let}\ x = M\ \mathit{in}\ \ldots$, and is similarly useful. However, unlike a "let" definition, $\{^{M}/_{x}\}$ floats and applies to any process that comes into contact with it. To control this contact, we may add a restriction: $\nu x.(\{^{M}/_{x}\} \mid P)$ corresponds exactly to $\mathit{let}\ x = M\ \mathit{in}\ P$. The substitution $\{^{M}/_{x}\}$ typically appears when the term M has been sent to the environment, but the environment may not have the atomic names that appear in M; the variable x is just a way to refer to M in this situation. Although the substitution $\{^{M}/_{x}\}$ concerns only one variable, we can build bigger substitutions by parallel composition, and may write

$$\{^{M_1}/_{x_1}, \ldots, ^{M_l}/_{x_l}\} \quad \text{for} \quad \{^{M_1}/_{x_1}\} \mid \cdots \mid \{^{M_l}/_{x_l}\}$$

We write σ, $\{^{M}/_{x}\}$, $\{^{\widetilde{M}}/_{\widetilde{x}}\}$ for substitutions, $x\sigma$ for the image of x by σ, and $T\sigma$ for the result of applying σ to the free variables of T. We identify the empty substitution and the null process $\mathbf{0}$.

As usual, names and variables have scopes, which are delimited by restrictions and by inputs. We write $\mathit{fv}(A)$ and $\mathit{fn}(A)$ for the sets of free variables and free names of A, respectively. These sets are inductively defined, as detailed in Figure 1. The domain $\mathit{dom}(A)$ of an extended process A is the set of variables that A exports (those variables x for which A contains a substitution $\{^{M}/_{x}\}$ not under a restriction on x). Figure 1 also defines $\mathit{dom}(A)$ formally. We consider that expressions (processes and extended processes) are equal modulo renaming of bound names and variables.

$$
\begin{aligned}
fv(x) &\stackrel{\text{def}}{=} \{x\} \\
fv(n) &\stackrel{\text{def}}{=} \emptyset \\
fv(f(M_1, \ldots, M_l)) &\stackrel{\text{def}}{=} fv(M_1) \cup \cdots \cup fv(M_l) \\
fv(\mathbf{0}) &\stackrel{\text{def}}{=} \emptyset \\
fv(P \mid Q) &\stackrel{\text{def}}{=} fv(P) \cup fv(Q) \\
fv(!P) &\stackrel{\text{def}}{=} fv(P) \\
fv(\nu n.P) &\stackrel{\text{def}}{=} fv(P) \\
fv(\mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q) &\stackrel{\text{def}}{=} fv(M) \cup fv(N) \cup fv(P) \cup fv(Q) \\
fv(N(x).P) &\stackrel{\text{def}}{=} fv(N) \cup (fv(P) \setminus \{x\}) \\
fv(\overline{N}\langle M\rangle.P) &\stackrel{\text{def}}{=} fv(N) \cup fv(M) \cup fv(P) \\
fv(A \mid B) &\stackrel{\text{def}}{=} fv(A) \cup fv(B) \\
fv(\nu n.A) &\stackrel{\text{def}}{=} fv(A) \\
fv(\nu x.A) &\stackrel{\text{def}}{=} fv(A) \setminus \{x\} \\
fv(\{^M/_x\}) &\stackrel{\text{def}}{=} fv(M) \cup \{x\}
\end{aligned}
$$

$fn(\cdot)$ is defined as $fv(\cdot)$, except that

$$
\begin{aligned}
fn(x) &\stackrel{\text{def}}{=} \emptyset \\
fn(n) &\stackrel{\text{def}}{=} \{n\} \\
fn(\nu n.P) &\stackrel{\text{def}}{=} fn(P) \setminus \{n\} \\
fn(N(x).P) &\stackrel{\text{def}}{=} fn(N) \cup fn(P) \\
fn(\nu n.A) &\stackrel{\text{def}}{=} fn(A) \setminus \{n\} \\
fn(\nu x.A) &\stackrel{\text{def}}{=} fn(A) \\
fn(\{^M/_x\}) &\stackrel{\text{def}}{=} fn(M)
\end{aligned}
$$

$$
\begin{aligned}
dom(P) &\stackrel{\text{def}}{=} \emptyset \\
dom(A \mid B) &\stackrel{\text{def}}{=} dom(A) \cup dom(B) \\
dom(\nu n.A) &\stackrel{\text{def}}{=} dom(A) \\
dom(\nu x.A) &\stackrel{\text{def}}{=} dom(A) \setminus \{x\} \\
dom(\{^M/_x\}) &\stackrel{\text{def}}{=} \{x\}
\end{aligned}
$$

Fig. 1. Free variables, free names, and domain

$$\frac{u : \tau}{\vdash u : \tau} \qquad \frac{f : \tau_1 \times \cdots \times \tau_l \to \tau \quad \vdash M_1 : \tau_1 \quad \ldots \quad \vdash M_l : \tau_l}{\vdash f(M_1, \ldots, M_l) : \tau}$$

$$\vdash 0 \qquad \frac{\vdash P \quad \vdash Q}{\vdash P \mid Q} \qquad \frac{\vdash P}{\vdash !P} \qquad \frac{\vdash P}{\vdash \nu n.P} \qquad \frac{\vdash M : \tau \quad \vdash N : \tau \quad \vdash P \quad \vdash Q}{\vdash \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q}$$

$$\frac{\vdash N : \mathsf{Channel} \quad \vdash P}{\vdash N(x).P} \qquad \frac{\vdash N : \mathsf{Channel} \quad \vdash M : \tau \quad \vdash P}{\vdash \overline{N}\langle M \rangle.P}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \mid B} \qquad \frac{\vdash A}{\vdash \nu u.A} \qquad \frac{x : \tau \quad \vdash M : \tau}{\vdash \{^{M}/_{x}\}}$$

Fig. 2. Sort system

We always assume that our substitutions are cycle-free, that is, by reordering, they can be written $\{^{M_1}/_{x_1}, \ldots, ^{M_l}/_{x_l}\}$ where $x_i \notin fv(M_j)$ for all $i \leq j \leq l$. For instance, we exclude substitutions such as $\{^{f(y)}/_{x}, ^{f(x)}/_{y}\}$. We also assume that, in an extended process, there is at most one substitution for each variable, and there is exactly one when the variable is restricted, that is, $dom(A) \cap dom(B) = \emptyset$ in every extended process $A \mid B$, and $x \in dom(A)$ in every extended process $\nu x.A$. An extended process A is *closed* when its free variables are all defined by an active substitution, that is, $dom(A) = fv(A)$. We use the abbreviation $\nu\widetilde{u}$ for the (possibly empty) series of pairwise distinct restrictions $\nu u_1.\nu u_2. \ldots \nu u_l$.

A *frame* is an extended process built up from $\mathbf{0}$ and active substitutions of the form $\{^{M}/_{x}\}$ by parallel composition and restriction. We let φ and ψ range over frames. Every extended process A can be mapped to a frame $\varphi(A)$ by replacing every plain process embedded in A with $\mathbf{0}$. The frame $\varphi(A)$ can be viewed as an approximation of A that accounts for the static knowledge exposed by A to its environment, but not for A's dynamic behavior. Assuming that all bound names and variables are pairwise distinct, and do not clash with free ones, one can ignore all restrictions in a frame, thus obtaining an underlying substitution; we require that, for each extended process, this resulting substitution be cycle-free.

We rely on a sort system for terms and processes. It includes a sort Channel for channels. It may also include other sorts such as Integer, Key, or simply a universal sort for data Data. Each variable and each name comes with a sort; we write $u : \tau$ to mean that u has sort τ. There are an infinite number of variables and an infinite number of names of each sort. We typically use a, b, and c as names of sort Channel, s and k as names of some other sort (e.g., Data), and m and n as names of any sort. Function symbols also come with the sorts of their arguments and of their result. We write $f : \tau_1 \times \cdots \times \tau_l \to \tau$ to mean that f has arguments of sorts $\tau_1, \ldots, \tau_l$ and a result of sort τ. Figure 2 gives the rules of the sort system. It defines the following judgments: $\vdash M : \tau$ means that M is a term of sort τ; $\vdash P$ means that the process P is well-sorted; $\vdash A$ means that the extended process A is well-sorted. This sort system enforces that function applications are well-sorted, that M and N are of the same sort in conditional expressions, that N has sort Channel in input and output expressions, that M is well-sorted (with an arbitrary sort τ) in output expressions, and that active substitutions preserve sorts. We always assume that expressions are well-sorted, and that substitutions preserve sorts.

2.2 Operational Semantics

We give an operational semantics for the applied pi calculus in the now customary "chemical style" [Berry and Boudol 1992; Milner 1992]. At the center of this operational semantics is a

reduction relation $\rightarrow$ on extended processes, which basically models the steps of computations. For example, $\overline{a}\langle M\rangle \mid a(x).\overline{b}\langle x\rangle \rightarrow \overline{b}\langle M\rangle$ represents the transmission of the message M on the channel a to a process that will forward the message on the channel b; the formal x is replaced with its actual value M in this reduction. The axioms for the reduction relation $\rightarrow$, which are remarkably simple, rely on auxiliary rules for a structural equivalence relation $\equiv$ that permits the rearrangement of processes, for example the use of commutativity and associativity of parallel composition. Furthermore, both structural equivalence and reduction depend on an underlying equational theory. Therefore, this section introduces equational theories, then defines structural equivalence and reduction.

Given a signature Σ, we equip it with an equational theory, that is, with a congruence relation on terms that is closed under substitution of terms for variables and names. (See for example Mitchell's textbook [Mitchell 1996, chapter 3] and its references for background on universal algebra and algebraic data types from a programming-language perspective.) We further require that this equational theory respect the sort system, that is, two equal terms are of the same sort, and that it be non-trivial, that is, there exist two different terms in each sort.

An equational theory may be generated from a finite set of equational axioms, or from rewrite rules, but this property is not essential for us. We tend to ignore the mechanics of specifying equational theories, but give several examples in Section 3.

We write $\Sigma \vdash M = N$ when the equation $M = N$ is in the theory associated with Σ. Here we keep the theory implicit, and we may even abbreviate $\Sigma \vdash M = N$ to $M = N$ when Σ is clear from context or unimportant. We write $\Sigma \nvdash M = N$ for the negation of $\Sigma \vdash M = N$.

As usual, a context is an expression with a hole. An *evaluation context* is a context whose hole is not under a replication, a conditional, an input, or an output. A context $E[\_]$ *closes* A when $E[A]$ is closed.

Structural equivalence $\equiv$ is the smallest equivalence relation on extended processes that is closed by application of evaluation contexts, and such that:

Par-**0**	A	$\equiv$	$A \mid \mathbf{0}$	
Par-A	$A \mid (B \mid C)$	$\equiv$	$(A \mid B) \mid C$	
Par-C	$A \mid B$	$\equiv$	$B \mid A$	
Repl	$!P$	$\equiv$	$P \mid !P$	
New-**0**	$\nu n.\mathbf{0}$	$\equiv$	$\mathbf{0}$	
New-C	$\nu u.\nu v.A$	$\equiv$	$\nu v.\nu u.A$	
New-Par	$A \mid \nu u.B$	$\equiv$	$\nu u.(A \mid B)$	when $u \notin fv(A) \cup fn(A)$
Alias	$\nu x.\{^M/_x\}$	$\equiv$	$\mathbf{0}$	
Subst	$\{^M/_x\} \mid A$	$\equiv$	$\{^M/_x\} \mid A\{^M/_x\}$	
Rewrite	$\{^M/_x\}$	$\equiv$	$\{^N/_x\}$	when $\Sigma \vdash M = N$

The rules for parallel composition and restriction are standard. Alias enables the introduction of an arbitrary active substitution. Subst describes the application of an active substitution to a process that is in contact with it. Rewrite deals with equational rewriting. Subst implicitly requires that $x : \tau$ and $\vdash M : \tau$ for some sort τ. In combination, Alias and Subst yield $A\{^M/_x\} \equiv \nu x.(\{^M/_x\} \mid A)$ for $x \notin fv(M)$:

$A\{^M/_x\}$	$\equiv$	$A\{^M/_x\} \mid \mathbf{0}$	by Par-**0**
	$\equiv$	$A\{^M/_x\} \mid \nu x.\{^M/_x\}$	by Alias
	$\equiv$	$\nu x.(A\{^M/_x\} \mid \{^M/_x\})$	by New-Par
	$\equiv$	$\nu x.(\{^M/_x\} \mid A\{^M/_x\})$	by Par-C
	$\equiv$	$\nu x.(\{^M/_x\} \mid A)$	by Subst

Using structural equivalence, every closed extended process A can be rewritten to consist of a substitution and a closed plain process with some restricted names:

$$A \quad \equiv \quad \nu \widetilde{n}.(\{^{\widetilde{M}}/_{\widetilde{x}}\} \mid P)$$

where $fv(P) = \emptyset$, $fv(\widetilde{M}) = \emptyset$, and $\{\widetilde{n}\} \subseteq fn(\widetilde{M})$. In particular, every closed frame φ can be rewritten to consist of a substitution with some restricted names:

$$\varphi \quad \equiv \quad \nu \widetilde{n}.\{^{\widetilde{M}}/_{\widetilde{x}}\}$$

where $fv(\widetilde{M}) = \emptyset$ and $\{\widetilde{n}\} \subseteq fn(\widetilde{M})$. The set $\{\widetilde{x}\}$ is the domain of φ.

Internal reduction $\rightarrow$ is the smallest relation on extended processes closed by structural equivalence and application of evaluation contexts such that:

$$\begin{array}{lrcl}
\textsc{Comm} & \overline{N}\langle x\rangle.P \mid N(x).Q & \rightarrow & P \mid Q \\
\textsc{Then} & \mathit{if}\ M = M\ \mathit{then}\ P\ \mathit{else}\ Q & \rightarrow & P \\
\textsc{Else} & \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q & \rightarrow & Q \\
 & \multicolumn{3}{l}{\text{for any ground terms } M \text{ and } N \text{ such that } \Sigma \nvdash M = N}
\end{array}$$

Communication (Comm) is remarkably simple because the message concerned is a variable; this simplicity entails no loss of generality because Alias and Subst can introduce a variable to stand for a term:

$$\begin{array}{rcll}
\overline{N}\langle M\rangle.P \mid N(x).Q & \equiv & \nu x.(\{^{M}/_{x}\} \mid \overline{N}\langle x\rangle.P \mid N(x).Q) & \\
 & \rightarrow & \nu x.(\{^{M}/_{x}\} \mid P \mid Q) & \text{by Comm} \\
 & \equiv & P \mid Q\{^{M}/_{x}\} &
\end{array}$$

(This derivation assumes that $x \notin fv(N) \cup fv(M) \cup fv(P)$, which can be established by renaming as needed.)

Comparisons (Then and Else) directly depend on the underlying equational theory. Using Else sometimes requires that active substitutions in the context be applied first, to yield ground terms M and N. For example, rule Else does not allow us to reduce $\{^{n}/_{x}\} \mid \mathit{if}\ x = n\ \mathit{then}\ P\ \mathit{else}\ Q$.

This use of the equational theory may be reminiscent of initial algebras. In an initial algebra, the principle of "no confusion" dictates that two elements are equal only if this is required by the corresponding equational theory. Similarly, $\mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$ reduces to P only if this is required by the equational theory, and reduces to Q otherwise. Initial algebras also obey the principle of "no junk", which says that all elements correspond to terms built exclusively from function symbols of the signature. In contrast, a fresh name need not equal any such term in the applied pi calculus.

2.3 Variants and Extensions

Several variants of the syntax of the applied pi calculus appear in the literature, and further variants may be considered. We discuss a few:

- In the conference paper, there are several sorts for channels: the sort Channel$\langle\tau\rangle$ is the sort of channels that convey messages of sort τ. The sort Channel without argument is more general, in the sense that all processes well-sorted with Channel$\langle\tau\rangle$ are also well-sorted with Channel. Having a single sort for channels simplifies some models, for instance when all public messages are sent on the same channel, even if they have different types. Moreover, by using Channel as only sort, we can encode an untyped version of the applied pi calculus. The tool ProVerif also uses the sort Channel without argument.

- In a more refined version of the sort system, we could allow names only in a distinguished set of sorts. For instance, we could consider a sort of booleans, containing as only values the constants true and false. Such a sort would not contain names. Sorts without names would have to be treated with special care in proofs, since our proofs often use fresh names.
  On the other hand, letting all sorts contain names does not prevent modeling booleans by a sort. For example, we can treat as false all terms of the sort different from true, including not only the constant false but also all names. Analogous treatments apply to other common datatypes.
- In the conference paper, channels in inputs and outputs are names or variables rather than any term. Allowing any term as channel yields a more general calculus and avoids some side conditions in theorems. It is also useful for some encodings [Abadi and Blanchet 2005b]. Finally, it is in line with the syntax of ProVerif, where this design choice was adopted in order to simplify the untyped version of the calculus.
  Nevertheless, the sort system can restrict the terms that appear as channels: if no function symbol returns a result of sort Channel, then channels can be only names or variables.
- Function symbols can also be defined by rewrite rules instead of an equational theory. This approach is taken in ProVerif [Blanchet 2009]: a destructor g is a partial function defined by rewrite rules $g(M_1, \ldots, M_l) \to M$; the destructor application $g(N_1, \ldots, N_l)$ fails when no rewrite rule applies, and this failure can be tested in the process calculus.
  A destructor $g : \tau_1 \times \cdots \times \tau_l \to \tau$ with rewrite rule $g(M_1, \ldots, M_l) \to M$ can be encoded in the applied pi calculus by function symbols $g : \tau_1 \times \cdots \times \tau_l \to \tau$ and $\mathsf{test}_g : \tau_1 \times \cdots \times \tau_l \to \mathsf{bool}$ with the equations
  $$\begin{aligned} g(M_1, \ldots, M_l) &= M \\ \mathsf{test}_g(M_1, \ldots, M_l) &= \mathsf{true} \end{aligned}$$
  The function test_g allows one to test whether $g(N_1, \ldots, N_l)$ is defined, by checking whether $\mathsf{test}_g(N_1, \ldots, N_l) = \mathsf{true}$ holds. (See Section 3 for examples of such test functions.) The function g may be applied even when its arguments are not instances of $(M_1, \ldots, M_l)$, thus yielding terms $g(N_1, \ldots, N_l)$ that do not exist in the calculus with rewrite rules. These "stuck" terms may be simulated with distinct fresh names in that variant of the calculus.
  Destructors are easy to implement in a tool. They also provide a built-in error-handling construct: the error handling is triggered when no rewrite rule applies. However, they complicate the semantics because they require a notion of evaluation of terms. Moreover, many useful functions can be defined by equations but not as destructors (for instance, encryption without redundancy, XOR, and modular exponentiation, which we use in the rest of this paper). Therefore, ProVerif supports both destructors and equations [Blanchet et al. 2008]. Thus, the language of ProVerif is a superset of the applied pi calculus as defined in this paper [Blanchet 2016, Chapter 4], with the caveat that ProVerif does not support all equational theories and that it considers only plain processes.
- An extension that combines the applied pi calculus with ambients and with a built-in construct for evaluating messages as programs has also been studied [Blanchet and Aziz 2003]. This extended calculus mixes many notions, so the corresponding proofs are complex. Considering a single notion at a time yields a simpler and more elegant calculus. Furthermore, although the applied pi calculus has few primitives, it supports various other constructs via encodings; in particular, the message-evaluation construct could be represented by defining an interpreter in the calculus.
- Our equational theories are closed under substitution of terms for names. This property yields a simple and uniform treatment of variables and names. An alternative definition, which may

suffice, assumes only that equational theories are closed under one-to-one renaming and do not equate names. That definition makes it possible to define a function that tests whether a term is a name.

Some other variations concern the definition of the semantics:

- As in other papers [Blanchet et al. 2008; Liu 2011], we can handle the replication by a reduction step $!P \rightarrow P \mid !P$ instead of the structural equivalence rule $!P \equiv P \mid !P$. This modification prevents transforming $P \mid !P$ into $!P$, and thus simplifies some proofs.
- As Section 2.2 indicates, we can rewrite extended processes by pulling restrictions to the top, so that every closed extended process A becomes an extended process A° such that

$$A \equiv A^\circ = \nu \widetilde{n}.(\{^{\widetilde{M}}/_{\widetilde{x}}\} \mid P_1 \mid \ldots \mid P_l)$$

  where $fv(P_1 \mid \ldots \mid P_l) = \emptyset$, $fv(\widetilde{M}) = \emptyset$, and $P_1, \ldots, P_l$ are replication, conditional, input, or output expressions. We can then modify the definitions of structural equivalence and internal reduction to act on processes in the form above. Structural equivalence says that the parallel composition $P_1 \mid \ldots \mid P_l$ is associative and commutative and that the names in $\widetilde{n}$ can be reordered. Internal reduction is the smallest relation on closed extended processes, closed by structural equivalence, such that:

$$\begin{array}{rcll} E[\overline{N}\langle M\rangle.P \mid N'(x).Q] & \rightarrow & E[P \mid Q\{^{M}/_{x}\}]^\circ & \text{if } \Sigma \vdash N = N' \\ E[\mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q] & \rightarrow & E[P]^\circ & \text{if } \Sigma \vdash M = N \\ E[\mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q] & \rightarrow & E[Q]^\circ & \text{if } \Sigma \nvdash M = N \\ E[!P] & \rightarrow & E[P \mid !P]^\circ & \end{array}$$

  for any evaluation context E. A similar idea appears in the intermediate applied pi calculus of Delaune et al. [2010] and Liu et al. [2011; 2012]. There, all restrictions not under replication are pulled to the top of processes, over conditionals, inputs, outputs, and parallel compositions; the processes $P_1, \ldots, P_l$ may be $\mathbf{0}$; and channels are names or variables.
- Pushing the previous idea further, we can represent the extended process

$$A \equiv \nu \widetilde{n}.(\{^{\widetilde{M}}/_{\widetilde{x}}\} \mid P_1 \mid \ldots \mid P_l)$$

  as a configuration $(\mathcal{N}, \sigma, \mathcal{P}) = (\{\widetilde{n}\}, \{^{\widetilde{M}}/_{\widetilde{x}}\}, \{P_1, \ldots, P_l\})$, where $\mathcal{N}$ is a set of names, σ is a substitution, and $\mathcal{P}$ is a multiset of processes. We can then define internal reduction on such configurations, without any structural equivalence. (Sets and multisets allow us to ignore the ordering of restrictions and parallel processes.) This idea is used in semantics of the calculus of ProVerif [Abadi and Blanchet 2005b; Allamigeon and Blanchet 2005; Blanchet 2009, 2016].

Semantics based on global configurations are closer to abstract machines. Such semantics simplify proofs, because they leave only few choices in reductions. They also make it easier to define further extensions of the calculus, such as tables and phases in ProVerif [Blanchet 2016]. However, our compositional semantics is more convenient in order to model interactions between a process and a context. It is also closer to the traditional semantics of the pi calculus. The two kinds of semantics are of course connected. In particular, [Blanchet 2016, Chapter 4] formally relates the semantics of ProVerif based on configurations to our semantics.

3 BRIEF EXAMPLES

This section collects several examples, focusing on signatures, equations, and some simple processes. We start with pairs; this trivial example serves to introduce some notations and issues. We then discuss lists, cryptographic hash functions, encryption functions, digital signatures, and the XOR

function [Menezes et al. 1996; Schneier 1996], as well as a form of multiplexing, which demonstrates the use of channels that are terms rather than names. Further examples appear in Sections 5 and 6. More examples, such as blind signatures [Kremer and Ryan 2005] and zero-knowledge proofs [Backes et al. 2008], have appeared in the literature since 2001.

Of course, at least some of these functions appear in most formalizations of cryptography and security protocols. In comparison with the spi calculus, the applied pi calculus permits a more uniform and versatile treatment of these functions, their variants, and their properties. Like the spi calculus, however, the applied pi calculus takes advantage of notations, concepts, and techniques from programming languages.

Pairs. Algebraic datatypes such as pairs, tuples, arrays, and lists occur in many examples. Encoding them in the pure pi calculus is not hard, but neither is representing them as primitive. For instance, the signature Σ may contain the binary function symbol pair and the unary function symbols fst and snd, with the abbreviation (M, N) for $\mathsf{pair}(M, N)$, and with the evident equations:

$$\mathsf{fst}((x, y)) = x \tag{1}$$
$$\mathsf{snd}((x, y)) = y \tag{2}$$

(So the equational theory consists of these equations, and all the equations obtained by reflexivity, symmetry, transitivity, applications of function symbols, and substitutions of terms for variables.) These function symbols may for instance be sorted as follows:

$$\begin{aligned} \mathsf{pair} &: \mathsf{Data} \times \mathsf{Data} \to \mathsf{Data} \\ \mathsf{fst} &: \mathsf{Data} \to \mathsf{Data} \\ \mathsf{snd} &: \mathsf{Data} \to \mathsf{Data} \end{aligned}$$

We may use the test $(\mathsf{fst}(M), \mathsf{snd}(M)) = M$ to check that M is a pair before using the values of $\mathsf{fst}(M)$ and $\mathsf{snd}(M)$. Alternatively, we may add a boolean function is_pair that recognizes pairs, defined by the equation:

$$\mathsf{is\_pair}((x, y)) = \mathsf{true}$$

With this equation, the conditional $\mathit{if}\ \mathsf{is\_pair}(M) = \mathsf{true}\ \mathit{then}\ P\ \mathit{else}\ Q$ runs P if M is a pair and Q otherwise. Using pairs, we may, for instance, define the process:

$$\nu s.\left(\overline{a}\langle (M, s)\rangle \mid a(z).\mathit{if}\ \mathsf{snd}(z) = s\ \mathit{then}\ \overline{b}\langle \mathsf{fst}(z)\rangle\right)$$

One of its components sends a pair consisting of a term M and a fresh name s on a channel a. The other receives a message on a and, if its second component is s, it forwards the first component on a channel b. Thus, we may say that s serves as a capability (or password) for the forwarding. However, this capability is not protected from eavesdroppers when it travels on a. Any other process can listen on a and can apply snd to the message received, thus learning s. We can represent such an attacker within the calculus, for example by the following process:

$$a(z).\overline{a}\langle (N, \mathsf{snd}(z))\rangle$$

which may receive (M, s) on a and send (N, s) on a. Composing this attacker in parallel with the process, we may obtain N instead of M on b.

Such attacks can be thwarted by the use of restricted channel names, as in the process

$$\nu a.\nu s.\left(\overline{a}\langle (M, s)\rangle \mid a(z).\mathit{if}\ \mathsf{snd}(z) = s\ \mathit{then}\ \overline{b}\langle \mathsf{fst}(z)\rangle\right)$$

Alternatively, they can be thwarted by the use of cryptography, as discussed below.

Lists. We may treat lists similarly, with the following function symbols and corresponding sorts:

$$\begin{array}{rcl} \mathsf{nil} & : & \mathsf{List} \\ \mathsf{cons} & : & \mathsf{Data} \times \mathsf{List} \to \mathsf{List} \\ \mathsf{hd} & : & \mathsf{List} \to \mathsf{Data} \\ \mathsf{tl} & : & \mathsf{List} \to \mathsf{List} \end{array}$$

The constant nil is the empty list; $\mathsf{cons}(x, y)$ represents the concatenation of the element x at the beginning of the list y, and we write it with infix notation as $x :: y$, where the symbol :: associates to the right; and hd and tl are head and tail functions with equations:

$$\mathsf{hd}(x :: y) = x \qquad \mathsf{tl}(x :: y) = y \tag{3}$$

Further, we write $M +\!\!+ N$ for the concatenation of an element N at the end of a list M, where the function $+\!\!+ : \mathsf{List} \times \mathsf{Data} \to \mathsf{List}$ associates to the left, and satisfies the equations:

$$\mathsf{nil} +\!\!+ x = x :: \mathsf{nil} \qquad (x :: y) +\!\!+ z = x :: (y +\!\!+ z) \tag{4}$$

Cryptographic Hash Functions. We represent a cryptographic hash function as a unary function symbol h with no equations. The absence of an inverse for h models the one-wayness of h. The fact that $\mathsf{h}(M) = \mathsf{h}(N)$ only when $M = N$ models that h is collision-free.

Modifying our first example, we may now write the process:

$$\nu s.\, \left(\overline{a}\langle(M, \mathsf{h}((s, M)))\rangle \mid a(x).\mathit{if}\ \mathsf{h}((s, \mathsf{fst}(x))) = \mathsf{snd}(x)\ \mathit{then}\ \overline{b}\langle\mathsf{fst}(x)\rangle\right)$$

Here the value M is authenticated by pairing it with the fresh name s and then hashing the pair. Although $(M, \mathsf{h}((s, M)))$ travels on the public channel a, no other process can extract s from this message, or produce $(N, \mathsf{h}((s, N)))$ for some other N using the available functions. Therefore, we may reason that this process will forward only the intended term M on channel b.

This example is a typical cryptographic application of hash functions. In light of the practical importance of those applications, our treatment of hash functions is attractively straightforward. Still, we may question whether our formal model of these functions is not too strong and simplistic in comparison with the properties of actual implementations based on algorithms such as SHA. In Section 6, we consider a somewhat weaker, subtler model for hash functions.

Symmetric Encryption. In order to model symmetric cryptography (that is, shared-key cryptography), we take binary function symbols enc and dec for encryption and decryption, respectively, with the equation:

$$\mathsf{dec}(\mathsf{enc}(x, y), y) \quad = \quad x$$

Here x represents the plaintext and y the key. We often use fresh names as keys in examples; for instance, the (useless) process:

$$\nu k.\overline{a}\langle\mathsf{enc}(M, k)\rangle$$

sends the term M encrypted under a fresh key k.

In applications of encryption, it is frequent to assume that each encrypted message comes with sufficient redundancy so that decryption with the "wrong" key is evident. Accordingly, we can test whether the decryption of M with the key k succeeds by testing whether $\mathsf{enc}(\mathsf{dec}(M, k), k) = M$. Alternatively, we could also add a test function $\mathsf{test}_{\mathsf{dec}}$ with the equation

$$\mathsf{test}_{\mathsf{dec}}(\mathsf{enc}(x, y), y) \quad = \quad \mathsf{true}$$

Provided that we check that decryption succeeds before using the decrypted message, this model of encryption basically yields the spi calculus [Abadi and Gordon 1999].

On the other hand, in modern cryptology, such redundancy is not usually viewed as part of the encryption function proper, but rather an addition. The redundancy can be implemented with message authentication codes. We can model an encryption scheme without redundancy with the two equations:

$$\begin{aligned} \mathsf{dec}(\mathsf{enc}(x, y), y) &= x \\ \mathsf{enc}(\mathsf{dec}(z, y), y) &= z \end{aligned}$$

These equations model that decryption is the inverse bijection of encryption, a property that is typically satisfied by block ciphers.

Asymmetric Encryption. It is only slightly harder to model asymmetric (public-key) cryptography, where the keys for encryption and decryption are different. We introduce two new unary function symbols pk and sk for generating public and private keys from a seed, and the equation:

$$\mathsf{dec}(\mathsf{enc}(x, \mathsf{pk}(y)), \mathsf{sk}(y)) = x$$

We may now write the process:

$$\nu s.\left(\overline{a}\langle \mathsf{pk}(s)\rangle \mid b(x).\overline{c}\langle \mathsf{dec}(x, \mathsf{sk}(s))\rangle\right)$$

The first component publishes the public key $\mathsf{pk}(s)$ by sending it on a. The second receives a message on b, uses the corresponding private key $\mathsf{sk}(s)$ to decrypt it, and forwards the resulting plaintext on c. As this example indicates, we essentially view name restriction (νs) as a generator of unguessable seeds. In some cases, those seeds may be directly used as passwords or keys; in others, some transformations are needed.

Some encryption schemes have additional properties. In particular, enc and dec may be the same function. This property matters in implementations, and sometimes permits attacks. Moreover, certain encryptions and decryptions commute in some schemes. For example, we have $\mathsf{dec}(\mathsf{enc}(x, y), z) = \mathsf{enc}(\mathsf{dec}(x, z), y)$ if the encryptions and decryptions are performed using RSA with the same modulus. The treatment of such properties is left open in the spi calculus [Abadi and Gordon 1999]. In contrast, it is easy to express the properties in the applied pi calculus, and to study the protocols and attacks that depend on them.

Non-Deterministic ("Probabilistic") Encryption. Going further, we may add a third argument to enc, so that the encryption of a plaintext with a key is not unique. This non-determinism is an essential property of probabilistic encryption [Goldwasser and Micali 1984]. The equation for decryption becomes:

$$\mathsf{dec}(\mathsf{enc}(x, \mathsf{pk}(y), z), \mathsf{sk}(y)) = x$$

With this variant, we may write the process:

$$a(x).\left(\nu m.\overline{b}\langle \mathsf{enc}(M, x, m)\rangle \mid \nu n.\overline{c}\langle \mathsf{enc}(N, x, n)\rangle\right)$$

which receives a message x and uses it as an encryption key for two messages, $\mathsf{enc}(M, x, m)$ and $\mathsf{enc}(N, x, n)$. An observer who does not have the corresponding decryption key cannot tell whether the underlying plaintexts M and N are identical by comparing the ciphertexts, because the ciphertexts rely on different fresh names m and n. Moreover, even if the observer learns x, M, and N (but not the decryption key), it cannot verify that the messages contain M and N because it does not know m and n.

Public-Key Digital Signatures. Like public-key encryption schemes, digital signature schemes rely on pairs of public and private keys. In each pair, the private key serves for computing signatures and the public key for verifying those signatures. In order to model key generation, we use again the two unary function symbols pk and sk for generating public and private keys from a seed. For signatures and their verification, we use a new binary function symbol sign, a ternary function symbol check, and a constant symbol ok, with the equation:

$$\mathsf{check}(x, \mathsf{sign}(x, \mathsf{sk}(y)), \mathsf{pk}(y)) \quad = \quad \mathsf{ok}$$

(Several variants are possible.)

Modifying once more our first example, we may now write the process:

$$\left(\nu s.\{^{\mathsf{pk}(s)}/_{y}\} \mid \overline{a}\langle(M, \mathsf{sign}(M, \mathsf{sk}(s)))\rangle\right) \mid$$
$$a(x).\mathit{if}\ \mathsf{check}(\mathsf{fst}(x), \mathsf{snd}(x), y) = \mathsf{ok}\ \mathit{then}\ \overline{b}\langle\mathsf{fst}(x)\rangle$$

Here the value M is signed using the private key $\mathsf{sk}(s)$. Although M and its signature travel on the public channel a, no other process can produce N and its signature for some other N. Therefore, again, we may reason that only the intended term M will be forwarded on channel b. This property holds despite the publication of $\mathsf{pk}(s)$ (but not $\mathsf{sk}(s)$), which is represented by the active substitution that maps y to $\mathsf{pk}(s)$. Despite the restriction on s, processes outside the restriction can use $\mathsf{pk}(s)$ through y. In particular, y refers to $\mathsf{pk}(s)$ in the process that checks the signature on M.

XOR. We may model the XOR function, some of its uses in cryptography, and some of the protocol flaws connected with it. Some of these flaws (e.g., [Ryan and Schneider 1998]) stem from the intrinsic equational properties of XOR, such as associativity, commutativity, the existence of a neutral element, and the cancellation property that we may write:

$$\begin{aligned} \mathsf{xor}(\mathsf{xor}(x, y), z) &= \mathsf{xor}(x, \mathsf{xor}(y, z)) \\ \mathsf{xor}(x, y) &= \mathsf{xor}(y, x) \\ \mathsf{xor}(x, 0) &= x \\ \mathsf{xor}(x, x) &= 0 \end{aligned}$$

Others arise because of the interactions between XOR and other operations (e.g., [Core SDI S.A. 1998; Stubblebine and Gligor 1992]). For example, CRCs (cyclic redundancy checks) can be poor proofs of integrity, partly because of the equation

$$\mathsf{crc}(\mathsf{xor}(x, y)) \quad = \quad \mathsf{xor}(\mathsf{crc}(x), \mathsf{crc}(y))$$

Multiplexing. Finally, we illustrate a possible usage of channels that are not names. Consider for instance a pairing function for building channels $\mathsf{pair} : \mathsf{Data} \times \mathsf{Port} \to \mathsf{Channel}$ with its associated projections $\mathsf{fst} : \mathsf{Channel} \to \mathsf{Data}$ and $\mathsf{snd} : \mathsf{Channel} \to \mathsf{Port}$, and equations (1) and (2) from our first example. We may use this function for multiplexing as follows:

$$\nu s.(\overline{\mathsf{pair}(s, \mathsf{port}_1)}\langle M_1\rangle \mid \overline{\mathsf{pair}(s, \mathsf{port}_2)}\langle M_2\rangle$$
$$\mid \mathsf{pair}(s, \mathsf{port}_1)(x_1) \mid \mathsf{pair}(s, \mathsf{port}_2)(x_2))$$

In this process, the first output can be received only by the first input, and the second output can be received only by the second input.

4 EQUIVALENCES AND PROOF TECHNIQUES

In examples, we frequently argue that two given processes cannot be distinguished by any context, that is, that the processes are observationally equivalent. The spi calculus developed the idea that the context represents an active attacker, and equivalences capture authenticity and secrecy

properties in the presence of the attacker. More broadly, a wide variety of security properties can be expressed as equivalences.

In this section we define observational equivalence for the applied pi calculus. We also introduce a notion of static equivalence for frames, a labelled semantics for processes, and a labelled equivalence relation. We prove that labelled equivalence and observational equivalence coincide, obtaining a convenient proof technique for observational equivalence.

4.1 Observational Equivalence

We write $A \Downarrow a$ when A can send a message on name a, that is, when $A \rightarrow^* \equiv E[\overline{a}\langle M\rangle.P]$ for some evaluation context $E[\_]$ that does not bind a.

Definition 4.1. An *observational bisimulation* is a symmetric relation $\mathcal{R}$ between closed extended processes with the same domain such that $A \mathcal{R} B$ implies:

(1) if $A \Downarrow a$, then $B \Downarrow a$;
(2) if $A \rightarrow^* A'$ and A' is closed, then $B \rightarrow^* B'$ and $A' \mathcal{R} B'$ for some B';
(3) $E[A] \mathcal{R} E[B]$ for all closing evaluation contexts $E[\_]$.

Observational equivalence ($\approx$) is the largest such relation.

For example, when h is a unary function symbol with no equations, we obtain that $\nu s.\overline{a}\langle s\rangle \approx \nu s.\overline{a}\langle \mathsf{h}(s)\rangle$.

These definitions are standard in the pi calculus, where $\Downarrow a$ is called a *barb* on a, and where $\approx$ is one of the two usual notions of weak barbed bisimulation congruence. (See Section 4.5 and [Fournet and Gonthier 1998] for a detailed discussion.) In the applied pi calculus, one could also define barbs on arbitrary terms, not just on names; we do not need that generalization for our purposes. The set of closing evaluation contexts for A depends only on A's domain; hence, in Definition 4.1, A and B have the same closing evaluation contexts. In Definition 4.1(2), since $\mathcal{R}$ is a relation between closed extended processes, we require that A' also be closed. Being closed is not preserved by all reductions, since structural equivalence may introduce free unused variables. For instance, we have $\mathbf{0} \equiv \nu x.\{^{y}/_{x}\}$ by Alias and $\{^{M}/_{x}\} \equiv \{^{\mathsf{fst}((M,y))}/_{x}\}$ by Rewrite using the equation $\mathsf{fst}((x,y)) = x$.

Although observational equivalence is undecidable in general, various tools support certain automatic proofs of observational equivalence and other equivalence relations, in the applied pi calculus and related languages (e.g., [Baudet 2005; Blanchet et al. 2008; Chadha et al. 2012; Cheval et al. 2013]).

4.2 Static Equivalence

Two substitutions may be seen as equivalent when they behave equivalently when applied to terms. We write $\approx_s$ for this notion of equivalence, and call it static equivalence. In the presence of the "new" construct, defining $\approx_s$ is somewhat delicate and interesting. For instance, consider two functions f and g with no equations (intuitively, two independent hash functions), and the three frames:

$$\varphi_0 \stackrel{\text{def}}{=} \nu k.\{^{k}/_{x}\} \mid \nu s.\{^{s}/_{y}\}$$
$$\varphi_1 \stackrel{\text{def}}{=} \nu k.\{^{\mathsf{f}(k)}/_{x}, ^{\mathsf{g}(k)}/_{y}\}$$
$$\varphi_2 \stackrel{\text{def}}{=} \nu k.\{^{k}/_{x}, ^{\mathsf{f}(k)}/_{y}\}$$

In φ_0, the variables x and y are mapped to two unrelated values that are different from any value that the context may build (since k and s are new). These properties also hold, but more subtly, for φ_1; although $\mathsf{f}(k)$ and $\mathsf{g}(k)$ are based on the same underlying fresh name, they look unrelated. (Analogously, it is common to derive apparently unrelated keys by hashing from a single underlying

secret, as in SSL and TLS [Dierks and Rescorla 2008; Freier et al. 1996].) Hence, a context that obtains the values for x and y cannot distinguish φ_0 and φ_1. On the other hand, the context can discriminate φ_2 by testing the predicate $\mathsf{f}(x) = y$. Therefore, we would like to define static equivalence so that $\varphi_0 \approx_s \varphi_1 \not\approx_s \varphi_2$.

This example relies on a concept of equality of terms in a frame, which the following definition captures.

Definition 4.2. Two terms M and N are equal in the frame φ, written $(M = N)\varphi$, if and only if $fv(M) \cup fv(N) \subseteq dom(\varphi)$, $\varphi \equiv \nu\widetilde{n}.\sigma$, $M\sigma = N\sigma$, and $\{\widetilde{n}\} \cap (fn(M) \cup fn(N)) = \emptyset$ for some names $\widetilde{n}$ and substitution σ.

In Definition 4.2, the equality $M\sigma = N\sigma$ is independent of the representative $\nu\widetilde{n}.\sigma$ chosen for the frame φ such that $\varphi \equiv \nu\widetilde{n}.\sigma$ and $\{\widetilde{n}\} \cap (fn(M) \cup fn(N)) = \emptyset$. (Lemma D.1 in Appendix D establishes this property.)

Definition 4.3. Two closed frames φ and ψ are *statically equivalent*, written $\varphi \approx_s \psi$, when $dom(\varphi) = dom(\psi)$ and when, for all terms M and N, we have $(M = N)\varphi$ if and only if $(M = N)\psi$.

Two closed extended processes are statically equivalent, written $A \approx_s B$, when their frames are statically equivalent.

For instance, in our example, we have $(\mathsf{f}(x) = y)\varphi_2$ but not $(\mathsf{f}(x) = y)\varphi_1$, hence $\varphi_1 \not\approx_s \varphi_2$.

Depending on Σ, static equivalence can be quite hard to check, but at least it does not depend on the dynamics of processes. Some simplifications are possible in common cases, in particular when terms can be put in normal forms (for example, in the proof of Theorems 6.1 and 6.3). Decisions procedures exist for static equivalence in large classes of equational theories [Abadi and Cortier 2006], some implemented in tools [Baudet et al. 2009a; Ciobâcă et al. 2012].

The next lemma establishes closure properties of static equivalence: it shows that static equivalence is invariant by structural equivalence and reduction, and closed by application of closing evaluation contexts. Its proof appears in Appendix A.

Lemma 4.4. *Let A and B be closed extended processes. If $A \equiv B$ or $A \to B$, then $A \approx_s B$. If $A \approx_s B$, then $E[A] \approx_s E[B]$ for all closing evaluation contexts $E[\_]$.*

As the next two lemmas demonstrate, static equivalence coincides with observational equivalence on frames, but is coarser on extended processes.

Lemma 4.5. *Observational equivalence and static equivalence coincide on frames.*

This lemma is an immediate corollary of Theorem 4.8 below. (See Corollary C.14 in Appendix C.3.)

Lemma 4.6. *Observational equivalence is strictly finer than static equivalence on extended processes: $\approx \subset \approx_s$.*

To see that observational equivalence implies static equivalence, note that if A and B are observationally equivalent then $A \mid C$ and $B \mid C$ have the same barbs for every C with $fv(C) \subseteq dom(A)$, and that they are statically equivalent when $A \mid C$ and $B \mid C$ have the same barb $\Downarrow a$ for every C of the special form *if* $M = N$ *then* $\overline{a}\langle n\rangle$, where a does not occur in A or B and $fv(C) \subseteq dom(A)$. (See Lemma C.9 in Appendix C.3.) The converse does not hold, as the following counter-example shows: letting $A = \overline{a}\langle n\rangle$ and $B = \overline{b}\langle n\rangle$, we have $A \not\approx B$, but $A \approx_s B$ because $\varphi(A) = \varphi(B) = \mathbf{0}$.

4.3 Labelled Operational Semantics and Equivalence

A labelled operational semantics extends the chemical semantics of Section 2.2, enabling us to reason about processes that interact with their context while keeping it implicit. The labelled semantics defines a relation $A \xrightarrow{\alpha} A'$, where α is a label of one of the following forms:

$$
\begin{array}{rl}
 & \nu k.\overline{a}\langle \mathrm{enc}(M,k)\rangle.\overline{a}\langle k\rangle.a(z).\mathit{if}\ z = M\ \mathit{then}\ \overline{c}\langle\,\mathrm{oops}\,!\,\rangle \\
\xrightarrow{\nu x.\overline{a}\langle x\rangle} & \nu k.\Big(\{^{\mathrm{enc}(M,k)}/_{x}\} \mid \overline{a}\langle k\rangle.a(z).\mathit{if}\ z = M\ \mathit{then}\ \overline{c}\langle\,\mathrm{oops}\,!\,\rangle\Big) \\
\xrightarrow{\nu y.\overline{a}\langle y\rangle} & \nu k.\Big(\{^{\mathrm{enc}(M,k)}/_{x}\} \mid \{^{k}/_{y}\} \mid a(z).\mathit{if}\ z = M\ \mathit{then}\ \overline{c}\langle\,\mathrm{oops}\,!\,\rangle\Big) \\
\xrightarrow{a(\mathrm{dec}(x,y))} & \nu k.\Big(\{^{\mathrm{enc}(M,k)}/_{x}\} \mid \{^{k}/_{y}\} \mid \mathit{if}\ \mathrm{dec}(x,y) = M\ \mathit{then}\ \overline{c}\langle\,\mathrm{oops}\,!\,\rangle\Big) \\
\rightarrow & \nu k.\Big(\{^{\mathrm{enc}(M,k)}/_{x}\} \mid \{^{k}/_{y}\}\Big) \mid \overline{c}\langle\,\mathrm{oops}\,!\,\rangle
\end{array}
$$

Fig. 3. Example transitions

- a label $N(M)$, which corresponds to an input of M on N;
- a label $\nu x.\overline{N}\langle x\rangle$, where x is a variable that must not occur in N, which corresponds to an output of x on N.

The variable x is bound in the label $\nu x.\overline{N}\langle x\rangle$, so we define the bound variables of labels by $bv(N(M)) \stackrel{\mathrm{def}}{=} \emptyset$ and $bv(\nu x.\overline{N}\langle x\rangle) \stackrel{\mathrm{def}}{=} \{x\}$. The free variables of labels are defined by $fv(N(M)) \stackrel{\mathrm{def}}{=} fv(N) \cup fv(M)$ and $fv(\nu x.\overline{N}\langle x\rangle) \stackrel{\mathrm{def}}{=} fv(N)$ (since x does not occur in N in the latter label).

In addition to the rules for structural equivalence and reduction of Section 2, we adopt the following rules:

$$\textsc{In} \qquad N(x).P \xrightarrow{N(M)} P\{^{M}/_{x}\}$$

$$\textsc{Out-Var} \qquad \frac{x \notin fv(\overline{N}\langle M\rangle.P)}{\overline{N}\langle M\rangle.P \xrightarrow{\nu x.\overline{N}\langle x\rangle} P \mid \{^{M}/_{x}\}}$$

$$\textsc{Scope} \qquad \frac{A \xrightarrow{\alpha} A' \qquad u \text{ does not occur in } \alpha}{\nu u.A \xrightarrow{\alpha} \nu u.A'}$$

$$\textsc{Par} \qquad \frac{A \xrightarrow{\alpha} A' \qquad bv(\alpha) \cap fv(B) = \emptyset}{A \mid B \xrightarrow{\alpha} A' \mid B}$$

$$\textsc{Struct} \qquad \frac{A \equiv B \qquad B \xrightarrow{\alpha} B' \qquad B' \equiv A'}{A \xrightarrow{\alpha} A'}$$

According to In, a term M may be input. On the other hand, Out-Var permits output for terms "by reference": a fresh variable is associated with the term in question and output.

For example, using the signature and equations for symmetric encryption, and the new constant symbol $\mathrm{oops}\,!$, we have the sequence of transitions of Figure 3. The first two transitions do not directly reveal the term M. However, they give enough information to the environment to compute M as $\mathrm{dec}(x,y)$, and to input it in the third transition.

The labelled operational semantics leads to an equivalence relation:

Definition 4.7. A *labelled bisimulation* is a symmetric relation $\mathcal{R}$ on closed extended processes such that $A\ \mathcal{R}\ B$ implies:

(1) $A \approx_s B$;
(2) if $A \rightarrow A'$ and A' is closed, then $B \rightarrow^* B'$ and $A'\ \mathcal{R}\ B'$ for some B';
(3) if $A \xrightarrow{\alpha} A'$, A' is closed, and $fv(\alpha) \subseteq dom(A)$, then $B \rightarrow^* \xrightarrow{\alpha} \rightarrow^* B'$ and $A'\ \mathcal{R}\ B'$ for some B'.

Labelled bisimilarity ($\approx_l$) is the largest such relation.

Conditions 2 and 3 are standard; condition 1, which requires that bisimilar processes be statically equivalent, is necessary for example in order to distinguish the frames φ_0 and φ_2 of Section 4.2. As in Definition 4.1, we explicitly require that A' be closed and $fv(\alpha) \subseteq dom(A)$ in order to exclude transitions that introduce free unused variables.

Our main result is that this relation coincides with observational equivalence. Although such results are fairly common in process calculi, they are important and non-trivial.

Theorem 4.8. *Observational equivalence is labelled bisimilarity:* $\approx = \approx_l$.

The proof of this theorem is outlined in Section 4.5 and completed in the appendix.

The theorem implies that $\approx_l$ is closed by application of closing evaluation contexts. However, unlike the definition of $\approx$, the definition of $\approx_l$ does not include a condition about contexts. It therefore permits simpler proofs.

In addition, labelled bisimilarity can probably be established via standard "bisimulation up to context" techniques [Sangiorgi 1998], which enable useful on-the-fly simplifications in frames after output steps. We do not develop the theory of "up to context" techniques, since we do not use them in this paper.

The following lemmas provide methods for simplifying frames:

Lemma 4.9 (Alias elimination). *Let A and B be closed extended processes, M be a term such that $fv(M) \subseteq dom(A)$, and x be a variable such that $x \notin dom(A)$. We have $A \approx_l B$ if and only if*

$$\{^M/_x\} \mid A \quad \approx_l \quad \{^M/_x\} \mid B$$

Proof. Both directions follow from context closure of $\approx_l$, for the contexts $\{^M/_x\} \mid \_$ and $\nu x.\_$, respectively. In the converse direction, since x is not free in A or B, we have $A \equiv \nu x.(\{^M/_x\} \mid A)$, $\nu x.(\{^M/_x\} \mid A) \approx_l \nu x.(\{^M/_x\} \mid B)$, and $\nu x.(\{^M/_x\} \mid B) \equiv B$ hence $A \approx_l B$. □

Lemma 4.10 (Name disclosure). *Let A and B be closed extended processes and x be a variable such that $x \notin dom(A)$. We have $A \approx_l B$ if and only if*

$$\nu n.(\{^n/_x\} \mid A) \quad \approx_l \quad \nu n.(\{^n/_x\} \mid B)$$

Proof. The direct implication follows from context closure of $\approx_l$. Conversely, we show that the relation $\mathcal{R}$ defined by $A \mathcal{R} B$ if and only if A and B are closed extended processes and $\nu n.(\{^n/_x\} \mid A) \approx_l \nu n.(\{^n/_x\} \mid B)$ for some $x \notin dom(A)$ is a labelled bisimulation. This proof is detailed in Appendix D. □

In Lemma 4.9, the substitution $\{^M/_x\}$ can affect only the context, since A and B are closed. However, the lemma implies that the substitution does not give or mask any information about A and B to the context. In Lemma 4.10, the restriction on n and the substitution $\{^n/_x\}$ mean that the context can access n only indirectly, through the free variable x. Intuitively, the lemma says that indirect access is equivalent to direct access in this case.

Our labelled operational semantics contrasts with a more naive semantics carried over from the pure pi calculus, with output labels of the form $\nu\widetilde{u}.\overline{N}\langle M\rangle$ and rules that permit direct output of any term, such as:

$$\textsc{Out-Term} \qquad \overline{N}\langle M\rangle.P \xrightarrow{\overline{N}\langle M\rangle} P$$

$$\textsc{Open} \qquad \frac{A \xrightarrow{\nu\widetilde{u}.\overline{N}\langle M\rangle} A' \qquad v \in fv(M) \cup fn(M) \setminus (fv(N) \cup fn(N) \cup \{\widetilde{u}\})}{\nu v.A \xrightarrow{\nu v,\widetilde{u}.\overline{N}\langle M\rangle} A'}$$

These rules lead to a different, finer equivalence relation, which for example would distinguish $\nu k, s.\overline{a}\langle(k, s)\rangle$ and $\nu k.\overline{a}\langle(\mathsf{f}(k), \mathsf{g}(k))\rangle$. This equivalence relation is often inadequate in applications (as in [Abadi and Gordon 1999, Section 5.2.1]), hence our definitions.

We have also studied intermediately liberal rules for output, which permit direct output of certain terms. In particular, the rules of the conference paper permit direct output of channel names. That feature implies that it is not necessary to export variables of channel types; as Section 4.5 explains, this property is needed for Theorem 4.8 for those rules. That feature makes little sense in the present calculus, in which arbitrary terms may be used as channels, so we abandon it in the rules above. Nevertheless, certain rules with more explicit labels can still be helpful. We explain those rules next.

4.4 Making the Output Labels More Explicit

In the labelled operational semantics of Section 4.3, the labels for outputs do not reveal anything about the terms being output: those terms are represented by fresh variables. Often, however, more explicit labels can be convenient in reasoning about protocols, and they do not cause harm as long as they only make explicit information that is immediately available to the environment. For instance, for the process $\nu k.\overline{a}\langle(\mathsf{Header}, \mathsf{enc}(M, k))\rangle$, the label $\nu y.\overline{a}\langle(\mathsf{Header}, y)\rangle$ is more informative than $\nu x.\overline{a}\langle x\rangle$. In this example, the environment could anyway observe that x is a pair such that $\mathsf{fst}(x) = \mathsf{Header}$ and use $\mathsf{snd}(x)$ for y. More generally, we rely on the following definition to characterize the information that the environment can derive.

Definition 4.11. Variables $\widetilde{x}$ *resolve to* $\widetilde{M}$ *in* A if and only if $A \equiv \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \nu\widetilde{x}.A$. They are *solvable* in A if and only if they resolve to some terms in A.

Hence, when variables $\widetilde{x}$ resolve to terms $\widetilde{M}$ in A, they are in $dom(A)$ and we can erase the restriction of $\nu\widetilde{x}.A$ by applying the context $\{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \_$ and by structural equivalence. Intuitively, A does not reveal more information than $\nu\widetilde{x}.A$, because the environment can build the terms $\widetilde{M}$ and use them instead of $\widetilde{x}$.

In general, when variables $\widetilde{x}$ are in $dom(A)$, there exist $\widetilde{n}$, $\widetilde{M}$, and A' such that $A \equiv \nu\widetilde{n}.(\{^{\widetilde{M}}/_{\widetilde{x}}\} \mid A')$. If variables $\widetilde{x}$ resolve to $\widetilde{M}$ in A, then $\widetilde{n}$ can be chosen empty, so that the terms $\widetilde{M}$ are not under restrictions. The following lemma provides two reformulations of Definition 4.11, including a converse to this observation. Its proof appears in Appendix E.

Lemma 4.12. *The following three properties are equivalent:*

(1) *the variables* $\widetilde{x}$ *resolve to* $\widetilde{M}$ *in* A*;*
(2) *there exists* A' *such that* $A \equiv \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid A'$*;*
(3) $(\widetilde{x} = \widetilde{M})\varphi(A)$ *and the substitution* $\{^{\widetilde{M}}/_{\widetilde{x}}\}$ *is cycle-free.*

For example, using pairs and symmetric encryption, we let:

$$\varphi \stackrel{\text{def}}{=} \nu k.\{^{M}/_{x}, {}^{\mathsf{enc}(x,k)}/_{y}, {}^{(\mathsf{Header}, y)}/_{z}\}$$

The variable y resolves to $\mathsf{snd}(z)$ in φ, since

$$\varphi \equiv \{^{\mathsf{snd}(z)}/_{y}\} \mid \nu k.\{^{M}/_{x}, {}^{(\mathsf{Header}, \mathsf{enc}(x,k))}/_{z}\}$$

and z resolves to (Header, y) in φ, since

$$\varphi \equiv \{^{(\mathsf{Header}, y)}/_{z}\} \mid \nu k.\{^{M}/_{x}, {}^{\mathsf{enc}(x,k)}/_{y}\}$$

In contrast, x is not always solvable in φ (for instance, when M is k).

A second lemma shows that Definition 4.11 is robust in the sense that it is preserved by static equivalence, so a fortiori by labelled bisimilarity:

Lemma 4.13. *If $A \approx_s B$ and $\widetilde{x}$ resolve to $\widetilde{M}$ in A, then $\widetilde{x}$ resolve to $\widetilde{M}$ in B.*

Proof. Static equivalence preserves property 3 of Lemma 4.12, so we conclude by Lemma 4.12. □

We introduce an alternative semantics in which the rules permit composite terms in output labels but require that every restricted variable that is exported be solvable. In this semantics, the label α in the relation $A \xrightarrow{\alpha} A'$ ranges over the same input labels $N(M)$ as in Section 4.3, and over generalized output labels of the form $\nu\widetilde{x}.\overline{N}\langle M\rangle$, where $\{\widetilde{x}\} \subseteq fv(M) \setminus fv(N)$. The label $\nu\widetilde{x}.\overline{N}\langle M\rangle$ corresponds to an output of M on N that reveals the variables $\widetilde{x}$. We retain the rules for structural equivalence and reduction, and rules In, Par, and Struct of Section 4.3. We also keep rule Scope, but only for labels with no extrusion, that is, for labels $N(M)$ and $\overline{N}\langle M\rangle$. This restriction is necessary because variables may not remain solvable after the application of a context $\nu u.\_$. As a replacement for the rule Out-Var, we use the rule Out-Term discussed in Section 4.3 and:

$$\textsc{Open-Var}\quad \frac{A \xrightarrow{\overline{N}\langle M\rangle} A' \qquad \{\widetilde{x}\} \subseteq fv(M) \setminus fv(N) \qquad \widetilde{x} \text{ solvable in } \{^{M}/_{z}\} \mid A' \text{ for some } z \notin fv(A') \cup \{\widetilde{x}\}}{\nu\widetilde{x}.A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'}$$

These rules are more liberal than those of Section 4.3. For instance, consider $A_1 = \nu k.\overline{a}\langle(\mathrm{f}(k), \mathrm{g}(k))\rangle$ and $A_2 = \nu k.\overline{a}\langle(k, \mathrm{f}(k))\rangle$. With the rules of Section 4.3, we have:

$$A_i \xrightarrow{\nu z.\overline{a}\langle z\rangle} \nu x, y.(\{^{(x,y)}/_{z}\} \mid \varphi_i)$$

where φ_i is as in Section 4.2. With the new rules, we also have:

$$A_i \xrightarrow{\nu x, y.\overline{a}\langle(x,y)\rangle} \varphi_i \tag{5}$$

Indeed, $A_i \equiv \nu x, y.(\overline{a}\langle(x, y)\rangle \mid \varphi_i)$ and the variables x, y are solvable in $\{^{(x,y)}/_{z}\} \mid \varphi_i$ because $\{^{(x,y}/_{z}\} \mid \varphi_i \equiv \{^{\mathrm{fst}(z)}/_{x}, ^{\mathrm{snd}(z)}/_{y}\} \mid \nu x, y.(\{^{(x,y)}/_{z}\} \mid \varphi_i)$, so we derive:

$$\overline{a}\langle(x, y)\rangle \xrightarrow{\overline{a}\langle(x,y)\rangle} \mathbf{0} \qquad \text{by Out-Term}$$

$$\overline{a}\langle(x, y)\rangle \mid \varphi_i \xrightarrow{\overline{a}\langle(x,y)\rangle} \varphi_i \qquad \text{by Par and Struct}$$

$$\nu x, y.(\overline{a}\langle(x, y)\rangle \mid \varphi_i) \xrightarrow{\nu x, y.\overline{a}\langle(x,y)\rangle} \varphi_i \qquad \text{by Open-Var}$$

$$A_i \xrightarrow{\nu x, y.\overline{a}\langle(x,y)\rangle} \varphi_i \qquad \text{by Struct}$$

Transition (5) is the most informative for A_1 since x and y behave like fresh, independent values in φ_1. For A_2, we also have the more informative transition:

$$A_2 \xrightarrow{\nu x.\overline{a}\langle(x,\mathrm{f}(x))\rangle} \nu k.\{^{k}/_{x}\}$$

that reveals the link between x and y, but not that x is a name. As in this example, several output transitions are sometimes possible, each transition leading to an extended process with a different frame. In reasoning (for example, in proving that a relation is included in labelled bisimilarity), it often suffices to consider any one of the transitions, so one may be chosen so as to limit the complexity of the resulting extended processes.

We name "simple semantics" the labelled semantics of Section 4.3 and "refined semantics" the semantics of this section, and "simple labels" and "refined labels" the corresponding labels. The next theorem states that the two labelled semantics yield the same notion of equivalence. Thus, making

the output labels more explicit only makes apparent some of the information that is otherwise kept in the static, equational part of labelled bisimilarity.

Theorem 4.14. *Let $\approx_L$ be the relation of labelled bisimilarity obtained by applying Definition 4.7 to the refined semantics. We have $\approx_l = \approx_L$.*

The proof of Theorem 4.14 relies on the next two lemmas, which relate simple and refined output transitions.

Lemma 4.15. *$A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$ if and only if, for some z that does not occur in any of A, A', $\widetilde{x}$, N, and M, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{{}^{M}/_{z}\} \mid A')$, $\{\widetilde{x}\} \subseteq fv(M) \setminus fv(N)$, and the variables $\widetilde{x}$ are solvable in $\{{}^{M}/_{z}\} \mid A'$.*

In Lemma 4.15, the transition $A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$ is performed in the refined semantics, while the transition $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{{}^{M}/_{z}\} \mid A')$ is performed in the simple semantics. However, Lemma 4.16 below shows that the choice of the semantics does not matter. Lemma 4.16 is a consequence of Lemma 4.15.

Lemma 4.16. *$A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ in the refined semantics if and only if $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ in the simple semantics.*

Theorem 4.14 is then proved as follows. By Lemma 4.16, $\approx_L$ is a simple-labelled bisimulation, and thus $\approx_L \subseteq \approx_l$. Conversely, to show that $\approx_l$ is a refined-labelled bisimulation, it suffices to prove its bisimulation property for any refined output label. This proof, which relies on Lemma 4.15, and the proofs of Lemmas 4.12, 4.15, and 4.16 are detailed in Appendix E.

4.5 Proving Theorem 4.8 ($\approx = \approx_l$)

A claim of Theorem 4.8 appears, without proof, in the conference version of this paper, for the calculus as presented in that version. There, the channels in labels cannot be variables. The claim neglects to include a corresponding hypothesis that exported variables must not be of channel type. This hypothesis is implicitly assumed, as it holds trivially for plain processes and is maintained, as an invariant, by output transitions. Without it, the two extended processes $\nu a.(\{{}^{a}/_{x}\})$ and $\nu a.(\{{}^{a}/_{x}\} \mid \overline{a}\langle N\rangle)$ (where the exported variable x stands for the channel a) would constitute a counterexample: they would not be observationally equivalent but they would be bisimilar in the labelled semantics, since neither could make a labelled transition. Delaune et al. [2007; 2010] included the hypothesis in their study of symbolic bisimulation. Avik Chaudhuri (private communication, 2007) pointed out this gap in the statement of the theorem, and Bengtson et al. [2011] discussed it as motivation for their work on alternative calculi, the psi calculi, with a more abstract treatment of terms and a mechanized metatheory. On the other hand, Liu [2011] presented a proof of the theorem, making explicit the necessary hypothesis. Her proof demonstrated that the theorem was basically right—no radical changes or new languages were needed. More recently, Liu and others have also developed an extension of the proof for a stateful variant of the applied pi calculus [Arapinis et al. 2014].

Theorem 4.8, in its present form, does not require that hypothesis because of some of the details of the calculus as we define it in this paper. Specifically, the labelled semantics allows variables that stand for channels in labels. Therefore, extended processes such as $\nu a.(\{{}^{a}/_{x}\} \mid \overline{a}\langle N\rangle)$ can make labelled transitions.

This section outlines the proof of Theorem 4.8. The appendix gives further details, including all proofs that this section omits. Those details are fairly long and technical. In particular, they rely on a definition of "partial normal forms" for extended processes, which are designed to simplify

reasoning about reductions. (In an extended process $A \mid B$, the frame of A may affect B and vice versa, so A and B may not reduce independently of each other; partial normal forms are designed to simplify the analysis of reductions in such situations.) We believe that these partial normal forms may be useful in other proofs on the applied pi calculus. In this section, we omit further specifics on partial normal forms, since they are not essential to understanding our main arguments.

The proof of Theorem 4.8 starts with a fairly traditional definition of "labelled bisimulation up to $\equiv$":

Definition 4.17. A relation $\mathcal{R}$ on closed extended processes is a *labelled bisimulation up to* $\equiv$ if and only if $\mathcal{R}$ is symmetric and $A \mathcal{R} B$ implies:

(1) $A \approx_s B$;
(2) if $A \rightarrow A'$ and A' is closed, then $B \rightarrow^* B'$ and $A' \equiv \mathcal{R} \equiv B'$ for some closed B';
(3) if $A \xrightarrow{\alpha} A'$, A' is closed, and $fv(\alpha) \subseteq dom(A)$, then $B \rightarrow^* \xrightarrow{\alpha} \rightarrow^* B'$ and $A' \equiv \mathcal{R} \equiv B'$ for some closed B'.

This definition implies that, if $\mathcal{R}$ is a labelled bisimulation up to $\equiv$, then $\equiv \mathcal{R} \equiv$ restricted to closed processes is a labelled bisimulation (since, by Lemma 4.4, static equivalence is invariant by structural equivalence).

We use the definition to establish the following lemma:

Lemma 4.18. *$\approx_l$ is closed by application of closing evaluation contexts.*

In the proof of this lemma (which is given in Appendix C.2), we show that we can restrict attention to contexts of the form $\nu \tilde{u}.(\_ \mid C)$. To every relation $\mathcal{R}$ on closed extended processes, we associate a relation $\mathcal{R}' = \{(\nu \tilde{u}.(A \mid C), \nu \tilde{u}.(B \mid C)) \mid A \mathcal{R} B, \nu \tilde{u}.(\_ \mid C) \text{ closing for } A \text{ and } B\}$. We prove that, if $\mathcal{R}$ is a labelled bisimulation, then $\mathcal{R}'$ is a labelled bisimulation up to $\equiv$, hence $\mathcal{R} \subseteq \equiv \mathcal{R}' \equiv \subseteq \approx_l$. For $\mathcal{R} = \approx_l$, this property entails that $\approx_l$ is closed by application of evaluation contexts $\nu \tilde{u}.(\_ \mid C)$.

Another lemma characterizes barbs in terms of labelled transitions:

Lemma 4.19. *Let A be a closed extended process. We have $A \Downarrow a$ if and only if $A \rightarrow^* \xrightarrow{\nu x.\overline{a}\langle x \rangle} A'$ for some fresh variable x and some A'.*

We then obtain Lemma 4.20, which is one direction of Theorem 4.8:

Lemma 4.20. $\approx_l \subseteq \approx$.

Proof. We show that $\approx_l$ satisfies the three properties of Definition 4.1, as follows.

(1) To show that $\approx_l$ preserves barbs, we apply Lemma 4.19 and use Properties 2 and 3 of Definition 4.7.
(2) Suppose that $A \approx_l B$, $A \rightarrow^* A'$, and A' is closed. Given the trace $A = A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n = A'$, we instantiate all variables in $\bigcup_{i=0}^{n} (fv(A_i) \setminus dom(A_i))$ with fresh names. This instantiation yields a trace in which all intermediate processes are closed. We can then conclude that $B \rightarrow^* B'$ and $A' \approx_l B'$ for some B' by Property 2 of Definition 4.7.
(3) $\approx_l$ is closed by application of closing evaluation contexts by Lemma 4.18.

Moreover, $\approx_l$ is symmetric. Since $\approx$ is the largest relation that satisfies these properties, we obtain $\approx_l \subseteq \approx$. □

The other direction of Theorem 4.8 relies on two lemmas that characterize input and output transitions. The first lemma characterizes inputs $N(M)$ using processes of the form $T^p_{N(M)} \stackrel{\text{def}}{=} \overline{p}\langle p \rangle \mid \overline{N}\langle M \rangle.p(x)$. Here, the use of p as a message in $\overline{p}\langle p \rangle$ is arbitrary: we could equally use processes of the form $\overline{p}\langle M' \rangle$ for any term M'.

Lemma 4.21. *Let A be a closed extended process. Let N and M be terms such that $fv(\overline{N}\langle M\rangle) \subseteq dom(A)$. Let p be a name that does not occur in A, M, and N.*

(1) *If $A \xrightarrow{N(M)} A'$ and p does not occur in A', then $A \mid T^{p}_{N(M)} \to\!\!\to A'$ and $A' \not\Downarrow p$.*

(2) *If $A \mid T^{p}_{N(M)} \to^{*} A'$ and $A' \not\Downarrow p$, then $A \to^{*} \xrightarrow{N(M)} \to^{*} A'$.*

The second lemma characterizes outputs $\nu x.\overline{N}\langle x\rangle$ using processes of the form $T^{p,q}_{\nu x.\overline{N}\langle x\rangle} \stackrel{\text{def}}{=} \overline{p}\langle p\rangle \mid N(x).p(y).\overline{q}\langle x\rangle$.

Lemma 4.22. *Let A be a closed extended process. Let N be a term such that $fv(N) \subseteq dom(A)$. Let p and q be names that do not occur in A and N.*

(1) *If $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ and p and q do not occur in A', then $A \mid T^{p,q}_{\nu x.\overline{N}\langle x\rangle} \to\!\!\to \nu x.(A' \mid \overline{q}\langle x\rangle)$, $\nu x.(A' \mid \overline{q}\langle x\rangle) \not\Downarrow p$, and $x \notin dom(A)$.*

(2) *Let x be a variable such that $x \notin dom(A)$. If $A \mid T^{p,q}_{\nu x.\overline{N}\langle x\rangle} \to^{*} A''$ and $A'' \not\Downarrow p$, then $A \to^{*} \xrightarrow{\nu x.\overline{N}\langle x\rangle} \to^{*} A'$ and $A'' \equiv \nu x.(A' \mid \overline{q}\langle x\rangle)$ for some A'.*

A further lemma provides a way of proving the equivalence of two extended processes with the same domain by putting them in a context that binds the variables in their domain and extrudes them. Given a family of processes P_i for i in a finite set I, we write $\prod_i P_i$ for the parallel composition of the processes P_i if I is not empty, and for $\mathbf{0}$ otherwise.

Lemma 4.23. *Let A and B be two closed extended processes with a same domain that contains $\widetilde{x}$. Let $E_{\widetilde{x}}[\_] \stackrel{def}{=} \nu\widetilde{x}.(\prod_{x\in\widetilde{x}} \overline{n_x}\langle x\rangle \mid \_)$ using names n_x that do not occur in A or B. If $E_{\widetilde{x}}[A] \approx E_{\widetilde{x}}[B]$, then $A \approx B$.*

The final lemma is the other direction of Theorem 4.8:

Lemma 4.24. *$\approx$ is a labelled bisimulation, and thus $\approx \subseteq \approx_l$.*

Proof. The relation $\approx$ is symmetric. We show that it satisfies the three properties of Definition 4.7.

(1) If $A \approx B$, then $A \approx_s B$, by Lemma 4.6.

(2) If $A \approx B$, $A \to A'$, and A' is closed, then $B \to^{*} B'$ and $A' \approx B'$ for some B', by Property 2 of the definition of $\approx$.

(3) If $A \approx B$, $A \xrightarrow{\alpha} A'$, A' is closed, and $fv(\alpha) \subseteq dom(A)$, then $B \to^{*} \xrightarrow{\alpha} \to^{*} B'$ and $A' \approx B'$ for some B'. To prove this property, we rely on characteristic contexts $\_ \mid T_\alpha$ that unambiguously test for a labelled transition $\xrightarrow{\alpha}$ using the disappearance of a barb $\Downarrow p$, and do not otherwise affect $\approx$.
Assume $A \approx B$, $A \xrightarrow{\alpha} A'$, A' is closed, and $fv(\alpha) \subseteq dom(A)$.
   (a) For input $\alpha = N(M)$ (where N and M may contain variables exported by A and B) and some fresh name p, we have $A \mid T^{p}_{N(M)} \to\!\!\to A' \not\Downarrow p$ by Lemma 4.21(1), hence $B \mid T^{p}_{N(M)} \to^{*} B' \not\Downarrow p$ with $A' \approx B'$, hence $B \to^{*} \xrightarrow{N(M)} \to^{*} B'$ by Lemma 4.21(2).
   (b) For output $\alpha = \nu x.\overline{N}\langle x\rangle$ and some fresh names p and q, we have $A \mid T^{p,q}_{\nu x.\overline{N}\langle x\rangle} \to\!\!\to \nu x.(A' \mid \overline{q}\langle x\rangle) \not\Downarrow p$ and $x \notin dom(A)$ by Lemma 4.22(1), hence $B \mid T^{p,q}_{\nu x.\overline{N}\langle x\rangle} \to^{*} B'' \not\Downarrow p$ for some B'', hence $B \to^{*} \xrightarrow{\nu x.\overline{N}\langle x\rangle} \to^{*} B'$ and $B'' \equiv \nu x.(B' \mid \overline{q}\langle x\rangle)$ for some B' by Lemma 4.22(2). We obtain a pair $\nu x.(A' \mid \overline{q}\langle x\rangle) \approx \nu x.(B' \mid \overline{q}\langle x\rangle)$, and conclude by applying Lemma 4.23.

Hence $\approx$ is a labelled bisimulation, and $\approx \subseteq \approx_l$, since $\approx_l$ is the largest labelled bisimulation. □

Theorem 4.8 is an immediate consequence of Lemmas 4.20 and 4.24.

Considering this proof of Theorem 4.8, we can explain further some aspects of our definition of observational equivalence (Definition 4.1). That definition includes conditions related to barbs, reductions, and evaluation contexts (Conditions (1) to (3), respectively), as is done in work on the ν-calculus [Honda and Yoshida 1995] and on the join calculus [Abadi et al. 1998]. In an alternative approach, used in CCS [Milner and Sangiorgi 1992] and in the pi calculus [Sangiorgi 1993], equivalence is defined in two stages:

(1) First, barbed bisimilarity is defined as the largest barbed bisimulation, that is, the largest symmetric relation $\mathcal{R}$ such that $A\ \mathcal{R}\ B$ implies Conditions (1) and (2) of Definition 4.1.
(2) Second, equivalence is defined as the largest congruence (that is, the largest relation $\mathcal{R}$ such that $A\ \mathcal{R}\ B$ implies Condition (3) of Definition 4.1) contained in barbed bisimilarity.

The two approaches do not necessarily yield the same equivalence relation; see [Fournet and Gonthier 1998] for positive and negative examples in variants of the pi calculus. The advantage of our approach is that, in reasoning about process equivalences, we can add a context at any point after reductions, as we do in the proof of Lemma 4.24. With the alternative approach, we can add a context only at the beginning, before any reduction, so we need to build contexts that test for all possible sequences of labelled transitions that the processes under consideration may make, and that manifest them as different combinations of barbs. This testing is not possible for all processes, so with the alternative approach, analogues of Theorem 4.8 would typically require a restriction to so-called *image finite* processes [Milner and Sangiorgi 1992]. Our definition of observational equivalence avoids this restriction.

It would be interesting to formalize the proofs of this section (and also those of the rest of the paper) with a theorem prover such as Coq. This formalization may perhaps benefit from past Coq developments on bisimulations for the pi calculus [Hirschkoff 1997; Honsell et al. 2001] and the spi calculus [Briais 2008]. However, the applied pi calculus introduces additional difficulties (because of the role of terms with equational theories), and proving our results with Coq would certainly require a major effort.

5 DIFFIE-HELLMAN KEY AGREEMENT

The fundamental Diffie-Hellman protocol allows two principals to establish a shared secret by exchanging messages over public channels [Diffie and Hellman 1976]. The principals need not have any shared secrets in advance. The basic protocol, on which we focus here as an example, does not provide authentication; therefore, a "bad" principal may play the role of either principal in the protocol. On the other hand, the two principals that follow the protocol will communicate securely with one another afterwards, even in the presence of active attackers. In extended protocols, such as the Station-to-Station protocol [Diffie et al. 1992] and SKEME [Krawczyk 1996], additional messages perform authentication.

We program the basic protocol in terms of the binary function symbol f and the unary function symbol g, with the equation:

$$\mathrm{f}(x, \mathrm{g}(y)) \quad = \quad \mathrm{f}(y, \mathrm{g}(x)) \tag{6}$$

Concretely, the functions are $\mathrm{f}(x, y) = y^x \ mod\ p$ and $\mathrm{g}(x) = \alpha^x \ mod\ p$ for a prime p and a generator α of $\mathbb{Z}_p^*$, and we have the equation $\mathrm{f}(x, \mathrm{g}(y)) = (\alpha^y)^x = \alpha^{y \times x} = \alpha^{x \times y} = (\alpha^x)^y = \mathrm{f}(y, \mathrm{g}(x))$. However, we ignore the underlying number theory, working abstractly with f and g.

The protocol has two symmetric participants, which we represent by the processes A_0 and A_1. The protocol establishes a shared key, then the participants respectively run P_0 and P_1 using the key. We use the public channel c_{01} for messages from A_0 to A_1 and the public channel c_{10} for

communication in the opposite direction. (Although the use of two distinct public channels is of no value for security, it avoids some trivial confusions, so makes for a cleaner presentation.) We assume that none of the values introduced in the protocol appears in P_0 and P_1, except for the key.

In order to establish the key, A_0 invents a name n_0, sends $\mathrm{g}(n_0)$ to A_1, and A_1 proceeds symmetrically. Then A_0 computes the key as $\mathrm{f}(n_0, \mathrm{g}(n_1))$ and A_1 computes it as $\mathrm{f}(n_1, \mathrm{g}(n_0))$, with the same result. We find it convenient to use the following substitutions for A_0's message and key:

$$\sigma_0 \stackrel{\text{def}}{=} \{^{\mathrm{g}(n_0)}/_{x_0}\}$$
$$\phi_0 \stackrel{\text{def}}{=} \{^{\mathrm{f}(n_0,x_1)}/_{y}\}$$

and the corresponding substitutions σ_1 and ϕ_1, as well as the frame:

$$\varphi \stackrel{\text{def}}{=} (\nu n_0.\ (\phi_0 \mid \sigma_0)) \mid (\nu n_1.\ \sigma_1)$$

With these notations, A_0 is:

$$A_0 \stackrel{\text{def}}{=} \nu n_0.(\overline{c_{01}}\langle x_0\sigma_0\rangle \mid c_{10}(x_1).P_0\phi_0)$$

and A_1 is analogous.

Two reductions represent a normal run of the protocol:

$$\begin{aligned} A_0 \mid A_1 &\longrightarrow\!\!\!\!\longrightarrow \nu x_0, x_1, n_0, n_1.\ (P_0\phi_0 \mid P_1\phi_1 \mid \sigma_0 \mid \sigma_1) && (7)\\ &\equiv \nu x_0, x_1, n_0, n_1, y.\ (P_0 \mid P_1 \mid \phi_0 \mid \sigma_0 \mid \sigma_1) && (8)\\ &\equiv \nu y.(P_0 \mid P_1 \mid \nu x_0, x_1.\ \varphi) && (9)\end{aligned}$$

The two communication steps (7) use structural equivalence to activate the substitutions σ_0 and σ_1 and extend the scope of the secret values n_0 and n_1. The structural equivalence (8) crucially relies on equation (6) in order to reuse the active substitution ϕ_0 instead of ϕ_1 after the reception of x_0 in A_1. The next structural equivalence (9) tightens the scope for restricted names and variables, then uses the definition of φ.

We model an eavesdropper as a process $c_{01}(x_0).\overline{c_{01}}\langle x_0\rangle.c_{10}(x_1).\overline{c_{10}}\langle x_1\rangle.P$ that intercepts messages on c_{01} and c_{10}, remembers them, but forwards them unmodified. Using the labelled semantics to represent the interaction of $A_0 \mid A_1$ with such a passive attacker, we obtain:

$$\begin{aligned} A_0 \mid A_1 &\xrightarrow{\overline{c_{01}}\langle x_0\rangle} \nu n_0.(\sigma_0 \mid c_{10}(x_1).P_0\phi_0) \mid A_1\\ &\xrightarrow{c_{01}(x_0)} \nu n_0.(\sigma_0 \mid c_{10}(x_1).P_0\phi_0) \mid \nu n_1.(\overline{c_{10}}\langle\sigma_1 x_1\rangle \mid P_1\phi_1)\\ &\xrightarrow{\overline{c_{10}}\langle x_1\rangle} \nu n_0.(\sigma_0 \mid c_{10}(x_1).P_0\phi_0) \mid \nu n_1.(\sigma_1 \mid P_1\phi_1)\\ &\xrightarrow{c_{10}(x_1)} \nu n_0.(\sigma_0 \mid P_0\phi_0) \mid \nu n_1.(\sigma_1 \mid P_1\phi_1)\\ &\equiv \nu n_0, n_1, y.\ (P_0 \mid P_1 \mid \phi_0 \mid \sigma_0 \mid \sigma_1)\\ &\equiv \nu y.(P_0 \mid P_1 \mid \varphi)\end{aligned}$$

The labelled transitions $\xrightarrow{\overline{c_{01}}\langle x_0\rangle}\xrightarrow{c_{01}(x_0)}$ show that the eavesdropper obtains the message sent on c_{01} by A_0, stores it in x_0, and forwards it to A_1. The transitions $\xrightarrow{\overline{c_{10}}\langle x_1\rangle}\xrightarrow{c_{10}(x_1)}$ deal with the message on c_{10} in a similar way. The absence of the restrictions on x_0 and x_1 corresponds to the fact that the eavesdropper has obtained the values of these variables.

The following theorem relates this process to

$$\nu k.(P_0 \mid P_1)\{^{k}/_{y}\}$$

which represents the bodies P_0 and P_1 of A_0 and A_1 sharing a key k. This key appears as a simple shared name, rather than as the result of communication and computation. Intuitively, we may read $\nu k.(P_0 \mid P_1)\{^k/_y\}$ as the ideal outcome of the protocol: P_0 and P_1 execute using a shared key, without concern for how the key was established, and without any side-effects from weaknesses in the establishment of the key. The theorem says that this ideal outcome is essentially achieved, up to some "noise". This "noise" is a substitution that maps x_0 and x_1 to unrelated, fresh names. It accounts for the fact that an attacker may have the key-exchange messages, and that they look just like unrelated values to the attacker. In particular, the key in use between P_0 and P_1 has no observable relation to those messages, or to any other left-over secrets. We view this independence of the shared key as an important forward-secrecy property.

Theorem 5.1. *Let P_0 and P_1 be processes with free variable y where the name k does not appear. We have:*

$$\nu y.(P_0 \mid P_1 \mid \varphi) \;\approx\; \nu k.(P_0 \mid P_1)\{^k/_y\} \mid \nu s_0.\{^{s_0}/_{x_0}\} \mid \nu s_1.\{^{s_1}/_{x_1}\}$$

Proof. The theorem follows from Lemma 4.5 and the static equivalence $\varphi \approx_s \nu s_0, s_1, k.\{^{s_0}/_{x_0}, {}^{s_1}/_{x_1}, {}^k/_y\}$, which says that the frame φ generated by the protocol execution is equivalent to one that maps variables to fresh names. This static equivalence is proved automatically by ProVerif, using the technique presented in [Blanchet et al. 2008]. We conclude by applying the context $\nu y.(P_0 \mid P_1 \mid \_)$. □

Extensions of the basic protocol add rounds of communication that confirm the key and authenticate the principals. We have studied one such extension with key confirmation. There, the shared secret $\mathsf{f}(n_0, \mathsf{g}(n_1))$ is used in confirmation messages. Because of these messages, the shared secret can no longer be equated with a virgin key for P_0 and P_1. Instead, the final key is computed by hashing the shared secret. This hashing guarantees the independence of the final key.

We have also studied more advanced protocols that rely on a Diffie-Hellman key exchange, such as the JFK protocol [Aiello et al. 2004]. The analysis of JFK in the applied pi calculus [Abadi et al. 2007] illustrates the composition of manual reasoning with invocations of ProVerif.

6 HASH FUNCTIONS AND MESSAGE AUTHENTICATION CODES

Section 3 briefly discusses cryptographic hash functions. In this section we continue their study, and also treat message authentication codes (MACs). We consider constructions of both hash functions and MACs. These examples provide a further illustration of the usefulness of equations in the applied pi calculus. On the other hand, some aspects of the constructions are rather low-level, and we would not expect to account for all their combinatorial details (e.g., the "birthday attacks" [Menezes et al. 1996]). A higher-level task is to express and reason about protocols treating hash functions and MACs as primitive; this is squarely within the scope of our approach.

6.1 Using MACs

MACs serve to authenticate messages using shared keys. When k is a key and M is a message, and k is known only to a certain principal A and to the recipient B of the message, B may take $\mathsf{mac}(k, M)$ as proof that M comes from A. More precisely, B can check $\mathsf{mac}(k, M)$ by recomputing it upon receipt of M and $\mathsf{mac}(k, M)$, and reason that A must be the sender of M. This property should hold even if A generates MACs for other messages as well; those MACs should not permit forging a MAC for M. In the worst case, it should hold even if A generates MACs for other messages on demand.

$$\begin{aligned} \nu k.(A \mid B) &\xrightarrow{a(M)} \nu k.(A \mid B \mid \overline{b}\langle (M, \mathsf{mac}(k, M))\rangle) \\ &\xrightarrow{\nu x.\overline{b}\langle x\rangle} \nu k.(A \mid B \mid \{^{(M,\mathsf{mac}(k,M))}/_{x}\}) \\ &\xrightarrow{b(x)} \nu k.(A \mid \overline{c}\langle M\rangle \mid \{^{(M,\mathsf{mac}(k,M))}/_{x}\}) \\ &\xrightarrow{\nu y.\overline{c}\langle y\rangle} \nu k.(A \mid \{^{(M,\mathsf{mac}(k,M))}/_{x}, {}^{M}/_{y}\}) \end{aligned}$$

Fig. 4. A correct trace

Using a new binary function symbol mac, we may describe this scenario by the following processes:

$$\begin{aligned} A &\stackrel{\text{def}}{=} !a(x).\overline{b}\langle (x, \mathsf{mac}(k, x))\rangle \\ B &\stackrel{\text{def}}{=} b(y).\mathit{if}\ \mathsf{mac}(k, \mathsf{fst}(y)) = \mathsf{snd}(y)\ \mathit{then}\ \overline{c}\langle \mathsf{fst}(y)\rangle \\ S &\stackrel{\text{def}}{=} \nu k.(A \mid B) \end{aligned}$$

The process S represents the complete system, composed of A and B; the restriction on k means that k is private to A and B. The process A receives messages on a public channel a and returns them MACed on the public channel b. When B receives a message on b, it checks its MAC and acts upon it, here simply by forwarding on a channel c. Intuitively, we would expect that B forwards on c only a message that A has MACed. In other words, although an attacker may intercept, modify, and inject messages on b, it should not be able to forge a MAC and trick B into forwarding some other message. Hence, every message output on c equals a preceding input on a, as illustrated in Figure 4.

This property can be expressed precisely in terms of the labelled semantics and it can be checked without too much difficulty when mac is a primitive function symbol with no equations. The property remains true even if there is a function extract that maps a MAC $\mathsf{mac}(x, y)$ to the underlying cleartext y, with the equation $\mathsf{extract}(\mathsf{mac}(x, y)) = y$. Since MACs are not supposed to guarantee secrecy, such a function may well exist, so it is safer to assume that it is available to the attacker.

The property is more delicate if mac is defined from other operations, as it invariably is in practice. In that case, the property may even be taken as *the* specification of MACs [Goldwasser and Bellare 1999]. Thus, a MAC implementation may be deemed correct if and only if the process S works as expected when mac is instantiated with that implementation. More specifically, the next section deals with the question of whether the property remains true when mac is defined from hash functions.

6.2 Constructing Hash Functions and MACs

In Section 3, we give no equations for hash functions. In practice, following Merkle and Damgård, hash functions are commonly defined by iterating a basic binary compression function, which maps two input blocks to one output block [Menezes et al. 1996]. Furthermore, keyed hash functions include a key as an additional argument. Thus, we may have:

$$\mathsf{h}(x, y_0 :: y_1 :: z) = \mathsf{h}(\mathsf{f}(x, y_0), y_1 :: z) \quad (10)$$

$$\mathsf{h}(x, y :: \mathsf{nil}) = \mathsf{f}(x, y) \quad (11)$$

Here, we use the sorts Block for blocks and BlockList for sequences of blocks, defined as lists as in Section 3, with sorts Block and BlockList instead of Data and List, respectively. The function

$$
\begin{array}{lll}
vk.(A \mid B) & \xrightarrow{a(M)} & vk.(A \mid B \mid \overline{b}\langle (M, \mathsf{mac}(k, M)) \rangle) \\
& \xrightarrow{\nu x.\overline{b}\langle x \rangle} & vk.(A \mid B \mid \{^{(M,\mathsf{mac}(k,M))}/_{x}\}) \\
& \xrightarrow{b((M+N,\mathsf{f}(\mathsf{snd}(x),N)))} & vk.(A \mid \overline{c}\langle M + N \rangle \mid \{^{(M,\mathsf{mac}(k,M))}/_{x}\}) \\
& \xrightarrow{\nu y.\overline{c}\langle y \rangle} & vk.(A \mid \{^{(M,\mathsf{mac}(k,M))}/_{x}, {}^{M+N}/_{y}\})
\end{array}
$$

Fig. 5. An attack scenario

$$
\begin{array}{lll}
vk.(A \mid B) & \xrightarrow{a(M)} & vk.(A \mid B \mid \overline{b}\langle (M, \mathsf{mac}(k, M)) \rangle) \\
& \xrightarrow{\nu x.\overline{b}\langle (M,x) \rangle} & vk.(A \mid B \mid \{^{\mathsf{mac}(k,M)}/_{x}\}) \\
& \xrightarrow{b((M+N,\mathsf{f}(x,N)))} & vk.(A \mid \overline{c}\langle M + N \rangle \mid \{^{\mathsf{mac}(k,M)}/_{x}\}) \\
& \xrightarrow{\overline{c}\langle M+N \rangle} & vk.(A \mid \{^{\mathsf{mac}(k,M)}/_{x}\})
\end{array}
$$

Fig. 6. An attack scenario (with refined labels)

$\mathsf{h} : \mathsf{Block} \times \mathsf{BlockList} \to \mathsf{Block}$ is the keyed hash function, $\mathsf{f} : \mathsf{Block} \times \mathsf{Block} \to \mathsf{Block}$ is the compression function.

In these equations we are rather abstract in our treatment of blocks, their sizes, and therefore of padding and other related issues. We also ignore two common twists: some functions use initialization vectors to start the iteration, and some append a length block to the input. Nevertheless, we can explain various MAC constructions, describing flaws in some and reasoning about the properties of others.

A first, classical definition of a MAC from a keyed hash function h is:

$$\mathsf{mac}(x, y) \stackrel{\mathrm{def}}{=} \mathsf{h}(x, y)$$

For instance, the MAC of a three-block message $M = M_1 :: M_2 :: M_3 :: \mathsf{nil}$ with key k is $\mathsf{mac}(k, M) = \mathsf{f}(\mathsf{f}(\mathsf{f}(k, M_1), M_2), M_3)$. More generally, the MAC of a n-block message $M = M_1 :: \ldots :: M_n :: \mathsf{nil}$ is $\mathsf{mac}(k, M) = \mathsf{f}(\ldots(\mathsf{f}(k, M_1), \ldots), M_n)$. This implementation is subject to a well-known extension attack. Given the MAC of $M = M_1 :: \ldots :: M_n :: \mathsf{nil}$, an attacker can compute the MAC of any extension $M + N = M_1 :: \ldots :: M_n :: N :: \mathsf{nil}$ without knowing the MAC key, since $\mathsf{mac}(k, M + N) = \mathsf{f}(\mathsf{mac}(k, M), N)$.

We describe the extension attack formally, through the operational semantics of the process S of Section 6.1, in Figures 5 and 6. These figures use the semantics of Sections 4.3 and 4.4 respectively. In both cases, we assume $k \notin fn(M) \cup fn(N)$. Additionally, we adopt the sorts $\mathsf{pair} : \mathsf{BlockList} \times \mathsf{Block} \to \mathsf{Data}$, $\mathsf{fst} : \mathsf{Data} \to \mathsf{BlockList}$, and $\mathsf{snd} : \mathsf{Data} \to \mathsf{Block}$, the abbreviation (M, N) for $\mathsf{pair}(M, N)$, and the equations (1) and (2) of Section 3. In Figures 5 and 6, we see that the message M that the system MACs differs from the message $M + N$ that it forwards on c. These transitions are not enabled with the primitive MAC of Section 6.1, hence S with the proposed MAC implementation is not labelled bisimilar to S with the primitive MAC.

There are several ways to address extension attacks, and indeed the literature contains many MAC constructions that are not subject to these attacks. We have considered some of them. Here

we describe a construction that uses the MAC key twice:

$$\mathsf{mac}(x, y) \stackrel{\text{def}}{=} \mathsf{f}(x, \mathsf{h}(x, y))$$

Under this definition, the MAC of $M = M_1 :: M_2 :: M_3 :: \mathsf{nil}$ with key k is $\mathsf{mac}(k, M) = \mathsf{f}(k, \mathsf{f}(\mathsf{f}(\mathsf{f}(k, M_1), M_2), M_3))$, and the process S forwards on c only a message that it has previously MACed, as desired.

Looking beyond the case of S, we can prove a more general result by comparing the situation where mac is primitive (and has no special equations) and one with the definition of $\mathsf{mac}(x, y)$ as $\mathsf{f}(x, \mathsf{h}(x, y))$. Given a name k and an extended process C that uses the symbol mac, we write $[\![C]\!]$ for the translation of C in which the definition of mac is expanded wherever the key k is used, with $\mathsf{f}(k, \mathsf{h}(k, M))$ replaced for $\mathsf{mac}(k, M)$. The theorem says that this translation yields an equivalent process (so, intuitively, the constructed MACs work as well as the primitive ones). It applies to a class of equational theories generated by rewrite rules.

Theorem 6.1. *Suppose that the signature Σ is equipped with an equational theory generated by a convergent rewrite system such that* mac *and* f *do not occur in the left-hand sides of rewrite rules; the only rewrite rules with* h *at the root of the left-hand side are those of* (10) *and* (11) *oriented from left to right; there are no rewrite rules with* :: *nor* nil *at the root of the left-hand side; and names do not occur in rewrite rules. Suppose that C is closed and the name k appears only as first argument of* mac *in C. Then $\nu k.C \approx \nu k.[\![C]\!]$.*

In the proof of this theorem (which is given in Appendix F), we use the same notion of partial normal form as in the proof of Theorem 4.8. We define a relation $\mathcal{R}$ by $A \mathcal{R} B$ if and only if A and B are closed, $A \equiv \nu k.C$, $B \equiv \nu k.[\![C]\!]$, C is a closed extended process in partial normal form, and the name k appears only as MAC key in C. We show that the relation $\mathcal{R} \cup \mathcal{R}^{-1}$ (that is, the union of $\mathcal{R}$ with its inverse relation) is a labelled bisimulation. Static equivalence follows from the preservation of equality by the translation $[\![\cdot]\!]$ for terms in which k occurs only as MAC key; reductions commute with the translation $[\![\cdot]\!]$ and preserve the restriction on the occurrences of the key k. We conclude by Theorem 4.8. An alternative proof of similar complexity would show that $\mathcal{R} \cup \mathcal{R}^{-1}$ is an observational bisimulation.

Theorem 6.1 considers a single MAC key at a time. For an extended process with several MAC keys $k_1, \ldots, k_n$, we can apply Theorem 6.1 once for each key k_i, using structural equivalence to move each restriction νk_i to the root of the extended process.

Theorem 6.1 allows cryptographic primitives other than hash functions and MACs, provided the assumptions on the equational theory are satisfied. The following corollary states a simple special case for the primitives mentioned in this section. It suffices for treating the system S.

Corollary 6.2. *Suppose that the signature Σ is equipped with the equational theory defined by the equations* (1), (2), (3), (4), (10), *and* (11). *Suppose that C is closed and the name k appears only as first argument of* mac *in C. Then $\nu k.C \approx \nu k.[\![C]\!]$.*

6.3 Constructing Robust Hash Functions

Constructions of hash functions, of the kind described in Section 6.2, typically impose constraints on the use of these functions. For example, some care is needed in order to thwart extension attacks in the definition of MACs. The possibility of such attacks stems from structural flaws in the constructions; details such as the iteration of a compression function are not completely hidden, lead to unwanted additional properties, and can be exploited.

A line of work in cryptography studies safer hash functions with stronger guarantees [Coron et al. 2005]. Although these functions are generally built much as in Section 6.2 by iterating a

compression function, their design conceals their inner structure. The functions thus aim to behave like abstract "random oracles" on inputs of arbitrary length. A notion of indifferentiability captures this goal.

In this section, as a final, more advanced example, we describe one design that strengthens the Merkle-Damgård approach, following [Coron et al. 2005, Section 3.4]. In this example, the attacker is given only indirect access to functions such as the hash function h. We model this restriction by inserting a private name k as the first argument of h. (Cryptographically, the name k may reflect the initial random sampling of h.) We refer to this argument as a key, of sort Key. We use sorts Block for blocks and BlockList for sequences of blocks, defined as lists as in Section 3, with sorts Block and BlockList instead of Data and List, respectively. We use sort Block2 for pairs of blocks, with pair : Block × Block → Block2, fst : Block2 → Block, and snd : Block2 → Block, the abbreviation (x, y) for pair(x, y), and the equations

$$\mathsf{fst}((x, y)) = x \qquad \mathsf{snd}((x, y)) = x \qquad (\mathsf{fst}(x), \mathsf{snd}(x)) = x \tag{12}$$

The third equation of (12) is not present in Section 3; it models that all elements of sort Block2 are pairs. We use sort Block3 for pairs of a Block2 and a Block defined in the same way with overloaded function symbols pair, fst, and snd, and sort Bool for booleans.

We define the hash function h : Key × BlockList → Block by:

$$\mathsf{h}(k, z) = \mathsf{h}_2(k, (0, 0), z) \tag{13}$$

where

$$\mathsf{h}_2(k, x, \mathsf{nil}) = \mathsf{fst}(x) \tag{14}$$

$$\mathsf{h}_2(k, x, y :: z) = \mathsf{h}_2(k, \mathsf{f}(k, (x, y)), z) \tag{15}$$

The function h_2 : Key×Block2×BlockList → Block uses a compression function f : Key×Block3 → Block2. In $\mathsf{h}_2(k, x, z)$, the variable x represents the fixed-size internal state of the hash function and z is the remainder of the input. The internal state starts at $(0, 0)$ and is updated by applications of the compression function f as input blocks are processed. Finally, only the first half of the internal state is returned.

For instance, the hash of a two-block message $M = M_1 :: M_2 :: \mathsf{nil}$ with key k is $\mathsf{h}(k, M) = \mathsf{fst}(\mathsf{f}(k, (\mathsf{f}(k, ((0, 0), M_1)), M_2)))$. More generally, we have

$$\mathsf{h}(k, M_1 :: \ldots :: M_n :: \mathsf{nil}) = \mathsf{fst}(\mathsf{f}(k, (\ldots \mathsf{f}(k, ((0, 0), M_1)) \ldots, M_n)))$$

Indifferentiability requires that the hash function behave like a black box (like a "random oracle"), even in interaction with an adversary that also has access to the underlying compression function. The compression function and the hash function are related, of course. However, as far as the adversary can tell, it is the compression function that may be defined from the hash function (in fact, from an ideal hash function without equations as in Section 3) rather than the other way around. Thus, we express indifferentiability as the equivalence of two systems, each of which provides access to the hash function and the compression function. In the applied pi calculus, one of the systems is:

$$\nu k.(A_h^0 \mid A_f^0)$$

where the processes

$$A_h^0 = !c_h(y).\mathit{if}\ \mathsf{ne\_list}(y) = \mathsf{true}\ \mathit{then}\ \overline{c'_h}\langle \mathsf{h}(k, y)\rangle$$

$$A_f^0 = !c_f(x).\overline{c'_f}\langle \mathsf{f}(k, x)\rangle$$

answer requests to evaluate h and f with key k. We restrict ourselves to hashes of non-empty sequences of blocks. In practice, one never hashes the empty string, because the input of the hash

function is padded to a non-zero multiple of the block length. This restriction is important in this example, because the definition of h yields $\mathsf{h}(k, \mathsf{nil}) = 0$, and this special hash value would break indifferentiability. In order to enforce this restriction, we use symbols $\mathsf{true} : \mathsf{Bool}$ and $\mathsf{ne\_list} : \mathsf{BlockList} \to \mathsf{Bool}$, with equations

$$\mathsf{ne\_list}(x :: \mathsf{nil}) = \mathsf{true} \qquad \mathsf{ne\_list}(x :: y :: z) = \mathsf{ne\_list}(y :: z) \tag{16}$$

The term $\mathsf{ne\_list}(M)$ is equal to true when M is a non-empty list.

The other system offers an analogous interface, for an ideal hash function $\mathsf{h}' : \mathsf{Key} \times \mathsf{BlockList} \to \mathsf{Block}$ and for a stateful compression function built from h':

$$\nu k.(A^1_h \mid A^1_f)$$

The process A^1_h answers requests to evaluate an ideal hash function h':

$$A^1_h = !c_h(y).\mathit{if}\ \mathsf{ne\_list}(y) = \mathsf{true}\ \mathit{then}\ \overline{c'_h}\langle \mathsf{h}'(k, y)\rangle$$

and A^1_f simulates the compression function using h'. The code for A^1_f, which is considerably more intricate, captures the core of the security argument as it might appear in the cryptography literature. (The paper by Coron et al. [2005] omits this argument and, as far as we know, this argument does not appear elsewhere.)

$$\begin{aligned}
A^1_f &= \nu \ell, c_s.(!c_s(s).c_f(x).\overline{\ell}\langle x, s, s\rangle \mid !Q \mid \overline{c_s}\langle((0,0), \mathsf{nil}) :: \mathsf{nil}\rangle)\\
Q &= \ell(x, t, s).\mathit{if}\ t = \mathsf{nil}\ \mathit{then}\ P_0\ \mathit{else}\\
&\quad \mathit{if}\ \mathsf{fst}(\mathsf{hd}(t)) = \mathsf{fst}(x)\ \mathit{then}\ P_1\ \mathit{else}\ \overline{\ell}\langle x, \mathsf{tl}(t), s\rangle\\
P_0 &= \overline{c'_f}\langle \mathsf{f}'(k, x)\rangle \mid \overline{c_s}\langle s\rangle\\
P_1 &= \mathit{let}\ z = \mathsf{snd}(\mathsf{hd}(t))\ \mathit{in}\\
&\quad \mathit{let}\ z' = z +\!\!+ \mathsf{snd}(x)\ \mathit{in}\\
&\quad \mathit{let}\ r = (\mathsf{h}'(k, z'), \mathsf{f}_c(k, z'))\ \mathit{in}\\
&\quad \overline{c'_f}\langle r\rangle \mid \overline{c_s}\langle(r, z') :: s\rangle
\end{aligned}$$

In this definition, $\mathit{let}\ x = M\ \mathit{in}\ P$ is syntactic sugar for $P\{^M/_x\}$, $\ell(x, t, s).P$ is syntactic sugar for $\ell(y).\mathit{let}\ x = \mathsf{fst}(y)\ \mathit{in}\ \mathit{let}\ t = \mathsf{fst}(\mathsf{snd}(y))\ \mathit{in}\ \mathit{let}\ s = \mathsf{snd}(\mathsf{snd}(y))\ \mathit{in}\ P$ where y is a fresh variable, and $\overline{\ell}\langle x, t, s\rangle$ is syntactic sugar for $\overline{\ell}\langle(x, (t, s))\rangle$, with the appropriate sorts and overloading of the function symbols for pairs. The function symbol $\mathsf{f}' : \mathsf{Key} \times \mathsf{Block3} \to \mathsf{Block2}$ represents the compression function outside the domain used for implementing the hash function, and the function symbol $\mathsf{f}_c : \mathsf{Key} \times \mathsf{BlockList} \to \mathsf{Block}$ represents the second projection of the compression function inside that domain. The channel c_s maintains global private state, a lookup table that maps each term $(\mathsf{h}'(k, M), \mathsf{f}_c(k, M))$ with $M = M_1 :: \ldots :: M_n :: \mathsf{nil}$ built as a result of previous compression requests to the term M, and initially maps $(0, 0)$ to nil. This lookup table is represented as a list of pairs. Each table element, of sort Block2Blocks, is a pair of a Block2 and a BlockList; the table, of sort Block2Blocks_List, is a list of Block2Blocks; we overload the function symbols for pairs and lists. Upon a compression request with input x, the process Q looks up $\mathsf{fst}(x)$ in the table: Q receives as input x, the initial state of the table s, and the tail t of the lookup table. It uses a local channel l for encoding the recursive call. The auxiliary processes P_0 and P_1 complete compression requests in the cases where lookups fail and succeed, respectively. When a lookup fails, the compression request is outside the domain used for implementing the hash function, so P_0 answers it using f', and leaves the table unchanged. When a lookup succeeds, we have either $\mathsf{fst}(x) = (\mathsf{h}'(k, M), \mathsf{f}_c(k, M))$ with $M = M_1 :: \ldots :: M_{n-1} :: \mathsf{nil}$ and $n > 0$ or $\mathsf{fst}(x) = (0, 0)$ and we let $M = \mathsf{nil}$. The lookup

yields $z = M$, P_1 computes $z' = z +\!\!+ \mathsf{snd}(x)$ and returns $r = (\mathsf{h}'(k, z'), \mathsf{f}_c(k, z'))$ as result of the compression request. The table is extended by adding the mapping from r to z'.

Let us now explain, informally, why this code ensures that the results of the compression function are consistent with those of hash computations. The result of a compression request with argument x needs to be made consistent with the hash function when

$$x = (\mathsf{f}(k, (\ldots \mathsf{f}(k, ((0, 0), M_1)) \ldots, M_{n-1})), M_n) \tag{17}$$

for some $M_1, \ldots, M_n$ ($n > 0$), because in that case

$$\mathsf{h}(k, M_1 :: \ldots :: M_n :: \mathsf{nil}) = \mathsf{fst}(\mathsf{f}(k, x)) \tag{18}$$

that is, in the system $\nu k.(A_h^0 \mid A_f^0)$, the result of the hash request with argument $M_1 :: \ldots :: M_n :: \mathsf{nil}$ computed by A_h^0 is equal to the first block of the result of the compression request with argument x computed by A_f^0. We need to have an analogous equality in the system $\nu k.(A_h^1 \mid A_f^1)$. In the system $\nu k.(A_h^0 \mid A_f^0)$, equality (17) holds exactly when $\mathsf{fst}(x)$ is the result of previous compression requests $\mathsf{f}(k, (\ldots \mathsf{f}(k, ((0, 0), M_1)) \ldots, M_{n-1}))$ for some $M_1, \ldots, M_{n-1}$. In the system $\nu k.(A_h^1 \mid A_f^1)$, the table lookup tests a corresponding condition and, when it succeeds, P_1 retrieves $z = M_1 :: \ldots :: M_{n-1} :: \mathsf{nil}$, computes $z' = M_1 :: \ldots :: M_{n-1} :: M_n :: \mathsf{nil}$ since $\mathsf{snd}(x) = M_n$, and returns $r = (\mathsf{h}'(k, z'), \mathsf{f}_c(k, z'))$. Hence, $\mathsf{fst}(r) = \mathsf{h}'(k, z')$ and the result of the hash request with argument $z' = M_1 :: \ldots :: M_n :: \mathsf{nil}$ computed by A_h^1 is equal to the first block of the result r of the compression request with argument x computed by A_f^1.

Formally, we obtain the following observational equivalence:

Theorem 6.3. $\nu k.(A_h^0 \mid A_f^0) \approx \nu k.(A_h^1 \mid A_f^1)$.

In the proof of this theorem (which is given in Appendix G), we define a relation $\mathcal{R}$ between configurations of the two systems, and show that $\mathcal{R} \cup \mathcal{R}^{-1}$ is a labelled bisimulation. A key step of this proof consists in proving static equivalence between related configurations; this step formalizes the informal explanation of the process A_1^f given above. We conclude by Theorem 4.8.

7 RELATED WORK

This section aims to position the applied pi calculus with respect to research on process calculi and on the analysis of security protocols. As discussed in Section 1, the applied pi calculus has been the basis for much further work since its initial publication; this section does not discuss many papers that build on the applied pi calculus. (Some of those papers, and others, are mentioned elsewhere in this paper.)

7.1 Process Calculi

The applied pi calculus has many commonalities with the original pi calculus [Milner 1999] and its relatives, such as the spi calculus [Abadi and Gordon 1999] (discussed in Sections 3 and 4). In particular, the model of communication adopted in the applied pi calculus is deliberately classical: communication is through named channels, and value computation is rather separate from communication.

Furthermore, active substitutions are reminiscent of the constraints of the fusion calculus [Victor 1998]. They are especially close to the substitution environments that Boreale et al. employ in their proof techniques for a variant of the spi calculus with a symmetric cryptosystem [Boreale et al. 1999]. We incorporate substitutions into processes, systematize them, and generalize from symmetric cryptosystems to arbitrary operations and equations.

Extensions of the pi calculus are not limited to modelling cryptography: many extensions and variants of the pi calculus have been designed for diverse applications. Examples include calculi for mobility, such as the ambient calculus [Cardelli and Gordon 2000], calculi for modelling biological processes, such as the enhanced pi calculus [Curti et al. 2004], and calculi for service-oriented computing, which model the contracts that services implement, the composition of services, and their protocols [Buscemi and Montanari 2007; Cruz-Filipe et al. 2014; Lapadula et al. 2007; Lucchi and Mazzara 2007]. The psi calculi [Bengtson et al. 2011] provide a general framework parameterized by nominal data types for terms, conditions (generalizing comparison between terms), and assertions (generalizing our notion of frames) and their operational semantics. They also give sufficient conditions on these parameters to ensure that the resulting observational equivalence coincides with labelled bisimilarity. The framework accommodates encodings of the pi calculus and several of its variants, for example ones with fusion [Wischik and Gardner 2004] and concurrent constraints [Buscemi and Montanari 2007]. In particular, Bengtson et al. give an encoding of the applied pi calculus into their framework. However, as they explain, the result of this encoding differs from the applied pi calculus in the way processes interact with contexts. In particular, an important difference is that, when an encoded process sends a ciphertext, the ciphertext appears on the label of the transition, and an agent that receives this message will immediately learn the cleartext and the key. In psi calculi, one can avoid such counter-intuitive disclosures by explicitly creating and using aliases. A recent extension of psi calculi [Borgström et al. 2014, 2016] addresses these difficulties with a new form of pattern matching. In contrast, the management of aliases is built into the applied pi calculus, facilitating the modelling of security-protocol attackers as contexts.

7.2 Analysis of Security Protocols

The analysis of a security protocol generally requires reasoning about its possible executions. However, the ways of talking about the executions and their properties vary greatly. We use a process calculus whose semantics provides a detailed specification for interactions with a context. Because the process calculus has a proper "new" construct (like the pi calculus but unlike CSP), it provides a direct account of the generation of new keys and other fresh quantities. It also enables reasoning with equivalence and implementation relations.

Reasoning with those relations is often more challenging than reasoning about trace properties, but it can be worthwhile. Equivalences are useful, in particular, for modeling privacy properties [Pfitzmann and Köhntopp 2001], for instance in electronic voting [Delaune et al. 2009]. While proofs of equivalences are difficult to automate in general—and observational equivalence is undecidable as noted in Section 4.1—, several tools support certain automatic proofs of equivalences in the applied pi calculus and similar languages: tools have focused on establishing particular kinds of equivalences such as trace equivalence for bounded processes (that is, processes without replication) [Chadha et al. 2012; Cheval et al. 2013; Tiu and Dawson 2010] or for restricted classes of unbounded processes [Chrétien et al. 2015a,b]. Although ProVerif initially focused on proofs of trace properties [Blanchet 2009], it also supports automatic proofs of diff-equivalences, which are equivalences between processes that share the same structure and differ only in the choice of terms [Blanchet et al. 2008]. A diff-equivalence between two processes requires that the two processes reduce in the same way, in the presence of any adversary. In particular, the two processes must have the same branching behaviour. Hence, diff-equivalence is much stronger than observational equivalence. Maude-NPA [Santiago et al. 2014] and Tamarin [Basin et al. 2015] also employ that notion. Baudet [2005; 2007] showed that diff-equivalence is decidable for bounded processes: he treated a class of equivalences that model security against off-line guessing attacks in [Baudet 2005] and proved the full result in [Baudet 2007].

Furthermore, the use of a process calculus permits treating security protocols as programs written in a programming notation—subject to typing [Abadi 1999; Cardelli 2000; Gordon and Jeffrey 2004], to other static analyses [Bodei et al. 1998], and to translations [Abadi 1998; Abadi et al. 1998, 2000]. Thus, language-based approaches have led to tools such as ProVerif where protocols can be described by programs, and analyzed using automated techniques that leverage type systems and Horn clauses [Abadi and Blanchet 2005a].

The applied pi calculus is also convenient as an intermediate language. Translations to ProVerif have been implemented from TulaFale (a language for standardized Web-services protocols) [Bhargavan et al. 2003], from F# [Bhargavan et al. 2008b], and from JavaScript in order to verify protocols, including TLS [Bhargavan et al. 2008a].

As in many other works (e.g., [Abadi and Gordon 1999; Amadio and Lugiez 2000; Armando et al. 2005; Cremers 2008; Dam 1998; DeMillo et al. 1982; Dolev and Yao 1983; Escobar et al. 2006; Kemmerer et al. 1994; Lowe 1996; Merritt 1983; Mitchell et al. 1997; Paulson 1998; Schmidt et al. 2012; Schneider 1996; Thayer Fábrega et al. 1998]), our use of the applied pi calculus conveniently avoids matters of computational complexity and probability. In contrast, other techniques for the analysis of security protocols employ more concrete computational models, where principals are basically Turing machines that manipulate bitstrings, and security depends on the computational limitations of attackers (e.g., [Bellare and Rogaway 1993; Goldwasser and Bellare 1999; Goldwasser and Micali 1984; Goldwasser et al. 1988; Yao 1982]).

Although these two approaches remained rather distinct during the 1980s and 1990s, fruitful connections have now been established (e.g., [Abadi et al. 2009; Abadi and Rogaway 2002; Backes et al. 2009; Barthe et al. 2010; Blanchet 2006; Blanchet and Pointcheval 2006; Comon-Lundh and Cortier 2008; Datta et al. 2005; Lincoln et al. 1998; Pfitzmann et al. 2000]). In particular, some work interprets symbolic proofs in terms of concrete, bitstring-based models [Abadi and Rogaway 2002], in some cases specifically studying the "computational soundness" of the applied pi calculus [Backes et al. 2009; Baudet et al. 2009b; Comon-Lundh and Cortier 2008]. Other work focuses directly on those concrete models but benefits from notations and ideas from process calculi and programming languages. For example, the tool CryptoVerif [Blanchet 2006; Blanchet and Pointcheval 2006] provides guarantees in terms bitstrings, running times, and probabilities, but its input language is strongly reminiscent of the applied pi calculus, which influenced it—rather than of Turing machines.

8 CONCLUSION

In this paper, we describe a uniform extension of the pi calculus, the applied pi calculus, in which messages may be compound values, not just channel names. We study its theory, developing its semantics and proof techniques. Although the calculus has no special, built-in features to deal with security, it has proved useful in the analysis of security protocols.

Famously, the pi calculus is the language of those lively social occasions where all conversations are exchanges of names. The applied pi calculus opens the possibility of more substantial, structured conversations; the cryptic character of some of these conversations can only add to their charm and to their tedium.

The previous paragraph closed the conference paper that introduced the applied pi calculus in 2001. We are now in a better position to assess the possibility to which it refers. As we hoped in 2001, the applied pi calculus has been extensively used for modeling and for reasoning about security protocols, particularly ones that rely heavily on cryptography (and less for ones that rely on simple capabilities). The flexibility of the applied pi calculus is a key enabler for those applications. This flexibility did not render unnecessary or uninteresting the exploration of variants and extensions. However, it did allow the applied pi calculus to remain a relevant core system—it was not displaced by an extended language with many more constructs.

We are also in a better position to comment on matters of charm and tedium, alas. It is debatable whether security protocols have become more charming or more tedious since 2001. It is clear, however, that they play an ever-growing role, and that their security remains problematic. The evolution of TLS exemplifies these points. The literature now contains many attacks on TLS, e.g., [Adrian et al. 2015; Aviram et al. 2016; Bhargavan et al. 2014a; Vanhoef and Piessens 2015], but also several partial specifications and proofs [Cremers et al. 2016; Jager et al. 2012; Krawczyk et al. 2013; Paterson et al. 2011], sometimes relying on the applied pi calculus [Bhargavan et al. 2017a, 2012], and sometimes with language-based methods of the kind that the applied pi calculus started to explore [Bhargavan et al. 2014b, 2013]. The state-of-the-art approaches [Bhargavan et al. 2017a,b, 2014b, 2013; Jager et al. 2012; Krawczyk et al. 2013; Paterson et al. 2011] rely on refined frameworks that consider matters of computational complexity and probability, which are beyond the (explicit) scope of the applied pi calculus, as explained above. In such applications, tools play a helpful role, often an essential one. Although they sometimes lead to important insights, manual proofs—in particular, manual proofs of equivalences—can be rather painful and tedious. (We may have suspected this fact in 2001; on the basis of our experience since then, we now know it with certainty.) On the other hand, the relative ease of use of ProVerif has contributed greatly to the spread of the applied pi calculus. The applied pi calculus has evolved through its implementation in ProVerif, and as a result of its use in this context. That evolution is, in our opinion, an improvement, so the present paper aims to reflect it.

Finally, independently of the merits and the future of the applied pi calculus, we believe that languages, and in particular the formalization of attackers as contexts, should continue to play a role in the analysis of security protocols. The alternatives (defining protocols as interacting Turing machines?) are not easier. Describing protocols in a programming notation not only makes them precise but also brings them into the realm where ideas and tools from programming can support analysis.

SUPPLEMENTARY MATERIAL: PROOFS

The appendix contains proofs and auxiliary definitions needed for these proofs. After introducing the notion of simple contexts and proving Lemma 4.4 (Section A), the bulk of the appendix is devoted to lemmas and definitions that contribute to the proof of Theorem 4.8 (Sections B and C). Section D presents the proof of Lemma 4.10. Section E presents the proofs related to refined labels. Finally, Sections F and G present the proofs related to the two constructions of hash functions given in Sections 6.2 and 6.3 respectively.

A SIMPLE CONTEXTS AND PROOF OF LEMMA 4.4

In order to work with definitions that refer to contexts, such as Definition 4.1, it is convenient to generalize structural equivalence from extended processes to contexts. For this generalization, we use the rules of Section 2.2, except that (1) we do not rename bound names and variables whose scope includes the hole; and (2) in rule New-Par, the hole is considered to contain any name and variable.

Further, in order to avoid special cases in proofs, we often adopt simplifying assumptions on contexts. We say that an evaluation context E is *simple* when (1) no name is both free in E and restricted above the hole; and (2) no variable is both in $dom(E)$ and restricted above the hole. We say that E is *simple for* A if, in addition, it is closing for A. These conditions on scopes exclude, for example, $\overline{a}\langle s\rangle \mid \nu s.(\_)$ and $\{^{s}/_{x}\} \mid \nu x.(\_)$.

Lemma A.1. *Let A be a closed extended process. Given a simple context E for A, there exists a context E' of the form $\nu\widetilde{u}.(\_ \mid B)$ such that $E \equiv E'$ and all subcontexts of E' are simple for A.*

Proof. We construct the context E' from E as follows:

(1) We rename all names and variables bound by restrictions that are not above the hole to distinct fresh names and variables.
(2) We move all restrictions above the hole in E to the root of E. These moves are possible because the names and variables bound by these restrictions do not occur elsewhere: they are not free since E is simple for A and they are not bound by other restrictions by the renaming of step 1.
(3) We reorganize parallel compositions by associativity and commutativity so that the obtained context is of the form $\nu\widetilde{u}.(\_ \mid B)$.

Hence we obtain a context $E' = \nu\widetilde{u}.(\_ \mid B)$ such that $E \equiv E'$ and E' is closing for A, that is, $E'[A]$ is closed.

The subcontexts of E' are $\_$, $\_ \mid B$, and contexts of the form $\nu\widetilde{u}'.(\_ \mid B)$ where $\widetilde{u}'$ is a suffix of $\widetilde{u}$. The contexts $\_$ and $\_ \mid B$ have no names and variables bound in the hole. In the contexts $\nu\widetilde{u}'.(\_ \mid B)$, the names and variables bound in the hole are $\widetilde{u}'$, and they are not free. So all these contexts are simple.

We show that all subcontexts of E' are closing for A. For the context $E'' = \_$, we have $E''[A] = A$ and we know by hypothesis that A is closed. For the other subcontexts, we proceed by removing one by one the elements of $\widetilde{u}'$.

- Suppose that $\nu x.E''[A]$ is closed where $E'' = \nu\widetilde{u}''.(\_ \mid B)$. We have $fv(E''[A]) \setminus dom(E''[A]) = fv(\nu x.E''[A]) \setminus dom(\nu x.E''[A]) = \emptyset$, so $E''[A]$ is closed.
- Suppose that $\nu n.E''[A]$ is closed where $E'' = \nu\widetilde{u}''.(\_ \mid B)$. Then $E''[A]$ is obviously also closed. □

Lemma A.2. *Let A and B be two closed extended processes.*

(1) *Let σ be a bijective renaming. We have $A \approx_s B$ if and only if $A\sigma \approx_s B\sigma$.*
(2) *Let A' and B' be obtained from A and B, respectively, by replacing all variables (including their occurrences in domains of active substitutions) with distinct variables. We have $A \approx_s B$ if and only if $A' \approx_s B'$.*

Proof. To show the first point, suppose that $A \approx_s B$. Hence for all terms M and N, $(M = N)\varphi(A)$ if and only if $(M = N)\varphi(B)$. So $(M\sigma = N\sigma)\varphi(A\sigma)$ if and only if $(M\sigma = N\sigma)\varphi(B\sigma)$, since the equational theory is closed under renaming. So $A\sigma \approx_s B\sigma$. The same argument also shows the converse, via the inverse renaming.

To show the second point, suppose that $A \approx_s B$. Hence for all terms M and N, $(M = N)\varphi(A)$ if and only if $(M = N)\varphi(B)$. We let M' and N' be obtained from M and N, respectively, by the same variable replacement as the one that transforms A and B into A' and B'. So $(M' = N')\varphi(A')$ if and only if $(M' = N')\varphi(B')$, since $M\sigma = M'\sigma'$ where $\varphi(A) \equiv \nu\widetilde{n}.\sigma$ and $\varphi(A') \equiv \nu\widetilde{n}.\sigma'$. So $A' \approx_s B'$. As above, the same argument also shows the converse, via the inverse variable replacement. □

Lemma 4.4. *Let A and B be closed extended processes. If $A \equiv B$ or $A \to B$, then $A \approx_s B$. If $A \approx_s B$, then $E[A] \approx_s E[B]$ for all closing evaluation contexts $E[\_]$.*

Proof. We show that, if $A \equiv B$, then $\varphi(A) \equiv \varphi(B)$, by an easy induction on the derivation of $A \equiv B$. We then show that, if $A \to B$, then $\varphi(A) \equiv \varphi(B)$ since the frame is not affected by Comm, Then, and Else. Since Definition 4.2 considers frames up to structural equivalence, we conclude that static equivalence is invariant by structural equivalence and reduction.

For the context-closure property, we suppose that $A \approx_s B$ and we show that for all closing evaluation contexts E, we have $E[A] \approx_s E[B]$. We first rename the free names and variables of E, so that the obtained context is simple, and apply Lemma A.2. Then by Lemma A.1, we construct

a context E' such that $E \equiv E'$ and all subcontexts of E' are simple. Since static equivalence is invariant by structural equivalence, it is sufficient to show that $E'[A] \approx_s E'[B]$. All subcontexts of E' are closing evaluation contexts, so we proceed by structural induction on E'. The cases of name restriction and variable restriction hold because they restrict the range of M and N in Definition 4.2. In the case of parallel composition, we have $E' \equiv \_ \mid \nu\tilde{n}.(\{^{\widetilde{M'}}/_{\tilde{x}}\} \mid P)$ with $fv(\widetilde{M'}) \cup fv(P) \subseteq dom(A) = dom(B)$ and $\{\tilde{x}\} \cap dom(A) = \emptyset$. By renaming the names $\tilde{n}$ so that they do not occur free in A and B, we have $\varphi(E'[A]) \equiv \nu\tilde{n}.(\varphi(A) \mid \{^{\widetilde{M'}}/_{\tilde{x}}\})$ and $\varphi(E'[B]) \equiv \nu\tilde{n}.(\varphi(B) \mid \{^{\widetilde{M'}}/_{\tilde{x}}\})$. Since we have already handled the case of name restriction, it suffices to consider the closing context $\_ \mid \{^{M'}/_{x}\}$ with $fv(M') \subseteq dom(A)$ and $x \notin dom(A)$. In this case, we apply the inductive hypothesis using $M\{^{M'}/_{x}\}$ and $N\{^{M'}/_{x}\}$ instead of M and N in Definition 4.2. □

B PROOF OF THEOREM 4.8: PARTIAL NORMAL FORMS

Our proof of Theorem 4.8, outlined in Section 4.5, requires a definition of partial normal forms, which we present in Section B.1. A semantics on partial normal forms corresponds to the standard semantics (Section B.2). We can soundly restrict attention to reductions between closed processes in the semantics of partial normal forms (Section B.3). Moreover, partial normal forms enable helpful compositions and decompositions of reductions (Section B.4).

B.1 Definition of Partial Normal Forms

In this section, we define partial normal forms and prove two of their basic properties.

We first define the normalization of the parallel composition of two substitutions. The composition $\sigma \uplus \sigma'$ of two substitutions σ and σ' such that $\sigma \mid \sigma'$ is cycle-free is defined as follows: we reorder $\sigma \mid \sigma'$ into $\{^{M_1}/_{x_1}, \ldots, ^{M_l}/_{x_l}\}$ where $x_i \notin fv(M_j)$ for all $i \leq j \leq l$; we let $\sigma_0 = \mathbf{0}$ and $\sigma_{i+1} = \sigma_i\{^{M_{i+1}}/_{x_{i+1}}\} \mid \{^{M_{i+1}}/_{x_{i+1}}\}$ for $0 \leq i \leq l-1$; then $\sigma \uplus \sigma' = \sigma_l$. By definition, we have $\sigma \uplus \sigma' \equiv \sigma \mid \sigma'$.

The partial normal form $\mathrm{pnf}(A)$ of an extended process A is an extended process of the form $\nu\tilde{n}.(\{^{\widetilde{M}}/_{\tilde{x}}\} \mid P)$ such that $(fv(P) \cup fv(\widetilde{M})) \cap \{\tilde{x}\} = \emptyset$. The sequence of restrictions $\nu\tilde{n}$ may be empty, in which case the partial normal form is written $\{^{\widetilde{M}}/_{\tilde{x}}\} \mid P$. The substitution $\{^{\widetilde{M}}/_{\tilde{x}}\}$ may be empty, in which case it is written $\mathbf{0}$. The partial normal form of A is defined by induction on A as follows:

$$
\begin{aligned}
\mathrm{pnf}(P) &= \mathbf{0} \mid P \\
\mathrm{pnf}(\{^{M}/_{x}\}) &= \{^{M}/_{x}\} \mid \mathbf{0} \\
\mathrm{pnf}(\nu n.A) &= \nu n, \tilde{n}.(\sigma \mid P) \text{ where } \mathrm{pnf}(A) = \nu\tilde{n}.(\sigma \mid P) \text{ and } n \notin \{\tilde{n}\} \\
\mathrm{pnf}(\nu x.A) &= \nu\tilde{n}.(\sigma_{|dom(\sigma)\setminus\{x\}} \mid P) \text{ where } \mathrm{pnf}(A) = \nu\tilde{n}.(\sigma \mid P) \\
\mathrm{pnf}(A \mid B) &= \nu\tilde{n}, \tilde{n}'.(\sigma \uplus \sigma' \mid (P \mid P')(\sigma \uplus \sigma')) \\
&\qquad \text{where } \mathrm{pnf}(A) = \nu\tilde{n}.(\sigma \mid P), \mathrm{pnf}(B) = \nu\tilde{n}'.(\sigma' \mid P') \text{ and } \tilde{n} \text{ and } \tilde{n}' \text{ are} \\
&\qquad \text{renamed so that they are disjoint, the names of } \tilde{n} \text{ are not free in } \sigma' \mid P', \\
&\qquad \text{and the names of } \tilde{n}' \text{ are not free in } \sigma \mid P.
\end{aligned}
$$

The last four cases apply only when the argument of pnf is not a plain process. We define a *normal process* as an extended process in partial normal form.

Two simple lemmas provide some basic properties of partial normal forms.

Lemma B.1. $A \equiv \mathrm{pnf}(A)$.

Proof. By induction on the syntax of A. □

Lemma B.2. *If A is closed, then* $\mathrm{pnf}(A)$ *is closed.*

Proof. We prove by induction on the syntax of A that $fv(\mathrm{pnf}(A)) \subseteq fv(A)$ and $dom(\mathrm{pnf}(A)) = dom(A)$. The result follows. □

B.2 Relation between the Standard Semantics and the Semantics on Partial Normal Forms

In this section, we define an operational semantics on partial normal forms, by defining structural equivalence, internal reduction, and labelled transitions. We relate this semantics to the standard semantics of the applied pi calculus given in Sections 2.2 and 4.3.

We begin with the definition of structural equivalence on partial normal forms. Let $\overset{\diamond}{\equiv}$ be the smallest equivalence relation on plain processes closed by application of evaluation contexts such that

$$
\begin{array}{lrcll}
\textsc{Par-0}' & P \mid \mathbf{0} & \overset{\diamond}{\equiv} & P & \\
\textsc{Par-A}' & P \mid (Q \mid R) & \overset{\diamond}{\equiv} & (P \mid Q) \mid R & \\
\textsc{Par-C}' & P \mid Q & \overset{\diamond}{\equiv} & Q \mid P & \\
\textsc{Repl}' & !P & \overset{\diamond}{\equiv} & P \mid !P & \\
\textsc{New-0}' & \nu n.\mathbf{0} & \overset{\diamond}{\equiv} & \mathbf{0} & \\
\textsc{New-C}' & \nu n.\nu n'.P & \overset{\diamond}{\equiv} & \nu n'.\nu n.P & \\
\textsc{New-Par}' & P \mid \nu n.Q & \overset{\diamond}{\equiv} & \nu n.(P \mid Q) & \text{when } n \notin fn(P) \\
\textsc{Rewrite}' & P\{^M/_x\} & \overset{\diamond}{\equiv} & P\{^N/_x\} & \text{when } \Sigma \vdash M = N
\end{array}
$$

and let $\overset{\circ}{\equiv}$ be the smallest equivalence relation on normal processes such that

$$
\begin{array}{lrcll}
\textsc{Plain}'' & \nu\tilde{n}.(\sigma \mid P) & \overset{\circ}{\equiv} & \nu\tilde{n}.(\sigma \mid P') & \\
 & \multicolumn{3}{l}{\text{when } P \overset{\diamond}{\equiv} P' \text{ and } (fv(P) \cup fv(P')) \cap dom(\sigma) = \emptyset} & \\
\textsc{New-C}'' & \nu\tilde{n}.(\sigma \mid P) & \overset{\circ}{\equiv} & \nu\tilde{n}'.(\sigma \mid P) & \text{when } \tilde{n}' \text{ is a reordering of } \tilde{n} \\
\textsc{New-Par}'' & \nu\tilde{n}.(\sigma \mid \nu n'.P) & \overset{\circ}{\equiv} & \nu\tilde{n}, n'.(\sigma \mid P) & \text{when } n' \notin fn(\sigma) \\
\textsc{Rewrite}'' & \nu\tilde{n}.(\sigma \mid P) & \overset{\circ}{\equiv} & \nu\tilde{n}.(\sigma' \mid P) & \\
 & \multicolumn{3}{l}{\text{when } dom(\sigma) = dom(\sigma')} & \\
 & \multicolumn{3}{l}{\text{and } \Sigma \vdash x\sigma = x\sigma' \text{ for all } x \in dom(\sigma)} & \\
 & \multicolumn{3}{l}{\text{and } (fv(x\sigma) \cup fv(x\sigma')) \cap dom(\sigma) = \emptyset \text{ for all } x \in dom(\sigma)} &
\end{array}
$$

In Plain'' and Rewrite'', the hypotheses on free variables ensure that the process remains normalized in case fresh variables are introduced (respectively, via Rewrite' and by rewriting the substitution σ to σ').

We also introduce the corresponding reduction relation. Let $\rightarrow_\diamond$ be the smallest relation on plain processes closed by $\overset{\diamond}{\equiv}$ and by application of evaluation contexts such that:

$$
\begin{array}{lrcl}
\textsc{Comm}' & \overline{N}\langle M\rangle.P \mid N(x).Q & \rightarrow_\diamond & P \mid Q\{^M/_x\} \\
\textsc{Then}' & \mathit{if}\ M = M\ \mathit{then}\ P\ \mathit{else}\ Q & \rightarrow_\diamond & P \\
\textsc{Else}' & \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q & \rightarrow_\diamond & Q \\
 & \multicolumn{3}{l}{\text{for any ground terms } M \text{ and } N \text{ such that } \Sigma \nvdash M = N}
\end{array}
$$

and let $\rightarrow_\circ$ be the smallest relation on normal processes closed by $\overset{\circ}{\equiv}$ such that $\nu\tilde{n}.(\sigma|P) \rightarrow_\circ \nu\tilde{n}.(\sigma|P')$ when $P \rightarrow_\diamond P'$.

Lemma B.3. (1) *If* $P \overset{\diamond}{\equiv} P'$, *then* $P\sigma \overset{\diamond}{\equiv} P'\sigma$.
(2) *If* $P \rightarrow_\diamond P'$, *then* $P\sigma \rightarrow_\diamond P'\sigma$.

Proof. These properties are immediate by induction on derivations. The proof of Property 2 relies on Property 1 in the case in which one applies $\overset{\diamond}{\equiv}$. Note that the change from Comm to Comm' is crucial for Property 2. □

Lemma B.4. *Assume that* $\nu\tilde{n}.(\sigma \mid P)$, $\nu\tilde{n}'.(\sigma' \mid P')$, *and* $\nu\tilde{n}''.(\sigma'' \mid P'')$ *are normal processes. If* $\nu\tilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\tilde{n}'.(\sigma' \mid P')$, *then*

(1) $\nu\widetilde{n}.(\sigma_{|dom(\sigma)\setminus\{x\}} \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}'.(\sigma'_{|dom(\sigma')\setminus\{x\}} \mid P')$;
(2) $\nu n, \widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu n, \widetilde{n}'.(\sigma' \mid P')$; *and*
(3) *if* $\sigma \mid \sigma''$ *and* $\sigma' \mid \sigma''$ *are cycle-free, then* $\nu\widetilde{n}, \widetilde{n}''.(\sigma \uplus \sigma'' \mid (P \mid P'')(\sigma \uplus \sigma'')) \overset{\circ}{\equiv} \nu\widetilde{n}', \widetilde{n}''.(\sigma' \uplus \sigma'' \mid (P' \mid P'')(\sigma' \uplus \sigma''))$.

If $\nu\widetilde{n}.(\sigma \mid P) \to_\circ \nu\widetilde{n}'.(\sigma' \mid P')$, *then*

(4) $\nu\widetilde{n}.(\sigma_{|dom(\sigma)\setminus\{x\}} \mid P) \to_\circ \nu\widetilde{n}'.(\sigma'_{|dom(\sigma')\setminus\{x\}} \mid P')$;
(5) $\nu n, \widetilde{n}.(\sigma \mid P) \to_\circ \nu n, \widetilde{n}'.(\sigma' \mid P')$; *and*
(6) *if* $\sigma \mid \sigma''$ *and* $\sigma' \mid \sigma''$ *are cycle-free, then* $\nu\widetilde{n}, \widetilde{n}''.(\sigma \uplus \sigma'' \mid (P \mid P'')(\sigma \uplus \sigma'')) \to_\circ \nu\widetilde{n}', \widetilde{n}''.(\sigma' \uplus \sigma'' \mid (P' \mid P'')(\sigma' \uplus \sigma''))$.

Proof. We establish these properties by induction on derivations. To prove Property 3 in the case $\overset{\diamond}{\equiv}$, we use that if $P \overset{\diamond}{\equiv} P'$ then $(P \mid P'')(\sigma \uplus \sigma'') \overset{\diamond}{\equiv} (P' \mid P'')(\sigma \uplus \sigma'')$, which follows from Lemma B.3(1). To prove Properties 4 to 6, we use Properties 1 to 3, respectively, in the case in which we apply $\overset{\circ}{\equiv}$. Additionally, to prove Property 6 in the case $\to_\diamond$, we use that if $P \to_\diamond P'$ then $(P \mid P'')(\sigma \uplus \sigma'') \to_\diamond (P' \mid P'')(\sigma \uplus \sigma'')$, which follows from Lemma B.3(2). □

Lemma B.5. *If* $A \equiv B$, *then* $\mathrm{pnf}(A) \overset{\circ}{\equiv} \mathrm{pnf}(B)$.

Proof. By induction on the derivation of $A \equiv B$. We first notice that, if P and Q are plain processes, then $\mathrm{pnf}(P \mid Q) = \mathbf{0} \mid (P \mid Q)$ can also be obtained by applying the definition of $\mathrm{pnf}(A \mid B)$ for extended processes, with the same result: $\mathrm{pnf}(P) = \mathbf{0} \mid P$, $\mathrm{pnf}(Q) = \mathbf{0} \mid Q$, so $\mathrm{pnf}(P \mid Q) = \mathbf{0} \mid (P \mid Q)$. We use this property to avoid distinguishing whether A, B, C are plain processes or not in the first three cases and in the last case.

- Case $A \equiv A \mid \mathbf{0}$.
  Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$. We have $\mathrm{pnf}(A \mid \mathbf{0}) = \nu\widetilde{n}.(\sigma \mid (P \mid \mathbf{0}))$. Since $P \overset{\diamond}{\equiv} P \mid \mathbf{0}$, we have $\mathrm{pnf}(A) \overset{\circ}{\equiv} \mathrm{pnf}(A \mid \mathbf{0})$.
- Case $A \mid (B \mid C) \equiv (A \mid B) \mid C$.
  Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$, $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' \mid P')$, and $\mathrm{pnf}(C) = \nu\widetilde{n}''.(\sigma'' \mid P'')$. To compute $\mathrm{pnf}(A \mid (B \mid C))$, we first rename $\widetilde{n}'$ and $\widetilde{n}''$ so that they are disjoint, the names of $\widetilde{n}'$ are not free in $\sigma'' \mid P''$, and the names of $\widetilde{n}''$ are not free in $\sigma' \mid P'$. Then $\mathrm{pnf}(B \mid C) = \nu\widetilde{n}', \widetilde{n}''.(\sigma' \uplus \sigma'' \mid (P' \mid P'')(\sigma' \uplus \sigma''))$. Then, we rename $\widetilde{n}$ and $\widetilde{n}', \widetilde{n}''$ so that they are disjoint, the names of $\widetilde{n}$ are not free in $\sigma' \uplus \sigma'' \mid (P' \mid P'')(\sigma' \uplus \sigma'')$, and the names of $\widetilde{n}', \widetilde{n}''$ are not free in $\sigma \mid P$. Hence, $\widetilde{n}, \widetilde{n}'$ and $\widetilde{n}''$ are renamed so that they are disjoint, the names of $\widetilde{n}$ are not free in $\sigma' \mid P'$ and $\sigma'' \mid P''$, the names of $\widetilde{n}'$ are not free in $\sigma \mid P$ and $\sigma'' \mid P''$, and the names of $\widetilde{n}''$ are not free in $\sigma \mid P$ and $\sigma' \mid P'$. This condition is the same as the one obtained when we compute $\mathrm{pnf}((A \mid B) \mid C)$. Let $\sigma_0 = \sigma \uplus (\sigma' \uplus \sigma'') = (\sigma \uplus \sigma') \uplus \sigma''$. So
  $$\begin{aligned}\mathrm{pnf}(A \mid (B \mid C)) &= \nu\widetilde{n}, \widetilde{n}', \widetilde{n}''.(\sigma_0 \mid (P \mid (P' \mid P'')(\sigma' \uplus \sigma''))\sigma_0) \\ &= \nu\widetilde{n}, \widetilde{n}', \widetilde{n}''.(\sigma_0 \mid (P\sigma_0 \mid (P'\sigma_0 \mid P''\sigma_0))) \\ &\overset{\circ}{\equiv} \nu\widetilde{n}, \widetilde{n}', \widetilde{n}''.(\sigma_0 \mid ((P\sigma_0 \mid P'\sigma_0) \mid P''\sigma_0)) = \mathrm{pnf}((A \mid B) \mid C)\end{aligned}$$
  since $P\sigma_0 \mid (P'\sigma_0 \mid P''\sigma_0) \overset{\diamond}{\equiv} (P\sigma_0 \mid P'\sigma_0) \mid P''\sigma_0$.
- Case $A \mid B \equiv B \mid A$.
  Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$ and $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' \mid P')$. We rename $\widetilde{n}$ and $\widetilde{n}'$ so that they are disjoint, the names of $\widetilde{n}$ are not free in $\sigma' \mid P'$, and the names of $\widetilde{n}'$ are not free in $\sigma \mid P$. Let $\sigma_0 = \sigma \uplus \sigma' = \sigma' \uplus \sigma$. We have $\mathrm{pnf}(A \mid B) = \nu\widetilde{n}, \widetilde{n}'.(\sigma_0 \mid (P\sigma_0 \mid P'\sigma_0)) \overset{\circ}{\equiv} \nu\widetilde{n}', \widetilde{n}.(\sigma_0 \mid (P'\sigma_0 \mid P\sigma_0)) = \mathrm{pnf}(B \mid A)$ since $P\sigma_0 \mid P'\sigma_0 \overset{\diamond}{\equiv} P'\sigma_0 \mid P\sigma_0$.
- Case $!P \equiv P \mid !P$.
  We have $\mathrm{pnf}(!P) = \mathbf{0} \mid !P \overset{\circ}{\equiv} \mathbf{0} \mid (P \mid !P) = \mathrm{pnf}(P \mid !P)$, since $!P \overset{\diamond}{\equiv} P \mid !P$.

- Case $\nu n.\mathbf{0} \equiv \mathbf{0}$.
  We have $\mathrm{pnf}(\nu n.\mathbf{0}) = \mathbf{0} \mid \nu n.\mathbf{0} \mathrel{\overset{\circ}{\equiv}} \mathbf{0} \mid \mathbf{0} = \mathrm{pnf}(\mathbf{0})$, since $\nu n.\mathbf{0} \mathrel{\overset{\diamond}{\equiv}} \mathbf{0}$.
- Case $\nu u.\nu v.A \equiv \nu v.\nu u.A$.
  If A is a plain process, then u and v are names. (If u or v were variables, these variables would be in the domain of A, so A would not be a plain process.) In this case, $\mathrm{pnf}(\nu u.\nu v.A) = \mathbf{0} \mid \nu u.\nu v.A \mathrel{\overset{\circ}{\equiv}} \mathbf{0} \mid \nu v.\nu u.A = \mathrm{pnf}(\nu v.\nu u.A)$ since $\nu u.\nu v.A \mathrel{\overset{\diamond}{\equiv}} \nu v.\nu u.A$.
  If A is not a plain process, let $\mathrm{pnf}(A) = \nu \tilde{n}.(\sigma \mid P)$. If u and v are names, then we have $\mathrm{pnf}(\nu u.\nu v.A) = \nu u, v, \tilde{n}.(\sigma \mid P) \mathrel{\overset{\circ}{\equiv}} \nu v, u, \tilde{n}.(\sigma \mid P) = \mathrm{pnf}(\nu v.\nu u.A)$. If u and v are variables, then $\mathrm{pnf}(\nu u.\nu v.A) = \nu \tilde{n}.(\sigma_{|dom(\sigma)\setminus\{u,v\}} \mid P) = \mathrm{pnf}(\nu v.\nu u.A)$. If u is a name and v is a variable, then $\mathrm{pnf}(\nu u.\nu v.A) = \nu u, \tilde{n}.(\sigma_{|dom(\sigma)\setminus\{v\}} \mid P) = \mathrm{pnf}(\nu v.\nu u.A)$. The remaining case is symmetric.
- Case $A \mid \nu u.B \equiv \nu u.(A \mid B)$ with $u \notin fv(A) \cup fn(A)$.
  If B is a plain process, then u is a name and $\mathrm{pnf}(\nu u.B) = \mathbf{0} \mid \nu u.B$.
  - If A is also a plain process, then $\mathrm{pnf}(A \mid \nu u.B) = \mathbf{0} \mid (A \mid \nu u.B) \mathrel{\overset{\circ}{\equiv}} \mathbf{0} \mid \nu u.(A \mid B) = \mathrm{pnf}(\nu u.(A \mid B))$ since $A \mid \nu u.B \mathrel{\overset{\diamond}{\equiv}} \nu u.(A \mid B)$.
  - If A is not a plain process, then let $\mathrm{pnf}(A) = \nu \tilde{n}.(\sigma \mid P)$. We rename $\tilde{n}$ so that the names of $\tilde{n}$ are not free in B and do not contain u. We have $\mathrm{pnf}(A \mid \nu u.B) = \nu \tilde{n}.(\sigma \mid (P \mid \nu u.B)\sigma) \mathrel{\overset{\circ}{\equiv}} \nu \tilde{n}.(\sigma \mid \nu u.(P \mid B)\sigma) \mathrel{\overset{\circ}{\equiv}} \nu u, \tilde{n}.(\sigma \mid (P \mid B)\sigma) = \mathrm{pnf}(\nu u.(A \mid B))$.
  If B is not a plain process, then let $\mathrm{pnf}(B) = \nu \tilde{n}'.(\sigma' \mid P')$. Let $\mathrm{pnf}(A) = \nu \tilde{n}.(\sigma \mid P)$. We rename $\tilde{n}$ and $\tilde{n}'$ so that $\tilde{n}$ and $\tilde{n}'$ are disjoint and do not contain u, the names of $\tilde{n}$ are not free in $\sigma' \mid P'$, and the names of $\tilde{n}'$ are not free in $\sigma \mid P$.
  - If u is a name, then $\mathrm{pnf}(A \mid \nu u.B) = \nu \tilde{n}, u, \tilde{n}'.(\sigma \uplus \sigma' \mid (P \mid P')(\sigma \uplus \sigma')) \mathrel{\overset{\circ}{\equiv}} \nu u, \tilde{n}, \tilde{n}'.(\sigma \uplus \sigma' \mid (P \mid P')(\sigma \uplus \sigma')) = \mathrm{pnf}(\nu u.(A \mid B))$.
  - If u is a variable, then

$$\begin{aligned}\mathrm{pnf}(A \mid \nu u.B) &= \nu \tilde{n}, \tilde{n}'.(\sigma \uplus (\sigma'_{|dom(\sigma')\setminus\{u\}}) \mid (P \mid P')(\sigma \uplus (\sigma'_{|dom(\sigma')\setminus\{u\}})))\\ &= \nu \tilde{n}, \tilde{n}'.((\sigma \uplus \sigma')_{|dom(\sigma\uplus\sigma')\setminus\{u\}} \mid (P \mid P')(\sigma \uplus \sigma'))\\ &= \mathrm{pnf}(\nu u.(A \mid B))\end{aligned}$$

- Case $\nu x.\{^M/_x\} \equiv \mathbf{0}$.
  We have $\mathrm{pnf}(\nu x.\{^M/_x\}) = \mathbf{0} \mid \mathbf{0} = \mathrm{pnf}(\mathbf{0})$.
- Case $\{^M/_x\} \mid A \equiv \{^M/_x\} \mid A\{^M/_x\}$.
  Let $\mathrm{pnf}(A) = \nu \tilde{n}.(\sigma \mid P)$. We rename $\tilde{n}$ so that these names do not occur in M. We have $\mathrm{pnf}(\{^M/_x\} \mid A) = \nu \tilde{n}.((\{^M/_x\} \uplus \sigma) \mid (\mathbf{0} \mid P)(\{^M/_x\} \uplus \sigma))$ and $\mathrm{pnf}(\{^M/_x\} \mid A\{^M/_x\}) = \nu \tilde{n}.((\{^M/_x\} \uplus \sigma\{^M/_x\}) \mid (\mathbf{0} \mid P\{^M/_x\})(\{^M/_x\} \uplus \sigma))$, so $\mathrm{pnf}(\{^M/_x\} \mid A) = \mathrm{pnf}(\{^M/_x\} \mid A\{^M/_x\})$.
- Case $\{^M/_x\} \equiv \{^N/_x\}$ with $\Sigma \vdash M = N$.
  We have $\mathrm{pnf}(\{^M/_x\}) = \{^M/_x\} \mid \mathbf{0} \mathrel{\overset{\circ}{\equiv}} \{^N/_x\} \mid \mathbf{0} = \mathrm{pnf}(\{^N/_x\})$.
- Case $\nu x.A \equiv \nu x.B$ knowing that $A \equiv B$.
  Let $\mathrm{pnf}(A) = \nu \tilde{n}.(\sigma \mid P)$ and $\mathrm{pnf}(B) = \nu \tilde{n}'.(\sigma' \mid P')$. By induction hypothesis, we have $\mathrm{pnf}(A) \mathrel{\overset{\circ}{\equiv}} \mathrm{pnf}(B)$, that is, $\nu \tilde{n}.(\sigma \mid P) \mathrel{\overset{\circ}{\equiv}} \nu \tilde{n}'.(\sigma' \mid P')$. By Lemma B.4(1), we conclude that

$$\mathrm{pnf}(\nu x.A) = \nu \tilde{n}.(\sigma_{|dom(\sigma)\setminus\{x\}} \mid P) \mathrel{\overset{\circ}{\equiv}} \nu \tilde{n}'.(\sigma'_{|dom(\sigma')\setminus\{x\}} \mid P') = \mathrm{pnf}(\nu x.B)$$

- Case $\nu n.A \equiv \nu n.B$ knowing that $A \equiv B$. By induction hypothesis, we have $\mathrm{pnf}(A) \mathrel{\overset{\circ}{\equiv}} \mathrm{pnf}(B)$. If A and B are plain processes, then $\mathrm{pnf}(A) = \mathbf{0} \mid A \mathrel{\overset{\circ}{\equiv}} \mathbf{0} \mid B = \mathrm{pnf}(B)$, so $\nu n.(\mathbf{0} \mid A) \mathrel{\overset{\circ}{\equiv}} \nu n.(\mathbf{0} \mid B)$ by Lemma B.4(2), so

$$\mathrm{pnf}(\nu n.A) = \mathbf{0} \mid \nu n.A \mathrel{\overset{\circ}{\equiv}} \mathbf{0} \mid \nu n.B = \mathrm{pnf}(\nu n.B)$$

  by New-Par″.

If A is a plain process and B is not a plain process, then let $\mathrm{pnf}(B) = \nu\widetilde{n}.(\sigma \mid P)$. We have $\mathrm{pnf}(A) = \mathbf{0} \mid A \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P) = \mathrm{pnf}(B)$, so $\nu n.(\mathbf{0} \mid A) \overset{\circ}{\equiv} \nu n, \widetilde{n}.(\sigma \mid P)$ by Lemma B.4(2), so

$$\mathrm{pnf}(\nu n.A) = \mathbf{0} \mid \nu n.A \overset{\circ}{\equiv} \nu n, \widetilde{n}.(\sigma \mid P) = \mathrm{pnf}(\nu n.B)$$

by New-Par″.
If A and B are not plain processes, then let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$ and $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' \mid P')$. We have

$$\mathrm{pnf}(\nu n.A) = \nu n, \widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu n, \widetilde{n}'.(\sigma' \mid P') = \mathrm{pnf}(\nu n.B)$$

by Lemma B.4(2).

- Case $A \mid A'' \equiv B \mid A''$ knowing that $A \equiv B$.
  Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$, $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' \mid P')$, and $\mathrm{pnf}(A'') = \nu\widetilde{n}''.(\sigma'' \mid P'')$. We rename $\widetilde{n}$, $\widetilde{n}'$, and $\widetilde{n}''$ so that $\widetilde{n}$ and $\widetilde{n}'$ are disjoint from $\widetilde{n}''$, the names of $\widetilde{n}$ and $\widetilde{n}'$ are not free in $\sigma'' \mid P''$, the names of $\widetilde{n}''$ are not free in $\sigma \mid P$ and $\sigma' \mid P'$. By induction hypothesis, we have $\mathrm{pnf}(A) \overset{\circ}{\equiv} \mathrm{pnf}(B)$, that is, $\nu\widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}'.(\sigma' \mid P')$, so

$$\begin{aligned}\mathrm{pnf}(A \mid A'') &= \nu\widetilde{n}, \widetilde{n}''.(\sigma \uplus \sigma'' \mid (P \mid P'')(\sigma \uplus \sigma'')) \\ &\overset{\circ}{\equiv} \nu\widetilde{n}', \widetilde{n}''.(\sigma' \uplus \sigma'' \mid (P' \mid P'')(\sigma' \uplus \sigma'')) = \mathrm{pnf}(B \mid A'')\end{aligned}$$

by Lemma B.4(3). □

Lemma B.6. *If $P \equiv Q$, then $P \overset{\diamond}{\equiv} Q$.*

Proof. By Lemma B.5, $P \equiv Q$ implies $\mathrm{pnf}(P) = \mathbf{0} \mid P \overset{\circ}{\equiv} \mathrm{pnf}(Q) = \mathbf{0} \mid Q$. We show that, if $\mathbf{0}|P \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma|Q)$, then $\sigma = \mathbf{0}$ and $P \overset{\diamond}{\equiv} \nu\widetilde{n}.Q$, by an easy induction on the derivation of $\mathbf{0}|P \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma|Q)$. By applying this result to $\mathbf{0} \mid P \overset{\circ}{\equiv} \mathbf{0} \mid Q$, we obtain $P \overset{\diamond}{\equiv} Q$. □

Lemma B.7. *If $P \overset{\diamond}{\equiv} Q$, then $P \equiv Q$. If $A \overset{\circ}{\equiv} B$, then $A \equiv B$.*

Proof. By induction on the derivations of $P \overset{\diamond}{\equiv} Q$ and of $A \overset{\circ}{\equiv} B$, respectively. □

Lemma B.8. *If $A \to B$, then $\mathrm{pnf}(A) \to_\circ \mathrm{pnf}(B)$.*

Proof. By induction on the derivation of $A \to B$.

- Cases Comm, Then, and Else. A and B are plain processes, so $\mathrm{pnf}(A) = \mathbf{0}|A \to_\circ \mathbf{0}|B = \mathrm{pnf}(B)$, since $A \to_\diamond B$ by Comm′, Then′, and Else′ respectively.
- Case $\nu x.A \to \nu x.B$ knowing $A \to B$. The result follows easily from Lemma B.4(4).
- Case $\nu n.A \to \nu n.B$ knowing $A \to B$. The result follows easily from Lemma B.4(5), by distinguishing cases depending on whether A and B are plain processes or not, as in the proof of Lemma B.5.
- Case $A \mid A'' \to B \mid A''$ knowing $A \to B$. The result follows easily from Lemma B.4(6), as in the proof of Lemma B.5.
- If we apply $\equiv$, the result follows immediately from Lemma B.5. □

Lemma B.9. *If $P \to_\diamond Q$, then $P \to Q$. If $A \to_\circ B$, then $A \to B$.*

Proof. By induction on the derivations of $P \to_\diamond Q$ and $A \to_\circ B$, respectively. In the cases in which we apply $\overset{\diamond}{\equiv}$ or $\overset{\circ}{\equiv}$, we rely on Lemma B.7. □

Similarly, we define restricted labelled transitions. First, for plain processes, we define $P \xrightarrow{\alpha}_\diamond A$ as follows:

$$\textsc{In}' \qquad N(x).P \xrightarrow{N(M)}_\diamond P\{^M/_x\}$$

$$\textsc{Out-Var}' \qquad \frac{x \notin fv(\overline{N}\langle M\rangle.P)}{\overline{N}\langle M\rangle.P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond P \mid \{^M/_x\}}$$

$$\textsc{Scope}' \qquad \frac{P \xrightarrow{\alpha}_\diamond A \qquad n \text{ does not occur in } \alpha}{\nu n.P \xrightarrow{\alpha}_\diamond \nu n.A}$$

$$\textsc{Par}' \qquad \frac{P \xrightarrow{\alpha}_\diamond A \qquad bv(\alpha) \cap fv(Q) = \emptyset}{P \mid Q \xrightarrow{\alpha}_\diamond A \mid Q}$$

$$\textsc{Struct}' \qquad \frac{P \stackrel{\diamond}{\equiv} Q \qquad Q \xrightarrow{\alpha}_\diamond B \qquad B \equiv A}{P \xrightarrow{\alpha}_\diamond A}$$

We define $A \xrightarrow{\alpha}_\circ B$, where A is a normal process and B is an extended process, as follows: there exist $\widetilde{n}$, σ, P, α', B' such that $A \stackrel{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, $P \xrightarrow{\alpha'}_\diamond B'$, $B \equiv \nu\widetilde{n}.(\sigma \mid B')$, $fv(\sigma) \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma = \alpha'$, and the elements of $\widetilde{n}$ do not occur in α.

We give below an alternative formulation of $\xrightarrow{\alpha}_\diamond$.

Lemma B.10. *We have $P \xrightarrow{\alpha}_\diamond A$ if and only if for some $\widetilde{n}$, P_1, P_2, A_1, N, M, P', x, we have $P \stackrel{\diamond}{\equiv} \nu\widetilde{n}.(P_1 \mid P_2)$, $A \equiv \nu\widetilde{n}.(A_1 \mid P_2)$, $\{\widetilde{n}\} \cap fn(\alpha) = \emptyset$, $bv(\alpha) \cap fv(P_1 \mid P_2) = \emptyset$, and one of the following two cases holds:*

(1) *$\alpha = N(M)$, $P_1 = N(x).P'$, and $A_1 = P'\{^M/_x\}$; or*
(2) *$\alpha = \nu x.\overline{N}\langle x\rangle$, $P_1 = \overline{N}\langle M\rangle.P'$, and $A_1 = P' \mid \{^M/_x\}$.*

Proof. For the implication from left to right, we proceed by induction on this derivation of $P \xrightarrow{\alpha}_\diamond A$.

- Case In′: We are in the first case with $P_2 = \mathbf{0}$ and $\widetilde{n} = \emptyset$.
- Case Out-Var′: We are in the second case with $P_2 = \mathbf{0}$ and $\widetilde{n} = \emptyset$.
- Case Scope′: $P \xrightarrow{\alpha}_\diamond A$ has been derived from $Q \xrightarrow{\alpha}_\diamond B$ with $P = \nu n.Q$, $A = \nu n.B$, and n does not occur in α. By induction hypothesis, $Q \stackrel{\diamond}{\equiv} \nu\widetilde{n}'.(Q_1 \mid Q_2)$, $B \equiv \nu\widetilde{n}'.(B_1 \mid Q_2)$, $\{\widetilde{n}'\} \cap fn(\alpha) = \emptyset$ and $bv(\alpha) \cap fv(Q_1 \mid Q_2) = \emptyset$. So $P \stackrel{\diamond}{\equiv} \nu n, \widetilde{n}'.(Q_1 \mid Q_2)$, $A \stackrel{\diamond}{\equiv} \nu n, \widetilde{n}'.(B_1 \mid Q_2)$, $\{n, \widetilde{n}'\} \cap fn(\alpha) = \emptyset$ and $bv(\alpha) \cap fv(Q_1 \mid Q_2) = \emptyset$.
- Case Par′: $P \xrightarrow{\alpha}_\diamond A$ has been derived from $Q \xrightarrow{\alpha}_\diamond B$ with $P = Q \mid Q'$, $A = B \mid Q'$, and $bv(\alpha) \cap fv(Q') = \emptyset$. By induction hypothesis, $Q \stackrel{\diamond}{\equiv} \nu\widetilde{n}'.(Q_1 \mid Q_2)$, $B \equiv \nu\widetilde{n}'.(B_1 \mid Q_2)$, $\{\widetilde{n}'\} \cap fn(\alpha) = \emptyset$ and $bv(\alpha) \cap fv(Q_1 \mid Q_2) = \emptyset$. So $P \stackrel{\diamond}{\equiv} \nu\widetilde{n}'.(Q_1 \mid Q_2) \mid Q' \stackrel{\diamond}{\equiv} \nu\widetilde{n}.(Q_1\{^{\widetilde{n}}/_{\widetilde{n}'}\} \mid (Q_2\{^{\widetilde{n}}/_{\widetilde{n}'}\} \mid Q'))$ and $A \equiv \nu\widetilde{n}'.(B_1 \mid Q_2) \mid Q' \equiv \nu\widetilde{n}.(B_1\{^{\widetilde{n}}/_{\widetilde{n}'}\} \mid (Q_2\{^{\widetilde{n}}/_{\widetilde{n}'}\} \mid Q'))$ where $\widetilde{n}$ consists of fresh names that do not occur in α nor Q'. Let $P_1 = Q_1\{^{\widetilde{n}}/_{\widetilde{n}'}\}$, $P_2 = Q_2\{^{\widetilde{n}}/_{\widetilde{n}'}\} \mid Q'$, and $A_1 = B_1\{^{\widetilde{n}}/_{\widetilde{n}'}\}$. Then $P \stackrel{\diamond}{\equiv} \nu\widetilde{n}.(P_1 \mid P_2)$, $A \equiv \nu\widetilde{n}.(A_1 \mid P_2)$, $\{\widetilde{n}\} \cap fn(\alpha) = \emptyset$, $bv(\alpha) \cap fv(P_1 \mid P_2) = \emptyset$, and the two cases are preserved because the renaming of $\widetilde{n}'$ into $\widetilde{n}$ leaves α unchanged, so in the first case, it leaves N and M unchanged, and just renames inside P', and in the second case, it leaves N unchanged and renames inside M and P'.
- Case Struct′: $P \xrightarrow{\alpha}_\diamond A$ has been derived from $Q \xrightarrow{\alpha}_\diamond B$ with $P \stackrel{\diamond}{\equiv} Q$ and $B \equiv A$. By induction hypothesis, $Q \stackrel{\diamond}{\equiv} \nu\widetilde{n}'.(Q_1 \mid Q_2)$, $B \equiv \nu\widetilde{n}'.(B_1 \mid Q_2)$, $\{\widetilde{n}'\} \cap fn(\alpha) = \emptyset$ and $bv(\alpha) \cap fv(Q_1 \mid Q_2) = \emptyset$. So $P \stackrel{\diamond}{\equiv} \nu\widetilde{n}'.(Q_1 \mid Q_2)$, $A \equiv \nu\widetilde{n}'.(B_1 \mid Q_2)$, $\{\widetilde{n}'\} \cap fn(\alpha) = \emptyset$ and $bv(\alpha) \cap fv(Q_1 \mid Q_2) = \emptyset$.

For the converse implication, we have $P_1 \xrightarrow{\alpha}_\diamond A_1$ by In′ in Case 1 and by Out-Var′ in Case 2. Then $P_1 \mid P_2 \xrightarrow{\alpha}_\diamond A_1 \mid P_2$ by Par′, $\nu\widetilde{n}.(P_1 \mid P_2) \xrightarrow{\alpha}_\diamond \nu\widetilde{n}.(A_1 \mid P_2)$ by Scope′, and $P \xrightarrow{\alpha}_\diamond A$ by Struct′. □

Lemma B.11. *If $P \xrightarrow{\alpha}_\diamond A$ and $fv(\sigma) \cap bv(\alpha) = \emptyset$, then $P\sigma \xrightarrow{\alpha\sigma}_\diamond A\sigma$.*

Proof. By Lemma B.3(1), if $P \overset{\diamond}{\equiv} P'$, then $P\sigma \overset{\diamond}{\equiv} P'\sigma$, We show that, if $A \equiv B$ and $dom(\sigma) \cap dom(A) = \emptyset$, then $A\sigma \equiv B\sigma$, by noticing that $A\sigma \equiv \nu\widetilde{x}.(A \mid \sigma)$ where $\{\widetilde{x}\} = dom(\sigma)$. Then, we use the characterization of Lemma B.10, after renaming the elements of $\widetilde{n}$ so that $\{\widetilde{n}\} \cap fn(\sigma) = \emptyset$. □

Lemma B.12. *If $A \xrightarrow{\alpha} B$, then $\mathrm{pnf}(A) \xrightarrow{\alpha}_\circ B$.*

Proof. By induction on the derivation of $A \xrightarrow{\alpha} B$.

- In all cases in which A is a plain process, we have $\mathrm{pnf}(A) = \mathbf{0} \mid A \xrightarrow{\alpha}_\diamond B$ since, for plain processes, the rules that define $A \xrightarrow{\alpha}_\diamond B$ are the same as those that define $A \xrightarrow{\alpha} B$. So $\mathrm{pnf}(A) = \mathbf{0} \mid A \xrightarrow{\alpha}_\circ B$, with $\alpha' = \alpha$, $\sigma = \mathbf{0}$, and $\widetilde{n} = \emptyset$.
- Case Scope with $u = n$. We have $A' \xrightarrow{\alpha} B'$, n does not occur in α, $A = \nu n.A'$, and $B = \nu n.B'$. By induction hypothesis, $\mathrm{pnf}(A') \xrightarrow{\alpha}_\circ B'$, so $\mathrm{pnf}(A') \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, $P \xrightarrow{\alpha'}_\diamond B''$, $B' \equiv \nu\widetilde{n}.(\sigma \mid B'')$, $fv(\sigma) \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma = \alpha'$, and the elements of $\widetilde{n}$ do not occur in α. So $\mathrm{pnf}(A) \overset{\circ}{\equiv} \nu n, \widetilde{n}.(\sigma \mid P)$ and $B \equiv \nu n, \widetilde{n}.(\sigma \mid B'')$, so $\mathrm{pnf}(A) \xrightarrow{\alpha}_\circ B$.
- Case Scope with $u = x$. We have $A' \xrightarrow{\alpha} B'$, x does not occur in α, $A = \nu x.A'$, and $B = \nu x.B'$. By induction hypothesis, $\mathrm{pnf}(A') \xrightarrow{\alpha}_\circ B'$, so $\mathrm{pnf}(A') \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, $P \xrightarrow{\alpha'}_\diamond B''$, $B' \equiv \nu\widetilde{n}.(\sigma \mid B'')$, $fv(\sigma) \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma = \alpha'$, and the elements of $\widetilde{n}$ do not occur in α. Let $\sigma' = \sigma_{|dom(\sigma)\setminus\{x\}}$. So $\mathrm{pnf}(A) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma' \mid P)$, $P \xrightarrow{\alpha'}_\diamond B''$, $B \equiv \nu\widetilde{n}.(\sigma' \mid B'')$, $fv(\sigma') \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma' = \alpha'$ since x does not occur in α, and the elements of $\widetilde{n}$ do not occur in α, so $\mathrm{pnf}(A) \xrightarrow{\alpha}_\circ B$.
- Case Par. We have $A' \xrightarrow{\alpha} B'$, $bv(\alpha) \cap fv(B_0) = \emptyset$, $A = A' \mid B_0$, and $B = B' \mid B_0$. By induction hypothesis, $\mathrm{pnf}(A') \xrightarrow{\alpha}_\circ B'$, so $\mathrm{pnf}(A') \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, $P \xrightarrow{\alpha'}_\diamond B''$, $B' \equiv \nu\widetilde{n}.(\sigma \mid B'')$, $fv(\sigma) \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma = \alpha'$, and the elements of $\widetilde{n}$ do not occur in α. Let $\mathrm{pnf}(B_0) = \nu\widetilde{n}'.(\sigma' \mid P')$, where $\widetilde{n}$ and $\widetilde{n}'$ are renamed so that they are disjoint, the names of $\widetilde{n}$ are not free in $\sigma' \mid P'$, and the names of $\widetilde{n}'$ are not free in $\sigma \mid P$, in α, nor in B''. Then $\mathrm{pnf}(A) \overset{\circ}{\equiv} \nu\widetilde{n}, \widetilde{n}'.(\sigma \uplus \sigma' \mid (P \mid P')(\sigma \uplus \sigma'))$. By Par′, $P \mid P' \xrightarrow{\alpha'}_\diamond B'' \mid P'$. (We have $bv(\alpha') \cap fv(P') = \emptyset$ because $fv(P') \subseteq fv(\mathrm{pnf}(B_0)) \subseteq fv(B_0)$, $bv(\alpha') = bv(\alpha)$, and $bv(\alpha) \cap fv(B_0) = \emptyset$.) By Lemma B.11, $(P \mid P')(\sigma \uplus \sigma') \xrightarrow{\alpha'(\sigma \uplus \sigma')}_\diamond (B'' \mid P')(\sigma \uplus \sigma')$. (We have $fv(\sigma \uplus \sigma') \cap bv(\alpha') = \emptyset$ because $fv(\sigma) \cap bv(\alpha') = \emptyset$ and $fv(\sigma') \cap bv(\alpha') = \emptyset$.) Moreover,

$$B = B' \mid B_0 \equiv \nu\widetilde{n}.(\sigma \mid B'') \mid \nu\widetilde{n}'.(\sigma' \mid P') \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \uplus \sigma' \mid (B'' \mid P')(\sigma \uplus \sigma'))$$
$$fv(\sigma \uplus \sigma') \cap bv(\alpha'(\sigma \uplus \sigma')) = fv(\sigma \uplus \sigma') \cap bv(\alpha') = \emptyset$$
$$\Sigma \vdash \alpha(\sigma \uplus \sigma') = \alpha\sigma(\sigma \uplus \sigma') = \alpha'(\sigma \uplus \sigma')$$

  and the elements of $\widetilde{n}, \widetilde{n}'$ do not occur in α, so $\mathrm{pnf}(A) \xrightarrow{\alpha}_\circ B$.
- Case Struct. We have $A' \xrightarrow{\alpha} B'$, $A \equiv A'$, and $B \equiv B'$. By induction hypothesis, $\mathrm{pnf}(A') \xrightarrow{\alpha} B'$, so $\mathrm{pnf}(A') \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, $P \xrightarrow{\alpha'}_\diamond B''$, $B' \equiv \nu\widetilde{n}.(\sigma \mid B'')$, $fv(\sigma) \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma = \alpha'$, and the elements of $\widetilde{n}$ do not occur in α. By Lemma B.5, $\mathrm{pnf}(A) \overset{\circ}{\equiv} \mathrm{pnf}(A')$, so $\mathrm{pnf}(A) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, and $B \equiv \nu\widetilde{n}.(\sigma \mid B'')$, hence $\mathrm{pnf}(A) \xrightarrow{\alpha}_\circ B$. □

Lemma B.13. *If $P \xrightarrow{\alpha}_\diamond A$, then $P \xrightarrow{\alpha} A$. If $A \xrightarrow{\alpha}_\circ B$, then $A \xrightarrow{\alpha} B$.*

Proof. The first point is proved by induction on the derivation of $P \xrightarrow{\alpha}_{\diamond} A$. In the case Struct′, we use Lemma B.7.

For the second point, we have $A \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, $P \xrightarrow{\alpha'}_{\diamond} B'$, $B \equiv \nu\widetilde{n}.(\sigma \mid B')$, $fv(\sigma) \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma = \alpha'$, and the elements of $\widetilde{n}$ do not occur in α, for some $\widetilde{n}$, σ, P, α', B'. By Lemma B.10, we have $P \overset{\diamond}{\equiv} \nu\widetilde{n}'.(P_1 \mid P_2)$, $B' \equiv \nu\widetilde{n}'.(B_1 \mid P_2)$, $\{\widetilde{n}'\} \cap fn(\alpha') = \emptyset$, $bv(\alpha') \cap fv(P_1 \mid P_2) = \emptyset$, and one of the following two cases holds:

(1) $\alpha' = N'(M')$, $P_1 = N'(x).P'$, and $B_1 = P'\{^{M'}/_{x}\}$;
(2) $\alpha' = \nu x.\overline{N'}\langle x\rangle$, $P_1 = \overline{N'}\langle M'\rangle.P'$, and $B_1 = P' \mid \{^{M'}/_{x}\}$

for some $\widetilde{n}'$, P_1, P_2, B_1, N', M', P', x. We rename the elements of $\widetilde{n}'$ so that $\{\widetilde{n}'\} \cap fn(\alpha) = \emptyset$.

In Case 1, $\alpha = N(M)$ for some N and M. We have

$$A \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \mid N'(x).P' \mid P_2) \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \mid N(x).P' \mid P_2)$$

using Lemma B.7 and Rewrite, since $\Sigma \vdash N\sigma = N'$. We have

$$B \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \mid P'\{^{M'}/_{x}\} \mid P_2) \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \mid P'\{^{M}/_{x}\} \mid P_2)$$

using Rewrite, since $\Sigma \vdash M\sigma = M'$. Hence, we derive

$$N(x).P' \xrightarrow{\alpha} P'\{^{M}/_{x}\} \qquad \text{by In}$$
$$N(x).P' \mid P_2 \mid \sigma \xrightarrow{\alpha} P'\{^{M}/_{x}\} \mid P_2 \mid \sigma \qquad \text{by Par}$$
$$\nu\widetilde{n}, \widetilde{n}'.(N(x).P' \mid P_2 \mid \sigma) \xrightarrow{\alpha} \nu\widetilde{n}, \widetilde{n}'.(P'\{^{M}/_{x}\} \mid P_2 \mid \sigma) \qquad \text{by Scope}$$
$$A \xrightarrow{\alpha} B \qquad \text{by Struct}$$

To apply Par, we notice that $fv(P_2 \mid \sigma) \cap bv(\alpha) = \emptyset$ since $bv(\alpha) = bv(\alpha')$.

In Case 2, $\alpha = \nu x.\overline{N}\langle x\rangle$ for some N. We have

$$A \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \mid \overline{N'}\langle M'\rangle.P' \mid P_2) \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \mid \overline{N}\langle M'\rangle.P' \mid P_2)$$

using Lemma B.7 and Rewrite, since $\Sigma \vdash N\sigma = N'$. We have

$$B \equiv \nu\widetilde{n}, \widetilde{n}'.(\sigma \mid P' \mid \{^{M'}/_{x}\} \mid P_2)$$

Hence, we derive

$$\overline{N}\langle M'\rangle.P' \xrightarrow{\alpha} P' \mid \{^{M'}/_{x}\} \qquad \text{by Out-Var}$$
$$\overline{N}\langle M'\rangle.P' \mid P_2 \mid \sigma \xrightarrow{\alpha} P' \mid \{^{M'}/_{x}\} \mid P_2 \mid \sigma \qquad \text{by Par}$$
$$\nu\widetilde{n}, \widetilde{n}'.(\overline{N}\langle M'\rangle.P' \mid P_2 \mid \sigma) \xrightarrow{\alpha} \nu\widetilde{n}, \widetilde{n}'.(P' \mid \{^{M'}/_{x}\} \mid P_2 \mid \sigma) \qquad \text{by Scope}$$
$$A \xrightarrow{\alpha} B \qquad \text{by Struct}$$

□

B.3 Restriction to Closed Processes

Next, we show that we can restrict ourselves to reductions between closed processes in the semantics on partial normal forms. Let $\mathcal{R}$ be an inductive relation on processes. We say that a derivation of $\mathcal{R}$ is *closed* when all processes that appear in the derivation are closed, and that a derivation of $\mathcal{R}$ is *closed on the left* when all processes that appear in the derivation before applying $\mathcal{R}$ are closed.

Let A and B be two normal processes. We write $\Sigma \vdash A = B$ when B is obtained from A by replacing some terms M with terms N such that $\Sigma \vdash M = N$. When $\Sigma \vdash P = Q$, we have $P \overset{\diamond}{\equiv} Q$ by (possibly several) applications of Rewrite′. When $\Sigma \vdash A = B$, we have $A \overset{\circ}{\equiv} B$ by (possibly several) applications of Rewrite′ and Rewrite″.

Lemma B.14. (1) *If $P \mathrel{\overset{\diamond}{\equiv}} Q$ and $\Sigma \vdash P = P\sigma$, then $P\sigma \mathrel{\overset{\diamond}{\equiv}} Q\sigma$ and $\Sigma \vdash Q = Q\sigma$.*
(2) *If $A \mathrel{\overset{\circ}{\equiv}} B$ and $\Sigma \vdash A = A\sigma$, then $A\sigma \mathrel{\overset{\circ}{\equiv}} B\sigma$ and $\Sigma \vdash B = B\sigma$.*

Proof. We prove these properties by induction on the derivations. All cases are straightforward. (When $y \in dom(A)$, we consider that $A\{^a/_y\} = A$.) In the cases Rewrite′ and Rewrite″, we use that the equational theory is closed under substitution of terms for variables. In the cases of transitivity of $\mathrel{\overset{\diamond}{\equiv}}$ and $\mathrel{\overset{\circ}{\equiv}}$, we use the induction hypothesis twice. □

Lemma B.15. *If $A \equiv B$, then $A\sigma \equiv B\sigma$.*

Proof. We prove this lemma by induction on the derivation of $A \equiv B$. □

Lemma B.16. *In all the cases below, Y is a set of variables, σ is a substitution from Y to pairwise distinct fresh names.*

(1) *If $P \mathrel{\overset{\diamond}{\equiv}} Q$, $Y \overset{def}{=} fv(P) \cup fv(Q)$, and $\Sigma \vdash P = P\sigma$, then $\Sigma \vdash Q = Q\sigma$ and $P\sigma \mathrel{\overset{\diamond}{\equiv}} Q\sigma$ by a closed derivation.*
(2) *If $P \to_\diamond Q$, $Y \overset{def}{=} fv(P) \cup fv(Q)$, and $\Sigma \vdash P = P\sigma$, then $\Sigma \vdash Q = Q\sigma$ and $P\sigma \to_\diamond Q\sigma$ by a closed derivation.*
(3) *If $P \xrightarrow{\alpha}_\diamond A$, $Y \overset{def}{=} fv(P) \cup fv(\alpha) \cup (fv(A) \setminus dom(A))$, and $\Sigma \vdash P = P\sigma$, then $\Sigma \vdash \alpha = \alpha\sigma$, $A \equiv A\sigma$, and $P\sigma \xrightarrow{\alpha\sigma}_\diamond A\sigma$ by a derivation closed on the left.*
(4) *If $A \mathrel{\overset{\circ}{\equiv}} B$, $Y \overset{def}{=} (fv(A) \setminus dom(A)) \cup (fv(B) \setminus dom(B))$, and $\Sigma \vdash A = A\sigma$, then $\Sigma \vdash B = B\sigma$ and $A\sigma \mathrel{\overset{\circ}{\equiv}} B\sigma$ by a closed derivation.*
(5) *If $A \to_\circ B$, $Y \overset{def}{=} (fv(A) \setminus dom(A)) \cup (fv(B) \setminus dom(B))$, and $\Sigma \vdash A = A\sigma$, then $\Sigma \vdash B = B\sigma$ and $A\sigma \to_\circ B\sigma$ by a closed derivation.*

Proof. We prove the lemma by induction on the derivations. All cases are straightforward. In the cases Rewrite′, Rewrite″, and Else′, we use that the equational theory is closed under substitution of names for variables. In the cases of transitivity of $\mathrel{\overset{\diamond}{\equiv}}$ and $\mathrel{\overset{\circ}{\equiv}}$, we use the induction hypothesis and notice that, if a variable does not occur free in a certain process, then we can substitute it or not without changing the result. We use a similar argument when we apply a structural equivalence step and a reduction step. In the case Struct′, we additionally use Lemma B.15. □

Lemma B.17. (1) *If $P \to_\diamond Q$ and P is closed, then $P \to_\diamond Q$ by a derivation closed on the left.*
(2) *If $A \to_\circ B$ and A is closed, then $A \to_\circ B$ by a derivation closed on the left.*
(3) *If $P \xrightarrow{\alpha}_\diamond A$, P is closed, and α is $\nu x.\overline{N}\langle x\rangle$ or $N(M)$ for some ground term N, then $P \xrightarrow{\alpha}_\diamond A$ by a derivation closed on the left.*
(4) *If $A \xrightarrow{\alpha}_\circ B$, A is closed, and $fv(\alpha) \subseteq dom(A)$, then $A \xrightarrow{\alpha}_\circ B$ by a derivation closed on the left, and the label α' of the transition $P \xrightarrow{\alpha'}_\diamond B'$ used in the definition of $A \xrightarrow{\alpha}_\circ B$ is closed.*

Proof. In the proof below, Y ranges over sets of variables and σ maps Y to pairwise distinct fresh names. The first two properties immediately follow from Lemma B.16. For instance, if $P \to_\diamond Q$ and P is closed, let $Y = fv(P) \cup fv(Q) = fv(Q)$. We have $P = P\sigma$, so a fortiori $\Sigma \vdash P = P\sigma$. By Lemma B.16(2), we have $\Sigma \vdash Q = Q\sigma$ and $P = P\sigma \to_\diamond Q\sigma$ by a closed derivation, so $Q \mathrel{\overset{\diamond}{\equiv}} Q\sigma$. Hence $P \to_\diamond Q$ by a derivation closed on the left.

Property 3: Suppose that $\alpha = N(M)$ where N is a ground term. By Lemma B.10, $P \mathrel{\overset{\diamond}{\equiv}} \nu\tilde{n}.(N(x).P_1 \mid P_2)$, $A \equiv \nu\tilde{n}.(P_1\{^M/_x\} \mid P_2)$, and $\{\tilde{n}\} \cap fn(\alpha) = \emptyset$. Let $Y = fv(N(x).P_1 \mid P_2)$. We rename x so that $x \notin Y$. Since P is closed, $P = P\sigma$, so a fortiori $\Sigma \vdash P = P\sigma$. By Lemma B.16(1), $P = P\sigma \mathrel{\overset{\diamond}{\equiv}} \nu\tilde{n}.(N(x).P_1\sigma \mid P_2\sigma)$ by a closed derivation and $\Sigma \vdash \nu\tilde{n}.(N(x).P_1 \mid P_2) = \nu\tilde{n}.(N(x).P_1\sigma \mid P_2\sigma)$, so $\Sigma \vdash P_1 = P_1\sigma$ and

$\Sigma \vdash P_2 = P_2\sigma$. Hence $A \equiv \nu\widetilde{n}.(P_1\{^M/_x\} \mid P_2) \equiv \nu\widetilde{n}.(P_1\sigma\{^M/_x\} \mid P_2\sigma)$. We derive

$$N(x).P_1\sigma \xrightarrow{N(M)}_\diamond P_1\sigma\{^M/_x\} \qquad \text{by In}'$$

$$N(x).P_1\sigma \mid P_2\sigma \xrightarrow{N(M)}_\diamond P_1\sigma\{^M/_x\} \mid P_2\sigma \qquad \text{by Par}'$$

$$\nu\widetilde{n}.(N(x).P_1\sigma \mid P_2\sigma) \xrightarrow{N(M)}_\diamond \nu\widetilde{n}.(P_1\sigma\{^M/_x\} \mid P_2\sigma) \qquad \text{by Scope}'$$

$$P \xrightarrow{N(M)}_\diamond A \qquad \text{by Struct}'$$

using the previous closed derivation of $P \overset{\diamond}{\equiv} \nu\widetilde{n}.(N(x).P_1\sigma \mid P_2\sigma)$. In the resulting derivation, all intermediate processes before $\xrightarrow{\alpha}_\diamond$ are closed. The case $\alpha = \nu x.\overline{N}\langle x\rangle$ where N is a ground term can be proved in a similar way, or by using Lemma B.16(3) since α is closed.

Property 4: Suppose that A is closed and $A \xrightarrow{\alpha}_\diamond B$. Then $A \overset{\diamond}{\equiv} \nu\widetilde{n}.(\sigma' \mid P)$, $P \xrightarrow{\alpha'}_\diamond B'$, $B \equiv \nu\widetilde{n}.(\sigma' \mid B')$, $fv(\sigma') \cap bv(\alpha') = \emptyset$, $\Sigma \vdash \alpha\sigma' = \alpha'$, and the names $\widetilde{n}$ do not occur in α. Let $Y = (fv(\sigma') \setminus dom(\sigma')) \cup fv(P) \cup fv(\alpha') \cup (fv(B') \setminus dom(B'))$. Then $\Sigma \vdash A = A\sigma$, so by Lemma B.16(4), $A\sigma \overset{\diamond}{\equiv} \nu\widetilde{n}.(\sigma'\sigma \mid P\sigma)$ by a closed derivation and $\Sigma \vdash \nu\widetilde{n}.(\sigma' \mid P) = \nu\widetilde{n}.(\sigma'\sigma \mid P\sigma)$, so $\Sigma \vdash \sigma' = \sigma'\sigma$ and $\Sigma \vdash P = P\sigma$. Hence by Lemma B.16(3), $P\sigma \xrightarrow{\alpha'\sigma}_\diamond B'\sigma$ by a derivation closed on the left, and $B' \equiv B'\sigma$. So $B \equiv \nu\widetilde{n}.(\sigma'\sigma \mid B'\sigma)$, $fv(\sigma'\sigma) \cap bv(\alpha'\sigma) = \emptyset$, $\Sigma \vdash \alpha\sigma'\sigma = \alpha'\sigma$, and the names $\widetilde{n}$ do not occur in α. Hence we obtain the desired derivation using $\alpha'\sigma$ instead of α', $\sigma'\sigma$ instead of σ', $P\sigma$ instead of P, and $B'\sigma$ instead of B'. □

B.4 Decomposition and Composition of Reductions on Partial Normal Forms

The next few lemmas allow us to analyze internal reductions and labelled transitions on partial normal forms. Most of these lemmas describe the possible reductions of a process. Lemma B.20 composes two reductions: if two processes perform labelled transitions, one an output transition and the other an input transition on the same channel, then their parallel composition performs an internal reduction.

Lemma B.18. *Suppose that P_0 is closed, α is $\nu x.\overline{N'}\langle x\rangle$ or $N'(M')$ for some ground term N', and $P_0 \xrightarrow{\alpha}_\diamond A$. Then one of the following cases holds:*

(1) *$P_0 = P \mid Q$ and either $P \xrightarrow{\alpha}_\diamond A'$ and $A \equiv A' \mid Q$, or $Q \xrightarrow{\alpha}_\diamond A'$ and $A \equiv P \mid A'$, for some P, Q, and A';*
(2) *$P_0 = \nu n.P$, $P \xrightarrow{\alpha}_\diamond A'$, and $A \equiv \nu n.A'$ for some P, A', and n that does not occur in α;*
(3) *$P_0 = !P$, $P \xrightarrow{\alpha}_\diamond A'$, and $A \equiv A' \mid !P$ for some P and A';*
(4) *$P_0 = N(x).P$, $\alpha = N'(M')$, $\Sigma \vdash N = N'$, and $A \equiv P\{^{M'}/_x\}$ for some N, x, P, N', and M';*
(5) *$P_0 = \overline{N}\langle M\rangle.P$, $\alpha = \nu x.\overline{N'}\langle x\rangle$, $\Sigma \vdash N = N'$, $x \notin fv(P_0)$, and $A \equiv P \mid \{^M/_x\}$ for some N, M, P, x, and N'.*

Proof. An obvious approach for proving this result is to proceed by induction on the derivation of $P_0 \xrightarrow{\alpha}_\diamond A$. However, the statement is not strong enough to provide an inductive invariant. For instance, in case $P_0 \xrightarrow{\alpha}_\diamond A$ is derived from $P_0 = \nu n.\nu n'.P \overset{\diamond}{\equiv} \nu n'.\nu n.P \xrightarrow{\alpha}_\diamond A$, we can apply the statement to $\nu n'.\nu n.P \xrightarrow{\alpha}_\diamond A$ by induction hypothesis, because $\nu n'.\nu n.P \xrightarrow{\alpha}_\diamond A$ is derived by a derivation smaller than that of $P_0 \xrightarrow{\alpha}_\diamond A$. Hence, we obtain that $\nu n.P \xrightarrow{\alpha}_\diamond A'$ and $A \equiv \nu n'.A'$ for some A'. However, we cannot apply the result to $\nu n.P \xrightarrow{\alpha}_\diamond A'$, because we are not sure that the derivation of $\nu n.P \xrightarrow{\alpha}_\diamond A'$ is smaller than that of $P_0 \xrightarrow{\alpha}_\diamond A$. For this reason, we strengthen the

induction hypothesis as shown below, to make sure that it can be applied to a labelled transition, such as $\nu n.P \xrightarrow{\alpha}_\diamond A'$, obtained by applying the desired result itself.

Let $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ be the greatest property such that $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ holds if and only if one of the following cases holds:

(1) $P_0 = P \mid Q$ and either $P \xrightarrow{\alpha}_\diamond A'$, $Prop(P \xrightarrow{\alpha}_\diamond A')$, and $A \equiv A' \mid Q$, or $Q \xrightarrow{\alpha}_\diamond A'$, $Prop(Q \xrightarrow{\alpha}_\diamond A')$, and $A \equiv P \mid A'$, for some P, Q, and A';
(2) $P_0 = \nu n.P$, $P \xrightarrow{\alpha}_\diamond A'$, $Prop(P \xrightarrow{\alpha}_\diamond A')$, and $A \equiv \nu n.A'$ for some P, A', and n that does not occur in α;
(3) $P_0 = !P$, $P \xrightarrow{\alpha}_\diamond A'$, $Prop(P \xrightarrow{\alpha}_\diamond A')$, and $A \equiv A' \mid !P$ for some P and A';
(4) $P_0 = N(x).P$, $\alpha = N'(M')$, $\Sigma \vdash N = N'$, and $A \equiv P\{^{M'}/_x\}$ for some N, x, P, N', and M';
(5) $P_0 = \overline{N}\langle M\rangle.P$, $\alpha = \nu x.\overline{N'}\langle x\rangle$, $\Sigma \vdash N = N'$, $x \notin fv(P_0)$, and $A \equiv P \mid \{^{M}/_x\}$ for some N, M, P, x, and N'.

Let us show that, if $P_0 \xrightarrow{\alpha}_\diamond A$ is derived by a derivation closed on the left, then $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$, by induction on the derivation of $P_0 \xrightarrow{\alpha}_\diamond A$.

- Case In′. We have $P_0 = N(x).P$, $\alpha = N(M)$, and $A = P\{^{M}/_x\}$, so we are in Case 4 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $N' = N$ and $M' = M$.
- Case Out-Var′. We have $P_0 = \overline{N}\langle M\rangle.P$, $\alpha = \nu x.\overline{N}\langle x\rangle$, $x \notin fv(P_0)$, and $A = P \mid \{^{M}/_x\}$, so we are in Case 5 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $N' = N$.
- Case Scope′. We have $P_0 = \nu n.P$, n does not occur in α, $P \xrightarrow{\alpha}_\diamond A'$, and $A = \nu n.A'$ for some A'. We obtain $Prop(P \xrightarrow{\alpha}_\diamond A')$ by induction hypothesis, so we are in Case 2 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.
- Case Par′. We have $P_0 = P \mid Q$, $P \xrightarrow{\alpha}_\diamond A'$ and $A = A' \mid Q$ for some P, Q, and A'. We obtain $Prop(P \xrightarrow{\alpha}_\diamond A')$ by induction hypothesis, so we are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.
- Case Struct′. We have $P_0 \overset{\diamond}{\equiv} Q_0 \xrightarrow{\alpha}_\diamond B \equiv A$. The case in which $P_0 = Q_0$ is obvious. Let us consider the case in which the structural equivalence $P_0 \overset{\diamond}{\equiv} Q_0$ consists of applying a single structural equivalence step. (The case in which it consists of several steps can be transformed into several applications of Struct′.) The process Q_0 is closed, and by induction hypothesis $Prop(Q_0 \xrightarrow{\alpha}_\diamond B)$. We show that, if $P_0 \overset{\diamond}{\equiv} Q_0 \xrightarrow{\alpha}_\diamond B \equiv A$, $Prop(Q_0 \xrightarrow{\alpha}_\diamond B)$, and all processes in the derivation of $P_0 \overset{\diamond}{\equiv} Q_0$ are closed, then $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$, by induction on the derivation of $P_0 \overset{\diamond}{\equiv} Q_0$.
  - Case $P_0 = Q_0 \mid \mathbf{0} \overset{\diamond}{\equiv} Q_0$. We have $Q_0 \xrightarrow{\alpha}_\diamond B$, $Prop(Q_0 \xrightarrow{\alpha}_\diamond B)$, and $A \equiv B \equiv B \mid \mathbf{0}$, so we are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.
  - Case $P_0 \overset{\diamond}{\equiv} P_0 \mid \mathbf{0}$. Since $Prop(P_0 \mid \mathbf{0} \xrightarrow{\alpha}_\diamond B)$, we have either $P_0 \xrightarrow{\alpha}_\diamond B'$, $Prop(P_0 \xrightarrow{\alpha}_\diamond B')$, and $B \equiv B' \mid \mathbf{0}$ for some B', or $\mathbf{0} \xrightarrow{\alpha}_\diamond B'$, $Prop(\mathbf{0} \xrightarrow{\alpha}_\diamond B')$, and $B \equiv P \mid B'$ for some B'. By definition of $Prop$, $Prop(\mathbf{0} \xrightarrow{\alpha}_\diamond B')$ is impossible, so we are in the first case: $Prop(P_0 \xrightarrow{\alpha}_\diamond B')$ and $A \equiv B'$. Since $Prop(P_0 \xrightarrow{\alpha}_\diamond B')$ is invariant by structural equivalence applied to B', we can then conclude that $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.
  - Case $P_0 = P \mid (Q \mid R) \overset{\diamond}{\equiv} (P \mid Q) \mid R$. We have $Prop((P \mid Q) \mid R \xrightarrow{\alpha}_\diamond B)$, so either $P \mid Q \xrightarrow{\alpha}_\diamond B'$, $Prop(P \mid Q \xrightarrow{\alpha}_\diamond B')$, and $B \equiv B' \mid R$ for some B', or $R \xrightarrow{\alpha}_\diamond B'$, $Prop(R \xrightarrow{\alpha}_\diamond B')$, and $B \equiv (P \mid Q) \mid B'$ for some B'. In the first case, either $P \xrightarrow{\alpha}_\diamond B''$, $Prop(P \xrightarrow{\alpha}_\diamond B'')$, and $B' \equiv B'' \mid Q$ for some B'', or $Q \xrightarrow{\alpha}_\diamond B''$, $Prop(Q \xrightarrow{\alpha}_\diamond B'')$, and $B' \equiv P \mid B''$ for some B''. Consider for instance the last case, in which Q reduces. The other two cases are similar.

Since R is closed, $bv(\alpha) \cap fv(R) = \emptyset$, so by Par′, $Q \mid R \xrightarrow{\alpha}_\diamond B'' \mid R$, $Prop(Q \mid R \xrightarrow{\alpha}_\diamond B'' \mid R)$, and $A \equiv B \equiv B' \mid R \equiv (P \mid B'') \mid R \equiv P \mid (B'' \mid R)$, so we are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.

- Case $P_0 = (P \mid Q) \mid R \overset{\diamond}{\equiv} P \mid (Q \mid R)$. This case is similar to the previous one.
- Case $P_0 = P \mid Q \overset{\diamond}{\equiv} Q \mid P$. (This case is its own symmetric.) This case is immediate, since the desired result is invariant by swapping P and Q.
- Case $P_0 = !P \overset{\diamond}{\equiv} P \mid !P$. Since $Prop(P \mid !P \xrightarrow{\alpha}_\diamond B)$, either $P \xrightarrow{\alpha}_\diamond B'$, $Prop(P \xrightarrow{\alpha}_\diamond B')$, and $B \equiv B' \mid !P$ for some B', or $!P \xrightarrow{\alpha}_\diamond B'$, $Prop(!P \xrightarrow{\alpha}_\diamond B')$, and $B \equiv P \mid B'$ for some B'. In the first case, we are in Case 3 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $A' = B'$. In the second case, since $Prop(!P \xrightarrow{\alpha}_\diamond B')$, we have $P \xrightarrow{\alpha}_\diamond B''$, $Prop(P \xrightarrow{\alpha}_\diamond B'')$, and $B' \equiv B'' \mid !P$ for some B''. Hence, $A \equiv B \equiv B' \mid P \equiv B'' \mid !P$, so we are in Case 3 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $A' = B''$.
- Case $P_0 = P \mid !P \overset{\diamond}{\equiv} !P$. Since $Prop(!P \xrightarrow{\alpha}_\diamond B)$, we have $P \xrightarrow{\alpha}_\diamond B'$, $Prop(P \xrightarrow{\alpha}_\diamond B')$, and $B \equiv B' \mid !P$ for some B'. We are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $A' = B'$.
- Case $P_0 = \nu n.\mathbf{0} \overset{\diamond}{\equiv} \mathbf{0}$. We have that $Prop(\mathbf{0} \xrightarrow{\alpha}_\diamond B)$ is impossible, so this case never happens.
- Case $P_0 = \mathbf{0} \overset{\diamond}{\equiv} \nu n.\mathbf{0}$. Since $Prop(\nu n.\mathbf{0} \xrightarrow{\alpha}_\diamond B)$, we have $Prop(\mathbf{0} \xrightarrow{\alpha}_\diamond B')$, which is impossible, so this case never happens.
- Case $P_0 = \nu n.\nu n'.P \overset{\diamond}{\equiv} \nu n'.\nu n.P$. (This case is its own symmetric.) We rename n and n' so that they do not occur in α. Since $Prop(\nu n'.\nu n.P \xrightarrow{\alpha}_\diamond B)$, we have $\nu n.P \xrightarrow{\alpha}_\diamond B'$, $Prop(\nu n.P \xrightarrow{\alpha}_\diamond B')$, and $B \equiv \nu n'.B'$ for some B', so $P \xrightarrow{\alpha}_\diamond B''$, $Prop(P \xrightarrow{\alpha}_\diamond B'')$, and $B' \equiv \nu n.B''$ for some B''. Hence, by Scope′, $\nu n'.P \xrightarrow{\alpha}_\diamond \nu n'.B''$, $Prop(\nu n'.P \xrightarrow{\alpha}_\diamond \nu n'.B'')$, and we have $A \equiv B \equiv \nu n'.B' \equiv \nu n'.\nu n.B'' \equiv \nu n.\nu n'.B''$, so we are in Case 2 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $A' = \nu n'.B''$.
- Case $P_0 = P \mid \nu n.Q \overset{\diamond}{\equiv} \nu n.(P \mid Q)$ and $n \notin fn(P)$. We rename n so that it does not occur in α. Since $Prop(\nu n.(P \mid Q) \xrightarrow{\alpha}_\diamond B)$, we have $P \mid Q \xrightarrow{\alpha}_\diamond B'$, $Prop(P \mid Q \xrightarrow{\alpha}_\diamond B')$, and $B \equiv \nu n.B'$ for some B', so either $P \xrightarrow{\alpha}_\diamond B''$, $Prop(P \xrightarrow{\alpha}_\diamond B'')$, and $B' \equiv B'' \mid Q$ for some B'', or $Q \xrightarrow{\alpha}_\diamond B''$, $Prop(Q \xrightarrow{\alpha}_\diamond B'')$, and $B' \equiv P \mid B''$ for some B''. In the first case, we rename n in Q so that $n \notin fn(B'')$, hence $A \equiv B \equiv \nu n.B' \equiv \nu n.(B'' \mid Q) \equiv B'' \mid \nu n.Q$, so we are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $A' = B''$. In the second case, by Scope′, $\nu n.Q \xrightarrow{\alpha}_\diamond \nu n.B''$, $Prop(\nu n.Q \xrightarrow{\alpha}_\diamond \nu n.B'')$, and $A \equiv B \equiv \nu n.B' \equiv \nu n.(P \mid B'') \equiv P \mid \nu n.B''$ since $n \notin fn(P)$, so we are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$ with $A' = \nu n.B''$.
- Case $P_0 = \nu n.(P \mid Q) \overset{\diamond}{\equiv} P \mid \nu n.Q$ and $n \notin fn(P)$. This case is fairly similar to the previous one.
- Case $P_0 = P_1\{^M/_x\} \overset{\diamond}{\equiv} P_1\{^N/_x\}$ and $\Sigma \vdash M = N$. (This case is its own symmetric.) We have $Prop(P_1\{^N/_x\} \xrightarrow{\alpha}_\diamond B)$. We show by induction on the syntax of P_1 that, if $Prop(P_1\{^N/_x\} \xrightarrow{\alpha}_\diamond B)$, $\Sigma \vdash M = N$, and $A \equiv B$, then $Prop(P_1\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$.
  - Case $P_1 = P \mid Q$. We have $Prop(P\{^N/_x\} \mid Q\{^N/_x\} \xrightarrow{\alpha}_\diamond B)$, so either $P\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(P\{^N/_x\} \xrightarrow{\alpha}_\diamond B')$, and $B \equiv B' \mid Q\{^N/_x\}$, or $Q\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(Q\{^N/_x\} \xrightarrow{\alpha}_\diamond B')$, and $B \equiv P\{^N/_x\} \mid B'$ for some B'. Hence $P_1\{^M/_x\} = P\{^M/_x\} \mid Q\{^M/_x\}$ and either $P\{^M/_x\} \overset{\diamond}{\equiv} P\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(P\{^M/_x\} \xrightarrow{\alpha}_\diamond B')$ by induction hypothesis, and $A \equiv B \equiv B' \mid Q\{^N/_x\} \equiv B' \mid Q\{^M/_x\}$, or $Q\{^M/_x\} \overset{\diamond}{\equiv} Q\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(Q\{^M/_x\} \xrightarrow{\alpha}_\diamond B')$ by induction hypothesis, and $A \equiv B \equiv P\{^N/_x\} \mid B' \equiv P\{^M/_x\} \mid B'$, so we are in Case 1 of $Prop(P_1\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$.
  - Case $P_1 = \nu n.P$. We rename n so that it does not occur in α. We have $Prop(\nu n.P\{^N/_x\} \xrightarrow{\alpha}_\diamond B)$, so $P\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(P\{^N/_x\} \xrightarrow{\alpha}_\diamond B')$, and $B \equiv \nu n.B'$ for some B'. Hence $P_1\{^M/_x\} =$

$vn.P\{^M/_x\}$, $P\{^M/_x\} \overset{\diamond}{\equiv} P\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(P\{^M/_x\} \xrightarrow{\alpha}_\diamond B')$ by induction hypothesis, and $A \equiv B \equiv vn.B'$, so we are in Case 2 of $Prop(P_1\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$.

- Case $P_1 = !P$. We have $Prop(!P\{^N/_x\} \xrightarrow{\alpha}_\diamond B)$, so $P\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(P\{^N/_x\} \xrightarrow{\alpha}_\diamond B')$, and $B \equiv B' \mid !P\{^N/_x\}$ for some B'. Hence $P_1\{^M/_x\} = !P\{^M/_x\}$, $P\{^M/_x\} \overset{\diamond}{\equiv} P\{^N/_x\} \xrightarrow{\alpha}_\diamond B'$, $Prop(P\{^M/_x\} \xrightarrow{\alpha}_\diamond B')$ by induction hypothesis, and $A \equiv B \equiv B' \mid !P\{^N/_x\} \equiv B' \mid !P\{^M/_x\}$, so we are in Case 3 of $Prop(P_1\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$.
- Case $P_1 = N_1(x_1).P$. We rename x_1 so that $x_1 \neq x$. We have $Prop(N_1\{^N/_x\}(x_1).P\{^N/_x\} \xrightarrow{\alpha}_\diamond B)$, so $\alpha = N'(M')$, $\Sigma \vdash N_1\{^N/_x\} = N'$, and $B \equiv P\{^N/_x\}\{^{M'}/_{x_1}\}$. So $\Sigma \vdash N_1\{^M/_x\} = N'$, and $A \equiv B \equiv P\{^M/_x\}\{^{M'}/_{x_1}\}$, hence $Prop(N_1\{^M/_x\}(x_1).P\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$, so we are in Case 4 of $Prop(P_1\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$.
- Case $P_1 = \overline{N_1}\langle M_1\rangle.P$. We have $Prop(\overline{N_1\{^N/_x\}}\langle M_1\{^N/_x\}\rangle.P\{^N/_x\} \xrightarrow{\alpha}_\diamond B)$, so $\alpha = vx_1.\overline{N'}\langle x_1\rangle$, $\Sigma \vdash N_1\{^N/_x\} = N'$, and $B \equiv P\{^N/_x\} \mid \{^{M_1\{^N/_x\}}/_{x_1}\}$. So $\Sigma \vdash N_1\{^M/_x\} = N'$, and $A \equiv B \equiv P\{^N/_x\} \mid \{^{M_1\{^M/_x\}}/_{x_1}\}$, hence $Prop(\overline{N_1\{^M/_x\}}\langle M_1\{^M/_x\}\rangle.P\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$, so we are in Case 5 of $Prop(P_1\{^M/_x\} \xrightarrow{\alpha}_\diamond A)$.

Using this result, we obtain $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.

- Case $P_0 = P \mid Q \overset{\diamond}{\equiv} P' \mid Q$ knowing $P \overset{\diamond}{\equiv} P'$. Since $Prop(P' \mid Q \xrightarrow{\alpha}_\diamond B)$, either $P' \xrightarrow{\alpha}_\diamond A'$, $Prop(P' \xrightarrow{\alpha}_\diamond A')$, and $A \equiv A' \mid Q$ for some A', or $Q \xrightarrow{\alpha}_\diamond A'$, $Prop(Q \xrightarrow{\alpha}_\diamond A')$, and $A \equiv P' \mid A'$ for some A'. In the first case, by Struct′, $P \xrightarrow{\alpha}_\diamond A'$. By induction hypothesis, $Prop(P \xrightarrow{\alpha}_\diamond A')$. Moreover, $A \equiv A' \mid Q$, so we are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$. In the second case, $Q \xrightarrow{\alpha}_\diamond A'$, $Prop(Q \xrightarrow{\alpha}_\diamond A')$, and $A \equiv P' \mid A' \equiv P \mid A'$, so we are in Case 1 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.
- Case $P_0 = vn.P \overset{\diamond}{\equiv} vn.P'$ knowing $P \overset{\diamond}{\equiv} P'$. We rename n so that it does not occur in α. Since $Prop(vn.P' \xrightarrow{\alpha}_\diamond B)$, we have $P' \xrightarrow{\alpha}_\diamond A'$, $Prop(P' \xrightarrow{\alpha}_\diamond A')$, and $A \equiv vn.A'$ for some A'. By Struct′, $P \xrightarrow{\alpha}_\diamond A'$. By induction hypothesis, $Prop(P \xrightarrow{\alpha}_\diamond A')$. Moreover, $A \equiv vn.A'$, so we are in Case 2 of $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$.

Since P_0 is closed and α is $vx.\overline{N'}\langle x\rangle$ or $N'(M')$ for some ground term N', by Lemma B.17(3), there exists a derivation of $P_0 \xrightarrow{\alpha}_\diamond A$ closed on the left. So by applying the previous result, $Prop(P_0 \xrightarrow{\alpha}_\diamond A)$, which yields the desired property. □

Lemma B.19. *If $v\tilde{n}.(\sigma \mid P)$ is a closed normal process, $v\tilde{n}.(\sigma \mid P) \xrightarrow{\alpha}_\circ A$, $fv(\alpha) \subseteq dom(\sigma)$, and the elements of $\tilde{n}$ do not occur in α, then $P \xrightarrow{\alpha\sigma}_\diamond A'$, $A \equiv v\tilde{n}.(\sigma \mid A')$, and $bv(\alpha) \cap dom(\sigma) = \emptyset$ for some A'.*

Proof. By Lemma B.17(4), we consider a derivation of $v\tilde{n}.(\sigma \mid P) \xrightarrow{\alpha}_\circ A$ closed on the left and the label α' below is closed. By definition of $\xrightarrow{\alpha}_\circ$, we have $v\tilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} v\tilde{n}'.(\sigma' \mid P')$, $P' \xrightarrow{\alpha'}_\diamond B$, $A \equiv v\tilde{n}'.(\sigma' \mid B)$, $\Sigma \vdash \alpha\sigma' = \alpha'$ for some $\tilde{n}'$, σ', P', α', B such that the elements of $\tilde{n}'$ do not occur in α and $fv(\sigma') \cap bv(\alpha') = \emptyset$. By applying Lemma B.10 back and forth, since $P' \xrightarrow{\alpha'}_\diamond B$ and $\Sigma \vdash \alpha\sigma' = \alpha'$, we have $P' \xrightarrow{\alpha\sigma'}_\diamond B$. We proceed by induction on the derivation of $v\tilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} v\tilde{n}'.(\sigma' \mid P')$.

- Base case: $v\tilde{n}.(\sigma \mid P) = v\tilde{n}'.(\sigma' \mid P')$, and the desired result holds.
- Transitivity: the result is proved by applying the induction hypothesis twice.
- Case Plain″: $\tilde{n} = \tilde{n}'$, $\sigma = \sigma'$, $P \overset{\diamond}{\equiv} P' \xrightarrow{\alpha\sigma}_\diamond B$, and $A \equiv v\tilde{n}.(\sigma \mid B)$, so the result holds.
- Case New-C″: $\tilde{n}'$ is a reordering of $\tilde{n}$, $v\tilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} v\tilde{n}'.(\sigma \mid P)$, $P \xrightarrow{\alpha\sigma}_\diamond B$ and $A \equiv v\tilde{n}'.(\sigma \mid B)$, so $P \xrightarrow{\alpha\sigma}_\diamond B$ and $A \equiv v\tilde{n}'.(\sigma \mid B) \equiv v\tilde{n}.(\sigma \mid B)$, hence the result holds.

- Case New-Par'': $P = \nu n'.P'$, $\nu\widetilde{n}.(\sigma \mid \nu n'.P') \overset{\circ}{\equiv} \nu\widetilde{n}, n'.(\sigma \mid P')$, $P' \xrightarrow{\alpha\sigma}_\diamond B$, and $A \equiv \nu\widetilde{n}, n'.(\sigma \mid B)$ where the elements of $\widetilde{n}, n'$ do not occur in α and $n' \notin fn(\sigma)$. By Scope', $P = \nu n'.P' \xrightarrow{\alpha\sigma}_\diamond \nu n'.B$ and by New-Par, $A \equiv \nu\widetilde{n}, n'.(\sigma \mid B) \equiv \nu\widetilde{n}.(\sigma \mid \nu n'.B)$, hence the result holds.
- Case New-Par'' reversed: $\nu\widetilde{n}, n'.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid \nu n'.P)$, $\nu n'.P \xrightarrow{\alpha\sigma}_\diamond B$ and $A \equiv \nu\widetilde{n}.(\sigma \mid B)$ where the elements of $\widetilde{n}$ do not occur in α and $n' \notin fn(\sigma)$. We rename n' so that it does not occur in α. By Lemma B.18, $P \xrightarrow{\alpha\sigma}_\diamond B'$ and $B \equiv \nu n'.B'$ for some B'. Hence, $P \xrightarrow{\alpha\sigma}_\diamond B'$ and $A \equiv \nu\widetilde{n}.(\sigma \mid B) \equiv \nu\widetilde{n}.(\sigma \mid \nu n'.B') \equiv \nu\widetilde{n}, n'.(\sigma \mid B')$ by New-Par, so the result holds.
- Case Rewrite'': $\nu\widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma' \mid P)$, $P \xrightarrow{\alpha\sigma'}_\diamond B$ and $A \equiv \nu\widetilde{n}.(\sigma' \mid B)$ where $dom(\sigma) = dom(\sigma')$, $\Sigma \vdash x\sigma = x\sigma'$ for all $x \in dom(\sigma)$, and $(fv(x\sigma) \cup fv(x\sigma')) \cap dom(\sigma) = \emptyset$ for all $x \in dom(\sigma)$. Hence $P \xrightarrow{\alpha\sigma'}_\diamond B$ and $\Sigma \vdash \alpha\sigma = \alpha\sigma'$, so by applying Lemma B.10 back and forth, $P \xrightarrow{\alpha\sigma}_\diamond B$. Moreover, $A \equiv \nu\widetilde{n}.(\sigma' \mid B) \equiv \nu\widetilde{n}.(\sigma \mid B)$ by several applications of Rewrite, so the result holds. □

Lemma B.20. *If P and Q are closed processes, N is a ground term, $P \xrightarrow{N(x)}_\diamond A$, and $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, then $P \mid Q \to_\diamond R$ and $R \equiv \nu x.(A \mid B)$ for some R.*

Proof. By Lemma B.10, we have $P \overset{\diamond}{\equiv} \nu\widetilde{n}.(N(y).P_1 \mid P_2)$, $A \equiv \nu\widetilde{n}.(P_1\{^x/_y\} \mid P_2)$, $\{\widetilde{n}\} \cap fn(N) = \emptyset$ and $Q \overset{\diamond}{\equiv} \nu\widetilde{n}'.(\overline{N}\langle M\rangle.Q_1 \mid Q_2)$, $B \equiv \nu\widetilde{n}'.(Q_1 \mid \{^M/_x\} \mid Q_2)$, $\{\widetilde{n}'\} \cap fn(N) = \emptyset$, $x \notin fv(\overline{N}\langle M\rangle.Q_1 \mid Q_2)$.

Let $Y = fv(\nu\widetilde{n}.(N(y).P_1 \mid P_2))$. We rename y so that $y \notin Y$. Let σ be a substitution from Y to pairwise distinct fresh names. Since P and N are closed, $P\sigma = P$ and $N\sigma = N$, so a fortiori $\Sigma \vdash P = P\sigma$. By Lemma B.16(1), $P \overset{\diamond}{\equiv} \nu\widetilde{n}.(N(y).P_1\sigma \mid P_2\sigma)$ and $\Sigma \vdash \nu\widetilde{n}.(N(y).P_1 \mid P_2) = \nu\widetilde{n}.(N(y).P_1\sigma \mid P_2\sigma)$. Then, by introducing fresh names $\widetilde{n}_1, \widetilde{n}_1'$,

$$\begin{aligned} P \mid Q \overset{\diamond}{\equiv}\ & \nu\widetilde{n}_1, \widetilde{n}_1'.(N(y).P_1\sigma\{^{\widetilde{n}_1}/_{\widetilde{n}}\} \mid P_2\sigma\{^{\widetilde{n}_1}/_{\widetilde{n}}\} \\ & \mid \overline{N}\langle M\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}\rangle.Q_1\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\} \mid Q_2\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}) \\ \to_\diamond\ & \nu\widetilde{n}_1, \widetilde{n}_1'.(P_1\sigma\{^{\widetilde{n}_1}/_{\widetilde{n}}\}\{^{M\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}}/_y\} \mid P_2\sigma\{^{\widetilde{n}_1}/_{\widetilde{n}}\} \mid Q_1\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\} \mid Q_2\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}) = R \end{aligned}$$

and

$$\begin{aligned} \nu x.(A \mid B) \equiv\ & \nu x, \widetilde{n}_1, \widetilde{n}_1'.(P_1\sigma\{^x/_y, ^{\widetilde{n}_1}/_{\widetilde{n}}\} \mid P_2\sigma\{^{\widetilde{n}_1}/_{\widetilde{n}}\} \\ & \mid Q_1\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\} \mid \{^{M\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}}/_x\} \mid Q_2\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}) \\ \equiv\ & \nu\widetilde{n}_1, \widetilde{n}_1'.(P_1\sigma\{^{M\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}}/_y, ^{\widetilde{n}_1}/_{\widetilde{n}}\} \mid P_2\sigma\{^{\widetilde{n}_1}/_{\widetilde{n}}\} \mid Q_1\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\} \mid Q_2\{^{\widetilde{n}_1'}/_{\widetilde{n}'}\}) \\ \equiv\ & R \end{aligned}$$

because x is not free in $P_1\sigma$, $P_2\sigma$, Q_1, Q_2, since $\nu\widetilde{n}.(N(y).P_1\sigma \mid P_2\sigma)$ is closed and $x \notin fv(\overline{N}\langle M\rangle.Q_1 \mid Q_2)$. □

Lemma B.21. *Suppose that P_0 is a closed process and $P_0 \to_\diamond R$. Then one of the following cases holds:*

(1) *$P_0 = P \mid Q$ for some P and Q, and one of the following cases holds:*
   (a) *$P \to_\diamond P'$ and $R \equiv P' \mid Q$ for some P',*
   (b) *$P \xrightarrow{N(x)}_\diamond A$, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N, and two symmetric cases obtained by swapping P and Q;*
(2) *$P_0 = \nu n.P$, $P \to_\diamond Q'$, and $R \equiv \nu n.Q'$ for some n, P, and Q';*
(3) *$P_0 = !P$, $P \mid P \to_\diamond Q'$, and $R \equiv Q' \mid !P$ for some P and Q';*
(4) *$P_0 =$ if $M = N$ then P else Q and either $\Sigma \vdash M = N$ and $R \equiv P$, or $\Sigma \vdash M \neq N$ and $R \equiv Q$, for some M, N, P, and Q.*

Proof. We proceed similarly to Lemma B.18. Let $Prop(P_0 \to_\diamond R_0)$ be the greatest property such that $Prop(P_0 \to_\diamond R_0)$ holds if and only if one of the following cases holds:

(1) $P_0 = P \mid Q$ for some P and Q, and one of the following cases holds:
   (a) $P \to_\diamond P'$, $Prop(P \to_\diamond P')$, and $R_0 \equiv P' \mid Q$ for some P',
   (b) $P \xrightarrow{N(x)}_\diamond A$, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_0 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N, and two symmetric cases obtained by swapping P and Q, named (a') and (b') respectively;
(2) $P_0 = \nu n.P$, $P \to_\diamond Q'$, $Prop(P \to_\diamond Q')$, and $R_0 \equiv \nu n.Q'$ for some n, P, and Q';
(3) $P_0 = !P$, $P \mid P \to_\diamond Q'$, $Prop(P \mid P \to_\diamond Q')$, and $R_0 \equiv Q' \mid !P$ for some P and Q';
(4) $P_0 = \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$ and either $\Sigma \vdash M = N$ and $R_0 \equiv P$, or $\Sigma \vdash M \neq N$ and $R_0 \equiv Q$, for some M, N, P, and Q.

Let us show that, if $P_0 \to_\diamond R_0$ is derived by a derivation closed on the left, then $Prop(P_0 \to_\diamond R_0)$, by induction on the derivation of $P_0 \to_\diamond R_0$.

- In the case Comm′, $P_0 = P \mid Q$ where $P = \overline{N}\langle M\rangle.P'$, $Q = N(x).Q'$, and $R_0 = P' \mid Q'\{^M/_x\}$, so by choosing a fresh variable y, $P \xrightarrow{\nu y.\overline{N}\langle y\rangle}_\diamond P' \mid \{^M/_y\}$, $Q \xrightarrow{N(y)}_\diamond Q'\{^y/_x\}$, $\nu y.(P' \mid \{^M/_y\} \mid Q'\{^y/_x\}) \equiv P' \mid Q'\{^M/_x\} \equiv R_0$, and N is ground since P_0 is closed. Therefore, we are in Case 1.(a') of $Prop(P_0 \to_\diamond R_0)$.
- In the case Then′, we are in Case 4 of $Prop(P_0 \to_\diamond R_0)$ with $P_0 = \mathit{if}\ M = M\ \mathit{then}\ P\ \mathit{else}\ Q$ and $R_0 = P$.
- In the case Else′, we are in Case 4 of $Prop(P_0 \to_\diamond R_0)$ with $P_0 = \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$, $\Sigma \vdash M \neq N$, and $R_0 = Q$.
- If we apply a reduction under an evaluation context E, then $P_0 = E[P] \to_\diamond E[P'] = R_0$ is derived from $P \to_\diamond P'$. By induction hypothesis, we have $Prop(P \to_\diamond P')$, and we are in Case 1.(a) (respectively, 1.(a') or 2) of $Prop(P_0 \to_\diamond R_0)$ when E is $E' \mid Q$ (respectively, $Q \mid E'$ or $\nu n.E'$).
- Finally, suppose that we use $\overset{\diamond}{\equiv}$. We have $P_0 \overset{\diamond}{\equiv} Q_0 \to_\diamond R_1 \overset{\diamond}{\equiv} R_0$. The case in which $P_0 = Q_0$ is obvious by induction, since $R_1 \overset{\diamond}{\equiv} R_0$ implies $R_1 \equiv R_0$ by Lemma B.7. Let us consider the case in which the structural equivalence $P_0 \overset{\diamond}{\equiv} Q_0$ consists of applying a single structural equivalence step. (The case in which it consists of several steps can be transformed into several applications of the rule.) The process Q_0 is closed, and by induction hypothesis $Prop(Q_0 \to_\diamond R_1)$. We show that, if $P_0 \overset{\diamond}{\equiv} Q_0 \to_\diamond R_1 \equiv R_0$, $Prop(Q_0 \to_\diamond R_1)$, and all processes in the derivation of $P_0 \overset{\diamond}{\equiv} Q_0$ are closed, then $Prop(P_0 \to_\diamond R_0)$, by induction on the derivation of $P_0 \overset{\diamond}{\equiv} Q_0$.
  - Case $P_0 = Q_0 \mid \mathbf{0} \overset{\diamond}{\equiv} Q_0$. We have $Q_0 \to_\diamond R_1$, $Prop(Q_0 \to_\diamond R_1)$, and $R_0 \equiv R_1 \equiv R_1 \mid \mathbf{0}$, so we are in Case 1.(a) of $Prop(P_0 \to_\diamond R_0)$.
  - Case $P_0 \overset{\diamond}{\equiv} P_0 \mid \mathbf{0}$. Since $Prop(P_0 \mid \mathbf{0} \to_\diamond R_1)$, we have either
    (1) $P_0 \to_\diamond R'$, $Prop(P_0 \to_\diamond R')$, and $R_1 \equiv R' \mid \mathbf{0}$ for some R';
    (2) $P_0 \xrightarrow{N(x)}_\diamond A$, $\mathbf{0} \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$;
    (3) $P_0 \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A$, $\mathbf{0} \xrightarrow{N(x)}_\diamond B$; or
    (4) $\mathbf{0} \to_\diamond R'$, $Prop(\mathbf{0} \to_\diamond R')$, and $R_1 \equiv P_0 \mid R'$ for some R'.
    Cases 2 and 3 are impossible by Lemma B.18. Case 4 is impossible since, by definition of $Prop$, $Prop(\mathbf{0} \to_\diamond R')$ does not hold. So we are in the first case: $Prop(P_0 \to_\diamond R')$ and $R_0 \equiv R_1 \equiv R' \mid \mathbf{0} \equiv R'$. Since $Prop(P_0 \to_\diamond R')$ is invariant by structural equivalence applied to R', we can then conclude that $Prop(P_0 \to_\diamond R_0)$.
  - Case $P_0 = P \mid (Q \mid R) \overset{\diamond}{\equiv} (P \mid Q) \mid R$. Since $Prop((P \mid Q) \mid R \to_\diamond R_1)$, we have four cases:
    * $P \mid Q \to_\diamond R_2$, $Prop(P \mid Q \to_\diamond R_2)$, and $R_1 \equiv R_2 \mid R$. We have again four cases.

· $P \to_\diamond R_3$, $Prop(P \to_\diamond R_3)$, and $R_2 \equiv R_3|Q$. Then $R_0 \equiv R_1 \equiv R_2|R \equiv (R_3|Q)|R \equiv R_3|(Q|R)$, so we are in Case 1.(a) of $Prop(P_0 \to_\diamond R_0)$.

· $Q \to_\diamond R_3$, $Prop(Q \to_\diamond R_3)$, and $R_2 \equiv P \mid R_3$. Then $Q \mid R \to_\diamond R_3 \mid R$, $Prop(Q \mid R \to_\diamond R_3 \mid R)$, and $R_0 \equiv R_1 \equiv R_2|R \equiv (P|R_3)|R \equiv P|(R_3|R)$, so we are in Case 1.(a') of $Prop(P_0 \to_\diamond R_0)$.

· $P \xrightarrow{N(x)}_\diamond A$, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_2 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. Then $Q|R \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B|R$ by Par′, and $R_0 \equiv R_1 \equiv R_2|R \equiv \nu x.(A|B)|R \equiv \nu x.(A|(B|R))$ since $x \notin fv(R)$, so we are in Case 1.(b) of $Prop(P_0 \to_\diamond R_0)$.

· $P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A$, $Q \xrightarrow{N(x)}_\diamond B$, and $R_2 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. This case can be handled similarly to the previous one.

∗ $R \to_\diamond R_2$, $Prop(R \to_\diamond R_2)$, and $R_1 \equiv (P|Q)|R_2$. Then $Q|R \to_\diamond Q|R_2$, $Prop(Q|R \to_\diamond Q|R_2)$, and $R_0 \equiv R_1 \equiv (P \mid Q) \mid R_2 \equiv P \mid (Q \mid R_2)$, so we are in Case 1.(a') of $Prop(P_0 \to_\diamond R_0)$.

∗ $P|Q \xrightarrow{N(x)}_\diamond A$, $R \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_1 \equiv \nu x.(A|B)$ for some A, B, x, and ground term N. By Lemma B.18, either $P \xrightarrow{N(x)}_\diamond A'$ and $A \equiv A' \mid Q$ for some A', or $Q \xrightarrow{N(x)}_\diamond A'$ and $A \equiv P \mid A'$ for some A'. In the first case, $P \xrightarrow{N(x)}_\diamond A'$, $Q \mid R \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond Q \mid B$ by Par′ and Struct′, and $R_0 \equiv R_1 \equiv \nu x.(A \mid B) \equiv \nu x.((A' \mid Q) \mid B) \equiv \nu x.(A' \mid (Q \mid B))$, so we are in Case 1.(b) of $Prop(P_0 \to_\diamond R_0)$. In the second case, by Lemma B.20, $Q \mid R \to_\diamond R_2$ and $R_2 \equiv \nu x.(A' \mid B)$ for some R_2, $Prop(Q \mid R \to_\diamond R_2)$, and $R_0 \equiv R_1 \equiv \nu x.(A|B) \equiv \nu x.((P|A') \mid B) \equiv P \mid \nu x.(A' \mid B) \equiv P \mid R_2$ since $x \notin fv(P)$. Hence, we are in Case 1.(a') of $Prop(P_0 \to_\diamond R_0)$.

∗ $P \mid Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A$, $R \xrightarrow{N(x)}_\diamond B$, and $R_1 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. This case can be handled similarly to the previous one.

– Case $P_0 = (P \mid Q) \mid R \overset{\diamond}{\equiv} P \mid (Q \mid R)$. This case is similar to the previous one.

– Case $P_0 = P \mid Q \overset{\diamond}{\equiv} Q \mid P$. (This case is its own symmetric.) This case is immediate, since the desired result is invariant by swapping P and Q.

– Case $P_0 = !P \overset{\diamond}{\equiv} P \mid !P$. Since $Prop(P \mid !P \to_\diamond R_1)$, we have four cases:

∗ $P \to_\diamond P'$, $Prop(P \to_\diamond P')$, and $R_1 \equiv P' \mid !P$ for some P'. Hence $P|P \to_\diamond P'|P$, $Prop(P|P \to_\diamond P' \mid P)$, and $R_0 \equiv R_1 \equiv P' \mid !P \equiv (P' \mid P) \mid !P$, so we are in Case 3 of $Prop(P_0 \to_\diamond R_0)$.

∗ $!P \to_\diamond P'$, $Prop(!P \to_\diamond P')$, and $R_1 \equiv P \mid P'$ for some P'. Since $Prop(!P \to_\diamond P')$, we have $P \mid P \to_\diamond R_2$, $Prop(P \mid P \to_\diamond R_2)$, and $P' \equiv R_2 \mid !P$. So $P \mid P \to_\diamond R_2$, $Prop(P \mid P \to_\diamond R_2)$, and $R_0 \equiv R_1 \equiv P \mid P' \equiv P \mid (R_2 \mid !P) \equiv R_2 \mid !P$, so we are in Case 3 of $Prop(P_0 \to_\diamond R_0)$.

∗ $P \xrightarrow{N(x)}_\diamond A$, $!P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_1 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. By Lemma B.18, $P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B'$ and $B \equiv B' \mid !P$ for some B'. So by Lemma B.20, $P \mid P \to_\diamond R_2$ and $R_2 \equiv \nu x.(A \mid B')$ for some R_2. So $P \mid P \to_\diamond R_2$, $Prop(P \mid P \to_\diamond R_2)$, and $R_0 \equiv R_1 \equiv \nu x.(A \mid B) \equiv \nu x.(A \mid (B' \mid !P)) \equiv \nu x.(A \mid B') \mid !P \equiv R_2 \mid !P$ since $x \notin fv(!P)$. So we are in Case 3 of $Prop(P_0 \to_\diamond R_0)$.

∗ $P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A$, $!P \xrightarrow{N(x)}_\diamond B$, and $R_1 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. This case can be handled similarly to the previous one.

– Case $P_0 = P \mid !P \overset{\diamond}{\equiv} !P$. Since $Prop(!P \to_\diamond R_1)$, we have $P \mid P \to_\diamond R_2$, $Prop(P \mid P \to_\diamond R_2)$, and $R_1 \equiv R_2 \mid !P$ for some R_2. Since $Prop(P \mid P \to_\diamond R_2)$, we have four cases, which reduce to two by symmetry:

∗ $P \to_\diamond P'$, $Prop(P \to_\diamond P')$, and $R_2 \equiv P' \mid P$ for some P'. Hence $R_0 \equiv R_1 \equiv R_2 \mid !P \equiv P' \mid P \mid !P \equiv P' \mid !P$, so we are in Case 1.(a) of $Prop(P_0 \to_\diamond R_0)$.

* $P \xrightarrow{N(x)}_\diamond A$, $P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_2 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. Hence $P \xrightarrow{N(x)}_\diamond A$, $!P \overset{\diamond}{\equiv} P \mid !P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B \mid !P$ by Par′ so $!P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B \mid !P$ by Struct′, and $R_0 \equiv R_1 \equiv R_2 \mid !P \equiv \nu x.(A \mid B) \mid !P \equiv \nu x.(A \mid (B \mid !P))$ since $x \notin fv(!P)$. So we are in Case 1.(b) of $Prop(P_0 \to_\diamond R_0)$.

– Case $P_0 = \nu n.\mathbf{0} \overset{\diamond}{\equiv} \mathbf{0}$. We have that $Prop(\mathbf{0} \to_\diamond R_1)$ is impossible, so this case never happens.
– Case $P_0 = \mathbf{0} \overset{\diamond}{\equiv} \nu n.\mathbf{0}$. Since $Prop(\nu n.\mathbf{0} \to_\diamond R_1)$, we have $Prop(\mathbf{0} \to_\diamond R_1')$, which is impossible, so this case never happens.
– Case $P_0 = \nu n.\nu n'.P \overset{\diamond}{\equiv} \nu n'.\nu n.P$. (This case is its own symmetric.) Since $Prop(\nu n'.\nu n.P \to_\diamond R_1)$, we have $\nu n.P \to_\diamond R_1'$, $Prop(\nu n.P \to_\diamond R_1')$, and $R_1 \equiv \nu n'.R_1'$ for some R_1', so $P \to_\diamond R_1''$, $Prop(P \to_\diamond R_1'')$, and $R_1' \equiv \nu n.R_1''$ for some R_1''. Hence, $\nu n'.P \to_\diamond \nu n'.R_1''$, $Prop(\nu n'.P \to_\diamond \nu n'.R_1'')$, and $R_0 \equiv R_1 \equiv \nu n'.R_1' \equiv \nu n'.\nu n.R_1'' \equiv \nu n.\nu n'.R_1''$, so we are in Case 2 of $Prop(P_0 \to_\diamond R_0)$ with $Q' = \nu n'.R_1''$.
– Case $P_0 = P \mid \nu n.Q \overset{\diamond}{\equiv} \nu n.(P \mid Q)$ and $n \notin fn(P)$. Since $Prop(\nu n.(P \mid Q) \to_\diamond R_1)$, we have $P \mid Q \to_\diamond R_2$, $Prop(P \mid Q \to_\diamond R_2)$, and $R_1 \equiv \nu n.R_2$ for some R_2. Since $Prop(P \mid Q \to_\diamond R_2)$, we have four cases:
  * $P \to_\diamond P'$, $Prop(P \to_\diamond P')$, and $R_2 \equiv P' \mid Q$ for some P'. We rename n in Q so that $n \notin fn(P')$. Hence $P \to_\diamond P'$, $Prop(P \to_\diamond P')$, and $R_0 \equiv R_1 \equiv \nu n.R_2 \equiv \nu n.(P' \mid Q) \equiv P' \mid \nu n.Q$ since $n \notin fn(P')$. So we are in Case 1.(a) of $Prop(P_0 \to_\diamond R_0)$.
  * $Q \to_\diamond Q'$, $Prop(Q \to_\diamond Q')$, and $R_2 \equiv P \mid Q'$ for some Q'. Then $\nu n.Q \to_\diamond \nu n.Q'$, $Prop(\nu n.Q \to_\diamond \nu n.Q')$, and $R_0 \equiv R_1 \equiv \nu n.R_2 \equiv \nu n.(P \mid Q') \equiv P \mid \nu n.Q'$ since $n \notin fn(P)$. So we are in Case 1.(a') of $Prop(P_0 \to_\diamond R_0)$.
  * $P \xrightarrow{N(x)}_\diamond A$, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_2 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N.
  We need to rename n so that $n \notin fn(N) \cup fn(A)$. To do that, we first show that, for all processes P, P', if $P \overset{\diamond}{\equiv} P'$ and $\Sigma \vdash P\{^{n'}/_{n}\} = P$, then $\Sigma \vdash P'\{^{n'}/_{n}\} = P'$, by induction on the derivation of $P \overset{\diamond}{\equiv} P'$.
  We also show that, if $A \equiv A'$, then $A\{^{n'}/_{n}\} \equiv A'\{^{n'}/_{n}\}$, by induction on the derivation of $A \equiv A'$.
  By Lemma B.10, since $P \xrightarrow{N(x)}_\diamond A$, we have $P \overset{\diamond}{\equiv} \nu\widetilde{n}.(N(y).P_1 \mid P_2)$, $A \equiv \nu\widetilde{n}.(P_1\{^{x}/_{y}\} \mid P_2)$, and $\{\widetilde{n}\} \cap fn(N) = \emptyset$, for some $\widetilde{n}$, P_1, P_2, y, and since $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, we have $Q \overset{\diamond}{\equiv} \nu\widetilde{n}'.(\overline{N}\langle M\rangle.Q_1 \mid Q_2)$, $A \equiv \nu\widetilde{n}'.(Q_1 \mid \{^{M}/_{x}\} \mid Q_2)$, $\{\widetilde{n}'\} \cap fn(N) = \emptyset$, and $x \notin fv(\overline{N}\langle M\rangle.Q_1 \mid Q_2)$, for some $\widetilde{n}'$, Q_1, Q_2, M. Let n' be a fresh name.
  · First case: $n \notin \widetilde{n}$. We have $P\{^{n'}/_{n}\} = P$ since $n \notin fn(P)$, so by the result shown above, $\Sigma \vdash (\nu\widetilde{n}.(N(y).P_1 \mid P_2))\{^{n'}/_{n}\} = \nu\widetilde{n}.(N(y).P_1 \mid P_2)$, so $\Sigma \vdash N\{^{n'}/_{n}\} = N$, $P \overset{\diamond}{\equiv} \nu\widetilde{n}.(N\{^{n'}/_{n}\}(y).P_1\{^{n'}/_{n}\} \mid P_2\{^{n'}/_{n}\})$, $A\{^{n'}/_{n}\} \equiv \nu\widetilde{n}.(P_1\{^{n'}/_{n}\}\{^{x}/_{y}\} \mid P_2\{^{n'}/_{n}\})$, and $fn(N\{^{n'}/_{n}\}) \cap \{\widetilde{n}\} = \emptyset$, so by Lemma B.10, $P \xrightarrow{N\{^{n'}/_{n}\}(x)}_\diamond A\{^{n'}/_{n}\}$.
  · Second case: $n \in \widetilde{n}$, so $n \notin fn(N)$. We have $N\{^{n'}/_{n}\} = N$. So $P \overset{\diamond}{\equiv} \nu\widetilde{n}.(N\{^{n'}/_{n}\}(y).P_1\} \mid P_2)$, $A\{^{n'}/_{n}\} \equiv \nu\widetilde{n}.(P_1\{^{x}/_{y}\} \mid P_2)$, and $\{\widetilde{n}\} \cap fn(N\{^{n'}/_{n}\}) = \emptyset$, so by Lemma B.10, $P \xrightarrow{N\{^{n'}/_{n}\}(x)}_\diamond A\{^{n'}/_{n}\}$.
  Let $N' = N\{^{n'}/_{n}\}$ and $A' = A\{^{n'}/_{n}\}$. Hence in both cases, $P \xrightarrow{N'(x)}_\diamond A'$ and $\Sigma \vdash N' = N$, so $Q \overset{\diamond}{\equiv} \nu\widetilde{n}'.(\overline{N'}\langle M\rangle.Q_1 \mid Q_2)$, $A \equiv \nu\widetilde{n}'.(Q_1 \mid \{^{M}/_{x}\} \mid Q_2)$, $\{\widetilde{n}'\} \cap fn(N') = \emptyset$, and $x \notin fv(\overline{N'}\langle M\rangle.Q_1 \mid Q_2)$, so by Lemma B.10, $Q \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\diamond B$. Hence $P \xrightarrow{N'(x)}_\diamond A'$, $\nu n.Q \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\diamond \nu n.B$ by Scope′, and $R_0 \equiv R_1 \equiv \nu n.R_2 \equiv \nu n.\nu x.(A' \mid B) \equiv \nu x.(A' \mid \nu n.B)$, so we are in Case 1.(b) of $Prop(P_0 \to_\diamond R_0)$.

* $P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A$, $Q \xrightarrow{N(x)}_\diamond B$, and $R_2 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. This case can be handled similarly to the previous one.

– Case $P_0 = \nu n.(P \mid Q) \overset{\diamond}{\equiv} P \mid \nu n.Q$ and $n \notin fn(P)$. This case is fairly similar to the previous one.

– Case $P_0 = P_1\{^M/_x\} \overset{\diamond}{\equiv} P_1\{^N/_x\}$ and $\Sigma \vdash M = N$. (This case is its own symmetric.) We have $Prop(P_1\{^N/_x\} \to_\diamond R_1)$. We show by induction on the syntax of P_1 that, if $Prop(P_1\{^N/_x\} \to_\diamond R_1)$, $\Sigma \vdash M = N$, and $R_0 \equiv R_1$, then $Prop(P_1\{^M/_x\} \to_\diamond R_0)$.

* Case $P_1 = P \mid Q$. We have $Prop(P\{^N/_x\} \mid Q\{^N/_x\} \to_\diamond R_1)$, so we have four cases:
  · $P\{^N/_x\} \to_\diamond P'$, $Prop(P\{^N/_x\} \to_\diamond P')$, and $R_1 \equiv P' \mid Q\{^N/_x\}$ for some P'. We have $P\{^M/_x\} \to_\diamond P'$, $Prop(P\{^M/_x\} \to_\diamond P')$ by induction hypothesis, and $R_0 \equiv R_1 \equiv P' \mid Q\{^N/_x\}$, so we are in Case 1.(a) of $Prop(P\{^M/_x\} \mid Q\{^M/_x\} \to_\diamond R_0)$.
  · $Q\{^N/_x\} \to_\diamond Q'$, $Prop(Q\{^N/_x\} \to_\diamond Q')$, and $R_1 \equiv P\{^N/_x\} \mid Q'$ for some Q'. This case is obtained from the previous one by swapping P and Q.
  · $P\{^N/_x\} \xrightarrow{N'(x)}_\diamond A$, $Q\{^N/_x\} \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\diamond B$, and $R_1 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N'. Then $P\{^M/_x\} \xrightarrow{N'(x)}_\diamond A$, $Q\{^M/_x\} \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\diamond B$ by Struct′, and $R_1 \equiv \nu x.(A \mid B)$, so $R_0 \equiv R_1 \equiv \nu x.(A \mid B)$. So we are in Case 1.(b) of $Prop(P\{^M/_x\} \mid Q\{^M/_x\} \to_\diamond R_0)$.
  · $P\{^N/_x\} \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\diamond A$, $Q\{^N/_x\} \xrightarrow{N'(x)}_\diamond B$, and $R_1 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N'. This case is obtained from the previous one by swapping P and Q.
* Case $P_1 = \nu n.P$. We have $Prop(\nu n.P\{^N/_x\} \to_\diamond R_1)$, so $P\{^N/_x\} \to_\diamond R_2$, $Prop(P\{^N/_x\} \to_\diamond R_2)$, and $R_1 \equiv \nu n.R_2$ for some R_2. Hence $P_1\{^M/_x\} = \nu n.P\{^M/_x\}$, $P\{^M/_x\} \overset{\diamond}{\equiv} P\{^N/_x\} \to_\diamond R_2$, $Prop(P\{^M/_x\} \to_\diamond R_2)$ by induction hypothesis, and $R_0 \equiv R_1 \equiv \nu n.R_2$, so we are in Case 2 of $Prop(P_1\{^M/_x\} \to_\diamond R_0)$.
* Case $P_1 = !P$. We have $Prop(!P\{^N/_x\} \to_\diamond R_1)$, so $P\{^N/_x\} \mid P\{^N/_x\} \to_\diamond R_2$, $Prop(P\{^N/_x\} \mid P\{^N/_x\} \to_\diamond R_2)$, and $R_1 \equiv R_2 \mid !P\{^N/_x\}$ for some R_2. Since $Prop(P\{^N/_x\} \mid P\{^N/_x\} \to_\diamond R_2)$, we have four cases, which reduce to two by symmetry:
  · $P\{^N/_x\} \to_\diamond P'$, $Prop(P\{^N/_x\} \to_\diamond P')$, and $R_2 \equiv P' \mid P\{^N/_x\}$ for some P'. We have $P\{^M/_x\} \to_\diamond P'$, $Prop(P\{^M/_x\} \to_\diamond P')$ by induction hypothesis, so $P\{^M/_x\} \mid P\{^M/_x\} \to_\diamond P' \mid P\{^M/_x\}$ and $R_0 \equiv R_1 \equiv R_2 \mid !P\{^N/_x\} \equiv P' \mid P\{^N/_x\} \mid !P\{^N/_x\} \equiv P' \mid P\{^M/_x\} \mid !P\{^M/_x\}$, so we are in Case 3 of $Prop(!P\{^M/_x\} \to_\diamond R_0)$.
  · $P\{^N/_x\} \xrightarrow{N'(x)}_\diamond A$, $P\{^N/_x\} \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\diamond B$, and $R_2 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N'. Hence $P\{^M/_x\} \xrightarrow{N'(x)}_\diamond A$, $P\{^M/_x\} \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\diamond B$ by Struct′, and $R_0 \equiv R_1 \equiv R_2 \mid !P\{^N/_x\} \equiv \nu x.(A \mid B) \mid !P\{^N/_x\}$. By Lemma B.20, $P\{^M/_x\} \mid P\{^M/_x\} \to_\diamond R_3$ and $R_3 \equiv \nu x.(A \mid B)$, so $Prop(P\{^M/_x\} \mid P\{^M/_x\} \to_\diamond R_3)$, and $R_0 \equiv R_3 \mid !P\{^M/_x\}$, so we are in Case 3 of $Prop(!P\{^M/_x\} \to_\diamond R_0)$.
* Case $P_1 = \mathit{if}\ M_1 = N_1\ \mathit{then}\ P\ \mathit{else}\ Q$. We have $Prop(\mathit{if}\ M_1\{^N/_x\} = N_1\{^N/_x\}\ \mathit{then}\ P\{^N/_x\}\ \mathit{else}\ Q\{^N/_x\} \to_\diamond R_1)$, so we have two cases:
  · $\Sigma \vdash M_1\{^N/_x\} = N_1\{^N/_x\}$ and $R_1 \equiv P\{^N/_x\}$. Hence, we have $\Sigma \vdash M_1\{^M/_x\} = N_1\{^M/_x\}$ and $R_0 \equiv R_1 \equiv P\{^M/_x\}$, so we are in Case 4 of $Prop(P_1\{^M/_x\} \to_\diamond R_0)$.
  · $\Sigma \vdash M_1\{^N/_x\} \neq N_1\{^N/_x\}$ and $R_1 \equiv Q\{^N/_x\}$. Hence, we have $\Sigma \vdash M_1\{^M/_x\} \neq N_1\{^M/_x\}$ and $R_0 \equiv R_1 \equiv Q\{^M/_x\}$, so we are in Case 4 of $Prop(P_1\{^M/_x\} \to_\diamond R_0)$.

Using this result, we obtain $Prop(P_0 \to_\diamond R_0)$.

– Case $P_0 = P \mid Q \overset{\diamond}{\equiv} P' \mid Q$ knowing $P \overset{\diamond}{\equiv} P'$. Since $Prop(P' \mid Q \to_\diamond R_1)$, we have four cases:

* $P' \to_\diamond P''$, $Prop(P' \to_\diamond P'')$, and $R_1 \equiv P'' \mid Q$ for some P''. Then $P \to_\diamond P''$, $Prop(P \to_\diamond P'')$ by induction hypothesis, and $R_0 \equiv R_1 \equiv P'' \mid Q$, so we are in Case 1.(a) of $Prop(P_0 \to_\diamond R_0)$.

  - $Q \to_\diamond Q'$, $Prop(Q \to_\diamond Q')$, and $R_1 \equiv P' \mid Q'$ for some Q'. Then $Q \to_\diamond Q'$, $Prop(Q \to_\diamond Q')$, and $R_0 \equiv R_1 \equiv P' \mid Q'$, so we are in Case 1.(a') of $Prop(P_0 \to_\diamond R_0)$.
  - $P' \xrightarrow{N(x)}_\diamond A$, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_1 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. Then $P \xrightarrow{N(x)}_\diamond A$ by Struct′, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R_0 \equiv R_1 \equiv \nu x.(A \mid B)$, so we are in Case 1.(b) of $Prop(P_0 \to_\diamond R_0)$.
  - $P' \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A$, $Q \xrightarrow{N(x)}_\diamond B$, and $R_1 \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N. This case can be handled similarly to the previous one.
- Case $P_0 = \nu n.P \overset{\diamond}{\equiv} \nu n.P'$ knowing $P \overset{\diamond}{\equiv} P'$. Since $Prop(\nu n.P' \to_\diamond R_1)$, $P' \to_\diamond R_1'$, $Prop(P' \to_\diamond R_1')$, and $R_1 \equiv \nu n.R_1'$ for some R_1'. Then, $P \to_\diamond R_1'$. By induction hypothesis, $Prop(P \to_\diamond R_1')$. Moreover, $R_0 \equiv R_1 \equiv \nu n.R_1'$, so we are in Case 2 of $Prop(P_0 \to_\diamond R_0)$.

If P_0 is closed and $P_0 \to_\diamond R_0$, then by Lemma B.17(1), there exists a derivation of $P_0 \to_\diamond R_0$ closed on the left. So by applying the previous result, $Prop(P_0 \to_\diamond R_0)$, which yields the desired property. □

Lemma B.22. *If $\nu\tilde{n}.(\sigma \mid P)$ is a closed normal process and $\nu\tilde{n}.(\sigma \mid P) \to_\circ A$, then $P \to_\diamond P'$ and $A \equiv \nu\tilde{n}.(\sigma \mid P')$ for some P'.*

Proof. We proceed similarly to Lemma B.19. By Lemma B.17(2), we consider a derivation of $\nu\tilde{n}.(\sigma \mid P) \to_\circ A$ closed on the left. By definition of $\to_\circ$, we have $\nu\tilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\tilde{n}'.(\sigma' \mid P')$, $P' \to_\diamond Q'$ and $A \overset{\circ}{\equiv} \nu\tilde{n}'.(\sigma' \mid Q')$ for some $\tilde{n}'$, σ', P', Q'. We proceed by induction on the derivation of $\nu\tilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\tilde{n}'.(\sigma' \mid P')$.

- Base case: $\nu\tilde{n}.(\sigma \mid P) = \nu\tilde{n}'.(\sigma' \mid P')$, and the desired result holds.
- Transitivity: the result is proved by applying the induction hypothesis twice.
- Case Plain″: $P \overset{\diamond}{\equiv} P' \to_\diamond Q'$ and $A \equiv \nu\tilde{n}.(\sigma \mid Q')$, so the result holds.
- Case New-C″: $\tilde{n}'$ is a reordering of $\tilde{n}$, $\nu\tilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\tilde{n}'.(\sigma \mid P)$, $P \to_\diamond Q'$ and $A \equiv \nu\tilde{n}'.(\sigma \mid Q')$, so $P \to_\diamond Q'$ and $A \equiv \nu\tilde{n}'.(\sigma \mid Q') \equiv \nu\tilde{n}.(\sigma \mid Q')$, hence the result holds.
- Case New-Par″: $P = \nu n'.P'$, $\nu\tilde{n}.(\sigma \mid \nu n'.P') \overset{\circ}{\equiv} \nu\tilde{n}, n'.(\sigma \mid P')$, $P' \to_\diamond Q'$, and $A \equiv \nu\tilde{n}, n'.(\sigma \mid Q')$ where $n' \notin fn(\sigma)$. Therefore, $P = \nu n'.P' \to_\diamond \nu n'.Q'$ and by New-Par, $A \equiv \nu\tilde{n}, n'.(\sigma \mid Q') \equiv \nu\tilde{n}.(\sigma \mid \nu n'.Q')$, hence the result holds.
- Case New-Par″ reversed: $\nu\tilde{n}, n'.(\sigma \mid P) \overset{\circ}{\equiv} \nu\tilde{n}.(\sigma \mid \nu n'.P)$, $\nu n'.P \to_\diamond Q'$ and $A \equiv \nu\tilde{n}.(\sigma \mid Q')$ where $n' \notin fn(\sigma)$. By Lemma B.21, $P \to_\diamond Q''$ and $Q' \equiv \nu n'.Q''$ for some Q''. Hence, $P \to_\diamond Q''$ and $A \equiv \nu\tilde{n}.(\sigma \mid Q') \equiv \nu\tilde{n}.(\sigma \mid \nu n'.Q'') \equiv \nu\tilde{n}, n'.(\sigma \mid Q'')$ by New-Par, so the result holds.
- Case Rewrite″: $\nu\tilde{n}.(\sigma|P) \overset{\circ}{\equiv} \nu\tilde{n}.(\sigma'|P)$, $P \to_\diamond Q$ and $A \equiv \nu\tilde{n}.(\sigma'|Q)$ where $dom(\sigma) = dom(\sigma')$, $\Sigma \vdash \sigma x = \sigma' x$ for all $x \in dom(\sigma)$, and $(fv(\sigma x) \cup fv(\sigma' x)) \cap dom(\sigma) = \emptyset$ for all $x \in dom(\sigma)$. Hence $P \to_\diamond Q$ and $A \equiv \nu\tilde{n}.(\sigma' \mid Q) \equiv \nu\tilde{n}.(\sigma \mid Q)$ by several applications of Rewrite, so the result holds. □

We prove the following strengthened version of Lemma B.22, in which the process P' is guaranteed to be closed.

Lemma B.23. *If $\nu\tilde{n}.(\sigma \mid P)$ is a closed normal process and $\nu\tilde{n}.(\sigma \mid P) \to_\circ A$, then $P \to_\diamond P'$ and $A \equiv \nu\tilde{n}.(\sigma \mid P')$ for some closed process P'.*

Proof. By Lemma B.22, we get the existence of a process P', which may not be closed. Let us apply Lemma B.16(2). Let $Y = fv(P) \cup fv(P') = fv(P')$. Let σ' be a substitution from Y to pairwise distinct fresh names. Since P is closed, $P = P\sigma'$, so a fortiori $\Sigma \vdash P = P\sigma'$. Hence $P\sigma' \to_\diamond P'\sigma'$ and $\Sigma \vdash P' = P'\sigma'$. So $P \to_\diamond P'\sigma'$ and $A \equiv \nu\tilde{n}.(\sigma \mid P') \equiv \nu\tilde{n}.(\sigma \mid P'\sigma')$, so we get the desired result by using the closed process $P'\sigma'$ instead of P'. □

The following strengthened version of Lemma B.21 is proved in a similar way.

Lemma B.24. *Suppose that P_0 is a closed process and $P_0 \rightarrow_\diamond R$. Then one of the following cases holds:*

(1) *$P_0 = P \mid Q$ for some P and Q, and one of the following cases holds:*
   (a) *$P \rightarrow_\diamond P'$ and $R \equiv P' \mid Q$ for some closed process P',*
   (b) *$P \xrightarrow{N(x)}_\diamond A$, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N, and two symmetric cases obtained by swapping P and Q;*
(2) *$P_0 = \nu n.P$, $P \rightarrow_\diamond Q'$, and $R \equiv \nu n.Q'$ for some n and some closed processes P and Q';*
(3) *$P_0 = !P$, $P \mid P \rightarrow_\diamond Q'$, and $R \equiv Q' \mid !P$ for some closed processes P and Q'.*
(4) *$P_0 = \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$ and either $\Sigma \vdash M = N$ and $R \equiv P$, or $\Sigma \vdash M \neq N$ and $R \equiv Q$, for some M, N, P, and Q.*

C PROOF OF THEOREM 4.8: MAIN LEMMAS

Relying on partial normal forms and their semantics, we prove the remaining lemmas needed for the proof of Theorem 4.8. Sections C.2 and C.3 establish the two directions of Theorem 4.8. The argument for the first direction employs lemmas about consequences of static equivalences; these lemmas are in Section C.1.

C.1 Exploiting Static Equivalence

The lemmas in this section rely on static equivalences in order to analyze and to establish structural equivalences or reductions. For all these lemmas, we consider the action of two equivalent frames $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$ on a process P' such that $fn(P') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$: we suppose a structural equivalence or reduction of a process P such that $\Sigma \vdash P'\sigma = P$, and prove a corresponding structural equivalence or reduction of the process $P'\sigma'$. Lemma C.1 deals with structural equivalence, Lemma C.2 with internal reduction, and Lemma C.3 with labelled transitions.

Lemma C.1. *Suppose that $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$, $fn(P') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, and $\Sigma \vdash P'\sigma = P$. If $P \mathrel{\overset{\diamond}{\equiv}} Q$, then $P'\sigma' \mathrel{\overset{\diamond}{\equiv}} Q'\sigma'$ for some Q' such that $fn(Q') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$; $\Sigma \vdash Q = Q'\sigma$; and, (*) if σ, σ', and $P'\sigma$ are closed, then $Q'\sigma$ is closed.*

Proof. We first prove the lemma without property (*), by induction on the derivation of $P \mathrel{\overset{\diamond}{\equiv}} Q$. The only rule that depends on terms is Rewrite′, and when $P \mathrel{\overset{\diamond}{\equiv}} Q$ by Rewrite′, $\Sigma \vdash P'\sigma = P = Q$, so taking $Q' = P'$, we have $P'\sigma' \mathrel{\overset{\diamond}{\equiv}} Q'\sigma'$, $fn(Q') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, and $\Sigma \vdash Q = Q'\sigma$. For all other base cases, the structural equivalence rule applied in $P \mathrel{\overset{\diamond}{\equiv}} Q$ also applies to P' and yields a process Q' such that $P' \mathrel{\overset{\diamond}{\equiv}} Q'$, $fn(Q') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, and $\Sigma \vdash Q = Q'\sigma$; by Lemma B.3(1) we conclude $P'\sigma' \mathrel{\overset{\diamond}{\equiv}} Q'\sigma'$. The case of transitivity is proved by applying the induction hypothesis twice.

We now prove the lemma with property (*) by applying Lemma B.16(1) to the structural equivalence $P'\sigma' \mathrel{\overset{\diamond}{\equiv}} Q'\sigma'$ for the process Q' obtained above. Let $Y = fv(P'\sigma) \cup fv(Q'\sigma) = fv(Q'\sigma) = fv(Q') \setminus dom(\sigma)$ and let σ'' map Y to pairwise distinct fresh names. We have $P'\sigma\sigma'' = P'\sigma$ so a fortiori $\Sigma \vdash P'\sigma = P'\sigma\sigma''$, then by Lemma B.16(1) $P'\sigma\sigma'' \mathrel{\overset{\diamond}{\equiv}} Q'\sigma\sigma''$ and $\Sigma \vdash Q'\sigma = Q'\sigma\sigma''$. So $\Sigma \vdash Q = Q'\sigma = (Q'\sigma'')\sigma$. Since $P'\sigma' \mathrel{\overset{\diamond}{\equiv}} Q'\sigma'$, we have $P'\sigma' = P'\sigma'\sigma'' \mathrel{\overset{\diamond}{\equiv}} Q'\sigma'\sigma'' = (Q'\sigma'')\sigma'$. Since $fn(Q') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, we have $fn(Q'\sigma'') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$. We also have $fv(Q'\sigma'') \subseteq dom(\sigma) = dom(\sigma')$, so we get the desired result by using $Q'\sigma''$ instead of Q'. □

Lemma C.2. *Suppose that $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$, $fn(P') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, and $\Sigma \vdash P'\sigma = P$. If $P \rightarrow_\diamond Q$, then $P'\sigma' \rightarrow_\diamond Q'\sigma'$ for some Q' such that $fn(Q') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$; $\Sigma \vdash Q = Q'\sigma$; and, (*) if σ, σ', and $P'\sigma$ are closed, then $Q'\sigma$ is closed.*

Proof. We first prove the lemma without property (*), by induction on the derivation of $P \rightarrow_\diamond Q$.

- Case Comm′. We have $P = \overline{N}\langle M\rangle.P_0 \mid N(x).Q_0 \rightarrow_\diamond P_0 \mid Q_0\{^M/_x\} = Q$. We rename x so that $x \notin dom(\sigma')$. Therefore, $P' = \overline{N'}\langle M'\rangle.P'_0 \mid N''(x).Q'_0$ for some N', M', P'_0, N'', Q'_0 such that

$\Sigma \vdash N'\sigma = N''\sigma = N$, $\Sigma \vdash M'\sigma = M$, $\Sigma \vdash P'_0\sigma = P_0$, and $\Sigma \vdash Q'_0\sigma = Q_0$. Let $Q' = P'_0 \mid Q'_0\{^{M'}/_x\}$. We have

$$\begin{aligned} P'\sigma' &= \overline{N'\sigma'}\langle M'\sigma'\rangle.P'_0\sigma' \mid N''\sigma'(x).Q'_0\sigma' \\ &\equiv \overline{N'\sigma'}\langle M'\sigma'\rangle.P'_0\sigma' \mid N'\sigma'(x).Q'_0\sigma' \\ &\to_\diamond P'_0\sigma' \mid Q'_0\sigma'\{^{M'\sigma'}/_x\} = Q'\sigma' \end{aligned}$$

since $\Sigma \vdash N'\sigma' = N''\sigma'$ because $\Sigma \vdash N'\sigma = N''\sigma$, $(fn(N') \cup fn(N'')) \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, and $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n'}.\sigma'$. Moreover, $fn(Q') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, and $\Sigma \vdash Q = P_0 \mid Q_0\{^M/_x\} = P'_0\sigma \mid Q'_0\sigma\{^{M'\sigma}/_x\} = Q'\sigma$.

- The cases Then′ and Else′ are similar: the equalities that trigger reductions happen both in P and in $P'\sigma$.
- The case in which we apply $\overset{\diamond}{\equiv}$ holds by Lemma C.1 and induction hypothesis.
- Case in which we apply a context. The reduction $P = E[P_0] \to_\diamond Q = E[Q_0]$ is derived from $P_0 \to_\diamond Q_0$. If E contains a restriction $\nu n.$ above the hole, we rename n so that $n \notin \{\widetilde{n}, \widetilde{n'}\}$. Hence $P' = E'[P'_0]$ with $\Sigma \vdash E'\sigma = E$, $\Sigma \vdash P'_0\sigma = P_0$, and $fn(P'_0) \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$. By induction hypothesis, $P'_0\sigma' \to_\diamond Q'_0\sigma'$, $fn(Q'_0) \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, and $\Sigma \vdash Q_0 = Q'_0\sigma$ for some Q'_0. Let $Q' = E'[Q'_0]$. Then $P'\sigma' = E'\sigma'[P'_0\sigma'] \to_\diamond E'\sigma'[Q'_0\sigma'] = Q'\sigma'$, $fn(Q') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, and $\Sigma \vdash Q = E[Q_0] = E'\sigma[Q'_0\sigma] = Q'\sigma$.

We now prove the lemma with property (*) by applying Lemma B.16(2) to the reduction $P'\sigma \to_\diamond Q'\sigma$ for the process Q' obtained above. Let $Y = fv(P'\sigma) \cup fv(Q'\sigma) = fv(Q'\sigma) = fv(Q') \setminus dom(\sigma)$ and let σ'' map Y to pairwise distinct fresh names. We have $P'\sigma\sigma'' = P'\sigma$ so a fortiori $\Sigma \vdash P'\sigma = P'\sigma\sigma''$, then by Lemma B.16(2), $P'\sigma\sigma'' \to_\diamond Q'\sigma\sigma''$ and $\Sigma \vdash Q'\sigma = Q'\sigma\sigma''$. So $\Sigma \vdash Q = Q'\sigma = (Q'\sigma'')\sigma$. Since $P'\sigma' \to_\diamond Q'\sigma'$, we have $P'\sigma' = P'\sigma'\sigma'' \to_\diamond Q'\sigma'\sigma'' = (Q'\sigma'')\sigma'$. Since $fn(Q') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, we have $fn(Q'\sigma'') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$. We also have $fv(Q'\sigma'') \subseteq dom(\sigma) = dom(\sigma')$, so we get the desired result by using $Q'\sigma''$ instead of Q'. □

Lemma C.3 gives two variants of the same result: if $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n'}.\sigma'$ and P such that $\Sigma \vdash P'\sigma = P$ has a labelled transition, then $P'\sigma'$ has a corresponding labelled transition. The two variants differ by the closure assumptions and conclusions.

Lemma C.3. *Suppose that $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n'}.\sigma'$, $fn(P') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash P'\sigma = P$, $P \xrightarrow{\alpha}_\diamond A$, and σ, σ', and $P'\sigma$ are closed.*

(1) *If α is an output or $\alpha = N(M)$ with some M' such that $\Sigma \vdash M'\sigma = M$, $M'\sigma$ is closed, and $fn(M') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$; then $P'\sigma' \xrightarrow{\alpha'\sigma'}_\diamond A'\sigma'$, $fn(A') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $A \equiv A'\sigma$, $fn(\alpha') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash \alpha = \alpha'\sigma$, and $A'\sigma$ is closed for some A', α'.*

(2) *If $\alpha = N(x)$ and $x \notin dom(\sigma)$, then $P'\sigma' \xrightarrow{N'\sigma'(x)}_\diamond A'\sigma'$, $fn(A') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $A \equiv A'\sigma$, $fn(N') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash N = N'\sigma$, $fv(A') \subseteq dom(\sigma) \cup \{x\}$, and $fv(N') \subseteq dom(\sigma)$ for some A', N'.*

Proof. *Property 1:* By induction on the derivation of $P \xrightarrow{\alpha}_\diamond A$.

- Case In′. We have $P = N(x).P_0 \xrightarrow{N(M)}_\diamond P_0\{^M/_x\} = A$ and there exists M' such that $\Sigma \vdash M'\sigma = M$ and $fn(M') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$. We rename x so that $x \notin dom(\sigma)$. So $P' = N'(x).P'_0$ with $\Sigma \vdash N = N'\sigma$ and $\Sigma \vdash P_0 = P'_0\sigma$. Let $A' = P'_0\{^{M'}/_x\}$ and $\alpha' = N'(M')$. Then we have $P'\sigma' = N'\sigma'(x).P'_0\sigma' \xrightarrow{N'\sigma'(M'\sigma')}_\diamond P'_0\sigma'\{^{M'\sigma'}/_x\} = A'\sigma'$, $fn(A') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash A = P'_0\sigma\{^{M'\sigma}/_x\} = A'\sigma$, $fn(\alpha') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, and $\Sigma \vdash \alpha = \alpha'\sigma$. Since $P'\sigma$ is closed, $fv(P'_0) \subseteq dom(\sigma) \cup \{x\}$; moreover $M'\sigma$ is closed, so $A'\sigma$ is closed.

- Case Out-Var′. We have $P = \overline{N}\langle M\rangle.P_0 \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond P_0 \mid \{^M/_x\} = A$ with $x \notin fv(\overline{N}\langle M\rangle.P_0)$. So $P' = \overline{N'}\langle M'\rangle.P'_0$ with $\Sigma \vdash N = N'\sigma$, $\Sigma \vdash M = M'\sigma$, and $\Sigma \vdash P_0 = P'_0\sigma$. Let $A' = P'_0 \mid \{^{M'}/_x\}$ and $\alpha' = \nu x.\overline{N'}\langle x\rangle$. We have $P'\sigma' = \overline{N'\sigma'}\langle M'\sigma'\rangle.P'_0\sigma' \xrightarrow{\nu x.\overline{N'\sigma'}\langle x\rangle}_\diamond P'_0\sigma' \mid \{^{M'\sigma'}/_x\} = A'\sigma'$, $fn(A') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash A = P'_0\sigma \mid \{^{M'\sigma}/_x\} = A'\sigma$, $fn(\alpha') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, and $\Sigma \vdash \alpha = \alpha'\sigma$. Since $P'\sigma$ is closed, $P'_0\sigma$ is closed; moreover $M'\sigma$ is closed, so $A'\sigma$ is closed.
- Case Scope′. The transition $P = \nu n.P_0 \xrightarrow{\alpha}_\diamond \nu n.A_0 = A$ is derived from $P_0 \xrightarrow{\alpha}_\diamond A_0$, where n does not occur in α. We rename n so that $n \notin \{\widetilde{n}, \widetilde{n'}\}$ and $n \notin fn(\sigma) \cup fn(\sigma')$. We have $P' = \nu n.P'_0$ for some P'_0, so $\Sigma \vdash P'_0\sigma = P_0$ and $P'_0\sigma$ is closed. By induction hypothesis, $P'_0\sigma' \xrightarrow{\alpha'\sigma'}_\diamond A'_0\sigma'$, $fn(A'_0) \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash A_0 = A'_0\sigma$, $fn(\alpha') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash \alpha = \alpha'\sigma$, and $A'_0\sigma$ is closed for some A'_0, α'. Let $A' = \nu n.A'_0$. Then $P'\sigma' \xrightarrow{\alpha'\sigma'}_\diamond A'\sigma'$ by Scope′, so we have the desired result.
- Case Par′. The transition $P = P_0 \mid Q_0 \xrightarrow{\alpha}_\diamond A_0 \mid Q_0 = A$ is derived from $P_0 \xrightarrow{\alpha}_\diamond A_0$, where $bv(\alpha) \cap fv(Q_0) = \emptyset$. We have $P' = P'_0 \mid Q'_0$ for some P'_0, Q'_0, so $\Sigma \vdash P'_0\sigma = P_0$, $\Sigma \vdash Q'_0\sigma = Q_0$, and $P'_0\sigma$ and $Q'_0\sigma$ are closed. By induction hypothesis, $P'_0\sigma' \xrightarrow{\alpha'\sigma'}_\diamond A'_0\sigma'$, $fn(A'_0) \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash A_0 = A'_0\sigma$, $fn(\alpha') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $\Sigma \vdash \alpha = \alpha'\sigma$, and $A'_0\sigma$ is closed for some A'_0, α'. Let $A' = A'_0 \mid Q'_0$. Then $P'\sigma' \xrightarrow{\alpha'\sigma'}_\diamond A'\sigma'$ by Par′, since $fv(Q'_0\sigma') = \emptyset$. Since $P'\sigma$ is closed, $Q'_0\sigma$ is closed, so $A'\sigma$ is closed. Therefore, we have the desired result.
- Case Struct′ follows by Lemma C.1 and induction hypothesis.

Property 2: By Lemma B.10, $P \overset{\diamond}{\equiv} \nu\widetilde{n''}.(N(y).P_1 \mid P_2)$, $A \equiv \nu\widetilde{n''}.(P_1\{^x/_y\} \mid P_2)$, and $\{\widetilde{n''}\} \cap fn(N) = \emptyset$, for some $\widetilde{n''}$, P_1, P_2, N, y. We rename $\widetilde{n''}$ so that $\{\widetilde{n''}\} \cap (fn(\sigma) \cup fn(\sigma')) = \emptyset$, and we rename y so that $y \notin dom(\sigma)$. By Lemma C.1, $P'\sigma' \overset{\diamond}{\equiv} Q'\sigma'$, $fn(Q') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, and $\Sigma \vdash \nu\widetilde{n''}.(N(y).P_1 \mid P_2) = Q'\sigma$ for some Q' such that $Q'\sigma$ is closed, so $fv(Q') \subseteq dom(\sigma) = dom(\sigma')$. Hence, Q' is of the form $Q' = \nu\widetilde{n''}.(N'(y).P'_1|P'_2)$ with $\Sigma \vdash N = N'\sigma$, $\Sigma \vdash P_1 = P'_1\sigma$, and $\Sigma \vdash P_2 = P'_2\sigma$. Hence, by Lemma B.10, $P'\sigma' \overset{\diamond}{\equiv} Q'\sigma' = \nu\widetilde{n''}.(N'\sigma'(y).P'_1\sigma'|P'_2\sigma') \xrightarrow{N'\sigma'(x)}_\diamond \nu\widetilde{n''}.(P'_1\sigma'\{^x/_y\}|P'_2\sigma')$. Let $A' = \nu\widetilde{n''}.(P'_1\{^x/_y\}|P'_2)$. Then $P'\sigma' \xrightarrow{N'\sigma'(x)}_\diamond A'\sigma'$, $fn(A') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$ and $fn(N') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$ because $fn(Q') \cap \{\widetilde{n}, \widetilde{n'}\} = \emptyset$, $A \equiv \nu\widetilde{n''}.(P_1\{^x/_y\} \mid P_2) \equiv \nu\widetilde{n''}.(P'_1\sigma\{^x/_y\} \mid P'_2\sigma) \equiv A'\sigma$, $\Sigma \vdash N = N'\sigma$, and $fv(A') \subseteq dom(\sigma) \cup \{x\}$ and $fv(N') \subseteq dom(\sigma) = dom(\sigma')$ because $fv(Q') \subseteq dom(\sigma) = dom(\sigma')$. □

C.2 Labelled Bisimilarity Implies Observational Equivalence

The goal of this section is to establish the lemmas needed in the outline of the argument that labelled bisimilarity implies observational equivalence in Section 4.5.

Lemma C.4. *Let A and B be two extended processes. Let σ be a bijective renaming (a substitution that is a bijection from names to names). We have:*

- *$A \equiv B$ if and only if $A\sigma \equiv B\sigma$,*
- *$A \to B$ if and only if $A\sigma \to B\sigma$,*
- *$A \xrightarrow{\alpha} B$ if and only if $A\sigma \xrightarrow{\alpha\sigma} B\sigma$.*

Let A', B', and α' be obtained from A, B, and α, respectively, by replacing all variables (including their occurrences in domains of active substitutions) with distinct variables. We have:

- *$A \equiv B$ if and only if $A' \equiv B'$,*
- *$A \to B$ if and only if $A' \to B'$,*
- *$A \xrightarrow{\alpha} B$ if and only if $A' \xrightarrow{\alpha'} B'$.*

Proof. The implications from left to right are proved by induction on the derivations. We use that the equational theory is closed under renaming of names and variables. The same argument also proves the converse implications, via the inverse renaming. □

Lemma C.5. *Let A and B be two closed extended processes.*

- *Let σ be a bijective renaming. We have $A \approx_l B$ if and only if $A\sigma \approx_l B\sigma$.*
- *Let A' and B' be obtained from A and B, respectively, by replacing all variables (including their occurrences in domains of active substitutions) with distinct variables. We have $A \approx_l B$ if and only if $A' \approx_l B'$.*

Proof. To prove the first point, we define a relation $\mathcal{R}$ by $A' \mathcal{R} B'$ if and only if $A' = A\sigma$, $B' = B\sigma$, and $A \approx_l B$ for some A and B. We show that $\mathcal{R}$ satisfies the three properties of Definition 4.7. Then $\mathcal{R} \subseteq \approx_l$, so if $A \approx_l B$, then $A' = A\sigma \approx_l B' = B\sigma$.

(1) Property 1 comes from Lemma A.2.
(2) If $A' \mathcal{R} B'$, $A' \to A'_1$, and A'_1 is closed, then by Lemma C.4, $A = A'\sigma^{-1} \to A'_1\sigma^{-1}$. We let $A'' = A'_1\sigma^{-1}$, which is also closed. So by definition of $\approx_l$, $B \to^* B''$ and $A'' \approx_l B''$ for some B''. By Lemma C.4, $B' = B\sigma \to^* B''\sigma$. We let $B'_1 = B''\sigma$. We have $A'_1 \mathcal{R} B'_1$ and $B' \to^* B'_1$. So Property 2 holds.
(3) The proof of Property 3 is similar to the proof of Property 2.

The same argument also proves the converse, via the inverse renaming.
The proof of the second point is similar. □

Lemma 4.18. *$\approx_l$ is closed by application of closing evaluation contexts.*

Proof. Let A and B be two closed extended processes such that $A \approx_l B$, and E be an evaluation context closing for A and B. Our goal is to show that $E[A] \approx_l E[B]$. We first rename the free names and variables of E by Lemma C.5, so that the obtained context is simple. Then by Lemma A.1, we construct a context E' of the form $\nu\widetilde{u}.(\_ \mid C'')$ such that $E \equiv E'$. Since $\approx_l$ is invariant by structural equivalence, it is sufficient to show that $E'[A] \approx_l E'[B]$. Hence, it is sufficient to consider evaluation contexts of the form $\nu\widetilde{u}.(\_ \mid C)$, such that $\nu\widetilde{u}.(A \mid C)$ and $\nu\widetilde{u}.(B \mid C)$ are closed.

To every relation $\mathcal{R}$ on closed extended processes, we associate the relation $\mathcal{R}' = \{(\nu\widetilde{u}.(A \mid C), \nu\widetilde{u}.(B \mid C)) \mid A \mathcal{R} B, \nu\widetilde{u}.(\_ \mid C)$ closing for A and $B\}$. We prove that, if $\mathcal{R}$ is a labelled bisimulation, then $\mathcal{R}'$ is a labelled bisimulation up to $\equiv$, hence $\mathcal{R} \subseteq \equiv\mathcal{R}'\equiv \subseteq \approx_l$. For $\mathcal{R} = \approx_l$, this establishes that $\approx_l$ is closed by application of evaluation contexts $\nu\widetilde{u}.(\_ \mid C)$.

Assume $S \mathcal{R}' T$, with $S = \nu\widetilde{u}.(A \mid C)$, $T = \nu\widetilde{u}.(B \mid C)$, and $A \mathcal{R} B$. Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$, $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' \mid P')$, $\mathrm{pnf}(C) = \nu\widetilde{n}''.(\sigma'' \mid P'')$, $\widetilde{u}$ consist of names $\widetilde{n}'''$ and variables $\widetilde{x}$. We suppose that A or C is not a plain process. (The case in which A and C are plain processes is simpler.) Since $A \mathcal{R} B$, we have $dom(A) = dom(B)$, so we also have that B or C is not a plain process. We rename $\widetilde{n}$, $\widetilde{n}'$, and $\widetilde{n}''$ so that they are disjoint, the names of $\widetilde{n}$ and of $\widetilde{n}'$ are not free in $\sigma'' \mid P''$, and the names of $\widetilde{n}''$ are not free in $\sigma \mid P$ nor in $\sigma' \mid P'$. Since A is closed, by Lemma B.2, $\mathrm{pnf}(A)$ is closed, so P and the image of σ have no free variables, so they are not modified by σ''. Similarly, P' and the image of σ' have no free variables, so they are not modified by σ''. Hence $\mathrm{pnf}(S) = \nu\widetilde{n}''', \widetilde{n}, \widetilde{n}''.((\sigma \mid \sigma''_{|dom(\sigma \mid \sigma''\sigma) \setminus \{\widetilde{x}\}} \mid P \mid P''\sigma)$ and $\mathrm{pnf}(T) = \nu\widetilde{n}''', \widetilde{n}', \widetilde{n}''.((\sigma' \mid \sigma''\sigma'_{|dom(\sigma' \mid \sigma''\sigma') \setminus \{\widetilde{x}\}} \mid P' \mid P''\sigma')$.

We argue that $\mathcal{R}'$ satisfies the three properties of a labelled bisimulation up to $\equiv$ (Definition 4.17). The proof of the first property is trivial; those of the last two properties (given in more detail below) go as follows. From a (labelled or internal) reduction of S, we infer a reduction of $\mathrm{pnf}(S)$, hence a reduction of $P \mid P''\sigma$ by a decomposition lemma (Lemma B.19 or B.22), hence reductions of P and/or $P''\sigma$ by another decomposition lemma (Lemma B.18 or B.24). From a reduction of P, we infer a reduction of A, hence a reduction of B since $\mathcal{R}$ is labelled bisimulation, so a reduction of

P' by a decomposition lemma. From a reduction of $P''\sigma$, we infer a reduction of $P''\sigma'$ using the static equivalence $A \approx_s B$, which means that $\nu n.\sigma \approx_s \nu n'.\sigma'$. Therefore, in all cases, we obtain a reduction of $P' \mid P''\sigma'$, hence a reduction of $\mathrm{pnf}(T)$, so a reduction of T. In more detail, the proof proceeds as follows.

(1) $S \approx_s T$ immediately follows from $A \approx_s B$ by Lemma 4.4.

(2) For every $S \to S'$ with S' closed, we prove that $T \to^* T'$ and $S' \equiv\mathcal{R}'\equiv T'$ for some T'. By Lemma B.8, $\mathrm{pnf}(S) \to_\circ \mathrm{pnf}(S')$. By Lemma B.22, $P \mid P''\sigma \to_\diamond Q$ and $\mathrm{pnf}(S') \equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid Q)$ for some Q. By Lemma B.24, we have four cases:

(a) $P \to_\diamond Q'$ and $Q \equiv Q' \mid P''\sigma$ for some closed process Q'. By Lemmas B.1 and B.9, $A \equiv \nu\widetilde{n}.(\sigma \mid P) \to A'$ where $A' = \nu\widetilde{n}.(\sigma \mid Q')$. Since $A \mathcal{R} B$ and A' is closed, we have $B \to^* B'$ and $A' \mathcal{R} B'$ for some B'. By Lemma B.8, $\mathrm{pnf}(B) \to_\circ^* \mathrm{pnf}(B')$, so by Lemma B.23, $P' \to_\diamond^* Q''$ and $\mathrm{pnf}(B') \equiv \nu\widetilde{n}'.(\sigma' \mid Q'')$ for some closed process Q''. We rename $\widetilde{n}''$ so that $\{\widetilde{n}''\} \cap fn(Q') = \emptyset$ and $\{\widetilde{n}''\} \cap fn(Q'') = \emptyset$. Hence, by Lemmas B.1 and B.9,

$$\begin{aligned} T &\equiv \nu\widetilde{n}'''.\widetilde{n}'.\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid P''\sigma') \\ &\to^* \nu\widetilde{n}'''.\widetilde{n}'.\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid Q'' \mid P''\sigma') \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}'.(\sigma' \mid Q'') \mid \nu\widetilde{n}''.(\sigma'' \mid P'')) \equiv \nu\widetilde{u}.(B' \mid C) \end{aligned}$$

If there is at least one reduction step in this trace, we let $T' = \nu\widetilde{u}.(B' \mid C)$; otherwise, we let $T' = T$. In all cases, $T \to^* T'$ and $T' \equiv \nu\widetilde{u}.(B' \mid C)$. Since $A' \mathcal{R} B'$,

$$\begin{aligned} S' &\equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid Q' \mid P''\sigma) \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}.(\sigma \mid Q') \mid \nu\widetilde{n}''.(\sigma'' \mid P'')) \\ &\equiv \nu\widetilde{u}.(A' \mid C)\,, \end{aligned}$$

and $\nu\widetilde{u}.(\_ \mid C)$ is closing for A' and B', we have $S' \equiv\mathcal{R}'\equiv T'$.

(b) $P''\sigma \to_\diamond Q'$ and $Q \equiv P \mid Q'$ for some closed process Q'. Since $A \mathcal{R} B$, we have $A \approx_s B$, that is, $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$. By Lemma C.2, $P''\sigma' \to_\diamond Q''\sigma'$, $fn(Q'') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, and $\Sigma \vdash Q' = Q''\sigma$ for some Q'' such that Q'' is closed, so $fv(Q'') \subseteq dom(\sigma) = dom(\sigma')$. So by Lemmas B.1 and B.9,

$$\begin{aligned} T &\equiv \nu\widetilde{n}'''.\widetilde{n}'.\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid P''\sigma') \\ &\to \nu\widetilde{n}'''.\widetilde{n}'.\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid Q''\sigma') \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}'.(\sigma' \mid P') \mid \nu\widetilde{n}''.(\sigma'' \mid Q'')) \equiv \nu\widetilde{u}.(B \mid C') \end{aligned}$$

where $C' = \nu\widetilde{n}''.(\sigma'' \mid Q'')$. Moreover,

$$\begin{aligned} S' &\equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid P \mid Q''\sigma) \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}.(\sigma \mid P) \mid \nu\widetilde{n}''.(\sigma'' \mid Q'')) \\ &\equiv \nu\widetilde{u}.(A \mid C') \end{aligned}$$

We let $T' = \nu\widetilde{u}.(B \mid C')$. We have $T \to T'$. Since $fv(Q'') \subseteq dom(\sigma) = dom(\sigma')$, $\nu\widetilde{u}.(\_ \mid C')$ is closing for A and B, and since $A \mathcal{R} B$, we have $S' \equiv\mathcal{R}' T'$.

(c) $P \xrightarrow{N(x)}_\diamond A_1$, $P''\sigma \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond C_1$, and $Q \equiv \nu x.(A_1 \mid C_1)$ for some A_1, C_1, x, and ground term N. We rename x so that $x \notin dom(\sigma) = dom(\sigma')$. By Lemma B.10 applied twice, $P \overset{\diamond}{\equiv} \nu\widetilde{n}_1.(N(y).P_1 \mid P_2)$, $A_1 \equiv \nu\widetilde{n}_1.(P_1\{^x/_y\} \mid P_2)$, $\{\widetilde{n}_1\} \cap fn(N) = \emptyset$ and $P''\sigma \overset{\diamond}{\equiv} \nu\widetilde{n}_2.(\overline{N}\langle M\rangle.P_3 \mid P_4)$, $C_1 \equiv \nu\widetilde{n}_2.(P_3 \mid \{^M/_x\} \mid P_4)$, $\{\widetilde{n}_2\} \cap fn(N) = \emptyset$, $x \notin fv(\overline{N}\langle M\rangle.P_3 \mid P_4)$. By Lemma B.16(1), we transform $\nu\widetilde{n}_1.(N(y).P_1 \mid P_2)$ and $\nu\widetilde{n}_2.(\overline{N}\langle M\rangle.P_3 \mid P_4)$ into closed processes that satisfy the same properties. Since $A \mathcal{R} B$, we have $A \approx_s B$, that is, $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$. By Lemma C.1,

$P''\sigma' \overset{\diamond}{\equiv} Q'\sigma'$, $fn(Q') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, and $\Sigma \vdash Q'\sigma = \nu\widetilde{n}_2.(\overline{N}\langle M\rangle.P_3 \mid P_4)$ for some Q' such that $Q'\sigma$ is closed, so $fv(Q') \subseteq dom(\sigma)$. Then Q' is of the form $Q' = \nu\widetilde{n}_2.(\overline{N'}\langle M'\rangle.P'_3 \mid P'_4)$, with $\Sigma \vdash N'\sigma = N$, $\Sigma \vdash M'\sigma = M$, $\Sigma \vdash P'_3\sigma = P_3$, $\Sigma \vdash P'_4\sigma = P_4$, and $fv(\overline{N'}\langle M'\rangle) \subseteq dom(\sigma) = dom(A)$. We rename $\widetilde{n}_1$ and $\widetilde{n}_2$ so that $\{\widetilde{n}_1\} \cap fn(M) = \emptyset$, $\{\widetilde{n}_1\} \cap \{\widetilde{n}_2\} = \emptyset$, $\{\widetilde{n}_1\} \cap (fn(P'_3\sigma) \cup fn(P'_4\sigma) \cup fn(\sigma \mid \sigma''\sigma)) = \emptyset$, $\{\widetilde{n}_2\} \cap (fn(P_1) \cup fn(P_2) \cup fn(\sigma \mid \sigma''\sigma)) = \emptyset$. By Lemma B.10, $P \overset{\diamond}{\equiv} \nu\widetilde{n}_1.(N(y).P_1 \mid P_2) \xrightarrow{N(M)}_\diamond \nu\widetilde{n}_1.(P_1\{^M/_y\} \mid P_2)$. By definition of $\xrightarrow{N'(M')}_\circ$, $A \equiv \nu\widetilde{n}.(\sigma \mid P) \xrightarrow{N'(M')}_\circ A'$ where $A' = \nu\widetilde{n}.(\sigma \mid \nu\widetilde{n}_1.(P_1\{^M/_y\} \mid P_2))$. (The elements of $\widetilde{n}$ do not occur in $N'(M')$ since $fn(Q') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$.) So by Lemma B.13, $A \xrightarrow{N'(M')} A'$. Since A' is closed and $A \mathcal{R} B$, we have $B \to^* \xrightarrow{N'(M')} B'' \to^* B'$ and $A' \mathcal{R} B'$ for some B', B''. By Lemmas B.8 and B.12, $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' \mid P') \to_\circ{}^* \xrightarrow{N'(M')}_\circ B''$. By Lemmas B.23 and B.19, $P' \to_\diamond{}^* \xrightarrow{N'\sigma'(M'\sigma')}_\diamond B'''$ and $B'' \equiv \nu\widetilde{n}'.(\sigma' \mid B''')$ for some B'''. By Lemma B.10, $P' \to_\diamond{}^* \overset{\diamond}{\equiv} \nu\widetilde{n}_3.(N'\sigma'(z).P_5 \mid P_6)$, $B''' \equiv \nu\widetilde{n}_3.(P_5\{^{M'\sigma'}/_z\} \mid P_6)$, and $\{\widetilde{n}_3\} \cap fn(N'\sigma'(M'\sigma')) = \emptyset$ for some $\widetilde{n}_3$, P_5, and P_6. We rename $\widetilde{n}''$ so that $\{\widetilde{n}''\} \cap fn(\nu\widetilde{n}_1.(P_1\{^M/_y\} \mid P_2)) = \emptyset$ and $\{\widetilde{n}''\} \cap fn(B''') = \emptyset$. Then we have

$$\begin{aligned}
S' &\equiv \nu\widetilde{n}''', \widetilde{n}, \widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid \nu x.(A_1 \mid C_1)) \\
&\equiv \nu\widetilde{u}.\nu\widetilde{n}.\nu\widetilde{n}''.(\sigma \mid \sigma''\sigma \mid \nu x.(\nu\widetilde{n}_1.(P_1\{^x/_y\} \mid P_2) \mid \nu\widetilde{n}_2.(P_3 \mid \{^M/_x\} \mid P_4)) \\
&\equiv \nu\widetilde{u}.\nu\widetilde{n}.\nu\widetilde{n}''.(\sigma \mid \sigma''\sigma \mid \nu x.(\nu\widetilde{n}_1.(P_1\{^x/_y\} \mid P_2) \mid \nu\widetilde{n}_2.(P'_3\sigma \mid \{^M/_x\} \mid P'_4\sigma)) \\
&\equiv \nu\widetilde{u}.\nu\widetilde{n}.\nu\widetilde{n}''.\nu\widetilde{n}_1.\nu\widetilde{n}_2.(\sigma \mid \sigma''\sigma \mid P_1\{^M/_y\} \mid P_2 \mid P'_3\sigma \mid P'_4\sigma) \\
&\equiv \nu\widetilde{u}.\nu\widetilde{n}.\nu\widetilde{n}''.\nu\widetilde{n}_1.\nu\widetilde{n}_2.(\sigma \mid \sigma'' \mid P_1\{^M/_y\} \mid P_2 \mid P'_3 \mid P'_4) \\
&\equiv \nu\widetilde{u}, \widetilde{n}_2.\nu\widetilde{n}.\nu\widetilde{n}''.(\sigma \mid \nu\widetilde{n}_1.(P_1\{^M/_y\} \mid P_2) \mid \sigma'' \mid (P'_3 \mid P'_4)) \\
&\equiv \nu\widetilde{u}, \widetilde{n}_2.(A' \mid C')
\end{aligned}$$

where $C' = \nu\widetilde{n}''.(\sigma'' \mid (P'_3 \mid P'_4))$. We have

$$\begin{aligned}
T &\equiv \nu\widetilde{n}''', \widetilde{n}', \widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid P''\sigma') \\
&\to^* \equiv \nu\widetilde{u}.\nu\widetilde{n}'.\nu\widetilde{n}''.(\sigma' \mid \sigma''\sigma' \mid \nu\widetilde{n}_3.(N'\sigma'(z).P_5 \mid P_6) \\
&\qquad\qquad \mid \nu\widetilde{n}_2.(\overline{N'\sigma'}\langle M'\sigma'\rangle.P'_3\sigma' \mid P'_4\sigma')) \\
&\to \nu\widetilde{u}, \widetilde{n}_2.\nu\widetilde{n}'.\nu\widetilde{n}''.(\sigma' \mid \sigma''\sigma' \mid \nu\widetilde{n}_3.(P_5\{^{M'\sigma'}/_z\} \mid P_6) \mid (P'_3\sigma' \mid P'_4\sigma')) \\
&\equiv \nu\widetilde{u}, \widetilde{n}_2.\nu\widetilde{n}'.\nu\widetilde{n}''.(\sigma' \mid \nu\widetilde{n}_3.(P_5\{^{M'\sigma'}/_z\} \mid P_6) \mid \sigma'' \mid (P'_3 \mid P'_4)) \\
&\equiv \nu\widetilde{u}, \widetilde{n}_2.(\nu\widetilde{n}'.(\sigma' \mid B''') \mid C') \\
&\equiv \nu\widetilde{u}, \widetilde{n}_2.(B'' \mid C') \to^* \nu\widetilde{u}.(B' \mid C')
\end{aligned}$$

We let $T' = \nu\widetilde{u}, \widetilde{n}_2.(B' \mid C')$. Hence, $T \to^* T'$. Since $fv(P'_3 \mid P'_4) \subseteq fv(Q') \subseteq dom(\sigma)$, $\nu\widetilde{u}, \widetilde{n}_2.(\_ \mid C')$ is closing for A' and B', and moreover $A' \mathcal{R} B'$, so $S' \equiv\mathcal{R}' T'$.

(d) $P \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A_1$, $P''\sigma \xrightarrow{N(x)}_\diamond C_1$, and $Q \equiv \nu x.(A_1 \mid C_1)$ for some A_1, C_1, x, and ground term N. We rename x so that $x \notin dom(\sigma) = dom(\sigma')$. By Lemma B.16(3), we transform A_1 into a closed extended process that satisfies the same properties. Since $A \mathcal{R} B$, we have $A \approx_s B$, that is, $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$. By Lemma C.3(2), $P''\sigma' \xrightarrow{N'\sigma'(x)}_\diamond C_2\sigma'$, $fn(C_2) \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, $C_1 \equiv C_2\sigma$, $fn(N') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, $\Sigma \vdash N = N'\sigma$, $fv(C_2) \subseteq dom(\sigma) \cup \{x\}$, and $fv(N') \subseteq dom(\sigma) = dom(A)$ for some C_2 and N'. By definition of $\xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\circ$, $A \equiv \nu\widetilde{n}.(\sigma \mid P) \xrightarrow{\nu x.\overline{N'}\langle x\rangle}_\circ A'$ where

$A' = \nu\widetilde{n}.(\sigma \mid A_1)$, so by Lemma B.13, $A \xrightarrow{\nu x.\overline{N'\langle x\rangle}} A'$. Since A' is closed and $A \mathcal{R} B$, we have $B \rightarrow^* \xrightarrow{\nu x.\overline{N'\langle x\rangle}} B'' \rightarrow^* B'$ and $A' \mathcal{R} B'$ for some B', B''. By Lemmas B.8 and B.12, $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' | P') \rightarrow_\circ{}^* \xrightarrow{\nu x.\overline{N'\langle x\rangle}}_\circ B''$. By Lemmas B.23 and B.19, $P' \rightarrow_\diamond{}^* \xrightarrow{\nu x.\overline{N'\sigma'\langle x\rangle}}_\diamond B'''$ and $B'' \equiv \nu\widetilde{n}'.(\sigma' \mid B''')$ for some B'''. By Lemma B.20, $P' \mid P''\sigma' \rightarrow_\diamond{}^* \rightarrow_\diamond \nu x.(B''' \mid C_2\sigma')$. We rename $\widetilde{n}''$ so that $\{\widetilde{n}''\} \cap fn(A_1) = \emptyset$ and $\{\widetilde{n}''\} \cap fn(B''') = \emptyset$. Moreover,

$$\begin{aligned} S' &\equiv \nu\widetilde{n}''', \widetilde{n}, \widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid \nu x.(A_1 \mid C_1)) \\ &\equiv \nu x, \widetilde{u}.\nu\widetilde{n}.\nu\widetilde{n}''.(\sigma \mid \sigma''\sigma \mid A_1 \mid C_2\sigma) \\ &\equiv \nu x, \widetilde{u}.(\nu\widetilde{n}.(\sigma \mid A_1) \mid \nu\widetilde{n}''.(\sigma'' \mid C_2)) \\ &\equiv \nu x, \widetilde{u}.(A' \mid C') \end{aligned}$$

where $C' = \nu\widetilde{n}''.(\sigma'' \mid C_2)$. We have

$$\begin{aligned} T &\equiv \nu\widetilde{n}''', \widetilde{n}', \widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid P''\sigma') \\ &\rightarrow^* \rightarrow \nu\widetilde{n}''', \widetilde{n}', \widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid \nu x.(B''' \mid C_2\sigma')) \\ &\equiv \nu x, \widetilde{u}.(\nu n'.(\sigma' \mid B''') \mid \nu n''.(\sigma'' \mid C_2)) \equiv \nu x, \widetilde{u}.(B'' \mid C') \\ &\rightarrow^* \nu x, \widetilde{u}.(B' \mid C') \end{aligned}$$

We let $T' = \nu x, \widetilde{u}.(B' \mid C')$. We have $T \rightarrow^* T'$ and, since $A' \mathcal{R} B'$ and $\nu x, \widetilde{u}.(\_ \mid C')$ is closing for A' and B', $S' \equiv\mathcal{R}' T'$.

(3) For every $S \xrightarrow{\alpha} S'$ with S' closed and $fv(\alpha) \subseteq dom(S)$, we prove that $T \rightarrow^* \xrightarrow{\alpha} \rightarrow^* T'$ and $S' \equiv\mathcal{R}'\equiv T'$ for some T'. We rename $\widetilde{n}'''$, $\widetilde{n}$, $\widetilde{n}'$, $\widetilde{n}''$ so that these names do not occur in α. By Lemma B.12, we have $\mathrm{pnf}(S) \xrightarrow{\alpha}_\circ S'$. By Lemma B.19, we have $P \mid P''\sigma \xrightarrow{\alpha'}_\diamond A_0$, $S' \equiv \nu\widetilde{n}'''.\widetilde{n}, \widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid A_0)$, and $bv(\alpha) \cap dom(\sigma \mid \sigma''\sigma) \setminus \{\widetilde{x}\} = \emptyset$ for $\alpha' = \alpha(\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} = \alpha\sigma''\sigma$ and some A_0. We rename $\widetilde{x}$ so that $bv(\alpha) \cap \{\widetilde{x}\} = \emptyset$. By Lemma B.18, we have two cases:

(a) $P \xrightarrow{\alpha'}_\diamond A_1$ and $A_0 \equiv A_1 \mid P''\sigma$ for some A_1. By Lemma B.16(3), we transform A_1 into a closed extended process that satisfies the same properties. We have $\alpha' = (\alpha\sigma'')\sigma$, the elements of $\widetilde{n}$ do not occur in $\alpha\sigma''$, because they do not occur in α nor in σ''. We also have $bv(\alpha') \cap dom(\sigma) = \emptyset$. By definition of $\xrightarrow{\alpha\sigma''}_\circ$, we have $A \equiv \nu\widetilde{n}.(\sigma \mid P) \xrightarrow{\alpha\sigma''}_\circ A'$ where $A' = \nu\widetilde{n}.(\sigma | A_1)$, so by Lemma B.13, $A \xrightarrow{\alpha\sigma''} A'$. Since $fv(\alpha) \subseteq dom(S) \subseteq dom(\sigma) \cup dom(\sigma'')$, we have $fv(\alpha\sigma'') \subseteq dom(\sigma) = dom(A)$. Since A' is closed and $A \mathcal{R} B$, we have $B \rightarrow^* \xrightarrow{\alpha\sigma''} B'' \rightarrow^* B'$ and $A' \mathcal{R} B'$ for some B' and B''. By Lemmas B.8 and B.12, $\mathrm{pnf}(B) = \nu\widetilde{n}'.(\sigma' \mid P') \rightarrow_\circ{}^* \xrightarrow{\alpha\sigma''}_\circ B''$. By Lemmas B.23 and B.19, $P' \rightarrow_\diamond{}^* \xrightarrow{\alpha\sigma''\sigma'}_\diamond B'''$, $B'' \equiv \nu\widetilde{n}'.(\sigma' \mid B''')$, and $bv(\alpha) \cap dom(\sigma') = \emptyset$ for some B'''. Hence by Par′, $P' \mid P''\sigma' \rightarrow_\diamond{}^* \xrightarrow{\alpha\sigma''\sigma'}_\diamond B''' \mid P''\sigma'$. By definition of $\xrightarrow{\alpha}_\circ$,

$$\begin{aligned} \mathrm{pnf}(T) &= \nu\widetilde{n}''', \widetilde{n}', \widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid P''\sigma') \\ &\rightarrow_\circ{}^* \xrightarrow{\alpha}_\circ \nu\widetilde{n}''', \widetilde{n}', \widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid B''' \mid P''\sigma') \,. \end{aligned}$$

(We have $bv(\alpha) \cap fv((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}}) = \emptyset$ because we have $bv(\alpha) = bv(\alpha')$, $fv((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}}) = dom(\sigma') \cup dom(\sigma'') \setminus \{\widetilde{x}\} = dom(\sigma) \cup dom(\sigma'') \setminus \{\widetilde{x}\}$, and $bv(\alpha') \cap (dom(\sigma) \cup dom(\sigma'') \setminus \{\widetilde{x}\}) = \emptyset$.) We rename $\widetilde{n}''$ so that $\{\widetilde{n}''\} \cap fn(A_1) = \emptyset$ and

$\{\widetilde{n}''\} \cap fn(B''') = \emptyset$. By Lemmas B.1, B.9, and B.13, we have

$$\begin{aligned} T &\equiv \mathrm{pnf}(T) \\ &\to^* \xrightarrow{\alpha} \nu\widetilde{n}'''.\nu\widetilde{n}'.\nu\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid B''' \mid P''\sigma') \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}'.(\sigma' \mid B''') \mid \nu\widetilde{n}''.(\sigma'' \mid P'')) \\ &\equiv \nu\widetilde{u}.(B'' \mid C) \\ &\to^* \nu\widetilde{u}.(B' \mid C) \end{aligned}$$

Moreover,

$$\begin{aligned} S' &\equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid A_0) \\ &\equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid A_1 \mid P''\sigma) \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}.(\sigma \mid A_1) \mid \nu\widetilde{n}''.(\sigma'' \mid P'')) \\ &\equiv \nu\widetilde{u}.(A' \mid C) \end{aligned}$$

We let $T' = \nu\widetilde{u}.(B' \mid C)$. Then we have $T \to^* \xrightarrow{\alpha} \to^* T'$ and $\nu\widetilde{u}.(\_ \mid C)$ is closing for A' and B' so $S' \equiv_{\mathcal{R}'} T'$.

(b) $P''\sigma \xrightarrow{\alpha'}_\circ A_1$ and $A_0 \equiv P \mid A_1$ for some A_1. Since $A \mathrel{\mathcal{R}} B$, we have $A \approx_s B$, that is, $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$. We have $\Sigma \vdash (\alpha\sigma'')\sigma = \alpha'$, so if $\alpha' = N(M)$, then we have $\alpha\sigma'' = N'(M')$, $\Sigma \vdash M'\sigma = M$, and $fn(M') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$ for some N', M'. By Lemma C.3(1), $P''\sigma' \xrightarrow{\alpha''\sigma'}_\circ A''\sigma'$, $fn(A'') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, $A_1 \equiv A''\sigma$, $fn(\alpha'') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, $\Sigma \vdash \alpha' = \alpha''\sigma$, and $A''\sigma$ is closed for some A'', α''. By Par', $P' \mid P''\sigma' \xrightarrow{\alpha''\sigma'}_\circ P' \mid A''\sigma'$. We have $\Sigma \vdash \alpha\sigma''\sigma = \alpha' = \alpha''\sigma$, $fn(\alpha\sigma'') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, $fn(\alpha'') \cap \{\widetilde{n}, \widetilde{n}'\} = \emptyset$, and $\nu\widetilde{n}.\sigma \approx_s \nu\widetilde{n}'.\sigma'$, so $\Sigma \vdash \alpha\sigma''\sigma' = \alpha''\sigma'$ by definition of static equivalence. By definition of $\xrightarrow{\alpha}_\circ$,

$$\begin{aligned} \mathrm{pnf}(T) = \nu\widetilde{n}'''.\widetilde{n}'.\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid P''\sigma') \\ \xrightarrow{\alpha}_\circ \nu\widetilde{n}'''.\widetilde{n}'.\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid A''\sigma') \,. \end{aligned}$$

(We have $bv(\alpha''\sigma') \cap fv((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}}) = \emptyset$ because $bv(\alpha''\sigma') = bv(\alpha'') = bv(\alpha')$, $fv((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}}) = dom(\sigma') \cup dom(\sigma'') \setminus \{\widetilde{x}\} = dom(\sigma) \cup dom(\sigma'') \setminus \{\widetilde{x}\}$, and $bv(\alpha') \cap (dom(\sigma) \cup dom(\sigma'') \setminus \{\widetilde{x}\}) = \emptyset$.) Hence, by Lemmas B.1, B.9, and B.13, we have

$$\begin{aligned} T &\equiv \mathrm{pnf}(T) \\ &\xrightarrow{\alpha} \nu\widetilde{n}'''.\widetilde{n}'.\widetilde{n}''.((\sigma' \mid \sigma''\sigma')_{|dom(\sigma'|\sigma''\sigma')\setminus\{\widetilde{x}\}} \mid P' \mid A''\sigma') \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}'.(\sigma' \mid P') \mid \nu\widetilde{n}''.(\sigma'' \mid A'')) \\ &\equiv \nu\widetilde{u}.(B \mid C') \end{aligned}$$

where $C' = \nu\widetilde{n}''.(\sigma'' \mid A'')$. Moreover,

$$\begin{aligned} S' &\equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid A_0) \\ &\equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid P \mid A_1) \\ &\equiv \nu\widetilde{n}'''.\widetilde{n}.\widetilde{n}''.((\sigma \mid \sigma''\sigma)_{|dom(\sigma|\sigma''\sigma)\setminus\{\widetilde{x}\}} \mid P \mid A''\sigma) \\ &\equiv \nu\widetilde{u}.(\nu\widetilde{n}.(\sigma \mid P) \mid \nu\widetilde{n}''.(\sigma'' \mid A'')) \\ &\equiv \nu\widetilde{u}.(A \mid C') \end{aligned}$$

We let $T' = \nu\widetilde{u}.(B \mid C')$. We have $T \xrightarrow{\alpha} T'$ and since $A \mathrel{\mathcal{R}} B$ and $\nu\widetilde{u}.(\_ \mid C')$ is closing for A and B, we have $S' \equiv_{\mathcal{R}'} T'$. □

Lemma 4.19. *Let A be a closed extended process. We have $A \Downarrow a$ if and only if $A \to^* \xrightarrow{\nu x.\overline{a}\langle x\rangle} A'$ for some fresh variable x and some A'.*

Proof. In order to establish this claim, we argue that $A \equiv E[\overline{a}\langle M\rangle.P]$ for some evaluation context $E[\_]$ that does not bind a if and only if $A \xrightarrow{\nu x.\overline{a}\langle x\rangle} A'$ for some fresh variable x and some A'.

For the implication from left to right, let x be a fresh variable. We derive

$$\overline{a}\langle M\rangle.P \xrightarrow{\nu x.\overline{a}\langle x\rangle} P \mid \{^M/_x\} \qquad \text{by Out-Var}$$

$$E[\overline{a}\langle M\rangle.P] \xrightarrow{\nu x.\overline{a}\langle x\rangle} E[P \mid \{^M/_x\}] \qquad \text{by Par and Scope}$$

$$A \xrightarrow{\nu x.\overline{a}\langle x\rangle} E[P \mid \{^M/_x\}] \qquad \text{by Struct, since } A \equiv E[\overline{a}\langle M\rangle.P]$$

Conversely, if $A \xrightarrow{\nu x.\overline{a}\langle x\rangle} A'$ for some fresh variable x and some A', then we show by induction on the derivation that $A \equiv E[\overline{a}\langle M\rangle.P]$ for some evaluation context $E[\_]$ that does not bind a. In case Out-Var, the context E is empty. In case Scope, a restriction that does not bind a is added to E. In case Par, a parallel composition is added to E. In case Struct, the context E is unchanged. □

C.3 Observational Equivalence Implies Labelled Bisimilarity

Finally, the goal of this section is to establish the lemmas needed in the outline of the argument that observational equivalence implies labelled bisimilarity in Section 4.5. The section also contains a corollary, namely that observational equivalence and static equivalence coincide on frames.

Lemma C.6. *Let P be a plain process. The existence of P' such that $\Sigma \vdash P = P'$ and $p \notin fn(P')$ is preserved by structural equivalence ($\overset{\diamond}{\equiv}$) and reduction ($\to_\diamond$) of P.*

Let A be a normal process. The existence of A' such that $\Sigma \vdash A = A'$ and $p \notin fn(A')$ is preserved by structural equivalence ($\overset{\circ}{\equiv}$) and reduction ($\to_\circ$) of A.

Proof. *Property 1:* Suppose that $P \overset{\diamond}{\equiv} Q$, $\Sigma \vdash P = P'$, and $p \notin fn(P')$. We show that there exists Q' such that $\Sigma \vdash Q = Q'$ and $p \notin fn(Q')$, by induction on the derivation of $P \overset{\diamond}{\equiv} Q$. We consider as base cases the application of each rule under an evaluation context, in the two directions, and use induction only for transitivity.

- Case Rewrite′, under an evaluation context E. We have $E[P_1\{^M/_x\}] \overset{\diamond}{\equiv} E[P_1\{^N/_x\}]$, $\Sigma \vdash M = N$, $\Sigma \vdash E[P_1\{^M/_x\}] = P'$, and $p \notin fn(P')$. Since $\Sigma \vdash E[P_1\{^N/_x\}] = E[P_1\{^M/_x\}] = P'$, we have the result with $Q' = Q$.
- Case Par-C′, under an evaluation context E. We have $E[P_1 | Q_1] \overset{\diamond}{\equiv} E[Q_1 | P_1]$, $\Sigma \vdash E[P_1 | Q_1] = P'$, and $p \notin fn(P')$. Since $\Sigma \vdash E[P_1 | Q_1] = P'$, we have $P' = E'[P_1' | Q_1']$ with $\Sigma \vdash E = E'$, $\Sigma \vdash P_1 = P_1'$, and $\Sigma \vdash Q_1 = Q_1'$. Let $Q' = E'[Q_1' \mid P_1']$. We have $\Sigma \vdash E[Q_1 \mid P_1] = Q'$ and $p \notin fn(Q') = fn(P')$.
- All other base cases are handled similarly to case Par-C′.
- The case of transitivity follows by applying the induction hypothesis twice.

Property 2: Suppose that $P \to_\diamond Q$, $\Sigma \vdash P = P'$, and $p \notin fn(P')$. We show that there exists Q' such that $\Sigma \vdash Q = Q'$ and $p \notin fn(Q')$, by induction on the derivation of $P \to_\diamond Q$. Again, we consider as base cases the application of each rule under an evaluation context, and use induction only for the application of $\overset{\diamond}{\equiv}$.

- Case Comm′, under an evaluation context E. We have $E[\overline{N}\langle M\rangle.P_1 \mid N(x).Q_1] \to_\diamond E[P_1 \mid Q_1\{^M/_x\}]$, $\Sigma \vdash E[\overline{N}\langle M\rangle.P_1 \mid N(x).Q_1] = P'$, and $p \notin fn(P')$. Since $\Sigma \vdash E[\overline{N}\langle M\rangle.P_1 \mid N(x).Q_1] = P'$, we have $P' = E'[\overline{N'}\langle M'\rangle.P_1' \mid N''(x).Q_1']$ with $\Sigma \vdash E = E'$, $\Sigma \vdash M = M'$, $\Sigma \vdash P_1 = P_1'$, and $\Sigma \vdash Q_1 = Q_1'$. Let $Q' = E'[P_1' \mid Q_1'\{^{M'}/_x\}]$. We have $\Sigma \vdash E[P_1 \mid Q_1\{^M/_x\}] = Q'$ and $p \notin fn(Q')$ since $fn(Q') \subseteq fn(P')$.

- Case Then′, under an evaluation context E. We have $E[\mathit{if}\ M = M\ \mathit{then}\ P_1\ \mathit{else}\ Q_1] \rightarrow_\diamond E[P_1]$, $\Sigma \vdash E[\mathit{if}\ M = M\ \mathit{then}\ P_1\ \mathit{else}\ Q_1] = P'$, and $p \notin fn(P')$. Since $\Sigma \vdash E[\mathit{if}\ M = M\ \mathit{then}\ P_1\ \mathit{else}\ Q_1] = P'$, we have $P' = E'[\mathit{if}\ M' = M''\ \mathit{then}\ P_1'\ \mathit{else}\ Q_1']$ with $\Sigma \vdash E = E'$ and $\Sigma \vdash P_1 = P_1'$. Let $Q' = E'[P_1']$. We have $\Sigma \vdash E[P_1] = Q'$ and $p \notin fn(Q')$ since $fn(Q') \subseteq fn(P')$.
- Case Else′ is handled similarly to case Then′.
- In case we additionally apply $\overset{\circ}{\equiv}$, we conclude using Property 1 and the induction hypothesis.

Property 3: Suppose that $A \overset{\circ}{\equiv} B$, $\Sigma \vdash A = A'$, and $p \notin fn(A')$. We show that there exists B' such that $\Sigma \vdash B = B'$ and $p \notin fn(B')$, by induction on the derivation of $A \overset{\circ}{\equiv} B$.

- Case Plain″ follows from Property 1.
- Case New-Par″. We have $\nu\widetilde{n}.(\sigma | \nu n'.P) \overset{\circ}{\equiv} \nu\widetilde{n}, n'.(\sigma | P)$ with $n' \notin fn(\sigma)$, $\Sigma \vdash \nu\widetilde{n}.(\sigma | \nu n'.P) = A'$, and $p \notin fn(A')$. If $p \in \{\widetilde{n}, n'\}$, then we have the result with $B' = \nu\widetilde{n}, n'.(\sigma \mid P)$. Otherwise, since $\Sigma \vdash \nu\widetilde{n}.(\sigma \mid \nu n'.P) = A'$, we have $A' = \nu\widetilde{n}.(\sigma' \mid \nu n'.P')$ with $\Sigma \vdash \sigma = \sigma'$ and $\Sigma \vdash P = P'$. Let $B' = \nu\widetilde{n}, n'.(\sigma' \mid P')$. We have $\Sigma \vdash \nu\widetilde{n}, n'.(\sigma \mid P) = B'$ and $p \notin fn(B')$ since $fn(B') \subseteq fn(A')$.
- Case New-Par″ reversed. We have $\nu\widetilde{n}, n'.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid \nu n'.P)$ with $n' \notin fn(\sigma)$, $\Sigma \vdash \nu\widetilde{n}, n'.(\sigma \mid P) = A'$, and $p \notin fn(A')$. If $p \in \{\widetilde{n}\}$, then we have the result with $B' = \nu\widetilde{n}.(\sigma \mid \nu n'.P)$. If $p = n'$, then we also have the result with $B' = \nu\widetilde{n}.(\sigma \mid \nu n'.P)$ because $n' \notin fn(\sigma)$. Otherwise, since $\Sigma \vdash \nu\widetilde{n}, n'.(\sigma \mid P) = A'$, we have $A' = \nu\widetilde{n}, n'.(\sigma' \mid P')$ with $\Sigma \vdash \sigma = \sigma'$ and $\Sigma \vdash P = P'$. Let $B' = \nu\widetilde{n}.(\sigma' \mid \nu n'.P')$. We have $\Sigma \vdash \nu\widetilde{n}.(\sigma \mid \nu n'.P) = B'$ and $p \notin fn(B')$.
- Case New-C″ is handled similarly to case New-Par″.
- Case Rewrite″. We have $\nu\widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma' \mid P)$ with $\Sigma \vdash \sigma = \sigma'$, $\Sigma \vdash \nu\widetilde{n}.(\sigma \mid P) = A'$, and $p \notin fn(A')$. Since $\Sigma \vdash \nu\widetilde{n}.(\sigma' \mid P) = \nu\widetilde{n}.(\sigma \mid P) = A'$, we have the result with $B' = A'$.
- The case of transitivity follows by applying the induction hypothesis twice.

Property 4: Suppose that $A \rightarrow_\circ B$, $\Sigma \vdash A = A'$, and $p \notin fn(A')$. We show that there exists B' such that $\Sigma \vdash B = B'$ and $p \notin fn(B')$. Suppose $\nu\widetilde{n}.(\sigma \mid P) \rightarrow_\circ \nu\widetilde{n}.(\sigma \mid Q)$ with $P \rightarrow_\diamond Q$, $\Sigma \vdash \nu\widetilde{n}.(\sigma \mid P) = A'$, and $p \notin fn(A')$. If $p \in \{\widetilde{n}\}$, then we have the result with $B' = \nu\widetilde{n}.(\sigma \mid Q)$. Otherwise, since $\Sigma \vdash \nu\widetilde{n}.(\sigma \mid P) = A'$, we have $A' = \nu\widetilde{n}.(\sigma' \mid P')$ with $\Sigma \vdash \sigma = \sigma'$ and $\Sigma \vdash P = P'$. Since $p \notin fn(A')$ and $p \notin \{\widetilde{n}\}$, $p \notin fn(\sigma') \cup fn(P')$. By Property 2, there exists Q' such that $\Sigma \vdash Q = Q'$ and $p \notin fn(Q')$. Let $B' = \nu\widetilde{n}.(\sigma' \mid Q')$. We have $\Sigma \vdash \nu\widetilde{n}.(\sigma \mid Q) = B'$ and $p \notin fn(B')$. In case we additionally apply $\overset{\circ}{\equiv}$, we conclude using Property 3 and the induction hypothesis. □

Lemma C.7. *If $p \notin fn(A)$, then $A \not\Downarrow p$.*

Proof. In order to obtain a contradiction, suppose that $A \Downarrow p$, that is, that $A \rightarrow^* \equiv E[\overline{p}\langle M\rangle.P]$ for some M, P, and evaluation context $E[\_]$ that does not bind p. Hence, $\mathrm{pnf}(A) \rightarrow_\circ{}^* \overset{\circ}{\equiv} E[\overline{p}\langle M\rangle.P]$ for some M, P, and evaluation context $E[\_]$ that does not bind p. Let $A_1 = \mathrm{pnf}(A)$. We have $p \notin fn(A_1)$. By Lemma C.6, the existence of A' such that $\Sigma \vdash A_1 = A'$ and $p \notin fn(A')$ is preserved by structural equivalence and reduction of A_1, so there exists A' such that $\Sigma \vdash E[\overline{p}\langle M\rangle.P] = A'$ and $p \notin fn(A')$. Hence, there exists N such that $\Sigma \vdash p = N$ and $p \notin fn(N)$. Since the equational theory is preserved by substitution of terms for names, for all N', $\Sigma \vdash p\{^{N'}/_p\} = N\{^{N'}/_p\}$, that is $\Sigma \vdash N' = N$, which contradicts the assumption that the equational theory is non-trivial. □

Lemma C.8. *If $p \notin fn(P)$ and $P \rightarrow_\diamond{}^* \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond A$ or $P \rightarrow_\diamond{}^* \xrightarrow{N(M)}_\diamond A$, then $\Sigma \vdash N \neq p$.*

Proof. The proof uses ideas similar to the proof of Lemma C.7. By Lemma B.10, $P \rightarrow_\diamond{}^* \overset{\diamond}{\equiv} \nu\widetilde{n}.(\overline{N}\langle M\rangle.P_1 \mid P_2)$ for some $\widetilde{n}$, M, P_1, P_2 with $\{\widetilde{n}\} \cap fn(N) = \emptyset$, or $P \rightarrow_\diamond{}^* \overset{\diamond}{\equiv} \nu\widetilde{n}.(N(x).P_1 \mid P_2)$ for some for some $\widetilde{n}$, x, P_1, P_2 with $\{\widetilde{n}\} \cap fn(N) = \emptyset$. By Lemma C.6, the existence of P' such that $\Sigma \vdash P = P'$ and $p \notin fn(P')$ is preserved by structural equivalence and reduction of P, so there exists N' such that $\Sigma \vdash N = N'$ and $p \notin fn(N')$. If we had $\Sigma \vdash N = p$, then we would have $\Sigma \vdash p = N'$ and $p \notin fn(N')$, which yields a contradiction as in the proof of Lemma C.7. So $\Sigma \vdash N \neq p$. □

Lemma C.9. $\approx \subseteq \approx_s$.

Proof. If A and B are observationally equivalent, then $A \mid C$ and $B \mid C$ have the same barbs for every C with $fv(C) \subseteq dom(A)$. In particular, $A \mid C$ and $B \mid C$ have the same barb $\Downarrow a$ for every C of the special form $\mathit{if}\ M = N\ \mathit{then}\ \overline{a}\langle s\rangle$, where a does not occur in A or B and $fv(C) \subseteq dom(A)$, that is, $fv(M) \cup fv(N) \subseteq dom(A)$. We obtain that A and B are statically equivalent, using the following property: assuming that A is closed, $fv(M) \cup fv(N) \subseteq dom(A)$, and a does not occur in A, we have $(M = N)\varphi(A)$ if and only if $A \mid \mathit{if}\ M = N\ \mathit{then}\ \overline{a}\langle s\rangle \Downarrow a$. We show this property below.

Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$. We rename $\widetilde{n}$ so that $\{\widetilde{n}\} \cap (fn(M) \cup fn(N) \cup \{a\}) = \emptyset$. If $(M = N)\varphi(A)$, then $M\sigma = N\sigma$, so $A \mid \mathit{if}\ M = N\ \mathit{then}\ \overline{a}\langle s\rangle \equiv \nu\widetilde{n}.(\sigma \mid P \mid \mathit{if}\ M\sigma = N\sigma\ \mathit{then}\ \overline{a}\langle s\rangle) \to \nu\widetilde{n}.(\sigma \mid P \mid \overline{a}\langle s\rangle)$, so we conclude that $A \mid \mathit{if}\ M = N\ \mathit{then}\ \overline{a}\langle s\rangle \Downarrow a$. Conversely, in order to obtain a contradiction, suppose that $(M \neq N)\varphi(A)$ and $A \mid \mathit{if}\ M = N\ \mathit{then}\ \overline{a}\langle s\rangle \Downarrow a$. Lemma 4.19 implies that $A \mid \mathit{if}\ M = N\ \mathit{then}\ \overline{a}\langle s\rangle \to^* \xrightarrow{\nu x.\overline{a}\langle x\rangle} A'$ for some fresh variable x and some A'. So $\mathrm{pnf}(A \mid \mathit{if}\ M = N\ \mathit{then}\ \overline{a}\langle s\rangle) = \nu\widetilde{n}.(\sigma \mid P \mid \mathit{if}\ M\sigma = N\sigma\ \mathit{then}\ \overline{a}\langle s\rangle) \to_\circ^* \xrightarrow{\nu x.\overline{a}\langle x\rangle}_\circ A'$ by Lemmas B.8 and B.12. Then $P \mid \mathit{if}\ M\sigma = N\sigma\ \mathit{then}\ \overline{a}\langle s\rangle \to_\diamond^* \xrightarrow{\nu x.\overline{a}\langle x\rangle}_\diamond A''$, $A' \equiv \nu\widetilde{n}.(\sigma \mid A'')$, and $x \notin dom(\sigma)$ for some A'' by Lemmas B.23 and B.19. We have $a \notin fn(\mathrm{pnf}(A))$, so $a \notin fn(P)$. We show by induction on the length of the trace, that it is impossible to have $P \mid \mathit{if}\ M\sigma = N\sigma\ \mathit{then}\ \overline{a}\langle s\rangle \to_\diamond^* \xrightarrow{\nu x.\overline{a}\langle x\rangle}_\diamond A''$.

- If this trace contains a single step, then $P \mid \mathit{if}\ M\sigma = N\sigma\ \mathit{then}\ \overline{a}\langle s\rangle \xrightarrow{\nu x.\overline{a}\langle x\rangle}_\diamond A''$, so by Lemma B.18, $P \xrightarrow{\nu x.\overline{a}\langle x\rangle}_\diamond$, which yields a contradiction by Lemma C.8.
- If this trace contains several steps, the first step is an internal reduction, so by Lemma B.24, either P reduces, and we conclude by induction hypothesis, or $\mathit{if}\ M\sigma = N\sigma\ \mathit{then}\ \overline{a}\langle s\rangle$ reduces to $\mathbf{0}$ and $P \mid \mathbf{0} \to_\diamond^* \xrightarrow{\nu x.\overline{a}\langle x\rangle}_\diamond A''$, which yields a contradiction by Lemma C.8. □

Lemma C.10. *Let $\widetilde{n}$ be pairwise distinct names. Let $\widetilde{n'}$ be pairwise distinct names that do not occur in P nor in P'.*
If $P \overset{\diamond}{\equiv} P'$ and $\Sigma \vdash P = P\{^{\widetilde{n'}}/_{\widetilde{n}}\}$, then $P\{^{\widetilde{n'}}/_{\widetilde{n}}\} \overset{\diamond}{\equiv} P'\{^{\widetilde{n'}}/_{\widetilde{n}}\}$ and $\Sigma \vdash P' = P'\{^{\widetilde{n'}}/_{\widetilde{n}}\}$.
If $P \to_\diamond P'$ and $\Sigma \vdash P = P\{^{\widetilde{n'}}/_{\widetilde{n}}\}$, then $P\{^{\widetilde{n'}}/_{\widetilde{n}}\} \to_\diamond P'\{^{\widetilde{n'}}/_{\widetilde{n}}\}$ and $\Sigma \vdash P' = P'\{^{\widetilde{n'}}/_{\widetilde{n}}\}$.

Proof. By induction on the derivations of $P \overset{\diamond}{\equiv} P'$ and $P \to_\diamond P'$, respectively. □

Lemma 4.21. *Let A be a closed extended process. Let N and M be terms such that $fv(\overline{N}\langle M\rangle) \subseteq dom(A)$. Let p be a name that does not occur in A, M, and N.*

(1) *If $A \xrightarrow{N(M)} A'$ and p does not occur in A', then $A \mid T^p_{N(M)} \to\!\!\to A'$ and $A' \not\Downarrow p$.*

(2) *If $A \mid T^p_{N(M)} \to^* A'$ and $A' \not\Downarrow p$, then $A \to^* \xrightarrow{N(M)} \to^* A'$.*

Proof. *Property 1:* Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$. We rename $\widetilde{n}$ so that these names do not occur in N, M, p. By Lemma B.12, $\mathrm{pnf}(A) \xrightarrow{N(M)}_\circ A'$. By Lemma B.19, $P \xrightarrow{N\sigma(M\sigma)}_\diamond A''$ and $A' \equiv \nu\widetilde{n}.(\sigma \mid A'')$ for some A'. By Lemma B.10, $P \overset{\diamond}{\equiv} \nu\widetilde{n'}.(N\sigma(x').P_1 \mid P_2)$, $A'' \equiv \nu\widetilde{n'}.(P_1\{^{M\sigma}/_{x'}\} \mid P_2)$, $\{\widetilde{n'}\} \cap fn(N\sigma(M\sigma)) = \emptyset$, for some $\widetilde{n'}$, P_1, P_2, x'. We rename $\widetilde{n'}$ so that $p \notin \{\widetilde{n'}\}$. Hence, by Lemmas B.1 and B.7,

$$\begin{aligned}
A \mid \overline{p}\langle p\rangle \mid \overline{N}\langle M\rangle.p(x) &\equiv \mathrm{pnf}(A) \mid \overline{p}\langle p\rangle \mid \overline{N}\langle M\rangle.p(x) \\
&\equiv \nu\widetilde{n}.(\sigma \mid \nu\widetilde{n'}.(N\sigma(x').P_1 \mid P_2)) \mid \overline{p}\langle p\rangle \mid \overline{N}\langle M\rangle.p(x) \\
&\equiv \nu\widetilde{n}.(\sigma \mid \nu\widetilde{n'}.(N\sigma(x').P_1 \mid P_2 \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x))) \\
&\to \nu\widetilde{n}.(\sigma \mid \nu\widetilde{n'}.(P_1\{^{M\sigma}/_{x'}\} \mid P_2 \mid \overline{p}\langle p\rangle \mid p(x)))
\end{aligned}$$

$$\begin{aligned}&\to \nu\tilde{n}.(\sigma \mid \nu\tilde{n}'.(P_1\{^{M\sigma}/_{x'}\} \mid P_2))\\ &\equiv \nu\tilde{n}.(\sigma \mid A'')\\ &\equiv A'\end{aligned}$$

Since $p \notin fn(A')$, we have $A' \not\Downarrow p$ by Lemma C.7.

Property 2: Let $\mathrm{pnf}(A) = \nu\tilde{n}.(\sigma \mid P)$. By Lemma B.2, $\mathrm{pnf}(A)$ is closed. We rename $\tilde{n}$ so that these names do not occur in N, M, p. Then $\mathrm{pnf}(A \mid \overline{p}\langle p\rangle \mid \overline{N}\langle M\rangle.p(x)) = \nu\tilde{n}.(\sigma \mid P \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x))$. By Lemma B.8, $\mathrm{pnf}(A \mid \overline{p}\langle p\rangle \mid \overline{N}\langle M\rangle.p(x)) \to_\circ^* \mathrm{pnf}(A')$. By Lemma B.23 applied several times, $P \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x) \to_\circ^* P'$ and $\mathrm{pnf}(A') \equiv \nu\tilde{n}.(\sigma \mid P')$ for some closed process P'. Since $A' \not\Downarrow p$, we have $P' \not\Downarrow p$. (If we had $P' \Downarrow p$, we would immediately obtain $A' \Downarrow p$ by definition of $\Downarrow p$.)

We prove that, if P is a closed process, $P \mid \overline{p}\langle p\rangle \mid p(x) \to_\circ^* P'$, $P' \not\Downarrow p$, and $p \notin fn(P)$, then $P \overset{\diamond}{\equiv} \to_\circ^* P'$, by induction on the length of the trace. Since $P \mid \overline{p}\langle p\rangle \mid p(x) \Downarrow p$, the trace $P \mid \overline{p}\langle p\rangle \mid p(x) \to_\circ^* P'$ has at least one step: $P \mid \overline{p}\langle p\rangle \mid p(x) \to_\circ P_1 \to_\circ^* P'$. By Lemmas B.24, B.18, and C.8, the only cases that can happen in the first step are:

- $P \to_\circ P''$ and $P'' \mid \overline{p}\langle p\rangle \mid p(x) \equiv P_1 \to_\circ^* P'$ for some closed process P''. As above this trace has at least one step, so $P'' \mid \overline{p}\langle p\rangle \mid p(x) \to_\circ^* P'$. By Lemma C.10, we rename p inside P'' so that $p \notin fn(P'')$, and we obtain the desired result by induction hypothesis.
- $\overline{p}\langle p\rangle \xrightarrow{\nu y.\overline{N}\langle y\rangle}_\circ A_1 \equiv \{^p/_y\}$, $p(x) \xrightarrow{N(y)}_\circ A_2 \equiv \mathbf{0}$, $\Sigma \vdash N = p$, $P_1 \equiv P \mid \nu y.(A_1 \mid A_2) \equiv P \mid \nu y.(\{^p/_y\} \mid \mathbf{0}) \equiv P$ so $P \mid \overline{p}\langle p\rangle \mid p(x) \to_\circ P \overset{\diamond}{\equiv} P_1 \to_\circ^* P'$, so we obtain $P \overset{\diamond}{\equiv} \to_\circ^* P'$ as desired.

Next, we prove that, if $P \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x)$ is a closed process, $P \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x) \to_\circ^* P'$, $P' \not\Downarrow p$, and $p \notin fn(P) \cup fn(N\sigma) \cup fn(M\sigma)$, then $P \to_\circ^* \xrightarrow{N\sigma(M\sigma)}_\circ \to_\circ^* P'$, by induction on the length of the trace. Since $P \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x) \Downarrow p$, the trace $P \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x) \to_\circ^* P'$ has at least one step: $P \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x) \to_\circ P_1 \to_\circ^* P'$. By Lemmas B.24, B.18, and C.8, the only cases that can happen in the first step are:

- $P \to_\circ P''$ and $P'' \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x) \equiv P_1 \to_\circ^* P'$ for some closed process P''. As above this trace has at least one step, so $P'' \mid \overline{p}\langle p\rangle \mid \overline{N\sigma}\langle M\sigma\rangle.p(x) \to_\circ^* P'$. By Lemma C.10, we rename p inside P'' so that $p \notin fn(P'')$, and we obtain the desired result by induction hypothesis.
- $P \xrightarrow{N'(y)}_\circ B$, $\overline{N\sigma}\langle M\sigma\rangle.p(x) \xrightarrow{\nu y.\overline{N'}\langle y\rangle}_\circ B'$, and $P_1 \equiv \nu y.(B \mid \overline{p}\langle p\rangle \mid B')$. By Lemma B.10, $P \overset{\diamond}{\equiv} \nu\tilde{n}'.(N'(z).P_2 \mid P_3)$, $B \equiv \nu\tilde{n}'.(P_2\{^y/_z\} \mid P_3)$, and $\{\tilde{n}'\} \cap fn(N') = \emptyset$ for some $\tilde{n}'$, z, P_2, and P_3. By Lemma B.18, $\Sigma \vdash N\sigma = N'$, $y \notin fv(\overline{N\sigma}\langle M\sigma\rangle.p(x))$, and $B' \equiv p(x) \mid \{^{M\sigma}/_y\}$. We rename $\tilde{n}'$ so that these names do not appear in $M\sigma$ and are distinct from p. By Lemma C.10, we rename p inside $\nu\tilde{n}'.(N'(z).P_2 \mid P_3)$ so that $p \notin fn(\nu\tilde{n}'.(N'(z).P_2 \mid P_3))$, so $p \notin fn(P_2) \cup fn(P_3)$. Hence $P_1 \equiv \nu y.(\nu\tilde{n}'.(P_2\{^y/_z\} \mid P_3) \mid \overline{p}\langle p\rangle \mid \{^{M\sigma}/_y\} \mid p(x)) \equiv \nu\tilde{n}'.(P_2\{^{M\sigma}/_z\} \mid P_3) \mid \overline{p}\langle p\rangle \mid p(x)$. We have $P \overset{\diamond}{\equiv} \nu\tilde{n}'.(N'(z).P_2 \mid P_3) \xrightarrow{N\sigma(M\sigma)}_\circ \nu\tilde{n}'.(P_2\{^{M\sigma}/_z\} \mid P_3)$. Let $P_4 = \nu\tilde{n}'.(P_2\{^{M\sigma}/_z\} \mid P_3)$. We have then $P \xrightarrow{N\sigma(M\sigma)}_\circ P_4$ and $P_4 \mid \overline{p}\langle p\rangle \mid p(x) \equiv P_1 \to_\circ^* P'$. By Lemma B.16(3), we transform P_4 into a closed process that satisfies the same properties. Since $P' \not\Downarrow p$, this trace has at least one step, so $P_4 \mid \overline{p}\langle p\rangle \mid p(x) \to_\circ^* P'$. Since $p \notin fn(P_4)$, by the property shown above, $P_4 \overset{\diamond}{\equiv} \to_\circ^* P'$, so $P \xrightarrow{N\sigma(M\sigma)}_\circ \to_\circ^* P'$.

To sum up, we have $A \equiv \mathrm{pnf}(A) = \nu\tilde{n}.(\sigma \mid P)$, $P \to_\circ^* P_5 \xrightarrow{N\sigma(M\sigma)}_\circ P_6 \to_\circ^* P'$, and $A' \equiv \mathrm{pnf}(A') \equiv \nu\tilde{n}.(\sigma \mid P')$. So $\nu\tilde{n}.(\sigma \mid P) \to_\circ^* \nu\tilde{n}.(\sigma \mid P_5) \xrightarrow{N(M)}_\circ \nu\tilde{n}.(\sigma \mid P_6) \to_\circ^* \nu\tilde{n}.(\sigma \mid P')$. Hence by Lemmas B.9 and B.13, $A \to^* \xrightarrow{N(M)} \to^* A'$. □

Lemma 4.22. *Let A be a closed extended process. Let N be a term such that $fv(N) \subseteq dom(A)$. Let p and q be names that do not occur in A and N.*

(1) *If $A \xrightarrow{vx.\overline{N}\langle x\rangle} A'$ and p and q do not occur in A', then $A \mid T^{p,q}_{vx.\overline{N}\langle x\rangle} \to\!\!\to vx.(A' \mid \overline{q}\langle x\rangle)$, $vx.(A' \mid \overline{q}\langle x\rangle) \not\Downarrow p$, and $x \notin dom(A)$.*

(2) *Let x be a variable such that $x \notin dom(A)$. If $A \mid T^{p,q}_{vx.\overline{N}\langle x\rangle} \to^* A''$ and $A'' \not\Downarrow p$, then $A \to^* \xrightarrow{vx.\overline{N}\langle x\rangle} \to^* A'$ and $A'' \equiv vx.(A' \mid \overline{q}\langle x\rangle)$ for some A'.*

Proof. *Property 1:* Let $\mathrm{pnf}(A) = v\widetilde{n}.(\sigma \mid P)$. By Lemma B.2, $\mathrm{pnf}(A)$ is closed. We rename $\widetilde{n}$ so that these names do not occur in N, p, and q. By Lemma B.12, $\mathrm{pnf}(A) \xrightarrow{vx.\overline{N}\langle x\rangle}_\circ A'$. By Lemma B.19, $P \xrightarrow{vx.\overline{N\sigma}\langle x\rangle}_\circ A''$, $A' \equiv v\widetilde{n}.(\sigma \mid A'')$, and $x \notin dom(\sigma)$ for some A''. By Lemma B.10, $P \overset{\diamond}{\equiv} v\widetilde{n}'.(\overline{N\sigma}\langle M\rangle.P_1 \mid P_2)$, $A'' \equiv v\widetilde{n}'.(P_1 \mid \{^M/_x\} \mid P_2)$, $\{\widetilde{n}'\} \cap fn(N\sigma) = \emptyset$, and $x \notin fv(\overline{N\sigma}\langle M\rangle.P_1 \mid P_2))$ for some $\widetilde{n}'$, P_1, P_2, M. We rename $\widetilde{n}'$ so that $p, q \notin \{\widetilde{n}'\}$ and y so that $y \notin fv(M)$. Hence, by Lemmas B.1 and B.7,

$$\begin{aligned}
A \mid \overline{p}\langle p\rangle \mid N(x).p(y).\overline{q}\langle x\rangle &\equiv \mathrm{pnf}(A) \mid \overline{p}\langle p\rangle \mid N(x).p(y).\overline{q}\langle x\rangle \\
&\equiv v\widetilde{n}.(\sigma \mid v\widetilde{n}'.(\overline{N\sigma}\langle M\rangle.P_1 \mid P_2)) \mid \overline{p}\langle p\rangle \mid N(x).p(y).\overline{q}\langle x\rangle \\
&\equiv v\widetilde{n}.(\sigma \mid v\widetilde{n}'.(\overline{N\sigma}\langle M\rangle.P_1 \mid P_2 \mid \overline{p}\langle p\rangle \mid N(x).p(y).\overline{q}\langle x\rangle)) \\
&\to v\widetilde{n}.(\sigma \mid v\widetilde{n}'.(P_1 \mid P_2 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M\rangle)) \\
&\to v\widetilde{n}.(\sigma \mid v\widetilde{n}'.(P_1 \mid P_2 \mid \overline{q}\langle M\rangle)) \\
&\equiv vx.v\widetilde{n}.(\sigma \mid v\widetilde{n}'.(P_1 \mid \{^M/_x\} \mid P_2 \mid \overline{q}\langle x\rangle)) \\
&\equiv vx.(v\widetilde{n}.(\sigma \mid A'') \mid \overline{q}\langle x\rangle) \\
&\equiv vx.(A' \mid \overline{q}\langle x\rangle)
\end{aligned}$$

Since $p \notin fn(vx.(A' \mid \overline{q}\langle x\rangle))$, we have $vx.(A' \mid \overline{q}\langle x\rangle) \not\Downarrow p$ by Lemma C.7.

Property 2: Let $\mathrm{pnf}(A) = v\widetilde{n}.(\sigma \mid P)$. By Lemma B.2, $\mathrm{pnf}(A)$ is closed. We rename $\widetilde{n}$ so that these names do not occur in N, p, q. Then $\mathrm{pnf}(A \mid \overline{p}\langle p\rangle \mid N(x).p(y).\overline{q}\langle x\rangle) = v\widetilde{n}.(\sigma \mid P \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle)$. By Lemma B.8, $\mathrm{pnf}(A \mid \overline{p}\langle p\rangle \mid N(x).p(y).\overline{q}\langle x\rangle) \to_\circ^* \mathrm{pnf}(A'')$. By Lemma B.23 applied several times, $P \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle \to_\diamond^* P''$ and $\mathrm{pnf}(A'') \equiv v\widetilde{n}.(\sigma \mid P'')$ for some closed process P''. Since $A'' \not\Downarrow p$, we have $P'' \not\Downarrow p$.

We prove that, if P_2 and M' are closed, $P_2 \mid \overline{q}\langle M'\rangle \overset{\diamond}{\equiv} \to_\diamond^* P''$, and $q \notin fn(P_2)$, then $P'' \equiv P_3 \mid \overline{q}\langle M'\rangle$ and $P_2 \to_\diamond^* P_3$ for some closed process P_3, by induction on the length of the trace $P_2 \mid \overline{q}\langle M'\rangle \overset{\diamond}{\equiv} \to_\diamond^* P''$. If this trace has zero reduction steps, then the result holds obviously with $P_3 = P_2$. If this trace has at least one reduction step, then $P_2 \mid \overline{q}\langle M'\rangle \to_\diamond P_4 \to_\diamond^* P''$, so by Lemmas B.24 and C.8, the only case that can happen is that $P_2 \to_\diamond P_2'$ and $P_2' \mid \overline{q}\langle M'\rangle \equiv P_4 \to_\diamond^* P''$ for some closed process P_2'. By Lemma C.10, we rename q inside P_2' so that $q \notin fn(P_2')$, and we obtain the desired result by induction hypothesis.

Next, we prove that, if P_1 and M' are closed, $P_1 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \to_\diamond^* P''$, $P'' \not\Downarrow p$, and $p, q \notin fn(P_1)$, then $P'' \equiv P_3 \mid \overline{q}\langle M'\rangle$ and $P_1 \to_\diamond^* P_3$ for some closed process P_3, by induction on the length of the trace. Since $P_1 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \Downarrow p$, the trace $P_1 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \to_\diamond^* P''$ has at least one step: $P_1 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \to_\diamond P_1' \to_\diamond^* P''$. By Lemmas B.24, B.18, and C.8, the only cases that can happen in the first step are:

- $P_1 \to_\diamond P_1''$ and $P_1'' \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \equiv P_1' \to_\diamond^* P''$ for some closed process P_1''. As above this trace has at least one step, so $P_1'' \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \to_\diamond^* P''$. By Lemma C.10, we rename p and q inside P_1'' so that $p, q \notin fn(P_1'')$, and we obtain the desired result by induction hypothesis.
- $\overline{p}\langle p\rangle \xrightarrow{\nu z.\overline{N}\langle z\rangle}_\diamond A_1 \equiv \{^p/_z\}, p(y).\overline{q}\langle M'\rangle \xrightarrow{N(z)}_\diamond A_2 \equiv \overline{q}\langle M'\rangle, P_1' \overset{\diamond}{\equiv} P_1 \mid \nu z.(A_1 \mid A_2) \equiv P_1 \mid \nu z.(\{^p/_z\} \mid \overline{q}\langle M'\rangle) \equiv P_1 \mid \overline{q}\langle M'\rangle$ so $P_1 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \to_\diamond P_1 \mid \overline{q}\langle M'\rangle \overset{\diamond}{\equiv} P_1' \to_\diamond^* P''$ and $q \notin fn(P_1)$, so by the property shown above, $P'' \equiv P_3 \mid \overline{q}\langle M'\rangle$ and $P_1 \to_\diamond^* P_3$ for some closed process P_3, as desired.

Finally, we prove that, if P and $N\sigma$ are closed, $P \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle \to_\diamond^* P''$, $P'' \not\Downarrow p$, and $p, q \notin fn(P) \cup fn(N\sigma)$, then $P \to_\diamond^* \xrightarrow{\nu x.\overline{N\sigma}\langle x\rangle}_\diamond \to^* B$ and $P'' \equiv \nu x.(B \mid \overline{q}\langle x\rangle)$ for some B, by induction on the length of the trace. Since $P \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle \Downarrow p$, the trace $P \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle \to_\diamond^* P''$ has at least one step: $P \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle \to_\diamond P_1 \to_\diamond^* P''$. By Lemmas B.24, B.18, and C.8, the only cases that can happen in the first step are:

- $P \to_\diamond P'$ and $P' \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle \equiv P_1 \to_\diamond^* P''$ for some closed process P'. As above this trace has at least one step, so $P' \mid \overline{p}\langle p\rangle \mid N\sigma(x).p(y).\overline{q}\langle x\rangle \to_\diamond^* P''$. By Lemma C.10, we rename p and q inside P' so that $p, q \notin fn(P')$, and we obtain the desired result by induction hypothesis.
- $P \xrightarrow{\nu z.\overline{N'}\langle z\rangle}_\diamond B'$, $N\sigma(x).p(y).\overline{q}\langle x\rangle \xrightarrow{N'(z)}_\diamond B''$, and $P_1 \equiv \nu z.(B' \mid \overline{p}\langle p\rangle \mid B'')$. By Lemma B.10, $P \overset{\diamond}{\equiv} \nu\widetilde{n}'.(\overline{N'}\langle M'\rangle.P_2 \mid P_3)$, $B' \equiv \nu\widetilde{n}'.(P_2 \mid \{^{M'}/_z\} \mid P_3)$, $\{\widetilde{n}'\} \cap fn(N') = \emptyset$, and $z \notin fv(\overline{N'}\langle M'\rangle.P_2 \mid P_3)$. By Lemma B.18, $\Sigma \vdash N\sigma = N'$ and $B'' \equiv p(y).\overline{q}\langle z\rangle$. Using Lemma B.16(1), we can guarantee that N', M', P_2, P_3 are closed. We rename $\widetilde{n}'$ so that these names are distinct from p and q. By Lemma C.10, we rename p and q inside $\nu\widetilde{n}'.(\overline{N'}\langle M'\rangle.P_2 \mid P_3)$ so that $p, q \notin fn(\nu\widetilde{n}'.(\overline{N'}\langle M'\rangle.P_2 \mid P_3))$. So $P_1 \equiv \nu z.(B' \mid \overline{p}\langle p\rangle \mid B'') \equiv \nu z.(\nu\widetilde{n}'.(P_2 \mid \{^{M'}/_z\} \mid P_3) \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle z\rangle) \equiv \nu\widetilde{n}'.(P_2 \mid P_3 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle)$. Since $P_1 \to_\diamond^* P''$ and this trace has at least one step because $P_1 \Downarrow p$ and $P'' \not\Downarrow p$, we have $\nu\widetilde{n}'.(P_2 \mid P_3 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle) \to_\diamond^* P''$, so by Lemma B.21, $P_2 \mid P_3 \mid \overline{p}\langle p\rangle \mid p(y).\overline{q}\langle M'\rangle \to_\diamond^* P_4$ and $P'' \equiv \nu\widetilde{n}'.P_4$ for some P_4. Since $p, q \notin fn(P_2 \mid P_3)$, by the previous result, $P_2 \mid P_3 \to_\diamond^* P_5$ and $P_4 \equiv P_5 \mid \overline{q}\langle M'\rangle$ for some closed process P_5. Therefore, we have $P'' \equiv \nu\widetilde{n}'.(P_5 \mid \overline{q}\langle M'\rangle) \equiv \nu x.(\nu\widetilde{n}'.(P_5 \mid \{^{M'}/_x\}) \mid \overline{q}\langle x\rangle)$ and $P \overset{\diamond}{\equiv} \nu\widetilde{n}'.(\overline{N'}\langle M'\rangle.P_2 \mid P_3) \xrightarrow{\nu x.\overline{N\sigma}\langle x\rangle}_\diamond \nu\widetilde{n}'.(P_2 \mid \{^{M'}/_x\} \mid P_3) \to^* \nu\widetilde{n}'.(P_5 \mid \{^{M'}/_x\})$. Let $B \overset{\text{def}}{=} \nu\widetilde{n}'.(P_5 \mid \{^{M'}/_x\})$. Then we have $P \xrightarrow{\nu x.\overline{N\sigma}\langle x\rangle}_\diamond \to^* B$ and $P'' \equiv \nu x.(B \mid \overline{q}\langle x\rangle)$.

To sum up, we have $A \equiv \text{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$, $P \to_\diamond^* \xrightarrow{\nu x.\overline{N\sigma}\langle x\rangle}_\diamond \to^* B$, and $P'' \equiv \nu x.(B \mid \overline{q}\langle x\rangle)$, so $A'' \equiv \text{pnf}(A'') \equiv \nu\widetilde{n}.(\sigma \mid P'') \equiv \nu\widetilde{n}.(\sigma \mid \nu x.(B \mid \overline{q}\langle x\rangle)) \equiv \nu x.(\nu\widetilde{n}.(\sigma \mid B) \mid \overline{q}\langle x\rangle)$ since $x \notin fv(\sigma)$. Let $A' \overset{\text{def}}{=} \nu\widetilde{n}.(\sigma \mid B)$. So $\text{pnf}(A) \to_\diamond^* \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond \to^* A'$. Hence by Lemmas B.9 and B.13, $A \to^* \xrightarrow{\nu x.\overline{N}\langle x\rangle} \to^* A'$ and $A'' \equiv \nu x.(A' \mid \overline{q}\langle x\rangle)$. □

Lemma C.11. *Let A and B be two closed extended processes.*

- *Let σ be a bijective renaming. We have $A \approx B$ if and only if $A\sigma \approx B\sigma$.*
- *Let A' and B' be obtained from A and B, respectively, by replacing all variables (including their occurrences in domains of active substitutions) with distinct variables. We have $A \approx B$ if and only if $A' \approx B'$.*

Proof. To prove the first point, we define a relation $\mathcal{R}$ by $A' \mathcal{R} B'$ if and only if $A' = A\sigma$, $B' = B\sigma$, and $A \approx B$ for some A and B. We show that $\mathcal{R}$ satisfies the three properties of Definition 4.1. Then $\mathcal{R} \subseteq \approx$, so if $A \approx B$, then $A' = A\sigma \approx B' = B\sigma$.

(1) If $A' \mathcal{R} B'$ and $A' \Downarrow a$, then $A' \to^* \equiv E[\overline{a}\langle M\rangle.P]$ for some evaluation context E that does not bind a. Then, by Lemma C.4, $A = A'\sigma^{-1} \to^* \equiv C\sigma^{-1}[\overline{a\sigma^{-1}}\langle M\sigma^{-1}\rangle.P\sigma^{-1}]$, so $A \Downarrow a\sigma^{-1}$. By definition of $\approx$, $B \Downarrow a\sigma^{-1}$, so $B' \Downarrow a$ as above.
(2) If $A' \mathcal{R} B'$, $A' \to A'_1$, and A'_1 is closed, then by Lemma C.4, $A = A'\sigma^{-1} \to A'_1\sigma^{-1}$. We let $A'' = A'_1\sigma^{-1}$, which is also closed. So by definition of $\approx$, $B \to^* B''$ and $A'' \approx B''$ for some B''. By Lemma C.4, $B' = B\sigma \to^* B''\sigma$. We let $B'_1 = B''\sigma$. We have $A'_1 \mathcal{R} B'_1$ and $B' \to^* B'_1$. So Property 2 holds.
(3) If $A' \mathcal{R} B'$, then $A = A'\sigma^{-1} \approx B'\sigma^{-1} = B$, so $E[A']\sigma^{-1} = E\sigma^{-1}[A] \approx E\sigma^{-1}[B] = E[B']\sigma^{-1}$, hence $E[A'] \mathcal{R} E[B']$.

The same argument also proves the converse, via the inverse renaming.
The proof of the second point is similar. □

Lemma C.12. *If M is ground, $fv(P) \subseteq \{x\}$, and $a \notin fn(P) \cup fn(M)$, then $va.(\overline{a}\langle M\rangle \mid a(x).P) \approx P\{^M/_x\}$.*

Proof. By Lemma 4.20, it is enough to prove that $va.(\overline{a}\langle M\rangle \mid a(x).P) \approx_l P\{^M/_x\}$. Let $A_1 = va.(\overline{a}\langle M\rangle \mid a(x).P)$ and $B_1 = P\{^M/_x\}$. Let $\mathcal{R} = \{(A, B) \mid A$ and B are closed extended processes, $A \equiv A_1$ and $B \equiv B_1$, or $A \equiv B_1$ and $B \equiv A_1\} \cup \{(A, B) \mid A$ and B are closed extended processes and $A \equiv B\}$. We show that $\mathcal{R}$ is a labelled bisimulation: $\mathcal{R}$ is symmetric and

(1) We have $A_1 \approx_s B_1$ since $\varphi(A_1) = \mathbf{0} = \varphi(B_1)$. Hence, if $A \mathcal{R} B$, then $A \approx_s B$.
(2) If $A_1 \to A'$ and A' is closed, then $A' \equiv B_1$. (This point can be proved in detail by using partial normal forms.)
Hence, if $A \mathcal{R} B$, $A \to A'$, and A' is closed, then
- either $A \equiv A_1$ and $B \equiv B_1$, so $A' \equiv B_1 \equiv B$, hence with $B' \stackrel{\text{def}}{=} B$, $B \to^* B'$ and $A' \mathcal{R} B'$.
- or $A \equiv B_1$ and $B \equiv A_1$, so $B \equiv A_1 \to B_1 \equiv A \to A'$, hence with $B' \stackrel{\text{def}}{=} A'$, $B \to^* B'$ and $A' \mathcal{R} B'$.
- or $A \equiv B$, so with $B' \stackrel{\text{def}}{=} A'$, $B \equiv A \to A' = B'$, and $A' \mathcal{R} B'$.

(3) A_1 does not reduce by $\stackrel{\alpha}{\to}$, for any α. (This point can be proved in detail by using partial normal forms.) Hence, if $A \mathcal{R} B$, $A \stackrel{\alpha}{\to} A'$, and A' is closed, then
- either $A \equiv A_1$ and $B \equiv B_1$, so $A_1 \stackrel{\alpha}{\to} A'$. This case is impossible.
- or $A \equiv B_1$ and $B \equiv A_1$, so $B \equiv A_1 \to B_1 \equiv A \stackrel{\alpha}{\to} A'$, hence with $B' \stackrel{\text{def}}{=} A'$, $B \to\stackrel{\alpha}{\to} B'$ and $A' \mathcal{R} B'$.
- or $A \equiv B$, so with $B' \stackrel{\text{def}}{=} A'$, $B \equiv A \stackrel{\alpha}{\to} A' = B'$, and $A' \mathcal{R} B'$.

Therefore, $\mathcal{R} \subseteq \approx_l$, so $A_1 \approx_l B_1$. □

Corollary C.13. *If A is a closed extended process, $x \in dom(A)$, $fv(P) \subseteq dom(A)$, and $a \notin fn(P)$, then $A \mid va.(\overline{a}\langle x\rangle \mid a(x).P) \approx A \mid P$.*

Proof. Let $\text{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P')$. We rename $\widetilde{n}$ so that $\{\widetilde{n}\} \cap fn(P) = \emptyset$. Let $\sigma' = \sigma_{|dom(\sigma)\setminus\{x\}}$. Let $a' \notin fn(P) \cup fn(\sigma)$. We have

$$\begin{aligned} A \mid va.(\overline{a}\langle x\rangle \mid a(x).P) &\equiv \nu\widetilde{n}.(\sigma \mid P' \mid va'.(\overline{a'}\langle x\sigma\rangle \mid a'(x).P\sigma') \\ &\approx \nu\widetilde{n}.(\sigma \mid P' \mid P\sigma'\{^{x\sigma}/_x\}) && \text{by Lemma C.12} \\ &= \nu\widetilde{n}.(\sigma \mid P' \mid P\sigma) \\ &\equiv A \mid P \end{aligned}$$

□

Lemma 4.23. *Let A and B be two closed extended processes with a same domain that contains $\widetilde{x}$. Let $E_{\widetilde{x}}[\_] \stackrel{def}{=} \nu\widetilde{x}.(\prod_{x\in\widetilde{x}} \overline{n_x}\langle x\rangle \mid \_)$ using names n_x that do not occur in A or B. If $E_{\widetilde{x}}[A] \approx E_{\widetilde{x}}[B]$, then $A \approx B$.*

Proof. We rely on the following property: if A is a closed extended process with $\{\widetilde{x}\} \subseteq dom(A)$ and $E_{\widetilde{x}}[A] \to C'$, then $A \to A'$ and $C' \equiv E_{\widetilde{x}}[A']$ for some closed extended process A', proved as follows. Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$. We rename $\widetilde{n}$ so that $\{\widetilde{n}\} \cap \{\widetilde{n}_x\} = \emptyset$. Then $\mathrm{pnf}(E_{\widetilde{x}}[A]) = \nu\widetilde{n}.(\sigma_{|dom(\sigma)\setminus\{\widetilde{x}\}} \mid \prod_{x\in\widetilde{x}} \overline{n_x}\langle x\sigma\rangle \mid P)$. By Lemma B.8, $\mathrm{pnf}(E_{\widetilde{x}}[A]) \to_\circ \mathrm{pnf}(C')$. By Lemma B.22, $\prod_{x\in\widetilde{x}} \overline{n_x}\langle x\sigma\rangle \mid P \to_\circ P'$ and $\mathrm{pnf}(C') \equiv \nu\widetilde{n}.(\sigma_{|dom(\sigma)\setminus\{\widetilde{x}\}} \mid P')$ for some P'. By Lemmas B.24 and C.8, since $\{\widetilde{n}_x\} \cap fn(P) = \emptyset$, the only case that can happen is $P \to_\circ P''$ and $P' \equiv \prod_{x\in\widetilde{x}} \overline{n_x}\langle x\sigma\rangle \mid P''$ for some closed process P''. Let $A' = \nu\widetilde{n}.(\sigma \mid P'')$. Then $A \equiv \mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P) \to \nu\widetilde{n}.(\sigma \mid P'') = A'$ and $C' \equiv \mathrm{pnf}(C') \equiv \nu\widetilde{n}.(\sigma_{|dom(\sigma)\setminus\{\widetilde{x}\}} \mid P') \equiv \nu\widetilde{n}.(\sigma_{|dom(\sigma)\setminus\{\widetilde{x}\}} \mid \prod_{x\in\widetilde{x}} \overline{n_x}\langle x\sigma\rangle \mid P'') \equiv \nu\widetilde{x}.(\prod_{x\in\widetilde{x}} \overline{n_x}\langle x\rangle \mid \nu\widetilde{n}.(\sigma \mid P'')) \equiv E_{\widetilde{x}}[A']$.

Let $\mathcal{R}$ be the relation that collects all closed extended processes A and B with a same domain that contains $\widetilde{x}$, such that $E_{\widetilde{x}}[A] \approx E_{\widetilde{x}}[B]$, for some $\widetilde{x}$ and some names $\widetilde{n}_x$ that do not occur in A or B. We show that $\mathcal{R}$ is an observational bisimulation.

Assume $A\ \mathcal{R}\ B$.

- If $A \to A'$ and A' is closed, then $E_{\widetilde{x}}[A] \to E_{\widetilde{x}}[A']$. By bisimulation hypothesis, $E_{\widetilde{x}}[B] \to^* C' \approx E_{\widetilde{x}}[A']$. By induction on the number of reductions and using partial normal forms, we build $B \to^* B'$ such that $C' \equiv E_{\widetilde{x}}[B']$ for some closed extended process B' and conclude using $A'\ \mathcal{R}\ B'$.
- We have $E_{\widetilde{x}}[A] \Downarrow n$ if and only if $n = n_x$ for some $x \in \widetilde{x}$ or $A \Downarrow n$, and similarly for B. Hence, if $A \Downarrow n$, then $E_{\widetilde{x}}[A] \Downarrow n$, so $E_{\widetilde{x}}[B] \Downarrow n$. By Lemma C.7, since $A \Downarrow n$, we have $n \neq n_x$ for all $x \in \widetilde{x}$, so $B \Downarrow n$.
- For the congruence property, we suppose that $A\ \mathcal{R}\ B$, and we want to show that $E[A]\ \mathcal{R}\ E[B]$ for all closing evaluation contexts E. Using Lemma C.11, we show that $\mathcal{R}$ is invariant by renaming of free names and variables, so we can rename the free names and variables of E, so that the obtained context is simple. Then by Lemma A.1, we construct a context E' of the form $\nu\widetilde{u}.(\_ \mid C'')$ such that $E \equiv E'$. Hence, it is sufficient to show that $E'[A]\ \mathcal{R}\ E'[B]$.
  Let $\mathrm{pnf}(C'') = \nu\widetilde{n}.(\sigma \mid P)$. Let $\widetilde{u} = \widetilde{m}\widetilde{z}$. We rename $\widetilde{n}$ so that $\{\widetilde{n}\} \cap (fn(A) \cup fn(B)) = \emptyset$. Since $A\ \mathcal{R}\ B$, $E_{\widetilde{x}}[A] \approx E_{\widetilde{x}}[B]$ for some $\widetilde{x}$. Using Lemma C.11, we rename $\widetilde{n}_x$ so that $\{\widetilde{n}_x\} \cap (\{\widetilde{n}, \widetilde{m}\} \cup fn(P) \cup fn(\sigma)) = \emptyset$. Let n'_x for $x \in (\widetilde{x} \cup dom(\sigma)) \setminus \widetilde{z}$ be fresh names. Let $E_1[\_] = \nu\widetilde{m}, \widetilde{n}, \widetilde{n}_x, \widetilde{z} \setminus (\widetilde{x} \cup dom(\sigma)).(\_ \mid \widetilde{n_x(x)}.(P \mid \prod_{x\in\widetilde{x}\setminus\widetilde{z}} \overline{n'_x}\langle x\rangle \mid \prod_{y\in dom(\sigma)\setminus\widetilde{z}} \overline{n'_y}\langle y\sigma\rangle))$, where $\widetilde{n_x(x)}$ stands for $n_{x_1}(x_1) \ldots n_{x_k}(x_k)$ when $\widetilde{x} = x_1, \ldots, x_k$.

$$
\begin{aligned}
&E_{(\widetilde{x}\cup dom(\sigma))\setminus\widetilde{z}}[E'[A]] \\
&\quad\equiv \nu(\widetilde{x} \cup dom(\sigma)) \setminus \widetilde{z}.(\nu\widetilde{m}.\nu\widetilde{z}.(A \mid \nu\widetilde{n}.(\sigma \mid P)) \mid \prod_{x\in\widetilde{x}\setminus\widetilde{z}} \overline{n'_x}\langle x\rangle \mid \prod_{y\in dom(\sigma)\setminus\widetilde{z}} \overline{n'_y}\langle y\rangle) \\
&\quad\equiv \nu\widetilde{m}, \widetilde{n}, \widetilde{z} \cup \widetilde{x} \cup dom(\sigma).(A \mid P \mid \sigma \mid \prod_{x\in\widetilde{x}\setminus\widetilde{z}} \overline{n'_x}\langle x\rangle \mid \prod_{y\in dom(\sigma)\setminus\widetilde{z}} \overline{n'_y}\langle y\sigma\rangle) \\
&\quad\equiv \nu\widetilde{m}, \widetilde{n}, \widetilde{z} \setminus (\widetilde{x} \cup dom(\sigma)).\nu\widetilde{x}.(A \mid P \mid \prod_{x\in\widetilde{x}\setminus\widetilde{z}} \overline{n'_x}\langle x\rangle \mid \prod_{y\in dom(\sigma)\setminus\widetilde{z}} \overline{n'_y}\langle y\sigma\rangle) \\
&\quad\approx \nu\widetilde{m}, \widetilde{n}, \widetilde{z} \setminus (\widetilde{x} \cup dom(\sigma)).\nu\widetilde{x}. \\
&\qquad (A \mid \nu\widetilde{n}_x.(\prod_{x\in\widetilde{x}} \overline{n_x}\langle x\rangle \mid \widetilde{n_x(x)}.(P \mid \prod_{x\in\widetilde{x}\setminus\widetilde{z}} \overline{n'_x}\langle x\rangle \mid \prod_{y\in dom(\sigma)\setminus\widetilde{z}} \overline{n'_y}\langle y\sigma\rangle)))
\end{aligned}
$$

by Corollary C.13 applied several times, so

$$\begin{aligned}
&E_{(\widetilde{x}\cup dom(\sigma))\setminus\widetilde{z}}[E'[A]]\\
&\quad\equiv \nu\widetilde{m},\widetilde{n},\widetilde{n}_x,\widetilde{z}\setminus(\widetilde{x}\cup dom(\sigma)).\\
&\qquad(\nu\widetilde{x}.(A \mid \prod_{x\in\widetilde{x}} \overline{n_x}\langle x\rangle) \mid \widetilde{n_x(x)}.(P \mid \prod_{x\in\widetilde{x}\setminus\widetilde{z}} \overline{n'_x}\langle x\rangle \mid \prod_{y\in dom(\sigma)\setminus\widetilde{z}} \overline{n'_y}\langle y\sigma\rangle))\\
&\quad\equiv E_1[E_{\widetilde{x}}[A]]
\end{aligned}$$

By the same argument, $E_{(\widetilde{x}\cup dom(\sigma))\setminus\widetilde{z}}[E'[B]] \approx E_1[E_{\widetilde{x}}[B]]$. Since $E_{\widetilde{x}}[A] \approx E_{\widetilde{x}}[B]$, we have $E_1[E_{\widetilde{x}}[A]] \approx E_1[E_{\widetilde{x}}[B]]$, so by transitivity of $\approx$, $E_{(\widetilde{x}\cup dom(\sigma))\setminus\widetilde{z}}[E'[A]] \approx E_{(\widetilde{x}\cup dom(\sigma))\setminus\widetilde{z}}[E'[B]]$. Hence, $E'[A]\ \mathcal{R}\ E'[B]$.

Since $\mathcal{R}$ is an observational bisimulation, $\mathcal{R}\subseteq\approx$, so $E_{\widetilde{x}}[A]\approx E_{\widetilde{x}}[B]$ implies $A\approx B$. □

Corollary C.14. *Observational equivalence and static equivalence coincide on frames.*

Proof. Since frames do not reduce, static equivalence and labelled bisimilarity coincide on frames. By Theorem 4.8, we can then conclude. □

D PROOF OF LEMMA 4.10

The *image* of a substitution $\sigma = \{^{M_1}/_{x_1},\ldots,^{M_n}/_{x_n}\}$ is the set of terms $\{M_1,\ldots,M_n\}$. We denote by ρ a bijective renaming. We denote by $\sigma\rho$ the substitution obtained by applying the renaming ρ to the terms in the image of σ, that is, when $\sigma = \{^{M_1}/_{x_1},\ldots,^{M_n}/_{x_n}\}$, $\sigma\rho = \{^{M_1\rho}/_{x_1},\ldots,^{M_n\rho}/_{x_n}\}$.

Lemma D.1. *Let $\nu\widetilde{n}.\sigma$ and $\nu\widetilde{n}'.\sigma'$ be two frames such that $\nu\widetilde{n}.\sigma \equiv \nu\widetilde{n}'.\sigma'$, and M and N be two terms such that $fv(M)\cup fv(N)\subseteq dom(\sigma) = dom(\sigma')$ and $\{\widetilde{n},\widetilde{n}'\}\cup(fn(M)\cup fn(N)) = \emptyset$. If $\Sigma\vdash M\sigma = N\sigma$, then $\Sigma\vdash M\sigma' = N\sigma'$.*

Proof. Let us prove the following result:

> Suppose $\nu\widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}'.(\sigma' \mid P')$ and $fv(M)\cup fv(N)\subseteq dom(\sigma) = dom(\sigma')$. Let ρ be a bijective renaming that maps names in $\widetilde{n}$ to names not in $fn(\nu\widetilde{n}.\sigma)\cup fn(M)\cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n}.\sigma)\cup fn(M)\cup fn(N))\setminus\{\widetilde{n}\}$ unchanged, and ρ' be a bijective renaming that maps names in $\widetilde{n}'$ to names not in $fn(\nu\widetilde{n}'.\sigma')\cup fn(M)\cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n}'.\sigma')\cup fn(M)\cup fn(N))\setminus\{\widetilde{n}'\}$ unchanged.
> We have $\Sigma\vdash M(\sigma\rho) = N(\sigma\rho)$ if and only if $\Sigma\vdash M(\sigma'\rho') = N(\sigma'\rho')$.

This result is proved by induction on the derivation of $\nu\widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}'.(\sigma' \mid P')$.

- Transitivity and symmetry: obvious.
- Reflexivity: The renamings ρ and ρ' map names in $\widetilde{n}$ to names not in $fn(\nu\widetilde{n}.\sigma)\cup fn(M)\cup fn(N)$ and leave names in $(fn(\nu\widetilde{n}.\sigma)\cup fn(M)\cup fn(N))\setminus\{\widetilde{n}\}$ unchanged. Let ρ'' be a bijective renaming that maps $\widetilde{n}\rho$ to $\widetilde{n}\rho'$ and leaves names in $fn(\nu\widetilde{n}.\sigma)\cup fn(M)\cup fn(N)$ unchanged. If $\Sigma\vdash M(\sigma\rho) = N(\sigma\rho)$, then $\Sigma\vdash M(\sigma\rho)\rho'' = N(\sigma\rho)\rho''$, so $\Sigma\vdash M(\sigma\rho') = N(\sigma\rho')$. The converse is proved is the same way, using ρ''^{-1} instead of ρ''.
- Cases Plain'' and New-C'': These cases are proved by the same proof as for reflexivity, since the desired property does not depend on the process P nor on the order of $\widetilde{n}$.
- Case New-Par'': $\nu\widetilde{n}.(\sigma \mid \nu n'.P) \overset{\circ}{\equiv} \nu\widetilde{n},n'.(\sigma \mid P)$ where $n'\notin fn(\sigma)$. Let ρ be a bijective renaming that maps names in $\widetilde{n}$ to names not in $fn(\nu\widetilde{n}.\sigma)\cup fn(M)\cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n}.\sigma)\cup fn(M)\cup fn(N))\setminus\{\widetilde{n}\}$ unchanged, and ρ' be a bijective renaming that maps names in $\widetilde{n},n'$ to names not in $fn(\nu\widetilde{n},n'.\sigma)\cup fn(M)\cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n},n'.\sigma)\cup fn(M)\cup fn(N))\setminus\{\widetilde{n},n'\}$ unchanged. Let ρ'' be a bijective renaming that maps $\widetilde{n}\rho$ to $\widetilde{n}\rho'$ and that leaves

names in $fn(\nu\widetilde{n}.\sigma) \cup fn(M) \cup fn(N)$ unchanged. (Since $n' \notin fn(\sigma)$, $fn(\nu\widetilde{n}.\sigma) = fn(\nu\widetilde{n}, n'.\sigma)$, so the names $\widetilde{n}\rho'$ do not collide with $fn(\nu\widetilde{n}.\sigma) \cup fn(M) \cup fn(N)$, hence ρ'' exists.)
If $\Sigma \vdash M(\sigma\rho) = N(\sigma\rho)$, then $\Sigma \vdash M(\sigma\rho)\rho'' = N(\sigma\rho)\rho''$, so $\Sigma \vdash M(\sigma\rho') = N(\sigma\rho')$. (We have $\sigma\rho\rho'' = \sigma\rho'$ because $n' \notin fn(\sigma)$.)
The converse is proved in the same way, using ρ''^{-1} instead of ρ''.

- Case Rewrite'': $\nu\widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma' \mid P)$ where $dom(\sigma) = dom(\sigma')$, $\Sigma \vdash x\sigma = x\sigma'$ for all $x \in dom(\sigma)$, and $(fv(x\sigma) \cup fv(x\sigma')) \cap dom(\sigma) = \emptyset$ for all $x \in dom(\sigma)$.
Let ρ be a bijective renaming that maps names in $\widetilde{n}$ to names not in $fn(\nu\widetilde{n}.\sigma) \cup fn(M) \cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n}.\sigma) \cup fn(M) \cup fn(N)) \setminus \{\widetilde{n}\}$ unchanged, and ρ' be a bijective renaming that maps names in $\widetilde{n}$ to names not in $fn(\nu\widetilde{n}.\sigma') \cup fn(M) \cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n}.\sigma') \cup fn(M) \cup fn(N)) \setminus \{\widetilde{n}\}$ unchanged.
Let ρ'' be a bijective renaming that maps names in $\widetilde{n}$ to names not in $fn(\nu\widetilde{n}.\sigma) \cup fn(\nu\widetilde{n}.\sigma') \cup fn(M) \cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n}.\sigma) \cup fn(\nu\widetilde{n}.\sigma') \cup fn(M) \cup fn(N)) \setminus \{\widetilde{n}\}$ unchanged.
The renaming ρ'' a fortiori maps names in $\widetilde{n}$ to names not in $fn(\nu\widetilde{n}.\sigma) \cup fn(M) \cup fn(N)$ and leaves names in $(fn(\nu\widetilde{n}.\sigma) \cup fn(M) \cup fn(N)) \setminus \{\widetilde{n}\}$ unchanged, so by the case of reflexivity $\nu\widetilde{n}.(\sigma \mid P) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, we have $\Sigma \vdash M(\sigma\rho) = N(\sigma\rho)$ if and only if $\Sigma \vdash M(\sigma\rho'') = N(\sigma\rho'')$.
Similarly, $\Sigma \vdash M(\sigma'\rho') = N(\sigma'\rho')$ if and only if $\Sigma \vdash M(\sigma'\rho'') = N(\sigma'\rho'')$.
Moreover, for all $x \in dom(\sigma)$, $\Sigma \vdash x\sigma = x\sigma'$, so $\Sigma \vdash x\sigma\rho'' = x\sigma'\rho''$, hence $\Sigma \vdash M(\sigma\rho') = M(\sigma'\rho'')$ and $\Sigma \vdash N(\sigma\rho') = N(\sigma'\rho'')$, therefore $\Sigma \vdash M(\sigma\rho'') = N(\sigma\rho'')$ if and only if $\Sigma \vdash M(\sigma'\rho'') = N(\sigma'\rho'')$.
We can then conclude that $\Sigma \vdash M(\sigma\rho) = N(\sigma\rho)$ if and only if $\Sigma \vdash M(\sigma'\rho') = N(\sigma'\rho')$.

The lemma is an easy consequence of this result: since $\nu\widetilde{n}.\sigma \equiv \nu\widetilde{n}'.\sigma'$, we have $\nu\widetilde{n}.(\sigma \mid \mathbf{0}) \overset{\circ}{\equiv} \nu\widetilde{n}'.(\sigma' \mid \mathbf{0})$ by Lemma B.5. We conclude by applying the previous result taking $P = P' = \mathbf{0}$ and the identity for ρ and ρ'. □

Lemma D.2. *Let A be a closed extended process. If $\nu s.(\{^s/_x\} \mid A) \to B'$, then there exists a closed extended process A' such that $A \to A'$ and $B' \equiv \nu s.(\{^s/_x\} \mid A')$.*

Proof. Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$. We rename $\widetilde{n}$ so that $s \notin \{\widetilde{n}\}$. By Lemma B.8, $\mathrm{pnf}(\nu s.(\{^s/_x\} \mid A)) = \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid P) \to_\circ \mathrm{pnf}(B')$. By Lemma B.23, $P \to_\circ P'$ and $\mathrm{pnf}(B') \equiv \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid P')$ for some closed process P'. Let $A' = \nu\widetilde{n}.(\sigma \mid P')$. Hence, $A \equiv \nu\widetilde{n}.(\sigma \mid P) \to \nu\widetilde{n}.(\sigma \mid P') = A'$ and $B' \equiv \mathrm{pnf}(B') \equiv \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid P') \equiv \nu s.(\{^s/_x\} \mid A')$. □

Lemma D.3. *Let A be a closed extended process and α be such that $fv(\alpha) \subseteq dom(A) \cup \{x\}$ and $s \notin fn(\alpha)$. If $\nu s.(\{^s/_x\} \mid A) \xrightarrow{\alpha} B'$, then there exists a closed extended process A' such that $A \xrightarrow{\alpha\{^s/_x\}} A'$ and $B' \equiv \nu s.(\{^s/_x\} \mid A')$.*

Proof. Let $\mathrm{pnf}(A) = \nu\widetilde{n}.(\sigma \mid P)$. We rename $\widetilde{n}$ so that $s \notin \{\widetilde{n}\}$ and the elements of $\widetilde{n}$ do not occur in α. By Lemma B.12, $\mathrm{pnf}(\nu s.(\{^s/_x\} \mid A)) = \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid P) \xrightarrow{\alpha}_\circ B'$. By Lemma B.19, $P \xrightarrow{\alpha(\{^s/_x\} \mid \sigma)}_\circ B''$ and $B' \equiv \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid B'')$ for some B''.

Next, we show that we can choose B'' so that it is closed. Let $\alpha' = \alpha(\{^s/_x\} \mid \sigma)$. By Lemma B.10, for some $\widetilde{n}'$, P_1, P_2, B_1, N, M, P', y, we have $P \overset{\circ}{\equiv} \nu\widetilde{n}'.(P_1 \mid P_2)$, $B'' \equiv \nu\widetilde{n}'.(B_1 \mid P_2)$, $\{\widetilde{n}'\} \cap fn(\alpha') = \emptyset$, $bv(\alpha) \cap fv(P_1 \mid P_2) = \emptyset$, and one of the following two cases holds:

(1) $\alpha' = N(M)$, $P_1 = N(y).P'$, and $B_1 = P'\{^M/_y\}$; or
(2) $\alpha' = \nu y.\overline{N}\langle y\rangle$, $P_1 = \overline{N}\langle M\rangle.P'$, and $B_1 = P' \mid \{^M/_y\}$.

Let σ' be a substitution that maps variables of $fv(B'') \setminus dom(B'')$ to distinct fresh names. We rename y so that $y \notin fv(B'') \setminus dom(B'')$. Since P and α' are closed, $P = P\sigma'$ and $\alpha' = \alpha'\sigma'$. By Lemma B.16(2),

$P \stackrel{\diamond}{\equiv} \nu\widetilde{n}'.(P_1\sigma' \mid P_2\sigma')$ and $\Sigma \vdash \nu\widetilde{n}'.(P_1 \mid P_2) = \nu\widetilde{n}'.(P_1\sigma' \mid P_2\sigma')$, so $\Sigma \vdash P_1 = P_1\sigma'$ and $\Sigma \vdash P_2 = P_2\sigma'$. By Lemma B.15, $B''\sigma' \equiv \nu\widetilde{n}'.(B_1\sigma' \mid P_2\sigma')$. Finally, one of the following two cases holds:

(1) $\alpha' = N(M)$, $P_1\sigma' = N(y).P'\sigma'$, and $B_1\sigma' = P'\sigma'\{^M/_y\}$; or
(2) $\alpha' = \nu y.\overline{N}\langle y\rangle$, $P_1\sigma' = \overline{N}\langle M\sigma'\rangle.P'\sigma'$, and $B_1\sigma' = P'\sigma' \mid \{^{M\sigma'}/_y\}$.

Hence, by Lemma B.10, $P \xrightarrow{\alpha'}_{\diamond} B''\sigma'$. Moreover, $\Sigma \vdash B_1 = B_1\sigma'$ because $\Sigma \vdash P_1 = P_1\sigma'$, so $\Sigma \vdash \nu\widetilde{n}'.(B_1 \mid P_2) = \nu\widetilde{n}'.(B_1\sigma' \mid P_2\sigma')$, hence $B''\sigma' \equiv \nu\widetilde{n}'.(B_1\sigma' \mid P_2\sigma') \equiv \nu\widetilde{n}'.(B_1 \mid P_2) \equiv B''$, so $B' \equiv \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid B''\sigma')$. Hence, by replacing B'' with $B''\sigma'$, we obtain the same properties as above, and additionally $B''\sigma'$ is closed.

Let $A' = \nu\widetilde{n}.(\sigma \mid B''\sigma')$. Hence, $A \equiv \nu\widetilde{n}.(\sigma \mid P) \xrightarrow{\alpha\{^s/_x\}}_{\circ} A'$ by definition of $\xrightarrow{\alpha\{^s/_x\}}_{\circ}$, so $A \xrightarrow{\alpha\{^s/_x\}} A'$ by Lemma B.13, and $B' \equiv \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid B''\sigma') \equiv \nu s.(\{^s/_x\} \mid A')$. □

Lemma 4.10 (Name disclosure). *Let A and B be closed extended processes and x be a variable such that $x \notin dom(A)$. We have $A \approx_l B$ if and only if*

$$\nu n.(\{^n/_x\} \mid A) \quad \approx_l \quad \nu n.(\{^n/_x\} \mid B)$$

Proof. The direct implication follows from context closure of $\approx_l$. Conversely, we show that the relation $\mathcal{R}$ defined by $A\ \mathcal{R}\ B$ if and only if A and B are closed extended processes and $\nu s.(\{^s/_x\} \mid A) \approx_l \nu s.(\{^s/_x\} \mid B)$ for some $x \notin dom(A)$ is a labelled bisimulation.

(1) The relation $\mathcal{R}$ is symmetric, because $\approx_l$ is.
(2) We suppose that $A\ \mathcal{R}\ B$ and show that $A \approx_s B$. Since $A\ \mathcal{R}\ B$, we have $\nu s.(\{^s/_x\} \mid A) \approx_l \nu s.(\{^s/_x\} \mid B)$ for some $x \notin dom(A)$, so $\nu s.(\{^s/_x\} \mid A) \approx_s \nu s.(\{^s/_x\} \mid B)$. We have $dom(A) = dom(\nu s.(\{^s/_x\} \mid A)) \setminus \{x\} = dom(\nu s.(\{^s/_x\} \mid B)) \setminus \{x\} = dom(B)$. Let M, N be two terms such that $fv(M) \cup fv(N) \subseteq dom(A)$. Let $M' = M\{^x/_s\}$ and $N' = N\{^x/_s\}$. We show that $(M = N)\varphi(A)$ if and only if $(M' = N')\varphi(\nu s.(\{^s/_x\} \mid A))$.
If $(M = N)\varphi(A)$, then $\varphi(A) \equiv \nu\widetilde{n}.\sigma$, $\Sigma \vdash M\sigma = N\sigma$, and $\{\widetilde{n}\} \cap (fn(M) \cup fn(N)) = \emptyset$ for some $\widetilde{n}$ and σ. If $s \in fn(M) \cup fn(N)$, we know that $s \notin \{\widetilde{n}\}$. Otherwise, we rename $\widetilde{n}$ so that $s \notin \{\widetilde{n}\}$, while preserving the previous properties. So $\varphi(\nu s.(\{^s/_x\} \mid A)) \equiv \nu s, \widetilde{n}.(\sigma \mid \{^s/_x\})$, $\Sigma \vdash M'(\sigma\{^s/_x\}) = N'(\sigma\{^s/_x\})$, and $\{s, \widetilde{n}\} \cap (fn(M') \cup fn(N')) = \emptyset$, so $(M' = N')\varphi(\nu s.(\{^s/_x\} \mid A))$. Conversely, if $(M' = N')\varphi(\nu s.(\{^s/_x\} \mid A))$, then $\varphi(\nu s.(\{^s/_x\} \mid A)) \equiv \nu\widetilde{n}'.\sigma'$, $\Sigma \vdash M'\sigma' = N'\sigma'$, and $\{\widetilde{n}'\} \cap (fn(M') \cup fn(N')) = \emptyset$ for some $\widetilde{n}'$ and σ'. We have $\varphi(A) \equiv \nu\widetilde{n}.\sigma$ for some $\widetilde{n}$ and σ. We rename $\widetilde{n}$ so that $(\{s\} \cup fn(N) \cup fn(M)) \cap \{\widetilde{n}\} = \emptyset$, so $(fn(N') \cup fn(M')) \cap \{\widetilde{n}\} = \emptyset$. Then $\varphi(\nu s.(\{^s/_x\} \mid A)) \equiv \nu s, \widetilde{n}.(\sigma \mid \{^s/_x\})$, so $\nu\widetilde{n}'.\sigma' \equiv \nu s, \widetilde{n}.(\sigma \mid \{^s/_x\})$. By Lemma D.1, $\Sigma \vdash M'(\sigma \mid \{^s/_x\}) = N'(\sigma \mid \{^s/_x\})$, so $\Sigma \vdash M\sigma = N\sigma$, hence $(M = N)\varphi(A)$.
Symmetrically, $(M = N)\varphi(B)$ if and only if $(M' = N')\varphi(\nu s.(\{^s/_x\} \mid B))$. Moreover, $(M' = N')\varphi(\nu s.(\{^s/_x\}|A))$ if and only if $(M' = N')\varphi(\nu s.(\{^s/_x\}|B))$, because $\nu s.(\{^s/_x\}|A) \approx_s \nu s.(\{^s/_x\}|B)$. Therefore, $(M = N)\varphi(A)$ if and only if $(M = N)\varphi(B)$, so $A \approx_s B$.
(3) We suppose that $A\ \mathcal{R}\ B$, $A \to A'$, and A' is closed, and we show that $B \to^* B'$ and $A'\ \mathcal{R}\ B'$ for some B'. For some $x \notin dom(A)$, we have $\nu s.(\{^s/_x\} \mid A) \approx_l \nu s.(\{^s/_x\} \mid B)$, $\nu s.(\{^s/_x\} \mid A) \to \nu s.(\{^s/_x\} \mid A')$, and $\nu s.(\{^s/_x\} \mid A')$ is closed, so $\nu s.(\{^s/_x\} \mid B) \to^* B''$ and $\nu s.(\{^s/_x\} \mid A') \approx_l B''$ for some B''.
By Lemma D.2 applied several times, $B \to^* B'$ and $B'' \equiv \nu s.(\{^s/_x\} \mid B')$ for some closed extended process B', so $\nu s.(\{^s/_x\} \mid A') \approx_l \nu s.(\{^s/_x\} \mid B')$, which shows that $A'\ \mathcal{R}\ B'$.
(4) We suppose that $A\ \mathcal{R}\ B$, $A \xrightarrow{\alpha} A'$, A' is closed, and $fv(\alpha) \subseteq dom(A)$, and we show that $B \to^* \xrightarrow{\alpha} \to^* B'$ and $A'\ \mathcal{R}\ B'$ for some B'.
For some $x \notin dom(A)$, we have $\nu s.(\{^s/_x\} \mid A) \approx_l \nu s.(\{^s/_x\} \mid B)$. First, we rename x in this equivalence so that $x \notin bv(\alpha)$, by Lemma C.5.

Let $\alpha' = \alpha\{^x/_s\}$. By Lemma B.12, we have $\mathrm{pnf}(A) \xrightarrow{\alpha}_\circ A'$, so there exist $\widetilde{n}$, σ, P, α'', and A'' such that $\mathrm{pnf}(A) \overset{\circ}{\equiv} \nu\widetilde{n}.(\sigma \mid P)$, $P \xrightarrow{\alpha''}_\circ A''$, $A' \equiv \nu\widetilde{n}.(\sigma \mid A'')$, $fv(\sigma) \cap bv(\alpha'') = \emptyset$, $\Sigma \vdash \alpha\sigma = \alpha''$, and the elements of $\widetilde{n}$ do not occur in α. We rename $\widetilde{n}$ so that $s \notin \{\widetilde{n}\}$. Since A is closed, $\mathrm{pnf}(A)$ is closed, so by Lemma B.16(1), we can arrange that $\nu\widetilde{n}.(\sigma \mid P)$ is also closed, by substituting fresh names for its free variables.
We have $\nu s.(\{^s/_x\} \mid A) \equiv \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid P)$, so by Lemma B.5, $\mathrm{pnf}(\nu s.(\{^s/_x\} \mid A)) \overset{\circ}{\equiv} \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid P)$ since $\nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid P)$ is in partial normal form, because $x \notin fv(P)$ and $x \notin fv(\sigma)$, since $\nu\widetilde{n}.(\sigma \mid P)$ is closed and $x \notin dom(A) = dom(\sigma)$. Moreover, $P \xrightarrow{\alpha''}_\circ A''$, $\nu s.(\{^s/_x\} \mid A') \equiv \nu s, \widetilde{n}.(\{^s/_x\} \mid \sigma \mid A'')$, $fv(\{^s/_x\} \mid \sigma) \cap bv(\alpha'') = \emptyset$ because $x \notin bv(\alpha'') = bv(\alpha)$, $\Sigma \vdash \alpha'(\{^s/_x\} \mid \sigma) = \alpha\sigma = \alpha''$, and the elements of $s, \widetilde{n}$ do not occur in α'. Therefore, $\mathrm{pnf}(\nu s.(\{^s/_x\} \mid A)) \xrightarrow{\alpha'}_\circ \nu s.(\{^s/_x\} \mid A')$.
So $\nu s.(\{^s/_x\} \mid A) \equiv \mathrm{pnf}(\nu s.(\{^s/_x\} \mid A)) \xrightarrow{\alpha'} \nu s.(\{^s/_x\} \mid A')$ using Lemmas B.1 and B.13, so $\nu s.(\{^s/_x\} \mid A) \xrightarrow{\alpha'} \nu s.(\{^s/_x\} \mid A')$ by Struct.
Since $\nu s.(\{^s/_x\} \mid A) \approx_l \nu s.(\{^s/_x\} \mid B)$, we have $\nu s.(\{^s/_x\} \mid B) \to^* \xrightarrow{\alpha'} \to^* B''$ and $\nu s.(\{^s/_x\} \mid A') \approx_l B''$ for some B''. By Lemma D.2 applied several times and Lemma D.3, $B \to^* \xrightarrow{\alpha} \to^* B'$ and $B'' \equiv \nu s.(\{^s/_x\} \mid B')$ for some closed extended process B', so $\nu s.(\{^s/_x\} \mid A') \approx_l \nu s.(\{^s/_x\} \mid B')$, which shows that $A' \mathcal{R} B'$.

Since $\mathcal{R}$ is a labelled bisimulation and $\approx_l$ is the largest labelled bisimulation, we have $\mathcal{R} \subseteq \approx_l$. If $\nu s.(\{^s/_x\} \mid A) \approx_l \nu s.(\{^s/_x\} \mid B)$, then $A \mathcal{R} B$, so $A \approx_l B$. □

E PROOFS FOR SECTION 4.4

Lemma 4.12. *The following three properties are equivalent:*

(1) *the variables $\widetilde{x}$ resolve to $\widetilde{M}$ in A;*
(2) *there exists A' such that $A \equiv \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid A'$;*
(3) *$(\widetilde{x} = \widetilde{M})\varphi(A)$ and the substitution $\{^{\widetilde{M}}/_{\widetilde{x}}\}$ is cycle-free.*

Proof. The implication from 1 to 2 is immediate, with $A' = \nu\widetilde{x}.A$. The implication from 2 to 3 is also obvious. Let us prove the implication from 3 to 1. Since $(\widetilde{x} = \widetilde{M})\varphi(A)$, we have $\{\widetilde{x}\} \subseteq dom(\varphi(A)) = dom(A)$, so $A \equiv \nu\widetilde{n}.(\{^{\widetilde{M}'}/_{\widetilde{x}}\} \mid \sigma \mid P)$ for some $\widetilde{n}$, $\widetilde{M}'$, σ, and P such that the variables of $dom(A)$ do not occur in $\widetilde{M}'$, the image of σ, nor P. We rename $\widetilde{n}$ so that these names do not occur in $\widetilde{M}$. Since $(\widetilde{x} = \widetilde{M})\varphi(A)$, we have $\widetilde{M}' = \widetilde{M}\{^{\widetilde{M}'}/_{\widetilde{x}}\}\sigma = \widetilde{M}\{^{\widetilde{M}}/_{\widetilde{x}}\}\sigma$ using that $\{^{\widetilde{M}}/_{\widetilde{x}}\}$ is cycle-free, so $A \equiv \nu\widetilde{n}.(\{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \sigma \mid P)$. Since the names $\widetilde{n}$ do not occur in $\widetilde{M}$, $A \equiv \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \nu\widetilde{n}.(\sigma \mid P) \equiv \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \nu\widetilde{x}.A$, which proves 1. □

Lemma 4.15. *$A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$ if and only if, for some z that does not occur in any of A, A', $\widetilde{x}$, N, and M, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid A')$, $\{\widetilde{x}\} \subseteq fv(M) \setminus fv(N)$, and the variables $\widetilde{x}$ are solvable in $\{^M/_z\} \mid A'$.*

Proof. We prove the implication from left to right by induction on the derivation of $A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$. Precisely, we prove the result for all z that do not occur in the derivation of $A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$.

- Case Out-Term. We have $A = \overline{N}\langle M\rangle.P \xrightarrow{\overline{N}\langle M\rangle} P = A'$ and $\widetilde{x}$ is empty. Let $z \notin fv(\overline{N}\langle M\rangle.P)$. By Out-Var, $A = \overline{N}\langle M\rangle.P \xrightarrow{\nu z.\overline{N}\langle z\rangle} P \mid \{^M/_z\} \equiv \{^M/_z\} \mid A'$, so by Struct, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \{^M/_z\} \mid A'$.

- Case Open-Var. The transition $A = \nu\widetilde{x}.B \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$ is derived from $B \xrightarrow{\overline{N}\langle M\rangle} A'$ with $\{\widetilde{x}\} \subseteq fv(M) \setminus fv(N)$ and $\widetilde{x}$ solvable in $\{^M/_{z'}\} \mid A'$ for some $z' \notin fv(A') \cup \{\widetilde{x}\}$. By induction hypothesis, $B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \{^M/_z\} \mid A'$ for all z that do not occur in the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} A'$, so z does not occur in $A = \nu\widetilde{x}.B \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$ since $\{\widetilde{x}\} \subseteq fv(M)$. By Scope, $A = \nu\widetilde{x}.B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid A')$, since $\{\widetilde{x}\} \cap fv(N) = \emptyset$.
- Case Scope. The transition $A = \nu u.B \xrightarrow{\overline{N}\langle M\rangle} \nu u.B' = A'$ is derived from $B \xrightarrow{\overline{N}\langle M\rangle} B'$, where u does not occur in $\overline{N}\langle M\rangle$. (The restriction of the rule Scope guarantees that $\widetilde{x}$ is empty.) By induction hypothesis, $B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \{^M/_z\} \mid B'$ for all z that do not occur in the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$. Let z be a variable that does not occur in the derivation of $A = \nu u.B \xrightarrow{\overline{N}\langle M\rangle} \nu u.B' = A'$. Since the derivation of $A = \nu u.B \xrightarrow{\overline{N}\langle M\rangle} \nu u.B' = A'$ includes the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$, z does not occur in the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$. Hence, we have $B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \{^M/_z\} \mid B'$, so by Scope, $A = \nu u.B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu u.(\{^M/_z\} \mid B')$, since u does not occur in $\nu z.\overline{N}\langle z\rangle$. Moreover, $\nu u.(\{^M/_z\} \mid B') \equiv \{^M/_z\} \mid \nu u.B' = \{^M/_z\} \mid A'$ since u does not occur in $\{^M/_z\}$. So by Struct, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \{^M/_z\} \mid A'$.
- Case Par. The transition $A = B \mid C \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} B' \mid C = A'$ is derived from $B \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} B'$, with $\{\widetilde{x}\} \cap fv(C) = \emptyset$. By induction hypothesis, $B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid B')$, $\{\widetilde{x}\} \subseteq fv(M) \setminus fv(N)$, and the variables $\widetilde{x}$ are solvable in $\{^M/_z\} \mid B'$, for all z that do not occur in the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$. Let z be a variable that does not occur in the derivation of $A = B \mid C \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} B' \mid C = A'$. Since the derivation of $A = B \mid C \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} B' \mid C = A'$ includes the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$, z does not occur in the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$. Hence, we have $B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid B')$, so by Par, $B \mid C \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid B') \mid C$, since $z \notin fv(C)$. Moreover, $\nu\widetilde{x}.(\{^M/_z\} \mid B') \mid C \equiv \nu\widetilde{x}.(\{^M/_z\} \mid (B' \mid C)) = \nu\widetilde{x}.(\{^M/_z\} \mid A')$ since $\{\widetilde{x}\} \cap fv(C) = \emptyset$, so by Struct, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid A')$. Moreover, the variables $\widetilde{x}$ are solvable in $\{^M/_z\} \mid A'$: assuming that the variables $\widetilde{x}$ resolve to $\widetilde{M}$ in $\{^M/_z\} \mid B'$, we have

$$\begin{aligned}
\{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \nu\widetilde{x}.(\{^M/_z\} \mid A') &\equiv \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \nu\widetilde{x}.(\{^M/_z\} \mid (B' \mid C)) \\
&\equiv \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \nu\widetilde{x}.(\{^M/_z\} \mid B') \mid C && \text{since } \{\widetilde{x}\} \cap fv(C) = \emptyset \\
&\equiv \{^M/_z\} \mid B' \mid C && \text{since } \widetilde{x} \text{ resolve to } \widetilde{M} \text{ in } \{^M/_z\} \mid B' \\
&\equiv \{^M/_z\} \mid A'
\end{aligned}$$

- Case Struct. The transition $A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$ is derived from $B \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} B'$, $A \equiv B$ and $A' \equiv B'$. By induction hypothesis, $B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid B')$, $\{\widetilde{x}\} \subseteq fv(M) \setminus fv(N)$, and the variables $\widetilde{x}$ are solvable in $\{^M/_z\} \mid B'$, for all z that do not occur in the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$. Let z be a variable that does not occur in the derivation of $A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$. Since the derivation of $A \xrightarrow{\nu\widetilde{x}.\overline{N}\langle M\rangle} A'$ includes the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$, z does not occur in the derivation of $B \xrightarrow{\overline{N}\langle M\rangle} B'$. Hence, we have $B \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\widetilde{x}.(\{^M/_z\} \mid B')$ and $\nu\widetilde{x}.(\{^M/_z\} \mid B') \equiv \nu\widetilde{x}.(\{^M/_z\} \mid A')$,

so by Struct, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\tilde{x}.(\{^M/_z\} \mid A')$. Moreover, the variables $\tilde{x}$ are solvable in $\{^M/_z\} \mid B'$ and $\{^M/_z\} \mid B' \equiv \{^M/_z\} \mid A'$, so by Definition 4.11, the variables $\tilde{x}$ are solvable in $\{^M/_z\} \mid A'$.

Let us now prove the implication from right to left. For this proof, we use the notion of partial normal form introduced in Appendix B. We have $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\tilde{x}.(\{^M/_z\} \mid A')$ where the variables $\tilde{x}$ are solvable in $\{^M/_z\} \mid A'$, $\{\tilde{x}\} \subseteq fv(M) \setminus fv(N)$, and z does not occur in $A, A', \tilde{x}, N, M$. By Lemma B.12, we have $\mathrm{pnf}(A) \xrightarrow{\nu z.\overline{N}\langle z\rangle}_\circ \nu\tilde{x}.(\{^M/_z\} \mid A')$. By definition of $\xrightarrow{\nu z.\overline{N}\langle z\rangle}_\circ$, we have $\mathrm{pnf}(A) \equiv \nu\tilde{n}.(\sigma \mid P)$, $P \xrightarrow{\nu z.\overline{N'}\langle z\rangle}_\diamond B'$, $\nu\tilde{x}.(\{^M/_z\} \mid A') \equiv \nu\tilde{n}.(\sigma \mid B')$, $z \notin fv(\sigma)$, $\Sigma \vdash N\sigma = N'$, and the elements of $\tilde{n}$ do not occur in N, for some $\tilde{n}, \sigma, P, N', B'$. By Lemma B.10, we have $P \overset{\diamond}{\equiv} \nu\tilde{n}'.(\overline{N'}\langle M'\rangle.P_1 \mid P_2)$, $B' \equiv \nu\tilde{n}'.(P_1 \mid \{^{M'}/_z\} \mid P_2)$, $\{\tilde{n}'\} \cap fn(N') = \emptyset$, $z \notin fv(P_1 \mid P_2)$ for some $\tilde{n}', P_1, P_2, N', M'$. Hence, we have

$$A \equiv \nu\tilde{n}.(\sigma \mid \nu\tilde{n}'.(\overline{N'}\langle M'\rangle.P_1 \mid P_2))$$
$$\nu\tilde{x}.(\{^M/_z\} \mid A') \equiv \nu\tilde{n}.(\sigma \mid \nu\tilde{n}'.(P_1 \mid \{^{M'}/_z\} \mid P_2))$$

We rename the names in $\tilde{n}'$ so that they do not occur in σ nor in N. Then

$$A \equiv \nu\tilde{n}, \tilde{n}'.(\sigma \mid \overline{N'}\langle M'\rangle.P_1 \mid P_2)$$
$$\nu\tilde{x}.(\{^M/_z\} \mid A') \equiv \nu\tilde{n}, \tilde{n}'.(\sigma \mid P_1 \mid \{^{M'}/_z\} \mid P_2)$$

We instantiate the variables using σ, so that the variables of $dom(\sigma)$ do not occur in the image of σ nor in N', M', P_1, P_2. Furthermore, let σ' be a substitution that maps $\tilde{x}$ to distinct fresh names. By Lemma B.5,

$$\mathrm{pnf}(\nu\tilde{x}.(\{^M/_z\} \mid A')) \overset{\circ}{\equiv} \nu\tilde{n}, \tilde{n}'.((\sigma \mid \{^{M'}/_z\}) \mid (P_1 \mid P_2))$$

Moreover, $\Sigma \vdash \mathrm{pnf}(\nu\tilde{x}.(\{^M/_z\} \mid A')) = \mathrm{pnf}(\nu\tilde{x}.(\{^M/_z\} \mid A'))\sigma'$ because $\{\tilde{x}\} \cap fv(\mathrm{pnf}(\nu\tilde{x}.(\{^M/_z\} \mid A'))) = \emptyset$. Therefore, by Lemma B.14, $\Sigma \vdash \nu\tilde{n}, \tilde{n}'.((\sigma \mid \{^{M'}/_z\}) \mid (P_1 \mid P_2)) = \nu\tilde{n}, \tilde{n}'.((\sigma \mid \{^{M'}/_z\}) \mid (P_1 \mid P_2))\sigma'$, so $\Sigma \vdash M' = M'\sigma'$, $\Sigma \vdash \sigma = \sigma\sigma'$, $\Sigma \vdash P_1 = P_1\sigma'$, and $\Sigma \vdash P_2 = P_2\sigma'$, so by replacing σ with $\sigma\sigma'$, M' with $M'\sigma'$, P_1 with $P_1\sigma'$, and P_2 with $P_2\sigma'$, we obtain

$$A \equiv \nu\tilde{n}, \tilde{n}'.(\sigma \mid \overline{N}\langle M'\rangle.P_1 \mid P_2)$$
$$\nu\tilde{x}.(\{^M/_z\} \mid A') \equiv \nu\tilde{n}, \tilde{n}'.(\sigma \mid P_1 \mid \{^{M'}/_z\} \mid P_2)$$

and the variables $\tilde{x}$ are not free in the right-hand sides of these equivalences.

The variables $\tilde{x}$ resolve to some $\tilde{M}$ in $\{^M/_z\} \mid A'$, so

$$\{^M/_z\} \mid A' \equiv \{^{\tilde{M}}/_{\tilde{x}}\} \mid \nu\tilde{x}.(\{^M/_z\} \mid A') \equiv \{^{\tilde{M}}/_{\tilde{x}}\} \mid \nu\tilde{n}, \tilde{n}'.(\sigma \mid P_1 \mid \{^{M'}/_z\} \mid P_2)$$

We rename the names $\tilde{n}, \tilde{n}'$ so that they do not occur in $\tilde{M}$. Hence

$$\{^M/_z\} \mid A' \equiv \nu\tilde{n}, \tilde{n}'.(\sigma \mid \{^{M'}/_z\} \mid \{^{\tilde{M}}/_{\tilde{x}}\} \mid P_1 \mid P_2)$$
$$A' \equiv \nu z.(\{^M/_z\} \mid A') \equiv \nu z, \tilde{n}, \tilde{n}'.(\sigma \mid \{^{M'}/_z\} \mid \{^{\tilde{M}}/_{\tilde{x}}\} \mid P_1 \mid P_2)$$

By Lemma 4.12, $(z = M)\varphi(\{^M/_z\} \mid A')$, so $(z = M)\nu\tilde{n}, \tilde{n}'.(\sigma \mid \{^{M'}/_z\} \mid \{^{\tilde{M}}/_{\tilde{x}}\})$. We rename the names $\tilde{n}, \tilde{n}'$ so that they do not occur in M. Therefore,

$$A \equiv \nu\tilde{x}, z, \tilde{n}, \tilde{n}'.(\sigma \mid \{^{M'}/_z\} \mid \{^{\tilde{M}}/_{\tilde{x}}\} \mid \overline{N}\langle z\rangle.P_1 \mid P_2)$$
$$\equiv \nu\tilde{x}, z, \tilde{n}, \tilde{n}'.(\sigma \mid \{^{M'}/_z\} \mid \{^{\tilde{M}}/_{\tilde{x}}\} \mid \overline{N}\langle M\rangle.P_1 \mid P_2)$$

So we derive

$$\overline{N}\langle M\rangle.P_1 \xrightarrow{\overline{N}\langle M\rangle} P_1 \qquad \text{by Out-Term}$$

$\overline{N}\langle M\rangle.P_1 \mid \sigma \mid \{^{M'}/_z\} \mid \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid P_2 \xrightarrow{\overline{N}\langle M\rangle} P_1 \mid \sigma \mid \{^{M'}/_z\} \mid \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid P_2$ by Par

$\nu z, \widetilde{n}, \widetilde{n}'.(\overline{N}\langle M\rangle.P_1 \mid \sigma \mid \{^{M'}/_z\} \mid \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid P_2) \xrightarrow{\overline{N}\langle M\rangle} \nu z, \widetilde{n}, \widetilde{n}'.(P_1 \mid \sigma \mid \{^{M'}/_z\} \mid \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid P_2)$
by Scope, since $z, \widetilde{n}, \widetilde{n}'$ do not occur in $\overline{N}\langle M\rangle$

$\nu z, \widetilde{n}, \widetilde{n}'.(\sigma \mid \{^{M'}/_z\} \mid \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \overline{N}\langle M\rangle.P_1 \mid P_2) \xrightarrow{\overline{N}\langle M\rangle} A'$ by Struct

$\nu \widetilde{x}, z, \widetilde{n}, \widetilde{n}'.(\sigma \mid \{^{M'}/_z\} \mid \{^{\widetilde{M}}/_{\widetilde{x}}\} \mid \overline{N}\langle M\rangle.P_1 \mid P_2) \xrightarrow{\nu \widetilde{x}.\overline{N}\langle M\rangle} A'$
by Open-Var, since $\{\widetilde{x}\} \subseteq fv(M) \setminus fv(N)$
and the variables $\widetilde{x}$ are solvable in $\{^{M}/_z\} \mid A'$

$A \xrightarrow{\nu \widetilde{x}.\overline{N}\langle M\rangle} A'$ by Struct

□

Lemma 4.16. *$A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ in the refined semantics if and only if $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ in the simple semantics.*

Proof. Suppose that $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ in the refined semantics. By Lemma 4.15, for some variable z that does not occur in this transition, we have $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu x.(\{^{x}/_z\} \mid A')$ in the simple semantics. Since $x \in dom(A')$, $A' \equiv \nu \widetilde{n}.(\{^{M}/_x\} \mid A'')$ for some $\widetilde{n}$ and some M and A'' that do not contain x nor z, so

$$\nu x.(\{^{x}/_z\} \mid A') \equiv \nu x.(\{^{x}/_z\} \mid \nu \widetilde{n}.(\{^{M}/_x\} \mid A'')) \equiv \nu \widetilde{n}.(\{^{M}/_z\} \mid A'')$$

Hence by Struct, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu \widetilde{n}.(\{^{M}/_z\} \mid A'')$. By renaming z into x and x into z everywhere in the derivation of this transition, we obtain $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} \nu \widetilde{n}.(\{^{M}/_x\} \mid A'')$, since z and x are not free in A, N, A'', M. Since we have $\nu \widetilde{n}.(\{^{M}/_x\} \mid A'') \equiv A'$, we obtain $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ by Struct in the simple semantics.

Conversely, suppose that $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ in the simple semantics. Since $x \in dom(A')$, $A' \equiv \nu \widetilde{n}.(\{^{M}/_x\} \mid A'')$ for some $\widetilde{n}$ and some M and A'' that do not contain x, so by Struct, $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} \nu \widetilde{n}.(\{^{M}/_x\} \mid A'')$. By renaming x into a fresh variable z everywhere in the derivation of this transition, $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu \widetilde{n}.(\{^{M}/_z\} \mid A'')$, since x is not free in A, M, A''. Moreover, $\nu \widetilde{n}.(\{^{M}/_z\} \mid A'') \equiv \nu x.(\{^{x}/_z\} \mid \nu \widetilde{n}.(\{^{M}/_x\} \mid A'')) \equiv \nu x.(\{^{x}/_z\} \mid A')$, so by Struct, we obtain $A \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu x.(\{^{x}/_z\} \mid A')$.

The variable x resolves to z in $\{^{x}/_z\} \mid A'$, because

$$\begin{aligned}
\{^{z}/_x\} \mid \nu x.(\{^{x}/_z\} \mid A') &\equiv \{^{z}/_x\} \mid \nu x.(\{^{x}/_z\} \mid \nu \widetilde{n}.(\{^{M}/_x\} \mid A'')) \\
&\equiv \{^{z}/_x\} \mid \nu \widetilde{n}.(\{^{M}/_z\} \mid A'') \\
&\equiv \nu \widetilde{n}.(\{^{z}/_x\} \mid \{^{M}/_z\} \mid A'') \\
&\equiv \nu \widetilde{n}.(\{^{M}/_x\} \mid \{^{x}/_z\} \mid A'') \\
&\equiv \{^{x}/_z\} \mid \nu \widetilde{n}.(\{^{M}/_x\} \mid A'') \\
&\equiv \{^{x}/_z\} \mid A'
\end{aligned}$$

Therefore, by Lemma 4.15, $A \xrightarrow{\nu x.\overline{N}\langle x\rangle} A'$ in the refined semantics. □

Theorem 4.14. *Let $\approx_L$ be the relation of labelled bisimilarity obtained by applying Definition 4.7 to the refined semantics. We have $\approx_l = \approx_L$.*

Proof. By Lemma 4.16, $\approx_L$ is a simple-labelled bisimulation, and thus $\approx_L \subseteq \approx_l$. Conversely, to show that $\approx_l$ is a refined-labelled bisimulation, it suffices to prove its bisimulation property for any refined output label.

Assume $A \approx_l B$, $A \xrightarrow{\nu\tilde{x}.\overline{N}\langle M\rangle} A'$, A' is closed, and $fv(\nu\tilde{x}.\overline{N}\langle M\rangle) \subseteq dom(A)$. By Lemma 4.15, we have

$$A \xrightarrow{\nu z.\overline{N}\langle z\rangle} A^\circ = \nu\tilde{x}.(\{^M/_z\} \mid A')$$

for some fresh variable z, where $\{\tilde{x}\} \subseteq fv(M) \setminus fv(N)$ and $\tilde{x}$ resolves to $\widetilde{M}$ in $\{^M/_z\} \mid A'$:

$$\{^M/_z\} \mid A' \equiv \{^{\widetilde{M}}/_{\tilde{x}}\} \mid \nu\tilde{x}.(\{^M/_z\} \mid A') \equiv \{^{\widetilde{M}}/_{\tilde{x}}\} \mid A^\circ \tag{19}$$

Let $E[\_] = \nu z.(\{^{\widetilde{M}}/_{\tilde{x}}\} \mid \_)$. Using the structural equivalence above and structural rearrangements, we obtain $E[A^\circ] \equiv \nu z.(\{^M/_z\} \mid A') \equiv A'$. By labelled bisimulation hypothesis on the simple output transition above, we have $B \to^* B_1 \xrightarrow{\nu z.\overline{N}\langle z\rangle} B_2 \to^* B^\circ$ with $A^\circ \approx_l B^\circ$ for some B_1, B_2, B°. By instantiating all variables in $fv(B_2) \setminus dom(B_2)$ with fresh names in the derivation of this reduction, we obtain the same property and additionally B_2 is closed. By Theorem 4.8, labelled bisimilarity is closed by application of closing contexts. Using $E[\_]$, we obtain $A' \approx_l E[B^\circ]$. Let $B' = E[B^\circ]$.

Let us first show that $B_2 \equiv \nu\tilde{x}.(\{^M/_z\} \mid E[B_2])$. By Lemma 4.12, we have $(z = M)\varphi(\{^M/_z\} \mid A')$ and by the structural equivalence (19), $(\tilde{x} = \widetilde{M})\varphi(\{^M/_z\} \mid A')$, so $(z = M\{^{\widetilde{M}}/_{\tilde{x}}\})\varphi(\{^M/_z\} \mid A')$, so $(z = M\{^{\widetilde{M}}/_{\tilde{x}}\})\varphi(\nu\tilde{x}.(\{^M/_z\} \mid A'))$ since the variables $\tilde{x}$ do not occur in $M\{^{\widetilde{M}}/_{\tilde{x}}\}$. Hence $(z = M\{^{\widetilde{M}}/_{\tilde{x}}\})\varphi(A^\circ)$. Since $A^\circ \approx_l B^\circ \approx_s B_2$, we have $A^\circ \approx_s B_2$, so $(z = M\{^{\widetilde{M}}/_{\tilde{x}}\})\varphi(B_2)$. Since $z \in dom(B_2)$, we have $B_2 \equiv \nu\tilde{n}.(\{^N/_z\} \mid B_3)$ for some $\tilde{n}$, N and B_3 such that z is not free in B_3. We rename $\tilde{n}$ so that these names do not occur in $\widetilde{M}$ nor in M. Then $E[B_2] \equiv \nu\tilde{n}.(\{^{\widetilde{M}\{^{N}/_{z}\}}/_{\tilde{x}}\} \mid B_3)$, so

$$\nu\tilde{x}.(\{^M/_z\} \mid E[B_2]) \equiv \nu\tilde{n}.(\{^{M\{^{\widetilde{M}\{^{N}/_{z}\}}/_{\tilde{x}}\}}/_z\} \mid B_3) \equiv \nu\tilde{n}.(\{^N/_z\} \mid B_3) \equiv B_2$$

because $(N = M\{^{\widetilde{M}\{^{N}/_{z}\}}/_{\tilde{x}}\})\varphi(B_3)$ since $(z = M\{^{\widetilde{M}}/_{\tilde{x}}\})\varphi(B_2)$. So we have the desired structural equivalence $B_2 \equiv \nu\tilde{x}.(\{^M/_z\} \mid E[B_2])$.

Then

$$B_1 \xrightarrow{\nu z.\overline{N}\langle z\rangle} \nu\tilde{x}.(\{^M/_z\} \mid E[B_2])$$

Moreover, $\tilde{x}$ resolves to $\widetilde{M}$ in $\{^M/_z\} \mid A'$ and

$$\{^M/_z\} \mid A' \equiv \{^M/_z\} \mid E[A^\circ] \approx_l \{^M/_z\} \mid E[B^\circ] \approx_s \{^M/_z\} \mid E[B_2]$$

so $\{^M/_z\} \mid A' \approx_s \{^M/_z\} \mid E[B_2]$, so by Lemma 4.13, $\tilde{x}$ resolves to $\widetilde{M}$ in $\{^M/_z\} \mid E[B_2]$. Hence, by Lemma 4.15, $B_1 \xrightarrow{\nu\tilde{x}.\overline{N}\langle M\rangle} E[B_2]$. Hence

$$B \to^* B_1 \xrightarrow{\nu\tilde{x}.\overline{N}\langle M\rangle} E[B_2] \to^* E[B^\circ] = B'$$

so we have $A' \approx_l B'$ and $B \to^* \xrightarrow{\nu\tilde{x}.\overline{N}\langle M\rangle} \to^* B'$, which concludes the proof. □

F PROOFS FOR SECTION 6.2

In this appendix, we suppose that the signature Σ satisfies the assumptions of Theorem 6.1 and write R for its convergent rewrite system. In particular, since R terminates, the left-hand sides of its rewrite rules cannot be variables.

In preparation for the proof of Theorem 6.1, we study the effect of the translation $[\![\cdot]\!]$ on the semantics of terms and processes where k occurs only as $\mathsf{mac}(k, \cdot)$, relying on the partial normal forms defined in Appendix B.

LEMMA F.1. *$\Sigma \vdash M_1 = M_2$ if and only if $\Sigma \vdash \mathsf{h}(k, M_1) = \mathsf{h}(k, M_2)$.*

PROOF. The implication from left to right is obvious. Conversely, suppose that $\Sigma \vdash \mathsf{h}(k, M_1) = \mathsf{h}(k, M_2)$. Let M_1' and M_2' be the normal forms under R of M_1 and M_2 respectively. Hence $\Sigma \vdash \mathsf{h}(k, M_1') = \mathsf{h}(k, M_2')$. For some $n, n' \geq 1$, we have $M_1' = N_1 :: \ldots :: N_n$ and $M_2' = N_1' :: \ldots :: N_{n'}'$ where the root symbols of N_n and $N_{n'}'$ are not ::. We compute the normal form of $\mathsf{h}(k, M_1')$:

- If $n = 1$, then $\mathsf{h}(k, M_1') = \mathsf{h}(k, N_1)$ is irreducible since N_1 is irreducible and the rewrite rules with h at the root of the left-hand side apply only to terms with :: at the root.
- If $n > 1$ and $N_n = \mathsf{nil}$, then $\mathsf{h}(k, M_1')$ reduces to $\mathsf{f}(\ldots(\mathsf{f}(k, N_1), \ldots), N_{n-1})$, and this term is irreducible since $N_1, \ldots, N_{n-1}$ are irreducible as subterms of an irreducible term, and no rewrite rule contains f in its left-hand side.
- If $n > 1$ and $N_n \neq \mathsf{nil}$, then $\mathsf{h}(k, M_1')$ reduces to $\mathsf{h}(\mathsf{f}(\ldots(\mathsf{f}(k, N_1), \ldots), N_{n-2}), N_{n-1} :: N_n)$ and this term is irreducible since $N_1, \ldots, N_n$ are irreducible, no rewrite rule contains f in its left-hand side, and no rewrite rule with h at the root applies since N_n is not nil and does not contain :: at the root.

and similarly compute a normal form of $\mathsf{h}(k, M_2')$. Their equality implies $n = n'$ and $N_i = N_i'$ for all $i \leq n$, hence $M_1' = M_2'$ and $\Sigma \vdash M_1 = M_2$. □

LEMMA F.2. *If $M_1 \to_R M_2$ and k occurs only as $\mathsf{mac}(k, \cdot)$ in M_1, then $[\![M_1]\!] \to_R [\![M_2]\!]$ and k occurs only as $\mathsf{mac}(k, \cdot)$ in M_2.*

PROOF. We have $M_1 = C[M_3\sigma]$ and $M_2 = C[M_4\sigma]$ for some rewrite rule $M_3 \to M_4$ of R, term context C, and substitution σ. Hence $[\![M_1]\!] = [\![C[M_3\sigma]]\!]$. Furthermore, mac and k do not occur in M_3 and M_3 is not a variable, so $[\![C[M_3\sigma]]\!] = [\![C]\!][M_3[\![\sigma]\!]]$. Since k occurs only as $\mathsf{mac}(k, \cdot)$ in M_1 and mac does not occur in M_3, k occurs only as $\mathsf{mac}(k, \cdot)$ in C and in the image of σ. Furthermore, k does not occur in M_4. Therefore, k occurs only as $\mathsf{mac}(k, \cdot)$ in $M_2 = C[M_4\sigma]$ and $[\![M_2]\!] = [\![C[M_4\sigma]]\!] = [\![C]\!][M_4[\![\sigma]\!]]$. We can then conclude that $[\![M_1]\!] \to_R [\![M_2]\!]$. □

LEMMA F.3. *Suppose that k occurs only as $\mathsf{mac}(k, \cdot)$ in M_1 and M_2. We have $\Sigma \vdash M_1 = M_2$ if and only if $\Sigma \vdash [\![M_1]\!] = [\![M_2]\!]$.*

PROOF. Let us first prove the implication from left to right. If $\Sigma \vdash M_1 = M_2$, then $M_1 \to_R^* M'$ and $M_2 \to_R^* M'$ for some M'. By Lemma F.2, $[\![M_1]\!] \to_R^* [\![M']\!]$ and $[\![M_2]\!] \to_R^* [\![M']\!]$, so $\Sigma \vdash [\![M_1]\!] = [\![M_2]\!]$.

Conversely, suppose that $\Sigma \vdash [\![M_1]\!] = [\![M_2]\!]$. Let M_1' and M_2' be the normal forms under R of M_1 and M_2, respectively. By Lemma F.2, k occurs only as $\mathsf{mac}(k, \cdot)$ in M_1' and M_2', $[\![M_1]\!] \to_R^* [\![M_1']\!]$, and $[\![M_2]\!] \to_R^* [\![M_2']\!]$, so $\Sigma \vdash [\![M_1']\!] = [\![M_2']\!]$. We show by induction on the total size of the terms M_1' and M_2' that, if k occurs only as $\mathsf{mac}(k, \cdot)$ in M_1' and M_2', M_1' and M_2' are irreducible under R, and $\Sigma \vdash [\![M_1']\!] = [\![M_2']\!]$, then $M_1' = M_2'$:

- First suppose that M_1' and M_2' are not of the form $\mathsf{mac}(k, \cdot)$.
  Since f does not occur on the left-hand sides of rewrite rules of R, if a rewrite rule of R could be applied at the root of $[\![M_1']\!]$ or $[\![M_2']\!]$, then it would match only symbols above

occurrences of $\mathsf{f}(\dots)$ in $[\![M_1']\!]$ or $[\![M_2']\!]$, hence, only symbols above occurrences of $\mathsf{mac}(k,\cdot)$ in M_1' or M_2'. Moreover, by induction hypothesis, if subterms of $[\![M_1']\!]$ or $[\![M_2']\!]$ are equal, the corresponding subterms of M_1' or M_2' are also equal. Hence, the same rewrite rule would also apply at the root of M_1' or M_2', which is impossible since M_1' and M_2' are irreducible.
Hence, the equality $\Sigma \vdash [\![M_1']\!] = [\![M_2']\!]$ is equivalent to the equality between the immediate subterms of $[\![M_1']\!]$ and $[\![M_2']\!]$, and we conclude by induction.

- Now suppose that $M_1' = \mathsf{mac}(k, M_1'')$ and $M_2' = \mathsf{mac}(k, M_2'')$. Then $[\![M_1']\!] = \mathsf{f}(k, \mathsf{h}(k, [\![M_1'']\!]))$ and $[\![M_2']\!] = \mathsf{f}(k, \mathsf{h}(k, [\![M_2'']\!]))$. Since $\Sigma \vdash [\![M_1']\!] = [\![M_2']\!]$, we have $\Sigma \vdash \mathsf{h}(k, [\![M_1'']\!]) = \mathsf{h}(k, [\![M_2'']\!])$ since no rewrite rule applies to $\mathsf{f}(\cdot,\cdot)$, so by Lemma F.1, $\Sigma \vdash [\![M_1'']\!] = [\![M_2'']\!]$. By induction hypothesis, $M_1'' = M_2''$, so $M_1' = M_2'$.
- Finally, if $M_1' = \mathsf{mac}(k, M_1'')$ and M_2' is not of the form $\mathsf{mac}(k,\cdot)$, then $[\![M_1']\!] = \mathsf{f}(k, \mathsf{h}(k, [\![M_1'']\!]))$ and $[\![M_2']\!]$ is not of the form $\mathsf{f}(k,\cdot)$ because k occurs only as $\mathsf{mac}(k,\cdot)$ in M_2', so $\Sigma \vdash [\![M_1']\!] \neq [\![M_2']\!]$: this case cannot happen. Symmetrically, the case $M_2' = \mathsf{mac}(k, M_2'')$ and M_1' is not of the form $\mathsf{mac}(k,\cdot)$ cannot happen.

From this result, we easily conclude that $\Sigma \vdash M_1 = M_2$. □

Lemma F.4. *Suppose that P_0 is closed, $\alpha = \nu x.\overline{N'}\langle x\rangle$ or $\alpha = N'(M')$ for some ground term N', and k occurs only as $\mathsf{mac}(k,\cdot)$ in P_0 and α.*

If $P_0 \xrightarrow{\alpha}_\diamond A$, then $[\![P_0]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![A']\!]$ and $A \equiv A'$ for some A' where k occurs only as $\mathsf{mac}(k,\cdot)$ and, moreover,

- *when $\alpha = \nu x.\overline{N'}\langle x\rangle$, $A' = E[\{^{M}/_{x}\}]$ where E is a closed plain evaluation context (with no active substitutions and no variable restrictions) and M is a ground term;*
- *when $\alpha = N'(M')$, A' is a plain process with $fv(A') \subseteq fv(M')$.*

Proof. We proceed by induction on the syntax of P_0 and apply Lemma B.18 to decompose $P_0 \xrightarrow{\alpha}_\diamond A$, with the following cases:

(1) $P_0 = P \mid Q$ and either $P \xrightarrow{\alpha}_\diamond A'$ and $A \equiv A' \mid Q$, or $Q \xrightarrow{\alpha}_\diamond A'$ and $A \equiv P \mid A'$, for some P, Q, and A'. In the first case, by induction hypothesis, $[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![A'']\!]$ and $A' \equiv A''$ for some A'' where k occurs only as $\mathsf{mac}(k,\cdot)$. By Par′, since $[\![Q]\!]$ is closed, $[\![P_0]\!] = [\![P]\!] \mid [\![Q]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![A'']\!] \mid [\![Q]\!] = [\![A'' \mid Q]\!]$ and $A'' \mid Q \equiv A' \mid Q \equiv A$. Furthermore, k occurs only as $\mathsf{mac}(k,\cdot)$ in $A'' \mid Q$. The second case is symmetric.

(2) $P_0 = \nu n.P$, $P \xrightarrow{\alpha}_\diamond A'$, and $A \equiv \nu n.A'$ for some P, A', and n that does not occur in α. We rename n so that $n \neq k$. By induction hypothesis, $[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![A'']\!]$ and $A' \equiv A''$ for some A'' where k occurs only as $\mathsf{mac}(k,\cdot)$. By Scope′, $[\![P_0]\!] = \nu n.[\![P]\!] \xrightarrow{\alpha}_\diamond \nu n.[\![A'']\!] = [\![\nu n.A'']\!]$ and $\nu n.A'' \equiv \nu n.A' \equiv A$. Furthermore, k occurs only as $\mathsf{mac}(k,\cdot)$ in $\nu n.A''$.

(3) $P_0 = !P$, $P \xrightarrow{\alpha}_\diamond A'$, and $A \equiv A' \mid !P$ for some P and A'. By induction hypothesis, $[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![A'']\!]$ and $A' \equiv A''$ for some A'' where k occurs only as $\mathsf{mac}(k,\cdot)$. We have $[\![P_0]\!] = ![\![P]\!] \overset{\diamond}{\equiv} [\![P]\!] \mid ![\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![A'']\!] \mid ![\![P]\!]$ by Par′, since $![\![P]\!]$ is closed. Hence by Struct′, $[\![P_0]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![A'' \mid !P]\!]$ and $A'' \mid !P \equiv A' \mid !P \equiv A$. Furthermore, k occurs only as $\mathsf{mac}(k,\cdot)$ in $A'' \mid !P$.

(4) $P_0 = N(x).P$, $\alpha = N'(M')$, $\Sigma \vdash N = N'$, and $A \equiv P\{^{M'}/_{x}\}$ for some N, x, P, N', and M'. By Lemma F.3, $\Sigma \vdash [\![N]\!] = [\![N']\!]$, so we have $[\![P_0]\!] = [\![N]\!](x).[\![P]\!] \overset{\diamond}{\equiv} [\![N']\!](x).[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![P]\!]\{^{[\![M']\!]}/_{x}\}$ by In′. Since k occurs only as $\mathsf{mac}(k,\cdot)$ in M', the substitution $[\![P]\!]\{^{[\![M']\!]}/_{x}\}$ does not create new occurrences of mac with key k, so $[\![P]\!]\{^{[\![M']\!]}/_{x}\} = [\![P\{^{M'}/_{x}\}]\!]$. By Struct′,

we obtain $[\![P_0]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![P\{^{M'}/_x\}]\!]$, and we have $A \equiv P\{^{M'}/_x\}$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $P\{^{M'}/_x\}$.

(5) $P_0 = \overline{N}\langle M\rangle.P$, $\alpha = \nu x.\overline{N'}\langle x\rangle$, $\Sigma \vdash N = N'$, $x \notin fv(P_0)$, and $A \equiv P \mid \{^M/_x\}$ for some N, M, P, x, and N'. By Lemma F.3, $\Sigma \vdash [\![N]\!] = [\![N']\!]$, so we have $[\![P_0]\!] = \overline{[\![N]\!]}\langle [\![M]\!]\rangle.[\![P]\!] \overset{\diamond}{\equiv} \overline{[\![N']\!]}\langle [\![M]\!]\rangle.[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![P]\!] \mid \{^{[\![M]\!]}/_x\} = [\![P \mid \{^M/_x\}]\!]$ by Out-Var′. By Struct′, we obtain $[\![P_0]\!] \xrightarrow{[\![\alpha]\!]}_\diamond [\![P \mid \{^M/_x\}]\!]$, and we have $A \equiv P \mid \{^M/_x\}$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $P \mid \{^M/_x\}$. □

Lemma F.5. *If $P_0 \to_\diamond R$ for some closed process P_0 where k occurs only as $\mathrm{mac}(k,\cdot)$, then $[\![P_0]\!] \to_\diamond [\![R']\!]$ and $R' \equiv R$ for some closed process R' where k occurs only as $\mathrm{mac}(k,\cdot)$.*

Proof. We define the size of processes by induction on the syntax, such that $size(!P) = 1 + 2 \times size(P)$ and, when P is not a replication, $size(P)$ is one plus the size of the immediate subprocesses of P. We proceed by induction on the size of P_0. By Lemma B.21, we decompose $P_0 \to_\diamond R$, with the following cases:

(1) $P_0 = P \mid Q$ for some P and Q, and one of the following cases holds:
   (a) $P \to_\diamond P'$ and $R \equiv P' \mid Q$ for some P',
   (b) $P \xrightarrow{N(x)}_\diamond A$, $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N,
   and two symmetric cases obtained by swapping P and Q.
   In case (a), by induction hypothesis, $[\![P]\!] \to_\diamond [\![P'']\!]$ and $P'' \equiv P'$ for some closed process P'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Hence $[\![P_0]\!] = [\![P]\!] \mid [\![Q]\!] \to_\diamond [\![P'']\!] \mid [\![Q]\!] = [\![P'' \mid Q]\!]$ and $P'' \mid Q \equiv P' \mid Q \equiv R$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $P'' \mid Q$.
   In case (b), by Lemma F.4, $[\![P]\!] \xrightarrow{[\![N]\!](x)}_\diamond [\![P_1]\!]$ and $A \equiv P_1$ for some P_1 where k occurs only as $\mathrm{mac}(k,\cdot)$ and $fv(P_1) \subseteq \{x\}$; and $[\![Q]\!] \xrightarrow{\nu x.\overline{[\![N]\!]}\langle x\rangle}_\diamond [\![B']\!]$ for some $B' = E_2[\{^{M_2}/_x\}]$ such that $B \equiv B'$, k occurs only as $\mathrm{mac}(k,\cdot)$ in B', E_2 is a closed plain evaluation context and M_2 is a ground term. By Lemma B.20, $[\![P_0]\!] = [\![P]\!] \mid [\![Q]\!] \to_\diamond R'$ and $R' \equiv \nu x.([\![P_1]\!] \mid [\![B']\!]) = \nu x.([\![P_1]\!] \mid [\![E_2]\!][\{^{[\![M_2]\!]}/_x\}])$ for some R'. We rename the bound names of E_2 so that they do not occur in P_1. Let $R'' = E_2[P_1\{^{M_2}/_x\}]$. The process R'' is closed and such that k occurs only as $\mathrm{mac}(k,\cdot)$. We have $R' \equiv [\![R'']\!]$, so $[\![P_0]\!] \to_\diamond [\![R'']\!]$ and $R'' \equiv \nu x.(P_1 \mid B') \equiv \nu x.(A \mid B) \equiv R$. The last two cases are symmetric.

(2) $P_0 = \nu n.P$, $P \to_\diamond Q'$, and $R \equiv \nu n.Q'$ for some n, P, and Q'. We rename n so that $n \neq k$. By induction hypothesis, $[\![P]\!] \to_\diamond [\![Q'']\!]$ and $Q' \equiv Q''$ for some closed process Q'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Hence $[\![P_0]\!] = \nu n.[\![P]\!] \to_\diamond \nu n.[\![Q'']\!] = [\![\nu n.Q'']\!]$ and $\nu n.Q'' \equiv \nu n.Q' \equiv R$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $\nu n.Q''$.

(3) $P_0 = !P$, $P \mid P \to_\diamond Q'$, and $R \equiv Q' \mid !P$ for some P and Q'. By induction hypothesis, $[\![P \mid P]\!] \to_\diamond [\![Q'']\!]$ and $Q' \equiv Q''$ for some closed process Q'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Hence $[\![P_0]\!] = ![\![P]\!] \overset{\diamond}{\equiv} [\![P]\!] \mid [\![P]\!] \mid ![\![P]\!] \to_\diamond [\![Q'']\!] \mid ![\![P]\!] = [\![Q'' \mid !P]\!]$ and $Q'' \mid !P \equiv Q' \mid !P \equiv R$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $Q'' \mid !P$.

(4) $P_0 = \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$ and either $\Sigma \vdash M = N$ and $R \equiv P$, or $\Sigma \vdash M \neq N$ and $R \equiv Q$, for some M, N, P, and Q.
   In the first case, by Lemma F.3, $\Sigma \vdash [\![M]\!] = [\![N]\!]$, so $[\![P_0]\!] = \mathit{if}\ [\![M]\!] = [\![N]\!]\ \mathit{then}\ [\![P]\!]\ \mathit{else}\ [\![Q]\!] \to_\diamond [\![P]\!]$ and we know that $P \equiv R$ and k occurs only as $\mathrm{mac}(k,\cdot)$ in P. In the second case, by Lemma F.3, $\Sigma \vdash [\![M]\!] \neq [\![N]\!]$, so $[\![P_0]\!] = \mathit{if}\ [\![M]\!] = [\![N]\!]\ \mathit{then}\ [\![P]\!]\ \mathit{else}\ [\![Q]\!] \to_\diamond [\![Q]\!]$ and we know that $Q \equiv R$ and k occurs only as $\mathrm{mac}(k,\cdot)$ in Q. □

Lemma F.6. *Suppose that P_0 is closed, $\alpha = \nu x.\overline{N'}\langle x\rangle$ or $\alpha = N'(M')$ for some ground term N', and k occurs only as $\mathrm{mac}(k,\cdot)$ in P_0 and α.*

If $[\![P_0]\!] \xrightarrow{[\![\alpha]\!]}_\diamond A$, then $P_0 \xrightarrow{\alpha}_\diamond A'$ and $A \equiv [\![A']\!]$ for some A' where k occurs only as $\mathrm{mac}(k,\cdot)$. Furthermore, when $\alpha = \nu x.\overline{N'}\langle x\rangle$, $A' = E[\{^{M}/_{x}\}]$ where E is a closed plain evaluation context and M is a ground term, and when $\alpha = N'(M')$, A' is a plain process with $fv(A') \subseteq fv(M')$.

Proof. We proceed by structural induction on P_0, with the following cases:

- $P_0 = P \mid Q$. Then $[\![P_0]\!] = [\![P]\!] \mid [\![Q]\!]$, so by Lemma B.18, either $[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond A'$ and $A \equiv A' \mid [\![Q]\!]$, or $[\![Q]\!] \xrightarrow{[\![\alpha]\!]}_\diamond A'$ and $A \equiv [\![P]\!] \mid A'$, for some A'. In the first case, by induction hypothesis, $P \xrightarrow{\alpha}_\diamond A''$ and $A' \equiv [\![A'']\!]$ for some A'' where k occurs only as $\mathrm{mac}(k,\cdot)$. By Par′, since Q is closed, we have $P_0 = P \mid Q \xrightarrow{\alpha}_\diamond A'' \mid Q$ and $[\![A'' \mid Q]\!] \equiv A' \mid [\![Q]\!] \equiv A$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $A'' \mid Q$. The second case is symmetric.
- $P_0 = \nu n.P$. We rename n so that $n \neq k$ and n does not occur in α. Then $[\![P_0]\!] = \nu n.[\![P]\!]$, so by Lemma B.18, we have $[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond A'$ and $A \equiv \nu n.A'$ for some A'. By induction hypothesis, $P \xrightarrow{\alpha}_\diamond A''$ and $A' \equiv [\![A'']\!]$ for some A'' where k occurs only as $\mathrm{mac}(k,\cdot)$. By Scope′, $P_0 = \nu n.P \xrightarrow{\alpha}_\diamond \nu n.A''$ and $[\![\nu n.A'']\!] = \nu n.[\![A'']\!] \equiv \nu n.A' \equiv A$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $\nu n.A''$.
- $P_0 = !P$. Then $[\![P_0]\!] = ![\![P]\!]$, so by Lemma B.18, we have $[\![P]\!] \xrightarrow{[\![\alpha]\!]}_\diamond A'$, and $A \equiv A' \mid ![\![P]\!]$ for some A'. By induction hypothesis, $P \xrightarrow{\alpha}_\diamond A''$ and $A' \equiv [\![A'']\!]$ for some A'' where k occurs only as $\mathrm{mac}(k,\cdot)$. We have $P_0 = !P \overset{\diamond}{\equiv} P \mid !P \xrightarrow{\alpha}_\diamond A'' \mid !P$ by Par′, since $!P$ is closed. Hence by Struct′, $P_0 \xrightarrow{\alpha}_\diamond A'' \mid !P$ and $[\![A'' \mid !P]\!] \equiv A' \mid ![\![P]\!] \equiv A$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $A'' \mid !P$.
- $P_0 = N(x).P$. Then $[\![P_0]\!] = [\![N]\!](x).[\![P]\!]$, so by Lemma B.18, we have $[\![\alpha]\!] = N'(M')$, $\Sigma \vdash [\![N]\!] = N'$, and $A \equiv [\![P]\!]\{^{M'}/_{x}\}$ for some N' and M'. Hence $\alpha = N''(M'')$, $N' = [\![N'']\!]$, and $M' = [\![M'']\!]$ for some N'' and M''. We have $\Sigma \vdash [\![N]\!] = [\![N'']\!]$, so by Lemma F.3, $\Sigma \vdash N = N''$, so we have $P_0 = N(x).P \overset{\diamond}{\equiv} N''(x).P \xrightarrow{\alpha}_\diamond P\{^{M''}/_{x}\}$ by In′. By Struct′, we obtain $P_0 \xrightarrow{\alpha}_\diamond P\{^{M''}/_{x}\}$. The name k occurs only as $\mathrm{mac}(k,\cdot)$ in M'', so the substitution $[\![P]\!]\{^{[\![M'']\!]}/_{x}\}$ does not create new occurrences of $\mathrm{mac}(k,\cdot)$, so $[\![P\{^{M''}/_{x}\}]\!] = [\![P]\!]\{^{[\![M'']\!]}/_{x}\} = [\![P]\!]\{^{M'}/_{x}\} \equiv A$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $P\{^{M''}/_{x}\}$.
- $P_0 = \overline{N}\langle M\rangle.P$. Then $[\![P_0]\!] = \overline{[\![N]\!]}\langle[\![M]\!]\rangle.[\![P]\!]$, so by Lemma B.18, $[\![\alpha]\!] = \nu x.\overline{N'}\langle x\rangle$, $\Sigma \vdash [\![N]\!] = N'$, $x \notin fv([\![P_0]\!]) = fv(P_0)$, and $A \equiv [\![P]\!] \mid \{^{[\![M]\!]}/_{x}\}$ for some x and N'. Hence $\alpha = \nu x.\overline{N''}\langle x\rangle$ and $N' = [\![N'']\!]$ for some N''. We have $\Sigma \vdash [\![N]\!] = [\![N'']\!]$, so by Lemma F.3, $\Sigma \vdash N = N''$, so we have $P_0 = \overline{N}\langle M\rangle.P \overset{\diamond}{\equiv} \overline{N''}\langle M\rangle.P \xrightarrow{\alpha}_\diamond P \mid \{^{M}/_{x}\}$ by Out-Var′. Hence by Struct′, we obtain $P_0 \xrightarrow{\alpha}_\diamond P \mid \{^{M}/_{x}\}$, and we have $[\![P \mid \{^{M}/_{x}\}]\!] = [\![P]\!] \mid \{^{[\![M]\!]}/_{x}\} \equiv A$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $P \mid \{^{M}/_{x}\}$.
- P_0 is neither $\mathbf{0}$ nor a conditional, because by Lemma B.18, $[\![P_0]\!]$ would not have a labelled transition. □

Lemma F.7. *Suppose that P_0 is a closed process where k occurs only as $\mathrm{mac}(k,\cdot)$. If $[\![P_0]\!] \xrightarrow{\alpha}_\diamond A$ with $\alpha = \nu x.\overline{N'}\langle x\rangle$ or $\alpha = N'(M')$, then $\Sigma \vdash N' = [\![N]\!]$ for some ground term N where k occurs only as $\mathrm{mac}(k,\cdot)$.*

Proof. We proceed by structural induction on P_0, with the following cases:

- $P_0 = P \mid Q$. Then $[\![P_0]\!] = [\![P]\!] \mid [\![Q]\!]$, so by Lemma B.18, either $[\![P]\!] \xrightarrow{\alpha}_\diamond A'$ and $A \equiv A' \mid [\![Q]\!]$, or $[\![Q]\!] \xrightarrow{\alpha}_\diamond A'$ and $A \equiv [\![P]\!] \mid A'$, for some A'. In both cases, the result follows immediately from the induction hypothesis.
- $P_0 = \nu n.P$. We rename n so that $n \neq k$ and n does not occur in α. Then $[\![P_0]\!] = \nu n.[\![P]\!]$, so by Lemma B.18, $[\![P]\!] \xrightarrow{\alpha}_\diamond A'$, and $A \equiv \nu n.A'$ for some A'. The result follows immediately from the induction hypothesis.
- $P_0 = !P$. Then $[\![P_0]\!] = ![\![P]\!]$, so by Lemma B.18, $[\![P]\!] \xrightarrow{\alpha}_\diamond A'$, and $A \equiv A' \mid ![\![P]\!]$ for some A'. The result follows immediately from the induction hypothesis.
- $P_0 = N(x).P$. Then $[\![P_0]\!] = [\![N]\!](x).[\![P]\!]$, so by Lemma B.18, $\alpha = N'(M')$, $\Sigma \vdash [\![N]\!] = N'$, and $A \equiv [\![P]\!]\{^{M'}/_x\}$ for some N' and M'. Moreover, since N occurs in P_0, N is ground and k occurs only as $\mathrm{mac}(k, \cdot)$ in N, so the result holds.
- $P_0 = \overline{N}\langle M\rangle.P$. Then $[\![P_0]\!] = \overline{[\![N]\!]}\langle[\![M]\!]\rangle.[\![P]\!]$, so by Lemma B.18, $\alpha = \nu x.\overline{N'}\langle x\rangle$, $\Sigma \vdash [\![N]\!] = N'$, $x \notin fv([\![P_0]\!]) = fv(P_0)$, and $A \equiv [\![P]\!] \mid \{^{[\![M]\!]}/_x\}$ for some x and N'. Moreover, since N occurs in P_0, N is ground and k occurs only as $\mathrm{mac}(k, \cdot)$ in N, so the result holds.
- P_0 is neither $\mathbf{0}$ nor a conditional, because by Lemma B.18, $[\![P_0]\!]$ would not have a labelled transition. □

Lemma F.8. *If $P \xrightarrow{\alpha}_\diamond A$ and $\Sigma \vdash \alpha = \alpha'$, then $P \xrightarrow{\alpha'}_\diamond A$.*

Proof. We proceed by induction on the derivation of $P \xrightarrow{\alpha}_\diamond A$.

- Case In′. We have $P = N(x).P'$, $\alpha = N(M)$, and $A = P'\{^M/_x\}$ for some N, M, x, and P'. Since $\Sigma \vdash \alpha = \alpha'$, we have $\alpha' = N'(M')$, $\Sigma \vdash N' = N$, and $\Sigma \vdash M' = M$ for some N' and M'. Hence $P \overset{\diamond}{\equiv} N'(x).P' \xrightarrow{\alpha'}_\diamond P'\{^{M'}/_x\} \equiv A$ by In′, so $P \xrightarrow{\alpha'}_\diamond A$ by Struct′.
- Case Out-Var′. We have $P = \overline{N}\langle M\rangle.P'$, $\alpha = \nu x.\overline{N}\langle x\rangle$, and $A = P \mid \{^M/_x\}$ for some N, M, x, and P'. Since $\Sigma \vdash \alpha = \alpha'$, we have $\alpha' = \nu x.\overline{N'}\langle x\rangle$ and $\Sigma \vdash N' = N$ for some N'. Hence $P \overset{\diamond}{\equiv} \overline{N'}\langle M\rangle.P' \xrightarrow{\alpha'}_\diamond P \mid \{^M/_x\}$ by Out-Var′, so $P \xrightarrow{\alpha'}_\diamond A$ by Struct′.
- The other cases follow easily from the induction hypothesis. In the case Scope′, we rename the bound name n so that it does not occur in α. In the case Par′, we use that $bv(\alpha') = bv(\alpha)$. □

Lemma F.9. *Suppose that P_0 is a closed process where k occurs only as $\mathrm{mac}(k, \cdot)$. If $[\![P_0]\!] \to_\diamond R$, then $P_0 \to_\diamond R'$ and $R \equiv [\![R']\!]$ for some closed process R' where k occurs only as $\mathrm{mac}(k, \cdot)$.*

Proof. We proceed by induction on the size of P_0, with the same definition of size as in the proof of Lemma F.5. The following cases may occur:

(1) $P_0 = P \mid Q$. Then $[\![P_0]\!] = [\![P]\!] \mid [\![Q]\!]$, so by Lemma B.21, one of the following cases holds:
   (a) $[\![P]\!] \to_\diamond P'$ and $R \equiv P' \mid [\![Q]\!]$ for some P',
   (b) $[\![P]\!] \xrightarrow{N(x)}_\diamond A$, $[\![Q]\!] \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B$, and $R \equiv \nu x.(A \mid B)$ for some A, B, x, and ground term N, and two symmetric cases obtained by swapping P and Q. In the first case, by induction hypothesis, $P \to_\diamond P''$ and $[\![P'']\!] \equiv P'$ for some closed process P'' where k occurs only as $\mathrm{mac}(k, \cdot)$. Hence $P_0 = P|Q \to_\diamond P''|Q$ and $[\![P''|Q]\!] = [\![P'']\!]|[\![Q]\!] \equiv P'|[\![Q]\!] \equiv R$. Furthermore, k occurs only as $\mathrm{mac}(k, \cdot)$ in $P'' \mid Q$. In the second case, by Lemma F.7, $\Sigma \vdash N = [\![N']\!]$ for some ground term N' where k occurs only as $\mathrm{mac}(k, \cdot)$. By Lemma F.8, $[\![P]\!] \xrightarrow{[\![N']\!](x)}_\diamond A$ and $[\![Q]\!] \xrightarrow{\nu x.\overline{[\![N']\!]}\langle x\rangle}_\diamond B$. By Lemma F.6, $P \xrightarrow{N(x)}_\diamond P_1$ and $A \equiv [\![P_1]\!]$ for some P_1 where k occurs only as $\mathrm{mac}(k, \cdot)$ and $fv(P_1) \subseteq \{x\}$; and $Q \xrightarrow{\nu x.\overline{N}\langle x\rangle}_\diamond B'$ and $B \equiv [\![B']\!]$ for some $B' = E_2[\{^{M_2}/_x\}]$ where k occurs only as $\mathrm{mac}(k, \cdot)$ in B', E_2 is a closed plain evaluation context, and M_2 is a ground term. By Lemma B.20, $P_0 = P \mid Q \to_\diamond R'$ and $R' \equiv \nu x.(P_1 \mid B') = \nu x.(P_1 \mid E_2[\{^{M_2}/_x\}])$

for some R'. We rename the bound names of E_2 so that they do not occur in P_1. Let $R'' = E_2[P_1\{^{M_2}/_x\}]$. The process R'' is closed and k occurs only as $\mathrm{mac}(k,\cdot)$ in R''. We have $R' \equiv R''$, so $P_0 \to_\diamond R''$ and $[\![R'']\!] \equiv [\![\nu x.(P_1 \mid B')]\!] \equiv \nu x.(A \mid B) \equiv R$. The last two cases are symmetric.

(2) $P_0 = \nu n.P$. We rename n so that $n \neq k$. Then $[\![P_0]\!] = \nu n.[\![P]\!]$, so by Lemma B.21, $[\![P]\!] \to_\diamond Q'$, and $R \equiv \nu n.Q'$ for some Q'. By induction hypothesis, $P \to_\diamond Q''$ and $Q' \equiv [\![Q'']\!]$ for some closed process Q'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Hence $P_0 = \nu n.P \to_\diamond \nu n.Q''$ and $[\![\nu n.Q'']\!] = \nu n.[\![Q'']\!] \equiv \nu n.Q' \equiv R$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $\nu n.Q''$.

(3) $P_0 = !P$. Then $[\![P_0]\!] = ![\![P]\!]$, so by Lemma B.21, $[\![P \mid P]\!] = [\![P]\!] \mid [\![P]\!] \to_\diamond Q'$, and $R \equiv Q' \mid ![\![P]\!]$ for some Q'. By induction hypothesis, $P \mid P \to_\diamond Q''$ and $Q' \underset{\diamond}{\equiv} [\![Q'']\!]$ for some closed process Q'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Hence $P_0 = !P \underset{\diamond}{\equiv} P \mid P \mid !P \to_\diamond Q'' \mid !P$ and $[\![Q'' \mid !P]\!] = [\![Q'']\!] \mid ![\![P]\!] \equiv Q' \mid ![\![P]\!] \equiv R$. Furthermore, k occurs only as $\mathrm{mac}(k,\cdot)$ in $Q'' \mid !P$.

(4) $P_0 = \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q$. Then $[\![P_0]\!] = \mathit{if}\ [\![M]\!] = [\![N]\!]\ \mathit{then}\ [\![P]\!]\ \mathit{else}\ [\![Q]\!]$, so by Lemma B.21, either $\Sigma \vdash [\![M]\!] = [\![N]\!]$ and $R \equiv [\![P]\!]$, or $\Sigma \vdash [\![M]\!] \neq [\![N]\!]$ and $R \equiv [\![Q]\!]$. In the first case, by Lemma F.3, $\Sigma \vdash M = N$, so $P_0 = \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q \to_\diamond P$ and we know that $[\![P]\!] \equiv R$ and k occurs only as $\mathrm{mac}(k,\cdot)$ in P. In the second case, by Lemma F.3, $\Sigma \vdash M \neq N$, so $P_0 = \mathit{if}\ M = N\ \mathit{then}\ P\ \mathit{else}\ Q \to_\diamond Q$ and we know that $[\![Q]\!] \equiv R$ and k occurs only as $\mathrm{mac}(k,\cdot)$ in Q.

(5) P_0 is not $\mathbf{0}$, an input, or an output, because by Lemma B.21, $[\![P_0]\!]$ would not reduce. □

Theorem 6.1. *Suppose that the signature Σ is equipped with an equational theory generated by a convergent rewrite system such that* mac *and* f *do not occur in the left-hand sides of rewrite rules; the only rewrite rules with* h *at the root of the left-hand side are those of* (10) *and* (11) *oriented from left to right; there are no rewrite rules with* :: *nor* nil *at the root of the left-hand side; and names do not occur in rewrite rules. Suppose that C is closed and the name k appears only as first argument of* mac *in C. Then $\nu k.C \approx \nu k.[\![C]\!]$.*

Proof. Let $\mathcal{R}$ relate all closed extended processes A and B such that $A \equiv \nu k.C$ and $B \equiv \nu k.[\![C]\!]$ for some closed normal process C where k occurs only as $\mathrm{mac}(k,\cdot)$.

We show that $\mathcal{R} \cup \mathcal{R}^{-1}$ is a labelled bisimulation. It is symmetric by construction. Assume that $A\ \mathcal{R}\ B$ for some $C = \nu\tilde{n}.(\sigma \mid P)$ where k occurs only as $\mathrm{mac}(k,\cdot)$. In particular, k does not occur in $\tilde{n}$, A and B are closed, $A \equiv \nu k.C$, and $B \equiv \nu k.[\![C]\!]$.

(1) We show that $A \approx_s B$.
Let M and N be two terms such that $fv(M) \cup fv(N) \subseteq dom(A) = dom(B) = dom(\sigma)$. We have $\varphi(A) \equiv \nu k, \tilde{n}.\sigma$ and $\varphi(B) \equiv \nu k, \tilde{n}.[\![\sigma]\!]$. We rename k and $\tilde{n}$ so that these names do not occur in M and N. Then

$$\begin{aligned}(M = N)\varphi(A) &\Leftrightarrow \Sigma \vdash M\sigma = N\sigma \\ &\Leftrightarrow \Sigma \vdash [\![M\sigma]\!] = [\![N\sigma]\!] && \text{by Lemma F.3} \\ &\Leftrightarrow \Sigma \vdash M[\![\sigma]\!] = N[\![\sigma]\!]\end{aligned}$$

since k does not occur in M and N and k occurs only as $\mathrm{mac}(k,\cdot)$ in σ, so

$$(M = N)\varphi(A) \Leftrightarrow (M = N)\varphi(B)$$

Therefore, $A \approx_s B$.

(2) We first show that, if $A \xrightarrow{\alpha} A'$, A' is closed, and $fv(\alpha) \subseteq dom(A)$, then $B \to^* \xrightarrow{\alpha} \to^* B'$ and $A'\ \mathcal{R}\ B'$ for some B'.
We have $\nu k, \tilde{n}.(\sigma \mid P) \xrightarrow{\alpha} A'$, so by Lemma B.12, $\nu k, \tilde{n}.(\sigma \mid P) \xrightarrow{\alpha}_\circ A'$. We rename k and $\tilde{n}$ so that these names do not occur in α. By Lemma B.19, $P \xrightarrow{\alpha\sigma}_\diamond A'_1$, $A' \equiv \nu k, \tilde{n}.(\sigma \mid A'_1)$, and $bv(\alpha) \cap dom(\sigma) = \emptyset$ for some A'_1. By Lemma F.4, $[\![P]\!] \xrightarrow{[\![\alpha\sigma]\!]}_\diamond [\![A''_1]\!]$ and $A'_1 \equiv A''_1$ for some

A_1'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Furthermore, when $\alpha\sigma = \nu x.\overline{N'}\langle x\rangle$, $A_1'' = E[\{^M/_x\}]$ where E is a closed plain evaluation context and M is a ground term, and when $\alpha\sigma = N'(M')$, A_1'' is a plain process with $fv(A_1'') \subseteq fv(M') = \emptyset$, so A_1'' is a closed plain process. Since k does not occur in α and k occurs only as $\mathrm{mac}(k,\cdot)$ in σ, we have $[\![\alpha\sigma]\!] = \alpha[\![\sigma]\!]$. Therefore, $B \equiv \nu k.[\![C]\!] \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![P]\!]) \xrightarrow{\alpha}_\circ \nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![A_1'']\!])$. Let $C' = \mathrm{pnf}(\nu\widetilde{n}.(\sigma \mid A_1''))$. The process C' is a closed normal process where k occurs only as $\mathrm{mac}(k,\cdot)$ We have $\nu k.C' \equiv \nu k, \widetilde{n}.(\sigma \mid A_1'') \equiv \nu k, \widetilde{n}.(\sigma \mid A_1') \equiv A'$. Let $B' = \nu k.[\![C']\!]$. We have $A' \mathcal{R} B'$. Given the form of A_1'', we can show that $[\![C']\!] \equiv \nu\widetilde{n}.([\![\sigma]\!] \mid [\![A_1'']\!])$, so $B \xrightarrow{\alpha} B'$.

Next, we show that, if $B \xrightarrow{\alpha} B'$, B' is closed, and $fv(\alpha) \subseteq dom(B)$, then $A \to^* \xrightarrow{\alpha} \to^* A'$ and $A' \mathcal{R} B'$ for some A'.

We have $\nu k, \widetilde{n}.([\![\sigma]\!] | [\![P]\!]) \xrightarrow{\alpha} B'$, so by Lemma B.12, $\nu k, \widetilde{n}.([\![\sigma]\!] | [\![P]\!]) \xrightarrow{\alpha}_\circ B'$. We rename k and $\widetilde{n}$ so that these names do not occur in α. By Lemma B.19, $[\![P]\!] \xrightarrow{\alpha[\![\sigma]\!]}_\diamond B_1'$, $B' \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid B_1')$, and $bv(\alpha) \cap dom([\![\sigma]\!]) = bv(\alpha) \cap dom(\sigma) = \emptyset$ for some B_1'. Since k does not occur in α and k occurs only as $\mathrm{mac}(k,\cdot)$ in σ, we have $[\![\alpha\sigma]\!] = \alpha[\![\sigma]\!]$. By Lemma F.6, $P \xrightarrow{\alpha\sigma}_\diamond A_1'$ and $B_1' \equiv [\![A_1']\!]$ for some A_1' where k occurs only as $\mathrm{mac}(k,\cdot)$. Furthermore, when $\alpha\sigma = \nu x.\overline{N'}\langle x\rangle$, $A_1' = E[\{^M/_x\}]$ where E is a closed plain evaluation context and M is a ground term, and when $\alpha\sigma = N'(M')$, A_1' is a plain process with $fv(A_1') \subseteq fv(M') = \emptyset$, so A_1' is a closed plain process. Therefore, $A \equiv \nu k.C \equiv \nu k, \widetilde{n}.(\sigma \mid P) \xrightarrow{\alpha}_\circ \nu k, \widetilde{n}.(\sigma \mid A_1')$. Let $C' = \mathrm{pnf}(\nu\widetilde{n}.(\sigma \mid A_1'))$. The process C' is a closed normal process where k occurs only as $\mathrm{mac}(k,\cdot)$. Given the form of A_1', we can show that $\nu k.[\![C']\!] \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![A_1']\!]) \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid B_1') \equiv B'$. Let $A' = \nu k.C'$. We have $A' \mathcal{R} B'$ and $\nu k, \widetilde{n}.(\sigma \mid A_1') \equiv \nu k.C' = A'$ so $A \xrightarrow{\alpha} A'$.

(3) We first show that, if $A \to A'$ for some closed A', then $B \to^* B'$ and $A' \mathcal{R} B'$ for some B'.

We have $\nu k, \widetilde{n}.(\sigma \mid P) \to A'$, so by Lemma B.8, $\nu k, \widetilde{n}.(\sigma \mid P) \to_\circ \mathrm{pnf}(A')$. By Lemma B.22, $P \to_\diamond P'$ and $\mathrm{pnf}(A') \equiv \nu k, \widetilde{n}.(\sigma \mid P')$ for some P'. By Lemma F.5, $[\![P]\!] \to_\diamond [\![P'']\!]$ and $P' \equiv P''$ for some closed process P'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Let $C' = \nu\widetilde{n}.(\sigma \mid P'')$. The process C' is a closed normal process where k occurs only as $\mathrm{mac}(k,\cdot)$. We have $\nu k.C' \equiv \nu k, \widetilde{n}.(\sigma \mid P'') \equiv \mathrm{pnf}(A') \equiv A'$. Let $B' = \nu k.[\![C']\!]$. We have $A' \mathcal{R} B'$ and $B \equiv \nu k.[\![C]\!] \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![P]\!]) \to_\circ \nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![P'']\!]) = \nu k.[\![C']\!] = B'$, so $B \to B'$.

Next, we show that, if $B \to B'$ for some closed B', then $A \to^* A'$ and $A' \mathcal{R} B'$ for some A'.

We have $\nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![P]\!]) \to B'$, so by Lemma B.8, $\nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![P]\!]) \to_\circ \mathrm{pnf}(B')$. By Lemma B.22, $[\![P]\!] \to_\diamond P'$ and $\mathrm{pnf}(B') \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid P')$ for some P'. By Lemma F.9, $P \to_\diamond P''$ and $P' \equiv [\![P'']\!]$ for some closed process P'' where k occurs only as $\mathrm{mac}(k,\cdot)$. Let $C' = \nu\widetilde{n}.(\sigma \mid P'')$. The process C' is a closed normal process where k occurs only as $\mathrm{mac}(k,\cdot)$. We have $\nu k.[\![C']\!] \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid [\![P'']\!]) \equiv \nu k, \widetilde{n}.([\![\sigma]\!] \mid P') \equiv \mathrm{pnf}(B') \equiv B'$. Let $A' = \nu k.C'$. We have $A' \mathcal{R} B'$ and $A \equiv \nu k.C \equiv \nu k, \widetilde{n}.(\sigma \mid P) \to_\circ \nu k, \widetilde{n}.(\sigma \mid P'') = \nu k.C' = A'$, so $A \to A'$.

Therefore, $\mathcal{R} \subseteq \approx_l$ and, by Theorem 4.8, $\mathcal{R} \subseteq \approx$.

Finally, when C is a closed extended process where k occurs only as $\mathrm{mac}(k,\cdot)$, we have $\nu k.C \mathcal{R} \nu k.[\![C]\!]$ because $\mathrm{pnf}(C)$ is a closed normal process where k occurs only as $\mathrm{mac}(k,\cdot)$ such that $\mathrm{pnf}(C) \equiv C$. We thus obtain $\nu k.C \approx \nu k.[\![C]\!]$. □

Corollary 6.2. *Suppose that the signature Σ is equipped with the equational theory defined by the equations (1), (2), (3), (4), (10), and (11). Suppose that C is closed and the name k appears only as first argument of* mac *in C. Then $\nu k.C \approx \nu k.[\![C]\!]$.*

Proof. We define the rewrite system R by orienting the equations (1), (2), (3), (4), (10), and (11) from left to right:

$$\begin{aligned}
\mathsf{fst}((x,y)) &\to x && (20)\\
\mathsf{snd}((x,y)) &\to y && (21)\\
\mathsf{hd}(x :: y) &\to x && (22)\\
\mathsf{tl}(x :: y) &\to y && (23)\\
\mathsf{nil} +\!\!+ x &\to x :: \mathsf{nil} && (24)\\
(x :: y) +\!\!+ z &\to x :: (y +\!\!+ z) && (25)\\
\mathsf{h}(x, y_0 :: y_1 :: z) &\to \mathsf{h}(\mathsf{f}(x, y_0), y_1 :: z) && (26)\\
\mathsf{h}(x, y :: \mathsf{nil}) &\to \mathsf{f}(x, y) && (27)
\end{aligned}$$

In order to prove that R terminates, we order terms M lexicographically, using:

(1) the size of M; then
(2) the number of occurrences of the $+\!\!+$ symbol in M; then
(3) the number of occurrences of the $::$ symbol in M; then
(4) the sum, over all occurrences of $+\!\!+$ in M, of the lengths of the first arguments of $+\!\!+$, computed as follows: $\mathit{length}(N_1 :: N_2) = 1 + \mathit{length}(N_2)$, $\mathit{length}(N_1 +\!\!+ N_2) = 1 + \mathit{length}(N_1)$, and $\mathit{length}(N) = 0$ for all other terms.

This ordering is well-founded. Rules (20), (21), (22), (23), and (27) decrease the size. Rule (24) preserves the size and decreases the number of occurrences of $+\!\!+$. Rule (25) preserves the size and the numbers of occurrences of $+\!\!+$ and $::$ but it decreases the sum above, because the length of the first argument decreases for the occurrence of $+\!\!+$ modified by rule (25) ($\mathit{length}(N) < \mathit{length}(M :: N)$) and is unchanged for all other occurrences of $+\!\!+$ in the term. Rule (26) preserves the size and the number of occurrences of $+\!\!+$; it decreases the number of occurrences of $::$. Therefore, if M reduces to M' by any of these rules, we have $M' < M$. This property shows that R terminates. (The termination of R can also be proved using well-known techniques. For instance, it can be proved using a lexicographic path ordering, provided the second argument of h, which decreases by (26), is considered before the first one, which increases, in the lexicographic ordering.)

The rewrite system R is confluent because there are no critical pairs between the rules. Hence R is convergent. Since R generates the equational theory under consideration, we conclude by Theorem 6.1. □

G PROOFS FOR SECTION 6.3

In Lemma G.1 and Corollary G.2, we suppose that the signature Σ is equipped with an equational theory generated by a convergent rewrite system R. Since R terminates, the left-hand side of its rewrite rules cannot be variables. We suppose that the rewrite rules of R do not contain names. We denote by θ a substitution and by ρ a variable renaming. We first study active substitutions from variables to hash computations, that is, terms whose root symbols range over functions that do not occur on the left-hand side of R.

Lemma G.1. *Suppose Σ is equipped with an equational theory generated by a convergent rewrite system R. Let θ be a closed substitution that ranges over pairwise distinct terms modulo Σ, each of the form $f(k, M)$ where f does not occur on the left-hand side of the rules of R. Let σ map the same variables to pairwise distinct names $\widetilde{a}$. We have $\nu k.\theta \approx_s \nu\widetilde{a}.\sigma$.*

Proof. More explicitly, let $\theta = \{({}^{M_i}/_{x_i})_{i=1..n}\}$, $\sigma = \{({}^{a_i}/_{x_i})_{i=1..n}\}$, and $\widetilde{a} = a_1, \ldots, a_n$.
We first prove the property

SubstInj: if, moreover, θ ranges over syntactically pairwise distinct terms, then $N_1\theta = N_2\theta$ and $k \notin fn(N_1) \cup fn(N_2)$ implies $N_1 = N_2$.

Let N_1' be obtained from N_1 by replacing the occurrences of $x_1, \ldots, x_n$ with pairwise distinct variables $y_1, \ldots, y_{n'}$, and let $(i_j)_{j=1..n'}$ and $\rho = \{({}^{x_{i_j}}/_{y_j})_{j=1..n'}\}$ be such that $N_1 = N_1'\rho$. We have $N_1'\rho\theta = N_2\theta$. Since k does not occur in N_2 or N_1', and k occurs as first argument of the root function symbol of M_i for $i = 1..n$ and M_{i_j} for $j = 1..n'$, the terms N_2 and N_1' are equal up to some variable renaming. Since each variable y_j occurs once in N_1', we have $N_2 = N_1'\rho'$ for some $(i_j')_{j=1..n'}$ and $\rho' = \{({}^{x_{i_j'}}/_{y_j})_{j=1..n'}\}$. We have $N_1'\rho'\theta = N_1'\rho\theta$, so for all $j = 1..n'$ we have $y_j\rho'\theta = y_j\rho\theta$, so $M_{i_j'} = M_{i_j}$. Since $M_1, \ldots, M_n$ are pairwise distinct, we have $i_j' = i_j$, so $\rho' = \rho$. Hence $N_1 = N_1'\rho = N_1'\rho' = N_2$.

Let us now prove the lemma itself. We first reduce $M_1, \ldots, M_n$ into irreducible form under R. By Lemma 4.4, it is enough to prove static equivalence on these reduced terms. Moreover, they are still of the form $f(k, M)$ with the same condition on f. (Indeed, the left-hand sides of rewrite rules do not contain f, so the rewrite rules apply to strict subterms of $f(k, M)$; and k is irreducible, so the rewrite rules apply only to the terms M within $f(k, M)$.)

Let N_1, N_2 be two terms with $fv(N_1) \cup fv(N_2) \subseteq \{x_1, \ldots, x_n\}$. We need to show that $(N_1 = N_2)\nu k.\theta$ if and only if $(N_1 = N_2)\nu\widetilde{a}.\sigma$. We rename $k, \widetilde{a}$ so that $(fn(N_1) \cup fn(N_2)) \cap \{k, \widetilde{a}\} = \emptyset$. We have $(N_1 = N_2)\nu k.\theta$ if and only if $\Sigma \vdash N_1\theta = N_2\theta$, and $(N_1 = N_2)\sigma$ if and only if $\Sigma \vdash N_1\sigma = N_2\sigma$. We show that $\Sigma \vdash N_1\theta = N_2\theta$ if and only if $\Sigma \vdash N_1 = N_2$ if and only if $\Sigma \vdash N_1\sigma = N_2\sigma$.

Since the equational theory is closed under substitution of terms for variables and names, we have that $\Sigma \vdash N_1 = N_2$ implies $\Sigma \vdash N_1\theta = N_2\theta$, $\Sigma \vdash N_1 = N_2$ implies $\Sigma \vdash N_1\sigma = N_2\sigma$, and $\Sigma \vdash N_1\sigma = N_2\sigma$ implies $\Sigma \vdash N_1 = N_2$ (by substituting x_i for a_i for $i = 1..n$). Hence, we just have to show that $\Sigma \vdash N_1\theta = N_2\theta$ implies $\Sigma \vdash N_1 = N_2$. We can restrict our attention to the case in which N_1 and N_2 are irreducible under R, since the equality of the initial terms is equivalent to the equality of their reduced forms.

Suppose that $\Sigma \vdash N_1\theta = N_2\theta$, with $N_1, N_2, M_1, \ldots, M_n$ irreducible under R. We first show that $N_1\theta$ is irreducible under R. In order to derive a contradiction, suppose that $N_1\theta$ is reducible by a rewrite rule $N_3 \to N_4$ of R. Then there exists a term context C and a substitution σ such that $C[N_3\sigma] = N_1\theta$. Let N_1' be obtained from N_1 by renaming the occurrences of $x_1, \ldots, x_n$ into pairwise distinct variables $y_1, \ldots, y_{n'}$, and let $(i_j)_{j=1..n'}$ and $\rho = \{({}^{x_{i_j}}/_{y_j})_{j=1..n'}\}$ such that $N_1 = N_1'\rho$. We have $C[N_3\sigma] = N_1'\rho\theta$. The position of the hole of C cannot be inside M_{i_j}, since otherwise M_{i_j} would be reducible by $N_3 \to N_4$. Hence, the position of the hole of C is inside N_1', so $N_3\sigma = N_1''\rho\theta$, $C = C'\rho\theta$, and $N_1' = C'[N_1'']$ for some subterm N_1'' of N_1' and term context C'.

Let ρ' be a variable renaming such that $N_3'\rho' = N_3$ and all variable occurrences in N_3' are fresh and pairwise distinct. We have $N_3'\rho'\sigma = N_1''\rho\theta$. Since the function symbols f at the root of M_{i_j} do not occur in N_3, all occurrences of M_{i_j} are in $z\rho'\sigma$ for some $z \in fv(N_3')$. Hence, for all $z \in fv(N_3')$, there exists a subterm N_z of N_1'' such that $z\rho'\sigma = N_z\rho\theta$ and $N_1'' = N_3'\{({}^{N_z}/_{z})_{z \in fv(N_3')}\}$. Furthermore, when z and z' are distinct variables of N_3' such that $z\rho' = z'\rho'$, we have $z\rho'\sigma = z'\rho'\sigma$, so $N_z\rho\theta = N_{z'}\rho\theta$ and, by SubstInj, $N_z\rho = N_{z'}\rho$.

For each variable y of N_3, let us choose one variable z_y of N_3' such that $z_y\rho' = y$. Let us define σ' by $y\sigma' = N_{z_y}\rho$. Since for all $z, z' \in fv(N_3')$, we have $z\rho' = z'\rho'$ implies $N_z\rho = N_{z'}\rho$, we have for all $z \in fv(N_3')$, $z\rho'\sigma' = N_z\rho$. Let $C'' = C'\rho$. We have

$$C''[N_3\sigma'] = C''[N_3'\rho'\sigma'] = C''[N_3'\{({}^{N_z}/_{z})_{z \in fv(N_3')}\}\rho] = C''[N_1''\rho] = C'[N_1'']\rho = N_1'\rho = N_1$$

Hence N_1 would be reducible by $N_3 \to N_4$, which is a contradiction. Therefore, $N_1\theta$ is irreducible. Similarly, $N_2\theta$ is irreducible. Hence $\Sigma \vdash N_1\theta = N_2\theta$ implies $N_1\theta = N_2\theta$. By SubstInj, $N_1 = N_2$, so a fortiori $\Sigma \vdash N_1 = N_2$. □

Corollary G.2. *Suppose Σ is equipped with an equational theory generated by a convergent rewrite system R. Let θ be a closed substitution that ranges over terms of the form $f(k, M)$ where each f does not occur on the left-hand side of the rules of R. Let σ map the same variables to names $\widetilde{a}$ such that, for all $x, y \in dom(\theta)$, we have $x\sigma = y\sigma$ if and only if $\Sigma \vdash x\theta = y\theta$. We have $\nu k.\theta \approx_s \nu\widetilde{a}.\sigma$.*

Proof. We factor θ and σ into $\rho\theta'$ and $\rho\sigma'$ where θ' and σ' range over pairwise distinct terms modulo Σ and ρ is a variable renaming. We apply Lemma G.1 and conclude by Lemma 4.4. □

Our next lemma confirms that, with the equations (12), all terms are pairs.

Lemma G.3. *Suppose Σ is equipped with an equational theory that contains the equations (12). We have $\nu a_1, a_2.\{^{(a_1,a_2)}/_x\} \approx_s \nu a.\{^{a}/_x\}$.*

Proof. Let M and N be two terms such that $fv(M) \cup fv(N) \subseteq \{x\}$. We rename a, a_1, a_2 so that $\{a, a_1, a_2\} \cap (fn(M) \cup fn(N)) = \emptyset$.

If $(M = N)\nu a_1, a_2.\{^{(a_1,a_2)}/_x\}$, then $\Sigma \vdash M\{^{(a_1,a_2)}/_x\} = N\{^{(a_1,a_2)}/_x\}$. Since the equational theory is closed under substitution of any term for names, we have $\Sigma \vdash M\{^{(a_1,a_2)}/_x\}\{^{\mathsf{fst}(a)}/_{a_1}, ^{\mathsf{snd}(a)}/_{a_2}\} = N\{^{(a_1,a_2)}/_x\}\{^{\mathsf{fst}(a)}/_{a_1}, ^{\mathsf{snd}(a)}/_{a_2}\}$, that is, $\Sigma \vdash M\{^{(\mathsf{fst}(a),\mathsf{snd}(a))}/_x\} = N\{^{(\mathsf{fst}(a),\mathsf{snd}(a))}/_x\}$, so $\Sigma \vdash M\{^{a}/_x\} = N\{^{a}/_x\}$ by the equation $(\mathsf{fst}(x), \mathsf{snd}(x)) = x$. Hence $(M = N)\nu a.\{^{a}/_x\}$.

Conversely, suppose that $(M = N)\nu a.\{^{a}/_x\}$. Hence $\Sigma \vdash M\{^{a}/_x\} = N\{^{a}/_x\}$. Since the equational theory is closed under substitution of any term for names, we have $\Sigma \vdash M\{^{a}/_x\}\{^{(a_1,a_2)}/_a\} = N\{^{a}/_x\}\{^{(a_1,a_2)}/_a\}$, that is, $\Sigma \vdash M\{^{(a_1,a_2)}/_x\} = N\{^{(a_1,a_2)}/_x\}$, so $(M = N)\nu a_1, a_2.\{^{(a_1,a_2)}/_x\}$.

Therefore, $(M = N)\nu a_1, a_2.\{^{(a_1,a_2)}/_x\}$ if and only if $(M = N)\nu a.\{^{a}/_x\}$, so $\nu a_1, a_2.\{^{(a_1,a_2)}/_x\} \approx_s \nu a.\{^{a}/_x\}$. □

Lemma G.4. *The equational theory defined by equations (3), (4), (12), (13), (14), (15), and (16) is generated by a convergent rewrite system R.*

Proof. We define R by orienting all equations from left to right, as follows:

$$\begin{aligned}
\mathsf{hd}(x :: y) &\to x && (28)\\
\mathsf{tl}(x :: y) &\to y && (29)\\
\mathsf{nil} +\!\!+ x &\to x :: \mathsf{nil} && (30)\\
(x :: y) +\!\!+ z &\to x :: (y +\!\!+ z) && (31)\\
\mathsf{ne_list}(x :: y :: z) &\to \mathsf{ne_list}(y :: z) && (32)\\
\mathsf{ne_list}(x :: \mathsf{nil}) &\to \mathsf{true} && (33)\\
\mathsf{fst}((x, y)) &\to x && (34)\\
\mathsf{snd}((x, y)) &\to y && (35)\\
(\mathsf{fst}(x), \mathsf{snd}(x)) &\to x && (36)\\
\mathsf{h}(k, z) &\to \mathsf{h}_2(k, (0, 0), z) && (37)\\
\mathsf{h}_2(k, x, \mathsf{nil}) &\to \mathsf{fst}(x) && (38)\\
\mathsf{h}_2(k, x, y :: z) &\to \mathsf{h}_2(k, \mathsf{f}(k, (x, y)), z) && (39)
\end{aligned}$$

To prove that R terminates, we order terms M lexicographically, as follows:

(1) by $hval(M)$, where $hval$ is defined by

$$\begin{aligned} hval(\mathsf{h}_2(M_1, M_2, M_3)) &= hval(M_2) + hval(M_3) + length(M_3) \\ hval(\mathsf{h}(M_1, M_2)) &= hval(M_2) + length(M_2) + 1 \\ hval(f(M_1, \ldots, M_n)) &= hval(M_1) + \cdots + hval(M_n) \\ &\qquad \text{for all other functions} \\ hval(M) &= 0 \text{ when } M \text{ is a variable or a name} \end{aligned}$$

and the $length$ of a term is defined by

$length(M :: N) = 1 + length(N)$
when the symbol :: has sort $\mathsf{Block} \times \mathsf{BlockList} \to \mathsf{BlockList}$
$length(M +\!\!+ N) = 1 + length(M)$
$length(f(M_1, \ldots, M_n)) = \max(length(M_1), \ldots, length(M_n))$
where f is a function symbol other than
$::$: $\mathsf{Block} \times \mathsf{BlockList} \to \mathsf{BlockList}$
$+\!\!+$: $\mathsf{BlockList} \times \mathsf{Block} \to \mathsf{BlockList}$
such that the sort of the result of f may contain BlockList, that is, this sort is BlockList, Block2Blocks, Block2Blocks_List, or one of the sorts of pairs used in the syntactic sugar for $\ell(x, t, s)$ and $\overline{\ell}\langle x, t, s\rangle$.
$length(M) = 0$ for all other terms M;

(2) then by the size of M;
(3) then by the number of occurrences of the $+\!\!+$ symbol in M;
(4) then by the sum of the lengths of the first arguments of $+\!\!+$ in M.

This ordering is well-founded. By induction on C, we show that, for all term contexts C,

- if $length(M') \leq length(M)$, then $length(C[M']) \leq length(C[M])$;
- if $hval(M') \leq hval(M)$ and $length(M') \leq length(M)$, then $hval(C[M']) \leq hval(C[M])$;
- if $hval(M') < hval(M)$ and $length(M') \leq length(M)$, then $hval(C[M']) < hval(C[M])$.

We notice that terms of sorts Bool, Block, Block2, and Block3 have length 0. For all rewrite rules $M \to M'$ above and all substitutions σ, we show that $length(M'\sigma) \leq length(M\sigma)$ by inspecting each rule. For all rules except (37), (38), and (39) and all substitutions σ, we have $hval(M'\sigma) \leq hval(M\sigma)$ because

$$hval(M\sigma) = \sum_{x \in fv(M)} hval(x\sigma) \times (\text{number of occurrences of } x \text{ in } M)$$

and similarly for M', and all variables x occur at least as many times in M as in M'. We have $hval(\mathsf{h}(M_1, M_2)) = hval(M_2) + length(M_2) + 1$ and $hval(\mathsf{h}_2(M_1, (0, 0), M_2)) = hval(M_2) + length(M_2)$, so rule (37) decreases $hval$. We have $hval(\mathsf{h}_2(M_1, M_2, \mathsf{nil})) = hval(M_2)$, so rule (38) preserves $hval$. We have $hval(\mathsf{h}_2(M_1, M_2, M_3 :: M_4)) = hval(M_2) + hval(M_3) + hval(M_4) + length(M_4) + 1$ and

$$\begin{aligned} &hval(\mathsf{h}_2(M_1, \mathsf{f}(M_1, (M_2, M_3)), M_4)) \\ &\quad = hval(\mathsf{f}(M_1, (M_2, M_3))) + hval(M_4) + length(M_4) \\ &\quad = hval(M_1) + hval(M_2) + hval(M_3) + hval(M_4) + length(M_4) \\ &\quad = hval(M_2) + hval(M_3) + hval(M_4) + length(M_4) \end{aligned}$$

since $hval(M_1) = 0$ because M_1 is a variable or a name since no function returns sort Key. Hence rule (39) decreases $hval$. Therefore, we have:

- Rules (28), (29), (32), (33), (34), (35), (36), (38) do not increase *hval* and decrease the size.
- Rule (30) does not increase *hval*, preserves the size and decreases the number of occurrences of $+\!\!+$.
- Rule (31) does not increase *hval*, preserves the size and the number of occurrences of $+\!\!+$, and decreases the sum because the length of the first argument decreases for the occurrence of $+\!\!+$ modified by rule (31) ($length(N) < length(M :: N)$) and is unchanged for all other occurrences of $+\!\!+$ in the term.
- Rules (37) and (39) decrease *hval*.

Therefore, if M reduces to M' by any of these rules, then M' is smaller than M in a well-founded lexicographic ordering, and thus R terminates. (The termination of R can also be proved using well-known techniques. For instance, it can be proved using a lexicographic path ordering, provided the third argument of h_2, which decreases by (39), is considered before the second one, which increases, in the lexicographic ordering.)

The only critical pairs between these rules are:

- between rules (34) and (36): $\mathsf{fst}((\mathsf{fst}(x), \mathsf{snd}(x)))$ reduces to $\mathsf{fst}(x)$ by both rules, and $(\mathsf{fst}((x, y)), \mathsf{snd}((x, y)))$ reduces to (x, y) by (36) or by (34) and (35), so these two critical pairs are joinable.
- between rules (35) and (36), symmetrically.

Since all critical pairs are joinable, R is confluent, so it is convergent. □

Lemma G.5. *Suppose that Σ is equipped with the equational theory of Lemma G.4. If $\Sigma \vdash \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n)) = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M'_1)) \ldots, M'_{n'}))$, then $n = n'$ and $\Sigma \vdash M_i = M'_i$ for all $i = 1..n$.*

Proof. We proceed by induction on n.

- If $n = n' = 0$, the result holds trivially.
- If $n = 0$ and $n' > 0$, then $\Sigma \vdash (0,0) = \mathsf{f}(k, M)$ for some term M and, after reducing under R of Lemma G.4, $(0,0) = \mathsf{f}(k, M')$ for some term M'. This equality does not hold, so this case is excluded. By symmetry, the case $n > 0$ and $n' = 0$ is also excluded.
- If $n > 0$ and $n' > 0$, then $\Sigma \vdash \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n)) = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M'_1)) \ldots, M'_{n'}))$ implies both $\Sigma \vdash \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_{n-1})) = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M'_1)) \ldots, M'_{n'-1}))$ and $\Sigma \vdash M_n = M'_{n'}$. By induction hypothesis, $n = n'$ and $\Sigma \vdash M_i = M'_i$ for all $i \leq n - 1$. □

Theorem 6.3. $\nu k.(A^0_h \mid A^0_f) \approx \nu k.(A^1_h \mid A^1_f)$.

Proof. In this proof, we use uppercase letters $X, Y, Z, S, \ldots$ for terms substituted for variables named with the corresponding lowercase letters $x, y, z, s, \ldots$ during execution. We first extend the notations of Section 6.3 with intermediate processes parametrized by terms, which we will use to define our candidate bisimulation.

$$A^0_{hi}(Y) = \mathit{if}\ \mathsf{ne\_list}(Y) = \mathsf{true}\ \mathit{then}\ \overline{c'_h}\langle \mathsf{h}(k, Y)\rangle$$
$$A^1_{hi}(Y) = \mathit{if}\ \mathsf{ne\_list}(Y) = \mathsf{true}\ \mathit{then}\ \overline{c'_h}\langle \mathsf{h}'(k, Y)\rangle$$
$$A^1_f(S) = \nu \ell, c_s.(!c_s(s).c_f(x).\overline{\ell}\langle x, s, s\rangle \mid !Q \mid \overline{c_s}\langle S\rangle)$$
$$A^1_{fi}(X, S) = \nu \ell, c_s.(!c_s(s).c_f(x).\overline{\ell}\langle x, s, s\rangle \mid !Q \mid \overline{\ell}\langle X, S, S\rangle)$$

Hence, for hash requests we have $A_h^0 \xrightarrow{c_h(Y)} A_h^0 \mid A_{hi}^0$ and $A_h^1 \xrightarrow{c_h(Y)} A_h^1 \mid A_{hi}^1$; and for compression requests we have $A_f^1(S) \rightarrow \xrightarrow{c_f(X)} A_{fi}^1(X, S) \rightarrow^* A_f^1(S') \mid \overline{c'_f}\langle X' \rangle$ for some S' and X' with, initially, $A_f^1 = A_f^1(((0,0), \mathsf{nil}) :: \mathsf{nil})$.

Consider traces that interleave inputs $c_f(X_i)$ for $i \in I$, outputs $\nu x_i.\overline{c'_f}\langle x_i \rangle$ for $i \in I_{done} \subseteq I$, inputs $c_h(Y_i)$ for $i \in J$, and outputs $\nu h_i.\overline{c'_h}\langle h_i \rangle$ for $i \in J_{done} \subseteq J$, for some disjoint index sets I and J, such that variables x_j or h_j may occur in X_i or Y_i only when $j < i$. We let $\mathcal{R}$ be the smallest relation closed by reductions within $A_f^1(S)$ or $A_{fi}^1(X_{i_0}, S)$, but not in A_{his}^0 or A_{his}^1, such that

$$\nu k, \widetilde{x}.(A_h^0 \mid A_{his}^0 \mid A_f^0 \mid \sigma^0 \mid O) \;\; \mathcal{R} \;\; \nu k, \widetilde{x}.(A_h^1 \mid A_{his}^1 \mid A_f^1(S) \mid \sigma^1 \mid O)$$
$$\text{and } \nu k, \widetilde{x}.(A_h^0 \mid A_{his}^0 \mid A_f^0 \mid \sigma^0 \mid O) \;\; \mathcal{R} \;\; \nu k, \widetilde{x}.(A_h^1 \mid A_{his}^1 \mid A_{fi}^1(X_{i_0}, S) \mid \sigma^1 \mid O')$$

where the following conditions hold:

- $J = J_{done} \uplus J_{out} \uplus J_{fail} \uplus J_{test}$, $I = I_{done} \uplus I_{out} = I'_h \uplus I'_{alt}$ and, in the second case of the definition of $\mathcal{R}$, $i_0 \in I_{out}$ is the greatest index in J and I.
  Intuitively, J collects the indices of all hash requests processed so far, partitioned into J_{test}, for requests before the test $\mathsf{ne\_list}(Y_i) = \mathsf{true}$; J_{fail}, for requests after failing the test; J_{out}, for requests after passing the test but before the output; and J_{done}, for requests after passing the test and performing the output. And I collects the indices of all compression requests processed so far, partitioned into I_{out}, for requests before the output and I_{done}, for requests after the output; and also into I'_h, for requests that must be made consistent with the hash function, and I'_{alt}, for unrelated requests; i_0 is the index of the current compression request.
- $A_{his}^0 = \prod_{i \in J_{test}} A_{hi}^0(Y_i)$ and $A_{his}^1 = \prod_{i \in J_{test}} A_{hi}^1(Y_i)$.
  These processes represent requests before the test $\mathsf{ne\_list}(Y_i) = \mathsf{true}$.
- $\widetilde{x} = \{x_i \mid i \in I_{out}\} \cup \{h_i \mid i \in J_{out}\}$ are pairwise distinct variables, and the name k and the variables $\widetilde{x}$ do not occur in any $(X_i)_{i \in I}$ or $(Y_i)_{i \in J}$.
- $\mathsf{ne\_list}(Y_i) = \mathsf{true}$ for $i \in J_{done} \cup J_{out}$, and $\mathsf{ne\_list}(Y_i) \neq \mathsf{true}$ for $i \in J_{fail}$.
- $O = \prod_{i \in I_{out}} \overline{c'_f}\langle x_i \rangle \mid \prod_{i \in J_{out}} \overline{c'_h}\langle h_i \rangle$ and
  $O' = \prod_{i \in I_{out} \setminus \{i_0\}} \overline{c'_f}\langle x_i \rangle \mid \prod_{i \in J_{out}} \overline{c'_h}\langle h_i \rangle$.
  These parallel compositions represent pending request outputs, and each output transition consists of removing one message from O and one restriction on the corresponding variable in $\widetilde{x}$.
- S is (any list representation of) a finite map from pairs of blocks to lists of blocks that maps $(0,0)$ to nil and $(\mathsf{h}'(k, M), \mathsf{f}_c(k, M))$ to M for some lists $M = M_1 :: \ldots :: M_n :: \mathsf{nil}$ with $n > 0$. The range of S is prefix-closed, that is, if S maps a pair to $M +\!\!+ M'$, then it also maps a pair to M.
  The variables x_i and h_i do not occur in S.
  For every $i \in I$, S maps $\mathsf{fst}(X_i\sigma^1)$ to some list M if and only if $i \in I'_h$; then S also maps $x_i\sigma^1$ to $M +\!\!+ \mathsf{snd}(X_i\sigma^1)$, except when $i = i_0$ in the second case in the definition of $\mathcal{R}$.
- $\sigma^0 = \sigma_h^0 \mid \sigma_f^0 \mid \sigma_{fo}^0$ and $\sigma^1 = \sigma_h^1 \mid \sigma_f^1 \mid \sigma_{fo}^1$, where
  $\sigma_h^0 = \{({}^{\mathsf{h}(k, Y_i)}/_{h_i})_{i \in J_{done} \cup J_{out}}\}$ and $\sigma_h^1 = \{({}^{\mathsf{h}'(k, Y_i)}/_{h_i})_{i \in J_{done} \cup J_{out}}\}$,
  $\sigma_f^0 = \{({}^{\mathsf{f}(k, X_i)}/_{x_i})_{i \in I'_h}\}$ and $\sigma_f^1 = \{({}^{(\mathsf{h}'(k, Z'_i), \mathsf{f}_c(k, Z'_i))}/_{x_i})_{i \in I'_h}\}$ where S maps $\mathsf{fst}(X_i\sigma^1)$ to Z_i and Z'_i is $Z_i +\!\!+ \mathsf{snd}(X_i)$,
  $\sigma_{fo}^0 = \{({}^{\mathsf{f}(k, X_i)}/_{x_i})_{i \in I'_{alt}}\}$ and $\sigma_{fo}^1 = \{({}^{\mathsf{f}'(k, X_i)}/_{x_i})_{i \in I'_{alt}}\}$.

With σ^1 defined in the second case of $\mathcal{R}$, for instance, we have

$$A^1_{fi}(X_{i_0}\sigma^1, S) \to^* A^1_f((x_{i_0}\sigma^1, M +\!\!+ \mathsf{snd}(X_{i_0}\sigma^1)) :: S) \mid \overline{c'_f}\langle x_{i_0}\sigma^1\rangle$$

when S maps $\mathsf{fst}(X_{i_0}\sigma^1)$ to M, and $A^1_{fi}(X_{i_0}\sigma^1, S) \to^* A^1_f(S) \mid \overline{c'_f}\langle x_{i_0}\sigma^1\rangle$ otherwise. Hence, the second case of the definition of $\mathcal{R}$ reduces to the first one. However, the first case is useful for the initial case, and the second case is useful after inputs $c_f(X_i)$. Taking $S = ((0,0), \mathsf{nil}) :: \mathsf{nil}$ and $I = J = \emptyset$, the first case yields $\nu k.(A^0_h \mid A^0_f)\ \mathcal{R}\ \nu k.(A^1_h \mid A^1_f)$, so $\mathcal{R}$ includes our target observational equivalence. We show that $\mathcal{R} \cup \mathcal{R}^{-1}$ is a labelled bisimulation.

(1) We show that, if $A\ \mathcal{R}\ B$, then $A \approx_s B$. To this end, we prove the two properties below by induction on the number of variables in the domain of σ^0 and σ^1.

P1. $\nu k.\sigma^0 \approx_s \nu k.\sigma^1$ and

P2. if $M = M_1 :: \ldots :: M_n :: \mathsf{nil}$ for some $n \geq 1$ contains neither k nor the variable with greatest index in $dom(\sigma^0) = dom(\sigma^1)$, then for all $i \in I$ we have $\Sigma \vdash \mathsf{fst}(x_i\sigma^0) = \mathsf{h}(k, M\sigma^0) \iff \Sigma \vdash \mathsf{fst}(x_i\sigma^1) = \mathsf{h}'(k, M\sigma^1)$.

For all $i \in I$, $x_i\sigma^0 = \mathsf{f}(k, X_i\sigma^0)$ and for all $i \in J_{done} \cup J_{out}$, $h_i\sigma^0 = \mathsf{fst}(\mathsf{f}(k, M))$ for some term M. Hence, by Corollary G.2,

$$\nu k.\sigma^0 \approx_s \nu \widetilde{a}.\{(^{x_i\sigma_0}/_{x_i})_{i\in I}, (^{\mathsf{fst}(h_i\sigma_0)}/_{h_i})_{i\in J_{done}\cup J_{out}}\}$$

where the following conditions hold:

- $x_i\sigma_0$ for $i \in I$ and $h_i\sigma_0$ for $i \in J_{done} \cup J_{out}$ are names in $\widetilde{a}$.
- For all $i, j \in I$, $x_i\sigma_0 = x_j\sigma_0$ if and only if $\Sigma \vdash \mathsf{f}(k, X_i\sigma^0) = \mathsf{f}(k, X_j\sigma^0)$, that is, $\Sigma \vdash X_i\sigma^0 = X_j\sigma^0$.
- For all $i, j \in J_{done} \cup J_{out}$, $h_i\sigma_0 = h_j\sigma_0$ if and only if $\Sigma \vdash \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n)) = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M'_1)) \ldots, M'_{n'}))$ where $Y_i\sigma^0 = M_1 :: \ldots :: M_n :: \mathsf{nil}$ and $Y_j\sigma^0 = M'_1 :: \ldots :: M'_n :: \mathsf{nil}$, that is, $\Sigma \vdash Y_i\sigma^0 = Y_j\sigma^0$, by Lemma G.5.
- For all $i \in I$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_0 = h_j\sigma_0$ if and only if $\Sigma \vdash \mathsf{f}(k, X_i\sigma^0) = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n))$ where $Y_j\sigma^0 = M_1 :: \ldots :: M_n :: \mathsf{nil}$. In this case, we have $\Sigma \vdash \mathsf{fst}(x_i\sigma^0) = \mathsf{fst}(\mathsf{f}(k, X_i\sigma^0)) = \mathsf{fst}(\mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n))) = \mathsf{h}(k, Y_j\sigma^0)$. Conversely, if $\Sigma \vdash \mathsf{fst}(x_i\sigma^0) = \mathsf{h}(k, Y_j\sigma^0)$, then $\Sigma \vdash \mathsf{fst}(\mathsf{f}(k, X_i\sigma^0)) = \mathsf{fst}(\mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n)))$. Since these terms do not reduce at the root under the rewrite system R of Lemma G.4, we have $\Sigma \vdash \mathsf{f}(k, X_i\sigma^0) = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n))$. Therefore, $x_i\sigma_0 = h_j\sigma_0$ if and only if $\Sigma \vdash \mathsf{fst}(x_i\sigma^0) = \mathsf{h}(k, Y_j\sigma^0)$.

By Lemma G.3, we can replace the names $x_i\sigma_0$ and $h_i\sigma_0$ with pairs $(x_i\sigma_1, x_i\sigma_2)$ and $(h_i\sigma_1, h_i\sigma_2)$ respectively. Thus

$$\nu k.\sigma^0 \approx_s \nu \widetilde{a}'.\{(^{(x_i\sigma_1, x_i\sigma_2)}/_{x_i})_{i\in I}, (^{h_i\sigma_1}/_{h_i})_{i\in J_{done}\cup J_{out}}\}$$

where the following conditions hold:

- $x_i\sigma_1, x_i\sigma_2$ for $i \in I$ and $h_i\sigma_1$ for $i \in J_{done} \cup J_{out}$ are names in $\widetilde{a}'$.
- For all $i, j \in I$, $x_i\sigma_1 \neq x_j\sigma_2$.
- For all $i \in I$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_2 \neq h_j\sigma_1$.
- For all $i, j \in I$, $x_i\sigma_1 = x_j\sigma_1 \iff x_i\sigma_2 = x_j\sigma_2 \iff \Sigma \vdash X_i\sigma^0 = X_j\sigma^0$.
- For all $i, j \in J_{done} \cup J_{out}$, $h_i\sigma_1 = h_j\sigma_1 \iff \Sigma \vdash Y_i\sigma^0 = Y_j\sigma^0$.
- For all $i \in I$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_1 = h_j\sigma_1 \iff \Sigma \vdash \mathsf{fst}(x_i\sigma^0) = \mathsf{h}(k, Y_j\sigma^0)$.

For all $i \in I'_{alt}$, $x_i\sigma^1 = \mathsf{f}'(k, X_i\sigma^1)$, for all $i \in I'_h$, $x_i\sigma^1 = (\mathsf{h}'(k, Z'_i\sigma^1), \mathsf{f}_c(k, Z'_i\sigma^1))$, and for all $i \in J_{done} \cup J_{out}$, $h_i\sigma^1 = \mathsf{h}'(k, Y_i\sigma^1)$. Hence, by Corollary G.2,

$$\nu k.\sigma^1 \approx_s \nu \widetilde{a}.\{(^{x_i\sigma_3}/_{x_i})_{i\in I'_{alt}}, (^{(x_i\sigma_4, x_i\sigma_5)}/_{x_i})_{i\in I'_h}, (^{h_i\sigma_4}/_{h_i})_{i\in J_{done}\cup J_{out}}\}$$

where the following conditions hold:

- $x_i\sigma_3$ for $i \in I'_{alt}$, $x_i\sigma_4$ and $x_i\sigma_5$ for $i \in I'_h$, and $h_i\sigma_4$ for $i \in J_{done} \cup J_{out}$ are names in $\widetilde{a}$.
- For all $i, j \in I'_h$, $x_i\sigma_4 \neq x_j\sigma_5$.
- For all $i \in I'_{alt}$ and $j \in I'_h$, $x_i\sigma_3 \neq x_j\sigma_4$ and $x_i\sigma_3 \neq x_j\sigma_5$.
- For all $i \in I'_{alt}$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_3 \neq h_j\sigma_4$.
- For all $i \in I'_h$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_5 \neq h_j\sigma_4$.
- For all $i, j \in I'_{alt}$, $x_i\sigma_3 = x_j\sigma_3 \iff \Sigma \vdash \mathsf{f}'(k, X_i\sigma^1) = \mathsf{f}'(k, X_j\sigma^1) \iff \Sigma \vdash X_i\sigma^1 = X_j\sigma^1$.
- For all $i, j \in J_{done} \cup J_{out}$, $h_i\sigma_4 = h_j\sigma_4 \iff \Sigma \vdash Y_i\sigma^1 = Y_j\sigma^1$.
- For all $i \in I'_h$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_4 = h_j\sigma_4 \iff \Sigma \vdash \mathsf{fst}(x_i\sigma^1) = \mathsf{h}'(k, Y_j\sigma^1)$.
- For all $i, j \in I'_h$, $x_i\sigma_4 = x_j\sigma_4 \iff x_i\sigma_5 = x_j\sigma_5 \iff \Sigma \vdash Z'_i\sigma^1 = Z'_j\sigma^1$. In this case, by definition of Z'_i, $\Sigma \vdash Z_i\sigma^1 = Z_j\sigma^1$ and $\Sigma \vdash \mathsf{snd}(X_i\sigma^1) = \mathsf{snd}(X_j\sigma^1)$. Since S maps $\mathsf{fst}(X_i\sigma^1)$ to Z_i and $\mathsf{fst}(X_j\sigma^1)$ to Z_j, we have $Z_i = Z_i\sigma^1$ and $Z_j = Z_j\sigma^1$, so either $\Sigma \vdash Z_i = Z_j = \mathsf{nil}$ and $\Sigma \vdash \mathsf{fst}(X_i\sigma^1) = (0,0) = \mathsf{fst}(X_j\sigma^1)$ or $\Sigma \vdash Z_i = Z_j \neq \mathsf{nil}$ and $\Sigma \vdash \mathsf{fst}(X_i\sigma^1) = (\mathsf{h}'(k, Z_i), \mathsf{f}_c(k, Z_i)) = (\mathsf{h}'(k, Z_j), \mathsf{f}_c(k, Z_j)) = \mathsf{fst}(X_j\sigma^1)$. So in both cases, $\Sigma \vdash X_i\sigma^1 = X_j\sigma^1$. Conversely, if $\Sigma \vdash X_i\sigma^1 = X_j\sigma^1$, then $\Sigma \vdash Z'_i\sigma^1 = Z'_j\sigma^1$, since Z'_i is computed from X_i. Therefore, for all $i, j \in I'_h$, $x_i\sigma_4 = x_j\sigma_4 \iff x_i\sigma_5 = x_j\sigma_5 \iff \Sigma \vdash X_i\sigma^1 = X_j\sigma^1$.

By Lemma G.3, we can replace the names $x_i\sigma_3$ for $i \in I'_{alt}$ with pairs $(x_i\sigma_4, x_i\sigma_5)$. Thus $\nu k.\sigma^1 \approx_s \nu\widetilde{a}'.\{(^{(x_i\sigma_4, x_i\sigma_5)}/_{x_i})_{i\in I}, (^{h_i\sigma_4}/_{h_i})_{i\in J_{done}\cup J_{out}}\}$ where the following conditions hold:

- $x_i\sigma_4$ and $x_i\sigma_5$ for $i \in I$, and $h_i\sigma_4$ for $i \in J_{done} \cup J_{out}$ are names in $\widetilde{a}'$.
- For all $i, j \in I$, $x_i\sigma_4 \neq x_j\sigma_5$.
- For all $i \in I$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_5 \neq h_j\sigma_4$.
- For all $i, j \in I$, $x_i\sigma_4 = x_j\sigma_4 \iff x_i\sigma_5 = x_j\sigma_5 \iff \Sigma \vdash X_i\sigma^1 = X_j\sigma^1$. Indeed, when $i \in I'_h$ and $j \in I'_{alt}$, we have $x_i\sigma_4 \neq x_j\sigma_4$, $x_i\sigma_5 \neq x_j\sigma_5$, and $\Sigma \vdash X_i\sigma^1 \neq X_j\sigma^1$ since $\mathsf{fst}(X_i\sigma^1)$ and $\mathsf{fst}(X_j\sigma^1)$ are not mapped to the same value by S. When i and j are both in I'_h or both in I'_{alt}, the result comes from the equivalences shown above.
- For all $i, j \in J_{done} \cup J_{out}$, $h_i\sigma_4 = h_j\sigma_4 \iff \Sigma \vdash Y_i\sigma^1 = Y_j\sigma^1$.
- For all $i \in I$ and $j \in J_{done} \cup J_{out}$, $x_i\sigma_4 = h_j\sigma_4 \iff \Sigma \vdash \mathsf{fst}(x_i\sigma^1) = \mathsf{h}'(k, Y_j\sigma^1)$. Indeed, if $i \in I'_{alt}$, we have $x_i\sigma_4 \neq h_j\sigma_4$ and $\Sigma \vdash \mathsf{fst}(x_i\sigma^1) \neq \mathsf{h}'(k, Y_j\sigma^1)$. When $i \in I'_h$, the result comes from an equivalence shown above.

Since X_i and Y_i contain variables x_j and h_j only with $j < i$, the variable of $dom(\sigma^0)$ with the greatest index does not occur in X_i and Y_i, so by induction hypothesis, we have $\Sigma \vdash X_i\sigma^0 = X_j\sigma^0 \iff \Sigma \vdash X_i\sigma^1 = X_j\sigma^1$ and $\Sigma \vdash Y_i\sigma^0 = Y_j\sigma^0 \iff \Sigma \vdash Y_i\sigma^1 = Y_j\sigma^1$, so it suffices to show property P2 to obtain $\nu k.\sigma^0 \approx_s \nu k.\sigma^1$.

Let us now show property P2, by induction on n. Let $M = M_1 :: \ldots :: M_n :: \mathsf{nil}$ be a term that does not contain k nor the variable with greatest index in $dom(\sigma^0) = dom(\sigma^1)$, $n \geq 1$, and $i \in I$.

Suppose that $\Sigma \vdash \mathsf{fst}(x_i\sigma^0) = \mathsf{h}(k, M\sigma^0)$. We have

$$\mathsf{fst}(x_i\sigma^0) = \mathsf{fst}(\mathsf{f}(k, X_i\sigma^0))$$
$$\mathsf{h}(k, M\sigma^0) = \mathsf{fst}(\mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n)))\sigma^0$$

so

$$\Sigma \vdash X_i\sigma^0 = (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_n)\sigma^0$$

If $n = 1$, we have $\Sigma \vdash X_i\sigma^0 = ((0,0), M_n\sigma^0)$. Since X_i and M_n do not contain the variable of $dom(\sigma^0)$ with the greatest index, we have $\Sigma \vdash X_i\sigma^1 = ((0,0), M_n\sigma^1)$ by induction hypothesis, so S maps $\mathsf{fst}(X_i\sigma^1) = (0,0)$ to $Z_i = \mathsf{nil}$, hence $Z'_i = Z_i +\!\!+ \mathsf{snd}(X_i)$, so $\Sigma \vdash Z'_i\sigma^1 = \mathsf{nil} +\!\!+ M_n\sigma^1 = M_n\sigma^1 :: \mathsf{nil} = M\sigma^1$ and $\Sigma \vdash \mathsf{fst}(x_i\sigma^1) = \mathsf{h}'(k, Z'_i\sigma^1) = \mathsf{h}'(k, M\sigma^1)$.

If $n > 1$, let $M' = M_1 :: \ldots :: M_{n-1} :: \mathsf{nil}$ and $H = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_{n-1}))$. We have $\Sigma \vdash X_i\sigma^0 = (H, M_n)\sigma^0$. Since k does not occur in X_i and $X_i\sigma^0$ is of the form $(H\sigma^0, M_n\sigma^0) = (\mathsf{f}(k, \cdot), \cdot)$, there exists $j_0 \in I$ such that $\Sigma \vdash H\sigma^0 = x_{j_0}\sigma^0$ and x_{j_0} occurs in X_i, so $j_0 < i$. Thus $\Sigma \vdash \mathsf{fst}(x_{j_0}\sigma^0) = \mathsf{fst}(H\sigma^0) = \mathsf{h}(k, M'\sigma^0)$, so by induction hypothesis,

$$\Sigma \vdash \mathsf{fst}(x_{j_0}\sigma^1) = \mathsf{h}'(k, M'\sigma^1)$$

By construction of σ^1, we have

$$\Sigma \vdash \mathsf{snd}(x_{j_0}\sigma^1) = \mathsf{f}_c(k, M'\sigma^1)$$

Moreover, we have $\Sigma \vdash X_i\sigma^0 = (x_{j_0}, M_n)\sigma^0$ and X_i, x_{j_0}, and M_n do not contain the variable of $dom(\sigma^0)$ with the greatest index, so we have $\Sigma \vdash X_i\sigma^1 = (x_{j_0}, M_n)\sigma^1$ by induction hypothesis. Hence $\Sigma \vdash \mathsf{fst}(X_i\sigma^1) = x_{j_0}\sigma^1 = (\mathsf{h}'(k, M'\sigma^1), \mathsf{f}_c(k, M'\sigma^1))$. Hence S maps $\mathsf{fst}(X_i\sigma^1)$ to Z_i such that $\Sigma \vdash Z_i = M'\sigma^1$, so $\Sigma \vdash Z_i'\sigma^1 = Z_i +\!\!+ \mathsf{snd}(X_i)\sigma^1 = M\sigma^1$, so $\Sigma \vdash \mathsf{fst}(x_i\sigma^1) = \mathsf{h}'(k, Z_i'\sigma^1) = \mathsf{h}'(k, M\sigma^1)$.

Conversely, suppose that $\Sigma \vdash \mathsf{fst}(x_i\sigma^1) = \mathsf{h}'(k, M\sigma^1)$. Thus $i \in I_h'$ and $\Sigma \vdash Z_i'\sigma^1 = M\sigma^1$. So S maps $\mathsf{fst}(X_i\sigma^1)$ to some Z_i.

If $Z_i = \mathsf{nil}$, then $\Sigma \vdash \mathsf{fst}(X_i\sigma^1) = (0,0)$. Since X_i does not contain the variable of $dom(\sigma^0)$ with the greatest index, we have $\Sigma \vdash \mathsf{fst}(X_i\sigma^0) = (0,0)$ by induction hypothesis. Moreover $Z_i' = Z_i +\!\!+ \mathsf{snd}(X_i)$, so $\Sigma \vdash M\sigma^1 = Z_i'\sigma^1 = \mathsf{snd}(X_i\sigma^1) :: \mathsf{nil}$. Since M and X_i do not contain the variable of $dom(\sigma^0)$ with the greatest index, we have $\Sigma \vdash M\sigma^0 = \mathsf{snd}(X_i\sigma^0) :: \mathsf{nil}$ by induction hypothesis. We obtain $\Sigma \vdash \mathsf{h}(k, M\sigma^0) = \mathsf{fst}(\mathsf{f}(k, ((0,0), \mathsf{snd}(X_i\sigma^0)))) = \mathsf{fst}(\mathsf{f}(k, X_i\sigma^0)) = \mathsf{fst}(x_i\sigma^0)$.

If $Z_i \neq \mathsf{nil}$, then $\Sigma \vdash \mathsf{fst}(X_i\sigma^1) = (\mathsf{h}'(k, Z_i), \mathsf{f}_c(k, Z_i))$ and $Z_i' = Z_i +\!\!+ \mathsf{snd}(X_i)$. Since $\Sigma \vdash Z_i'\sigma^1 = M\sigma^1$, we have

$$\Sigma \vdash Z_i = (M_1 :: \ldots :: M_{n-1} :: \mathsf{nil})\sigma^1$$
$$\Sigma \vdash \mathsf{snd}(X_i\sigma^1) = M_n\sigma^1$$

Since k does not occur in X_i, there exists $j_0 \in I$ such that $\Sigma \vdash x_{j_0}\sigma^1 = (\mathsf{h}'(k, Z_i), \mathsf{f}_c(k, Z_i)) = \mathsf{fst}(X_i\sigma^1)$. Hence

$$\Sigma \vdash \mathsf{fst}(x_{j_0}\sigma^1) = \mathsf{h}'(k, Z_i) = \mathsf{h}'(k, (M_1 :: \ldots :: M_{n-1} :: \mathsf{nil})\sigma^1)$$

By induction hypothesis,

$$\begin{aligned}\Sigma \vdash \mathsf{fst}(x_{j_0}\sigma^0) &= \mathsf{h}(k, (M_1 :: \ldots :: M_{n-1} :: \mathsf{nil})\sigma^0) \\ &= \mathsf{fst}(\mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_{n-1}))\sigma^0)\end{aligned}$$

Since $x_{j_0}\sigma^0$ is of the form $\mathsf{f}(k, \cdot)$, we obtain

$$\Sigma \vdash x_{j_0}\sigma^0 = \mathsf{f}(k, (\ldots \mathsf{f}(k, ((0,0), M_1)) \ldots, M_{n-1}))\sigma^0$$

We have $\Sigma \vdash \mathsf{fst}(X_i\sigma^1) = x_{j_0}\sigma^1$ and $\Sigma \vdash \mathsf{snd}(X_i\sigma^1) = M_n\sigma^1$, so $\Sigma \vdash X_i\sigma^1 = (x_{j_0}\sigma^1, M_n\sigma^1)$ Since X_i, x_{j_0}, and M_n do not contain the variable of $dom(\sigma^0)$ with the greatest index, we have $\Sigma \vdash X_i\sigma^0 = (x_{j_0}\sigma^0, M_n\sigma^0)$ by induction hypothesis. So $\Sigma \vdash \mathsf{fst}(x_i\sigma^0) = \mathsf{fst}(\mathsf{f}(k, X_i\sigma^0)) = \mathsf{fst}(\mathsf{f}(k, (x_{j_0}\sigma^0, M_n\sigma^0))) = \mathsf{h}(k, M\sigma^0)$.

(2) We first show that, if $A \mathcal{R} B$, $A \xrightarrow{\alpha} A'$, A' is closed, and $fv(\alpha) \subseteq dom(A)$, then $B \to^* \xrightarrow{\alpha} \to^* B'$ and $A' \mathcal{R} B'$ for some B'. The only possible labelled transitions in A are as follows:

- A_h^0 performs an input with label $\alpha = c_h(Y_i)$, with $fv(Y_i) \subseteq dom(\sigma^0) \setminus \{\widetilde{x}\}$, creating a process $A_{hi}^0(Y_i)$. The process A_h^1 can perform the same input, creating a process $A_{hi}^1(Y_i)$. The resulting extended processes are still in $\mathcal{R}$, by adding to J_{test} an index i greater than those already in I and J.

- A^0_f performs an input with label $\alpha = c_f(X_i)$, with $fv(X_i) \subseteq dom(\sigma^0) \setminus \{\widetilde{x}\}$, creating a process $\overline{c'_f}\langle \mathsf{f}(k, X_i)\rangle \equiv \nu x_i.(\overline{c'_f}\langle x_i\rangle \mid \{^{\mathsf{f}(k,X_i)}/_{x_i}\})$. A reduced form of $A^1_f(S)$ or $A^1_{fi}(X_{i_0}, S)$ can perform the same input (possibly after internal reductions). We keep performing internal reductions after the input, until the output on c'_f is enabled. Hence a new process

$$\overline{c'_f}\langle \mathsf{f}'(k, X_i)\rangle \equiv \nu x_i.(\overline{c'_f}\langle x_i\rangle \mid \{^{\mathsf{f}'(k,X_i)}/_{x_i}\})$$

  or

$$\overline{c'_f}\langle (\mathsf{h}'(k, Z'_i), \mathsf{f}_c(k, Z'_i))\rangle \equiv \nu x_i.(\overline{c'_f}\langle x_i\rangle \mid \{^{(\mathsf{h}'(k,Z'_i),\mathsf{f}_c(k,Z'_i))}/_{x_i}\})$$

  appears, depending on whether S maps $\mathrm{fst}(X_i)$ to some Z_i or not, and for Z'_i given in the definition of $\mathcal{R}$. The resulting extended processes are still in $\mathcal{R}$, by adding to I_{out} an index greater than those already in I and J.
- O performs an output with label $\alpha = \nu h_i.\overline{c'_h}\langle h_i\rangle$. (We arrange that the bound variable of α has the same name as the variable used internally by the output that we perform.) In this case, h_i is removed from $\widetilde{x}$ and $\overline{c'_h}\langle h_i\rangle$ is removed from O. The process O can perform the same output on the right-hand side, hence we remain in $\mathcal{R}$ by moving the index i from J_{out} to J_{done}.
- O performs an output with label $\alpha = \nu x_i.\overline{c'_f}\langle x_i\rangle$. In this case, x_i is removed from $\widetilde{x}$ and $\overline{c'_f}\langle x_i\rangle$ is removed from O. If we are in the second case of the definition of $\mathcal{R}$ with $i = i_0$, we first reduce $A^1_{fi}(X_{i_0}, S)$ until we arrive at the first case of the definition of $\mathcal{R}$. The process O can then perform the same output on the right-hand side, hence we remain in $\mathcal{R}$ by moving the index i from I_{out} to I_{done}.

A detailed proof that these are the only possible labelled transitions of A uses the partial normal forms introduced in Appendix B and the decomposition lemmas proved in Appendix B.4. This comment also applies to other case distinctions below in the proof of Theorem 6.3.

Conversely, we show that, if $A \;\mathcal{R}\; B$, $B \xrightarrow{\alpha} B'$, B' is closed, and $fv(\alpha) \subseteq dom(B)$, then $A \rightarrow^* \xrightarrow{\alpha} \rightarrow^* A'$ and $A' \;\mathcal{R}\; B'$ for some A'. The only possible labelled transitions in B are as follows:

- A^1_h performs an input with label $\alpha = c_h(Y_i)$, with $fv(Y_i) \subseteq dom(\sigma^1) \setminus \{\widetilde{x}\}$. The process A^1_h can perform the same input, and we remain in $\mathcal{R}$ by adding to J_{test} an index i greater than those already in I and J.
- (A reduced form of) $A^1_f(S)$ performs an input with label $\alpha = c_f(X_i)$, with $fv(Y_i) \subseteq dom(\sigma^1) \setminus \{\widetilde{x}\}$. After the input, $A^1_f(S)$ is transformed into the process $A^1_{fi}(X_i, S)$. The process A^0_f can perform the same input. A new process $\overline{c'_f}\langle \mathsf{f}(k, X_i)\rangle \equiv \nu x_i.(\overline{c'_f}\langle x_i\rangle \mid \{^{\mathsf{f}(k,X_i)}/_{x_i}\})$ appears on left-hand side. We remain in $\mathcal{R}$ by adding to I_{out} an index i_0 greater than those already in $I \cup J$. (On the right-hand side, the variable x_{i_0} is defined but not used.)
  When a reduced form of $A^1_{fi}(X_{i_0}, S)$ performs an input with label $\alpha = c_f(X_i)$, $A^1_{fi}(X_{i_0}, S)$ has first been reduced so that the configuration is in the first case of the definition of $\mathcal{R}$, with the considered input in $A^1_f(S)$, so this case is already treated above.
- O performs an output with label $\alpha = \nu h_i.\overline{c'_h}\langle h_i\rangle$. The process O performs the same output on the left-hand side and we remain in $\mathcal{R}$ by moving the index i from J_{out} to J_{done}.
- O performs an output with label $\alpha = \nu x_i.\overline{c'_f}\langle x_i\rangle$. The process O performs the same output on the left-hand side, and we remain in $\mathcal{R}$, by moving the index i from I_{out} to I_{done}.

When a reduced form of $A^1_{fi}(X_{i_0}, S)$ performs an output with label $\alpha = \nu x_i.\overline{c'_f}\langle x_i\rangle$, $A^1_{fi}(X_{i_0}, S)$ has first been reduced so that the configuration is in the first case of the definition of $\mathcal{R}$, with the considered output included in O, so this case is already treated above.

(3) We first show that, if $A\ \mathcal{R}\ B$, $A \to A'$, and A' is closed, then $B \to^* B'$ and $A'\ \mathcal{R}\ B'$ for some B'. The only processes that can be reduced in A are processes $A^0_{hi}(Y_i)$ inside A^0_{his}. If $\mathsf{ne\_list}(Y_i) = \mathsf{true}$, then $A^0_{hi}(Y_i)$ reduces to

$$\overline{c'_h}\langle \mathsf{h}(k, Y_i)\rangle \equiv \nu h_i.(\overline{c'_h}\langle h_i\rangle \mid \{^{\mathsf{h}(k,Y_i)}/_{h_i}\})$$

and similarly $A^1_{hi}(Y_i)$ reduces to

$$\overline{c'_h}\langle \mathsf{h}'(k, Y_i)\rangle \equiv \nu h_i.(\overline{c'_h}\langle h_i\rangle \mid \{^{\mathsf{h}'(k,Y_i)}/_{h_i}\})$$

and we remain in $\mathcal{R}$ by moving i from J_{test} to J_{out}. (The value of $\mathsf{ne\_list}(Y_i)$ remains unchanged when we instantiate Y_i with σ^0 or σ^1 because the image of these substitutions does not contain lists.) If $\mathsf{ne\_list}(Y_i) \neq \mathsf{true}$, then $A^0_{hi}(Y_i)$ reduces to $\mathbf{0}$ and similarly $A^1_{hi}(Y_i)$ reduces to $\mathbf{0}$, and we remain in $\mathcal{R}$ by moving i from J_{test} to J_{fail}.

Conversely, we show that, if $A\ \mathcal{R}\ B$, $B \to B'$, and B' is closed, then $A \to^* A'$ and $A'\ \mathcal{R}\ B'$ for some A'.

The only reductions in B are due to processes $A^1_{hi}(Y_i)$ within A^1_{his}, and $A^1_f(S)$ or $A^1_{fi}(X_{i_0}, S)$. The first case can be handled similarly to the case in which $A^0_{hi}(Y_i)$ reduces. In the second case, we remain in $\mathcal{R}$ with $A' = A$.

Therefore, $\mathcal{R} \subseteq \approx_l$. By Theorem 4.8, $\mathcal{R} \subseteq \approx$. So $\nu k.(A^0_h \mid A^0_f) \approx \nu k.(A^1_h \mid A^1_f)$. □

ACKNOWLEDGMENTS

We thank Rocco De Nicola, Andy Gordon, Tony Hoare, and Phil Rogaway for discussions that contributed to this work. Georges Gonthier and Jan Jürjens suggested improvements to the presentation of the conference version of this paper. Steve Kremer and Ben Smyth provided helpful comments on a draft of this paper.

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Received October 2015; revised July 2017; accepted July 2017